Inverse scattering solution of the weak noise theory of the Kardar-Parisi-Zhang equation with flat and Brownian initial conditions

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We present the solution of the weak noise theory (WNT) for the Kardar-Parisi-Zhang equation in one dimension at short time for flat initial condition (IC). The nonlinear hydrodynamic equations of the WNT are solved analytically through a connection to the Zakharov-Shabat (ZS) system using its classical integrability. This approach is based on a recently developed Fredholm determinant framework previously applied to the droplet IC. The flat IC provides the case for a nonvanishing boundary condition of the ZS system and yields a richer solitonic structure comprising the appearance of multiple branches of the Lambert function. As a byproduct, we obtain the explicit solution of the WNT for the Brownian IC, which undergoes a dynamical phase transition. We elucidate its mechanism by showing that the related spontaneous breaking of the spatial symmetry arises from the interplay between two solitons with different rapidities.

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I. INTRODUCTION AND AIM OF THE PAPER

Nonlinear stochastic equations are a central tool in nonequilibrium physics [1]. They are often studied using optimal fluctuation theory and instanton methods [2–4]. This usually amounts to performing a saddle point evaluation on the action of the associated dynamical field theory [5,6]. In the favorable situations this approximation is controlled by a small parameter. This is often the case when describing rare large fluctuations, i.e., large deviations [7,8]. The resulting saddle point equations are typically a set of coupled nonlinear equations, which can only be solved in some special limits, or numerically. It is rare that there are exact solutions, and even more remarkable when this set of equations is fully integrable.

Recently we showed [9] that the saddle point equations which describe the large deviations at short time for the Kardar-Parisi-Zhang (KPZ) stochastic growth equation in one space dimension, the so-called weak noise theory (WNT) [10–21], can be solved exactly. As noted in Ref. [13], the basic system of nonlinear equations is the so-called $\{P, Q\}$ system, a cousin of the nonlinear Schrödinger equation. Using inverse scattering methods coupled to a recently developed Fredholm determinant framework [22,23], we showed how to construct general solutions of this system and obtained an explicit solution for the so-called droplet initial condition (IC) which is localized in space and decays at infinity. In this paper we extend the method and present solutions in the case of initial conditions which are nonvanishing at infinity. We first treat the case of the flat IC for the KPZ equation, from which, in a second stage, we obtain the solution for the Brownian IC.

The KPZ equation [24] describes the stochastic growth in time τ of the height field $h(y, \tau)$ of an interface, here in one space dimension $y \in \mathbb{R}$:

$$\partial_\tau h(y, \tau) = \partial_y^2 h(y, \tau) + (\partial_y h(y, \tau))^2 + \sqrt{2}\eta(y, \tau), \tag{1}$$

where $\eta(y, \tau)$ is a standard space-time white noise, i.e., $\overline{\eta(y, \tau)\eta(y', \tau')} = \delta(\tau - \tau')\delta(y - y')$. In this work we first focus on the solution of Eq. (1) with the flat IC,

$$h(y, \tau = 0) = 0. \tag{2}$$

Because of the nonlinear term in Eq. (1), the growth at late times belongs to a different universality class (the so-called KPZ class) than its simpler version, the Edwards-Wilkinson equation (without the nonlinear term). Interestingly, this nonlinearity has a profound effect already at short time, not for the typical height fluctuations, which are Gaussian with Edwards-Wilkinson scaling $\delta h \sim \tau^{1/4}$, but for the rare but much larger fluctuations $\delta h = \mathcal{O}(1)$. For example, the probability $P(H, T)$ to observe the value of the field $h(0, T) = H$ at some time $\tau = T$ takes, for $T \ll 1$ and $H = \mathcal{O}(1)$, the following large deviation form:

$$P(H, T) \sim \exp(-\Phi(H)/\sqrt{T}). \tag{3}$$

The rate function $\Phi(H)$ was obtained analytically in a few cases where Bethe ansatz solutions of the KPZ equation are available [25–31] and confirmed numerically through importance sampling simulations in Refs. [32,33]. In our previous work [9] we showed how to obtain $\Phi(H)$ by solving exactly the weak noise theory equations. This is a completely different route, which until now was limited to numerical or asymptotic

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solutions [10–20]. Being classically integrable, the $\{P, Q\}$ system has an infinite number of conserved quantities, and we showed that $\Phi(H)$ is obtained from one of them. Solving the full equations gives much more information beyond the rate function, since it determines the exact "optimal" KPZ height and noise space-time fields producing the rare fluctuations. Here we obtain these fields for the flat and Brownian IC as well as the rate functions. We follow the same outline as in Ref. [9] and indicate how the present case differs in crucial ways.

Let us recall how the $\{P, Q\}$ system arises. It is more convenient to work with the exponential field $Z = e^{h(y,\tau)}$. It is also equal to the partition sum of a directed polymer $x(\tau)$ at equilibrium in a random potential $\eta(x(\tau), \tau)$ (the KPZ noise) in dimension $d = 1 + 1$. The equivalence of the two problems is quite convenient, e.g., for numerical simulations [32–35]. We have introduced the rescaled time and space variables as $t = \tau/T$, $x = y/\sqrt{T}$, where T, the observation time, is fixed [36]. The field $Z(x, t)$, expressed in these coordinates, satisfies the (rescaled) stochastic heat equation (SHE) in the Ito sense,

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \sqrt{2}T^{1/4}\tilde{\eta}(x,t)Z(x,t). \tag{4}$$

Here $\tilde{\eta}(x, t)$ is another standard space-time white noise. This equation is now studied for $t \in [0, 1]$. The noise amplitude being now $\mathcal{O}(T^{1/4})$, a short observation time $T \ll 1$ corresponds to a weak noise. As in Refs. [25–31] and in Ref. [9], it is convenient to study the following generating function which admits a large deviation principle at short time $T \ll 1$, with $z \geqslant 0$:

$$\overline{\exp(-ze^H/\sqrt{T})} \sim \exp(-\Psi(z)/\sqrt{T}). \tag{5}$$

Inserting Eq. (3) into the expectation value over $P(H, T)$ in the left-hand side (lhs), we see that for $T \ll 1$, $\Psi(z)$ and $\Phi(H)$ are related through a Legendre transform:

$$\Psi(z) = \min_H (ze^H + \Phi(H)). \tag{6}$$

As detailed in Ref. [9], in the short time limit $T \ll 1$ the expectation value (5) over the stochastic dynamics (4) can be obtained from saddle point equations, which take the form of the $\{P, Q\}_g$ system,

$$\begin{aligned} \partial_t Q &= \partial_x^2 Q + 2gPQ^2, \\ -\partial_t P &= \partial_x^2 P + 2gP^2 Q. \end{aligned} \tag{7}$$

These equations for $P(x, t)$ and $Q(x, t)$ must be solved for $x \in \mathbb{R}$ and $t \in [0, 1]$ with mixed boundary conditions, which for the flat IC read

$$Q(x,0) = 1, \quad P(x,1) = \delta(x), \tag{8}$$

and with the coupling set to $g = -z$. The new feature, as compared to Ref. [9], is that $Q(x, t) \to 1$ for $x \to \pm\infty$. The function P, however, as well as the product PQ, still decay at infinity. The solution of Eqs. (7) and (8) determines the optimal height via $Z_{\rm opt}(x, t) = e^{h_{\rm opt}(y,\tau)} = Q(x, t)$ while the optimal noise is $\tilde{\eta}_{\rm opt}(x, t) = P(x, t)Q(x, t)$. As in Ref. [9] we will calculate from the solution the value $C_1(g)$ of the first conserved quantity, $C_1 = g\int_{\mathbb{R}} dxPQ$, and from $C_1(g)$ obtain the rate function $\Psi(z)$. Indeed C_1 being time independent, at $t = 1$ one has $C_1(g) = gQ(0, 1) = ge^H$. On the other hand, differentiating the Legendre transform in Eq. (6) with respect to z gives $\Psi'(z) = e^H$. Since $g = -z$ this gives $C_1(-z) = -z\Psi'(z)$, from which we obtain $\Psi(z)$ by integration and, in a second stage, $\Phi(H)$ by Legendre inversion of Eq. (6).

II. SCATTERING APPROACH TO THE LARGE DEVIATIONS

A. Setting up the scattering problem

As in Ref. [9], to solve the nonlinear system (7) and (8), one proceeds in two stages: the direct and the inverse scattering problems. First one studies an auxiliary scattering problem [37], in which the scattering amplitudes obey a linear time evolution and exhibit a very simple time dependence. In a second stage, from these scattering amplitudes, one constructs the solution of Eqs. (7) and (8). The $\{P, Q\}_g$ system belongs to the Ablowitz-Kaup-Newell-Segur (AKNS) class [38], for which there exists a Lax pair, i.e., a pair of linear differential equations whose compatibility conditions are equivalent to Eq. (7). Here the system reads $\partial_x \vec{v} = U_1\vec{v}$, $\partial_t \vec{v} = U_2\vec{v}$, where $\vec{v} = (v_1, v_2)^\intercal$ is a two-component vector (depending on space, time, and spectral variables x, t, k) where

$$U_1 = \begin{pmatrix} -\mathrm{i}k/2 & -gP(x,t) \\ Q(x,t) & \mathrm{i}k/2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A} \end{pmatrix}, \tag{9}$$

where $\mathsf{A} = k^2/2 - gPQ$, $\mathsf{B} = g(\partial_x - \mathrm{i}k)P$, and $\mathsf{C} = (\partial_x + \mathrm{i}k)Q$. One can check that the compatibility condition, $\partial_t U_1 - \partial_x U_2 + [U_1, U_2] = 0$, recovers system (7). In particular, the Lax pair implies the existence of an infinite number of conserved quantities. The new feature as compared to Ref. [9] is that for the flat IC we have $Q(\pm\infty, t) = c$ (we set $c = 1$ later); hence the eigenvectors at $x = \pm\infty$ of the matrix U_1 are now $(1, c/(-\mathrm{i}k))^\intercal$ and $(0, 1)^\intercal$ with eigenvalues $-\mathrm{i}k/2$ and $\mathrm{i}k/2$, respectively. We define two linearly independent pairs of solutions of the x member of the Lax pair as $\vec{v} = e^{k^2 t/2}\phi$ with $\phi = (\phi_1, \phi_2)^\intercal$ and $\vec{v} = e^{-k^2 t/2}\bar{\phi}$ with $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2)^\intercal$ for the first pair, and $\phi, \bar{\phi}$ replaced by $\psi, \bar{\psi}$ for the second pair. The pair $\phi, \bar{\phi}$ is such that, at $x \to -\infty$, $\phi \simeq e^{-\mathrm{i}kx/2}(1, c/(-\mathrm{i}k))^\intercal$ and $\bar{\phi} \simeq e^{\mathrm{i}kx/2}(0, -1)^\intercal$. The pair $\psi, \bar{\psi}$ is such that, at $x \to +\infty$, $\psi \simeq e^{\mathrm{i}kx/2}(0, 1)^\intercal$ and $\bar{\psi} \simeq e^{-\mathrm{i}kx/2}(1, c/(-\mathrm{i}k))^\intercal$. The linear relation between the two independent pairs of solutions defines the four scattering amplitudes

$$\begin{aligned} \phi(x,t,k) &= a(k,t)\bar{\psi}(x,t,k) + b(k,t)\psi(x,t,k), \\ \bar{\phi}(x,t,k) &= \tilde{b}(k,t)\bar{\psi}(x,t,k) - \tilde{a}(k,t)\psi(x,k,t). \end{aligned} \tag{10}$$

Equivalently, this implies the following asymptotics for $\phi, \bar{\phi}$ at $x = +\infty$:

$$\begin{aligned} \phi &\underset{x\to+\infty}{\simeq} \begin{pmatrix} a(k,t)e^{-\frac{\mathrm{i}kx}{2}} \\ b(k,t)e^{\frac{\mathrm{i}kx}{2}} + \frac{c}{-\mathrm{i}k}a(k,t)e^{-\frac{\mathrm{i}kx}{2}} \end{pmatrix}, \\ \bar{\phi} &\underset{x\to+\infty}{\simeq} \begin{pmatrix} \tilde{b}(k,t)e^{-\frac{\mathrm{i}kx}{2}} \\ -\tilde{a}(k,t)e^{\frac{\mathrm{i}kx}{2}} + \frac{c}{-\mathrm{i}k}\tilde{b}(k,t)e^{-\frac{\mathrm{i}kx}{2}} \end{pmatrix}. \end{aligned} \tag{11}$$

Plugging this form into the ∂_t equation of the Lax pair at $x \to +\infty$, one finds a very simple time dependence, $a(k, t) = a(k)$ and $b(k, t) = b(k)e^{-k^2 t}$, $\tilde{a}(k, t) = \tilde{a}(k)$ and $\tilde{b}(k, t) = \tilde{b}(k)e^{k^2 t}$. The Wronskian $W = \phi_1\bar{\phi}_2 - \phi_2\bar{\phi}_1$ is space and time independent since $\partial_x W = \mathrm{Tr}(U_1)W = 0$ and $\partial_t W = \mathrm{Tr}(U_2)W = 0$. It is $W = -1$ at $x = -\infty$ and evaluating it using Eqs. (11) at $x = +\infty$ leads to the relation

$$a(k)\tilde{a}(k) + b(k)\tilde{b}(k) = 1 \tag{12}$$

as in the case $c = 0$.

B. Solving the direct scattering

Let us now make use of the boundary data in Eq. (8), and characterize the scattering amplitudes. Integrating the ∂_x equation of the Lax pair at $t = 1$ for $\bar{\phi}$ using Eq. (8) allows to obtain (see Appendix C) $\tilde{b}(k) = ge^{-k^2}$, together with some relations between $a(k)$ and $\tilde{a}(k)$ and $Q(x, 1)$ (which is yet unknown). Using that $Q(x, 0) = c = 1$ is even in x then $\tilde{a}(k) = a(-k) = [a(k^*)]^*$ and $b(k)$ is real and even. This leads to the form

$$a(k) = e^{-\mathbf{i}\varphi(k)}\sqrt{1 - gb(k)e^{-k^2}}, \tag{13}$$

where we still have two unknown functions, a phase $\varphi(k)$, which is odd $\varphi(k) = -\varphi(-k)$, and $b(k)$.

The form for the amplitudes obtained at this stage is still quite similar to the general solution for decaying IC (i.e., of the droplet type) obtained in Ref. [9]. For the droplet IC we obtained $b(k) = 1$. Here we obtain $b(k)$ for the flat IC as follows. Let us return to the ∂_x equation at $t = 0$ using that $Q(x, 0) = c$. It reads

$$\partial_x\phi_1 = -\mathbf{i}\frac{k}{2}\phi_1 - gP(x, 0)\phi_2, \quad \partial_x\phi_2 = \mathbf{i}\frac{k}{2}\phi_2 + c\phi_1. \tag{14}$$

Eliminating ϕ_1 we obtain

$$\partial_x^2\phi_2 + \left(cgP(x, 0) + \frac{k^2}{4}\right)\phi_2 = 0. \tag{15}$$

Unlike the general case, it is a Schrödinger equation with a *real* potential. Hence if ϕ_2 is solution, so is ϕ_2^*. Note that $\bar{\phi}_2$ satisfies also Eqs. (14) and (15). For $x \to -\infty$, from the aforementioned asymptotics, one has $\phi_2^* = \frac{c}{-\mathbf{i}k}\bar{\phi}_2$. Hence the same relation should hold for any x, including $x \to +\infty$. From Eqs. (11) one then obtains $a^*(k^*) = \tilde{a}(k)$, which we already knew, and

$$b(k) = -\frac{c^2}{k^2}\tilde{b}(k) = -\frac{g}{k^2}e^{-k^2}, \tag{16}$$

where we set $c = 1$ in the last identity.

It remains to obtain $\varphi(k)$. Here we will rely on Ref. [9], where for a general $b(k)$ we obtained

$$\varphi(k) = \fint_{\mathbb{R}} \frac{dq}{2\pi}\frac{k}{q^2 - k^2}\log(1 - gb(q)e^{-q^2}). \tag{17}$$

The proof presented there was based on a random walk representation which assumes that $b(q)$ has a proper inverse Fourier transform. It thus cannot be readily applied here. We believe that this is a technical issue (which maybe can be resolved using proper regularizations) and we will here *conjecture* that Eq. (17) extends to the present case. This conjecture will be abundantly confirmed by the results below. Importantly, note also that Eq. (17) follows if one assumes that $a(k)$ is analytic in the upper half plane [38], from Kramers-Kronig relations (see Appendix F).

C. Solving the inverse scattering

Having determined the scattering amplitudes we now follow Ref. [9] to perform the inverse-scattering transform and obtain the solution of the $\{P, Q\}_g$ system (7) for the flat IC (8) as

$$Q(x, t) = \langle\delta|\mathcal{A}_{xt}(I + g\mathcal{B}_{xt}\mathcal{A}_{xt})^{-1}|\delta\rangle,$$
$$P(x, t) = \langle\delta|\mathcal{B}_{xt}(I + g\mathcal{A}_{xt}\mathcal{B}_{xt})^{-1}|\delta\rangle, \tag{18}$$

where $|\delta\rangle$ is the vector with component $\delta(v)$ so that $\langle\delta|\mathcal{O}|\delta\rangle = \mathcal{O}(0, 0)$ for any operator $\mathcal{O}$. Here $\mathcal{A}_{xt}$, $\mathcal{B}_{xt}$ are two linear operators from $\mathbb{L}^2(\mathbb{R}^+)$ to $\mathbb{L}^2(\mathbb{R}^+)$ with respective kernels

$$\mathcal{A}_{xt}(v, v') = A_t(x + v + v'),\ \mathcal{B}_{xt}(v, v') = B_t(x + v + v'), \tag{19}$$

where the two functions $A_t(x)$ and $B_t(x)$ are the Fourier transforms of the time-dependent reflection coefficients and obey the heat equation (and, respectively, its time reverse) and are given for $g < 0$ by

$$A_t(x) = -g\int_{\mathbb{R}} \frac{dk}{2\pi}\frac{e^{\mathbf{i}kx - k^2(1+t) + \mathbf{i}\varphi(k)}}{k^2\sqrt{1 + g^2k^{-2}e^{-2k^2}}} + \frac{1}{2}, \tag{20}$$

$$B_t(x) = \int_{\mathbb{R}} \frac{dk}{2\pi}\frac{e^{-\mathbf{i}kx - k^2(1-t) - \mathbf{i}\varphi(k)}}{\sqrt{1 + g^2k^{-2}e^{-2k^2}}}. \tag{21}$$

Here the phase reads

$$\varphi(k) = \fint_{\mathbb{R}} \frac{dq}{2\pi}\frac{k}{q^2 - k^2}\log(1 + g^2q^{-2}e^{-2q^2}). \tag{22}$$

We used Eq. (11) of Ref. [9], inserting the scattering data obtained above. Note, however, the additional constant $1/2$ in Eq. (20). Indeed, since the product $\mathcal{A}_{xt}\mathcal{B}_{xt}$ vanishes for $x \to +\infty$, one must have $Q(x, t) \simeq A_t(x)$ for $x \to +\infty$. Since $Q(\pm\infty, x) = 1$ we must have $\lim_{x\to+\infty} A_t(x) = 1$. We have checked that this is indeed the case from Eq. (20), the $1/2$ constant being crucial. Its origin can be traced to the pole in the integrand of Eq. (20), following the general scheme in Ref. [37]. The functions $\varphi(k)$ and $A_x(t), B_x(t)$ are plotted in Appendix D for various values of t, g. Note that $\varphi(0^\pm) = \mp\frac{\pi}{2}$ for any $g \neq 0$ so that the integrand in A_t behaves as $-\text{sgn}(g)/(\mathbf{i}k) = \frac{1}{\mathbf{i}k}$ since Eqs. (20) and (21) are valid only for $g < 0$, at small k. We further note the unexpected relation $A_t''(x) = gB_{-t}(x)$.

III. SOLVING THE LARGE DEVIATIONS VIA THE CONSERVED QUANTITIES

A. Conserved quantities and the main branch of the large deviation function

We can now examine the conserved quantities C_n and obtain $\Psi(z)$ from C_1. The C_n for the $\{P, Q\}_g$ system were obtained in Ref. [9], with $C_1 = g\int_{\mathbb{R}} dxPQ$, $C_2 = g\int_{\mathbb{R}} dxP\partial_xQ$, $C_3 = g(\int_{\mathbb{R}} dxP\partial_x^2Q + gP^2Q^2)$, and so on. Since the product PQ still vanishes at infinity, these remain valid in the present case. As before, the values $C_n(g)$ of these conserved charges can be extracted (see Appendix B) by expanding $-\mathbf{i}\varphi(k) = \sum_{n\geqslant 1}\frac{C_n(g)}{(\mathbf{i}k)^n}$ in powers of $1/k$ in Eq. (22). This leads to $C_{2m+1}(g) = (-1)^{m-1}\int_{\mathbb{R}}\frac{dq}{2\pi}q^{2m}\log(1 + g^2q^{-2}e^{-2q^2})$.

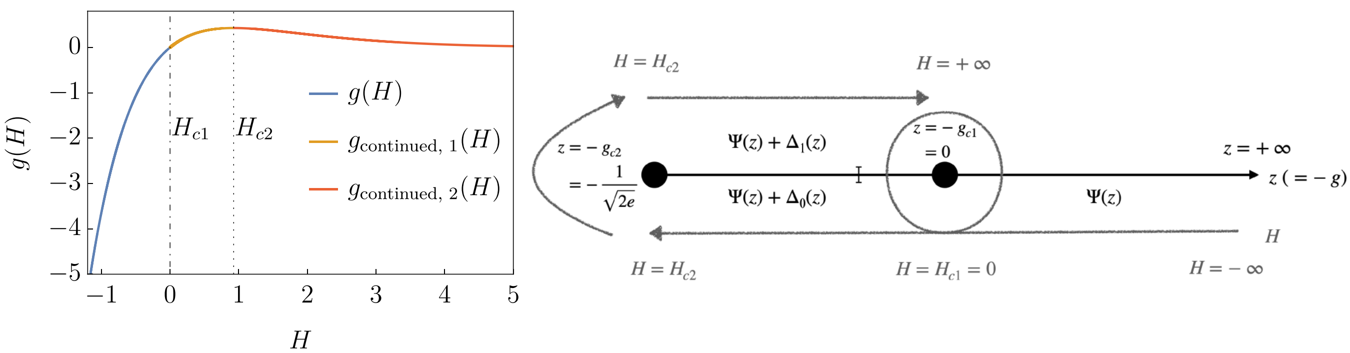

FIG. 1. Left: Plot of the coupling $g = g(H)$ of the $\{P, Q\}$ system (7) and (8) to be used to obtain $\Phi(H)$ for a given H. It is obtained from $H = \log \Psi'(z = -g)$. From left to right are the main branch, the first continuation, and the second continuation. The fields $H_{c1} = 0$ and $H_{c2} = 0.926581$ correspond to the limits of the three branches of solutions discussed in the text (with $g_{c1} = 0$ and $g_{c2} = 1/\sqrt{2e}$). Right: Schematic plot of the branches for $\Psi(z)$ as $z = -g$ is varied, and the corresponding ranges of values for H. For $H < H_{c_1} = 0$ one uses $\Psi(z) = \Psi_0(z)$ given in Eq. (23). At $H = H_{c_1} = 0$, one needs to turn around the branching point of $\Psi_0(z)$ at $z = 0$, and change the Riemann sheet. This leads to the continuation $\Psi(z) = \Psi_0(z) + \Delta_0(z)$ which, using Eqs. (24), determines $\Phi(H)$ for all $H_{c1} = 0 < H < H_{c2}$ by decreasing z from zero to $-g_{c2}$. A second continuation $\Psi(z) = \Psi_0(z) + \Delta_1(z)$ is obtained similarly by rotating around $z = -g_{c2}$, which, using Eqs. (24), determines $\Phi(H)$ for all $H_{c2} < H$ by increasing z from $-g_{c2}$ back to zero.

Since $-z\Psi'(z) = C_1(-z)$, with $g = -z$, we obtain $-z\Psi'(z) = \int_{\mathbb{R}} \frac{dq}{2\pi} \mathrm{Li}_1(-\frac{z^2}{q^2} e^{-2q^2})$, where $\mathrm{Li}_1(y) = -\log(1 - y)$. Using the relation between polylogarithmic functions, $z\partial_z \mathrm{Li}_n = \mathrm{Li}_{n-1}$, we obtain upon integration our final result for the flat IC,

$$\Psi(z) = \Psi_0(z) := -\int_{\mathbb{R}} \frac{dq}{4\pi} \mathrm{Li}_2\left(-\frac{z^2}{q^2} e^{-2q^2}\right). \tag{23}$$

Taking a derivative of Eq. (6) one obtains the rate function $\Phi(H)$ in a parametric form:

$$e^H = \Psi'(z), \quad \Phi(H) = \Psi(z) - z\Psi'(z). \tag{24}$$

As in Ref. [9] this is valid only for $z > 0$ (i.e., $g < 0$) since the right-hand side in Eq. (5) diverges for $z < 0$. Since $\Phi'(H) = -ze^H$, the range $z > 0$ corresponds to H in $(-\infty, 0]$, where $H = 0$ is the most probable value of H defined by $\Phi'(H) = 0$. Thus up to now we have solved the case $g < 0$, i.e., $z > 0$, which corresponds to the left-hand side of $P(H, t)$ and to the *main branch* for $\Psi(z)$.

B. Continuations of the conserved quantities via the generation of solitons

To obtain the right-hand side $H > 0$ we proceed as in Ref. [9]. Equations (7) also hold for any $H > 0$, corresponding to the attractive regime $g > 0$ of the $\{P, Q\}_g$ system. Indeed, $\Psi(z)$ can be analytically continued to $z < 0$, allowing to determine $\Phi(H)$ for any H. By contrast with the droplet IC, the flat IC requires a continuation in two steps. Since $\Psi_0(z)$ has a branch cut on the negative real axis, for $H \in [0, H_{c2}]$, with $H_{c2} = 0.926581$ [see Eq. (A8)], a first continuation is needed, with $\Psi(z) = \Psi_0(z) + \Delta_0(z)$ (second branch), where $\Delta_0(z)$ is obtained from the cut of $\Psi_0(z)$. In that branch, $g = -z$ increases from zero to g_{c2}, with $g_{c2} = 1/\sqrt{2e} = 0.428882$. This is further explained in Fig. 1.

For $H \in [H_{c2}, +\infty]$, a third branch is required, $\Psi(z) = \Psi_0(z) + \Delta_1(z)$ and $g = -z$ now decreases from g_{c2} back to zero (see Fig. 1). These continuations correspond to two branches of solutions of the $\{P, Q\}_g$ system for $0 < g \leqslant g_{c2}$. As in Ref. [9] these branches have a very nice physical origin, and one finds that the second branch corresponds to the spontaneous generation of a soliton while the third one is interpreted as a modification of the rapidity of the soliton. In all branches, the rate function $\Phi(H)$ is obtained from Eqs. (24) by inserting the corresponding result for $\Psi(z)$, i.e., Ψ_0 for the main branch, and $\Psi_0 + \Delta_0$ and $\Psi_0 + \Delta_1$ for the second and third branches.

Technically, the second branch arises from the fact that, for $g > 0$, the logarithm inside $\varphi(k)$ has a cut for the integration variable in Eq. (22) located at $q = \pm \mathbf{i}\kappa_0$ with

$$\kappa_0^2 e^{-2\kappa_0^2} = 1/g^2, \quad \kappa_0^2 = -\tfrac{1}{2} W_0(-2g^2), \tag{25}$$

where W_0 is the Lambert function [40] and κ_0 is the positive root of Eqs. (25). This cut exists only if $0 < g \leqslant g_{c2}$. The third branch arises from the continuation of the Lambert function W_0 to W_{-1} (see Appendix E), so that the position of the cut is located at $q = \pm \mathbf{i}\kappa_1$ with $\kappa_1^2 = -\frac{1}{2} W_{-1}(-2g^2)$. The contribution of the cuts give rise to a pole in the integrand of A_t (B_t) in the upper (lower) half plane which according to the general construction of Ref. [37] simply generates solitons.

Practically, the cuts of the phase φ modify the expression of A_t and B_t by adding rational factors providing poles whose residues generate the solitons (see Sec. S-K of the Supplemental Material of Ref. [9]). For the second branch, $0 < g < g_{c2}$ and $0 < H < H_{c2}$, one finds

$$\begin{aligned} A_t(x) &= -g \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{e^{\mathbf{i}kx - k^2(1+t) + \mathbf{i}\varphi(k)}}{k^2\sqrt{1 + g^2 k^{-2} e^{-2k^2}}} \frac{k + \mathbf{i}\kappa_0}{k - \mathbf{i}\kappa_0} \\ &\quad + \frac{1}{2} + \frac{2g}{\kappa_0} e^{-\kappa_0 x + \kappa_0^2(1+t) + \mathbf{i}\varphi(\mathbf{i}\kappa_0)}, \\ B_t(x) &= \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{e^{-\mathbf{i}kx - k^2(1-t) - \mathbf{i}\varphi(k)}}{\sqrt{1 + g^2 k^{-2} e^{-2k^2}}} \frac{k - \mathbf{i}\kappa_0}{k + \mathbf{i}\kappa_0} \\ &\quad + 2\kappa_0 e^{-\kappa_0 x + \kappa_0^2(1-t) - \mathbf{i}\varphi(-\mathbf{i}\kappa_0)}, \end{aligned} \tag{26}$$

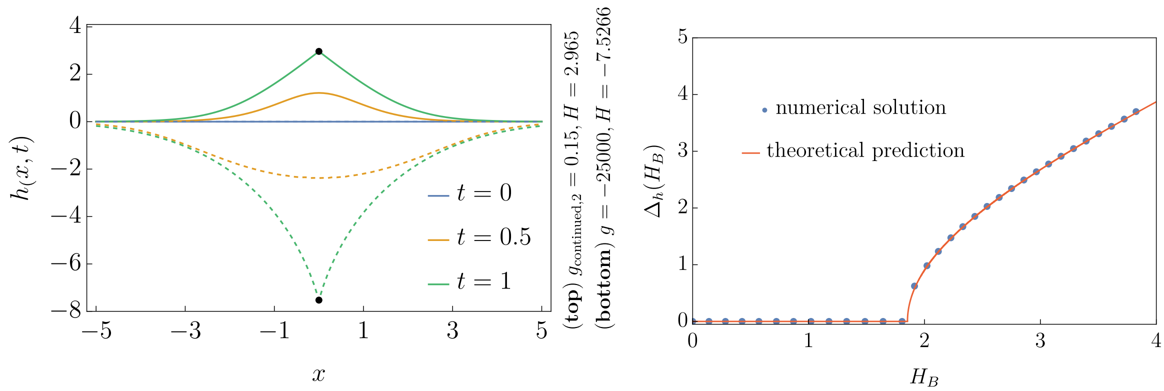

FIG. 2. Left: The optimal height $h_{\text{opt}}(x,t)$ for flat initial conditions plotted for various times t and for two values of H indicated by the black dots [one in the main branch, $H=-7.5266$ (dashed line), and the other in the third branch, $H=2.965$ (solid line)]. Right: Plot of the order parameter Δ_h of the parity-breaking transition for the Brownian IC as a function of H_B predicted here in Eq. (31), as compared to $-\Delta/2$, where Δ is defined and obtained numerically in Ref. [13]. Courtesy of B. Meerson for the data of the numerical solution of the WNT equations.

where $\varphi(k)$ is given in Eq. (22). The cuts also modify the conserved quantities by adding a solitonic contribution $\Delta C_n(g)=\frac{2}{n}\kappa_0^n$ for n odd and zero even charges [9]. Integrating $-z\Delta_0'(z)=\Delta C_1(g=-z)$ one finds

$$\Delta_0(z)=\frac{\sqrt{2}}{3}[-W_0(-2z^2)]^{3/2}-\sqrt{2}[-W_0(-2z^2)]^{1/2}. \quad (27)$$

The third branch, $0<g<g_{c2}$ and $H>H_{c2}$, is obtained by the minimal replacement of κ_0 by κ_1 in both functions $A_t(x)$ and $B_t(x)$ in Eqs. (26). This leads again to $\Delta C_1(g)=2\kappa_1$ and, by integration, to $\Delta_1(z)$ given by the same equation as Eq. (27) with $W_0\to W_{-1}$. As in Ref. [9] the solitonic part dominates the large deviations for $H\to+\infty$.

From the above exact solutions for $A_t(x)$ and $B_t(x)$ we obtain the solutions to the $\{P,Q\}_g$ system through the Fredholm operator inversion formula (18) for various values of H and g. We use the numerical method in Sec. S-L of Ref. [9]. We have performed several numerical checks of some highly nontrivial consequences of the formulas, which validate our conjecture: (i) the functions P,Q are even in x, (ii) $Q(x,t=1)=A_1(|x|)$, (iii) $Q(0,t=1)=e^H=\Psi'(-g)$, and (iv) $Q(\pm\infty,t)=1$. The results for the optimal height $h_{\text{opt}}(x,t)=\log Q(x,t)$ are plotted in Fig. 2.

The above results provide the first direct analytical derivation of $\Phi(H)$ for the flat IC. Note that they are in agreement with those of Ref. [16] which were cleverly inferred, using various symmetries of the weak noise theory together with the known rate function of the Brownian initial condition calculated from the Bethe ansatz in Ref. [26].

IV. SOLVING THE BROWNIAN INITIAL CONDITION USING THE FLAT ONE

Conversely, starting from the flat IC, a remarkable byproduct of our results is the solution of the WNT for the KPZ equation with the Brownian (i.e., stationary) IC. It is defined as the solution of Eq. (1) with $h(y,0)=W(y)$ where $W(y)$ is a two-sided standard Brownian motion with zero drift with $W(0)=0$. This corresponds to Eq. (4) with initial condition $Z(x,0)=e^{T^{1/4}W(x)}$. We are interested in the probability $P(H_B,\tau)$ that $h(0,\tau)=H_B$, which behaves at small t as $P(H_B,T)\sim\exp(-\Phi(H_B)/\sqrt{T})$. To obtain the solution in that case we first notice that our solution for the flat IC for $P(x,t)$, $Q(x,t)$ defined in Eqs. (18) is also well defined in the extended interval $t\in[-1,1]$, since Eqs. (20) and (21) are also well defined in this interval. In the spirit of Ref. [16], let us now define the functions P_B and Q_B for $t\in[0,1]$ as

$$Q_B(x,t)=e^{\frac{H_B}{2}}Q(\sqrt{2}x,2t-1),$$
$$P_B(x,t)=\sqrt{2}P(\sqrt{2}x,2t-1). \quad (28)$$

One can check that P_B, Q_B satisfy the $\{P,Q\}_{g_B}$ system (7) with coupling constant $g_B=\sqrt{2}ge^{-H_B/2}$ and $Q_B(0,1)=e^{H_B}$ with $H_B=2H$. As we show in Appendix A, they obey the boundary conditions (i) $Q_B(0,0)=1$, (ii) $P_B(x,1)=\delta(x)$, (iii)

$$g_BP_B(x,0)Q_B(x,0)+\partial_x^2\log Q_B(x,0)=g_Be^{H_B}\delta(x), \quad (29)$$

as well as (iv) $Q_B(0,1)=e^{H_B}$. As shown in Ref. [13] the boundary conditions (i)–(iv) for the $\{P,Q\}_{g_B}$ system are the ones corresponding to the Brownian IC, hence P_B, Q_B constructed as above provide the solution of the WNT in that case.

We have thus obtained through Eqs. (28) the solution for the Brownian IC in terms of our solution P,Q for the flat IC (extended in $t\in[-1,1]$). Let us discuss now what happens for the different branches as $H_B=2H$ is varied. In the main branch, $H\leqslant 0$, the functions A_t and B_t are given by Eqs. (20) and (21). A consequence (see Appendix A) is that $\Phi(H)=\frac{1}{2\sqrt{2}}\Phi_B(2H)$ for $H\leqslant 0$ (in fact for $H\leqslant H_{c_2}$ see below).

The discussion of the other branches is a bit more involved in the Brownian case. For the second branch $0<H=\frac{H_B}{2}<H_{c2}$ the construction is exactly the same as for the flat IC, i.e., one uses Eqs. (28) and in P,Q one chooses the continuations for A_t, B_t given in Eqs. (26) which includes the solitonic part with rapidity κ_0. For the third branch $H=\frac{H_B}{2}>H_{c2}$ it is in principle allowed to proceed to the change $\kappa_0\to\kappa_1$ solely in one of the functions A_t or B_t: hence there exist two additional distinct asymmetric solutions that we did not consider for the flat IC. In that case the solutions P,Q will not be even in x, providing a mechanism for a spontaneous breaking of the symmetry $x \to -x$. This was forbidden for the flat IC, which is why one must choose $\kappa_0 \to \kappa_1$ in both A_t and B_t, leading to an even solution. For the Brownian IC, however, it was shown numerically [13] and analytically [26] that the large deviation function $\Phi_B(H_B)$ has a second-order phase transition at precisely this value $H = H_{c2}$. The solution obtained here provides a mechanism for this transition. As was observed in Refs. [13,15] this phase transition is indeed accompanied by a spontaneous symmetry breaking of the spatial parity in the $\{P_B, Q_B\}_{g_B}$ solution, although no analytical results were obtained there for $H_B \approx 2H_{c_2}$. Hence for the Brownian IC we claim that there are two equivalent solutions, denoted $\pm$, related by parity, i.e., $P_B^-(x,t) = P_B^+(-x,t)$, $Q_B^-(x,t) = Q_B^+(-x,t)$, and which are obtained by replacing solely one κ_0 into a κ_1 inside either A_t $(+)$ or B_t $(-)$ and using Eqs. (28).

This is further understood from the solitonic contributions to the conserved quantities of the $\{P, Q\}_g$ system given for all n in this case as (see Eq. (S59) in the Supplemenal Material of Ref. [9])

$$\Delta C_n^{\pm} = \pm\frac{1}{n}(\kappa_0^n - (-\kappa_1)^n). \tag{30}$$

For $n = 1$ this implies that the corresponding value of $\Psi(z)$ for this asymmetric solution is $\Psi(z) = \Psi_0(z) + \frac{\Delta_0(z)+\Delta_1(z)}{2}$, which gives $g(H)$ in that branch from $\Psi'(z=-g) = e^H$, in agreement with Ref. [26] (see Appendix A). Note that now the even conserved quantities are nonzero, indicating the breaking of the spatial parity together with the presence of a nonzero current in the solutions. Such continuation corresponds to a true phase transition, since the conserved quantities are not smooth functions of the coupling parameter g at g_{c2} [26]. Indeed, as was noticed numerically in Ref. [13] the conserved quantity $\Delta_h = h_{\text{opt},B}(+\infty,t) - h_{\text{opt},B}(-\infty,t) = h_{\text{opt}}(+\infty,t) - h_{\text{opt}}(-\infty,t) = \int_{\mathbb{R}} dx \partial_x Q(x,t)/Q(x,t)$ [where $h_{\text{opt,B}}(x,t)$ is the optimal height for the Brownian IC] can be considered as an order parameter since it is nonzero for $H = \frac{H_B}{2} > H_{c2}$ and vanishes for $H \leqslant H_{c2}$. Here we conjecture (see Appendix B) that Δ_h can be obtained analytically and is equal to

$$\Delta_h = 2\log\frac{\kappa_1}{\kappa_0}\bigg|_{g=g(H=H_B/2)} \tag{31}$$

for $g \in (0, 1/\sqrt{2e}]$ and $H = \frac{H_B}{2} \geqslant H_{c2}$ and $\Delta_h = 0$ for $H \leqslant H_{c2}$. Note that Eq. (31) can be seen as the $n \to 0$ limit of Eq. (30) and is not part of the standard ZS conserved quantities [41]. The prediction (31) is compared to the numerical results of Ref. [13] in Fig. 2.

V. CONCLUSION

In this work, we constructed the explicit solution to the weak noise theory of the KPZ equation for the flat and Brownian initial conditions, and obtained the exact optimal height and noise fields. The structure of the solution is richer than in the case of the droplet IC recently solved in Ref. [9]. We have shown that the interplay between solitons with different rapidities provides a mechanism for obtaining a phase transition in the large deviation.

ACKNOWLEDGMENTS

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APPENDIX A: RELATION BROWNIAN-FLAT

1. Previous results for the Brownian IC

Let us recall the results of Ref. [26], which were obtained by a completely different method making use of the exact determinantal solution available for the stationary KPZ equation at any finite time. There, the following generating function was computed (see Eq, (119) in the Supplemental Material [39] or Eq. (18) in the limit $\tilde{w} \to 0^+$; see also discussion around formula (7.3.21) of Ref. [30]), together with its small time large deviation form, for $\tilde{z} > 0$:

$$\int_{\mathbb{R}} dH_B P(H_B, T)\exp\left(-\frac{2\sqrt{\tilde{z}}}{\sqrt{T}}e^{H_B/2}\right) \sim \exp\left(-\frac{\Psi_B(\tilde{z})}{\sqrt{T}}\right). \tag{A1}$$

Note that the argument in Eq. (A1) is $2\sqrt{\tilde{z}}$, for technical reasons. The result for $\Psi_B(\tilde{z})$ obtained in Ref. [26] reads

$$\Psi_B(\tilde{z}) = \Psi_{B,0}(\tilde{z}) = -\int_{\mathbb{R}}\frac{dq}{2\pi}\text{Li}_2\left(-\frac{\tilde{z}}{q^2}e^{-q^2}\right) \tag{A2}$$

corresponding to the main branch. One defines the continuation of this function in the two other branches,

$$\Psi_B(\tilde{z}) = \Psi_{B,0}(\tilde{z}) + \Delta_{B,0}(\tilde{z}) \quad \text{(second branch)}, \tag{A3}$$

$$\Psi_B(\tilde{z}) = \Psi_{B,0}(\tilde{z}) + \frac{\Delta_{B,0}(\tilde{z}) + \Delta_{B,1}(\tilde{z})}{2} \quad \text{(third branch)}, \tag{A4}$$

where the jump functions are expressed in terms of the Lambert functions W_0, W_{-1} [40] as

$$\Delta_{B,0}(\tilde{z}) = \tfrac{4}{3}[-W_0(-\tilde{z})]^{3/2} - 4[-W_0(-\tilde{z})]^{1/2}, \tag{A5}$$

$$\Delta_{B,1}(\tilde{z}) = \tfrac{4}{3}[-W_{-1}(-\tilde{z})]^{3/2} - 4[-W_{-1}(-\tilde{z})]^{1/2}. \tag{A6}$$

Once the function $\Psi_B(\tilde{z})$ is known the rate function $\Phi_B(H_B)$ is obtained via a Legendre transform, which reads explicitly

$$\Phi_B(H_B) = \begin{cases} \max_{\tilde{z}\in[0,+\infty[}[\Psi_{B,0}(\tilde{z}) - 2\sqrt{\tilde{z}e^{H_B}}], & H_B \leqslant H_{c,B} = 0 \\ \max_{\tilde{z}\in[0,e^{-1}]}[\Psi_{B,0}(\tilde{z}) + \Delta_{0,B}(\tilde{z}) + 2\sqrt{\tilde{z}e^{H_B}}], & H_{c,B} \leqslant H_B \leqslant H_{c2,B} \\ \min_{\tilde{z}\in]0,e^{-1}]}\left[\Psi_{B,0}(\tilde{z}) + \frac{\Delta_{0,B}(\tilde{z})+\Delta_{1,B}(\tilde{z})}{2} + 2\sqrt{\tilde{z}e^{H_B}}\right], & H_B \geqslant H_{c2,B}, \end{cases} \tag{A7}$$

with $H_{c2,B} = 2H_{c2}$, where

$$H_{c2} = \log \Psi'(z)|_{z=-g_{c2}} = \log\left(\Psi_0'\left(-\frac{1}{\sqrt{2e}}\right) + \Delta_0'\left(-\frac{1}{\sqrt{2e}}\right)\right) \approx 0.926581 \tag{A8}$$

is defined in the text. Note that one can understand the change of sign in front of $2\sqrt{\tilde{z}e^{H_B}}$ in Eqs. (A7) as follows: We first decrease $\tilde{z}$ from $+\infty$ to zero and then increase it to e^{-1}. In the complex $\tilde{z}$ plane, turning around zero induces a branch change in the square root function $\sqrt{\tilde{z}} \to -\sqrt{\tilde{z}}$. The change from a maximum to a minimum can be seen from a change of convexity in the argument of the variational problem.

2. From the exact solution of the WNT for flat IC to the one for the Brownian IC

In the paper [16] the symmetries of the WNT action were studied in the case of the Brownian IC. The authors cleverly noticed that they imply that at time $t_B = 1/2$ the KPZ height field is flat, i.e., $Q_B(x, 1/2)$ is independent of x (where for clarity we denote $t_B \in [0, 1]$ the time for the Brownian IC). From this they concluded that one can deduce the WNT solution for the flat IC if one knows the solution for the Brownian IC. Using our result in Ref. [26], recalled in the previous section, they displayed the solution for the flat IC, expected from these symmetries. They obtained the following relation between the rate functions, which read in our notations

$$\Phi(H) = \frac{1}{2\sqrt{2}}\Phi_B(2H), \quad H < H_{c2}, \tag{A9}$$

valid for the main and second branch. In the third branch, there are in fact three solutions to the WNT equations: one is relevant for the flat IC, and the two others for the Brownian IC, as discussed in the text.

In the text we have done the converse: we have obtained directly the solution for the flat IC (which had not been obtained directly before), denoted $P(x,t), Q(x,t)$ in the text. We noticed that it can be extended for $t \in [-1, 1]$ instead of the original interval [0,1]. From this extension we constructed using Eqs. (28) the solution $P_B(x, t_B), Q_B(x, t_B)$ (with $t_B = 2t - 1$) for the Brownian IC.

Let us now give the arguments in support of this construction. The method makes use of the nontrivial "fluctuation dissipation" symmetry of the dynamical action for the KPZ equation, and of its implementation on the saddle point equations of the WNT, used in Ref. [16] (for earlier applications of this symmetry see [42]). We first recall the following general property of the $\{P, Q\}$ system. Let us define $\tilde{Q}(x,t)$ and $\tilde{P}(x,t)$ via the relations

$$\tilde{Q}(x,t) = 1/Q(-x,-t) \tag{A10}$$

and

$$2g\tilde{P}(x,t)\tilde{Q}(x,t) + \partial_x^2 \log \tilde{Q}(x,t) = 2gP(-x,-t)Q(-x,-t) + \partial_x^2 \log Q(-x,-t). \tag{A11}$$

One can show that if P, Q are solutions of Eqs. (7) (with coupling g) in some time interval, $\tilde{P}, \tilde{Q}$ are also solutions of Eqs. (7) (with the same coupling g) in the mirror image interval.

We now use this symmetry to define an extended solution of the $\{P, Q\}_g$ system (7), P_F, Q_F on the interval $t \in [-1, 1]$, such that

$$Q_F(x,t) = \begin{cases} Q(x,t) & \text{for } t \in [0,1] \\ \tilde{Q}(x,t) & \text{for } t \in [-1,0] \end{cases} \tag{A12}$$

and similarly for P_F. Let us now define the functions P_B and Q_B for $t \in [0, 1]$ as

$$Q_B(x,t) = e^{\frac{H_B}{2}} Q_F(\sqrt{2}x, 2t-1),$$
$$P_B(x,t) = \sqrt{2}P_F(\sqrt{2}x, 2t-1). \tag{A13}$$

One can check that P_B, Q_B satisfy the $\{P, Q\}_{g_B}$ system (7) with coupling constant $g_B = \sqrt{2}ge^{-H_B/2}$ and $Q_B(0, 1) = e^{H_B}$ with $H_B = 2H$. The important point for us now is that if P, Q satisfy the boundary conditions for the flat IC,

$$Q(x,0) = 1, \quad P(x,1) = \delta(x), \tag{A14}$$

then, P_B, Q_B constructed as above satisfy the boundary conditions for the Brownian IC, which read [13]

(i) $Q_B(0, 0) = 1$,
(ii) $P_B(x, 1) = \delta(x)$,
(iii) $g_B P_B(x, 0)Q_B(x, 0) + \partial_x^2 \log Q_B(x, 0) = g_B e^{H_B}\delta(x)$,
(iv) $Q_B(0, 1) = e^{H_B}$.

This can be checked using all the above definitions. For condition (i) one has

$$Q_B(0,0) = e^{\frac{H_B}{2}} Q_F(0,-1) = e^{\frac{H_B}{2}} \tilde{Q}(0,-1) = e^{\frac{H_B}{2}}/Q(0,1) = e^{\frac{H_B}{2} - H} = 1. \tag{A15}$$

For condition (ii) it is obvious. For condition (iii), denoting $y = \sqrt{2}x$ and using $g_B = \sqrt{2}ge^{-\frac{H_B}{2}}$,

$$\begin{aligned} &g_B P_B(x,0)Q_B(x,0) + \partial_x^2 \log Q_B(x,0) \\ &= \sqrt{2}g_B e^{\frac{H_B}{2}} \tilde{P}(y,-1)\tilde{Q}(y,-1) + 2\partial_y^2 \log \tilde{Q}(y,-1) \\ &= 2g\tilde{P}(y,-1)\tilde{Q}(y,-1) + 2\partial_y^2 \log \tilde{Q}(y,-1) \\ &= 2gP(-y,1)Q(-y,1) \\ &= 2ge^H \delta(y) \\ &= \sqrt{2}g_B e^{H_B}\delta(\sqrt{2}x) \\ &= g_B e^{H_B}\delta(x) \end{aligned} \tag{A16}$$

where in the third line we have used the symmetry (A11) and the flat IC. For condition (iv), $Q_B(0, 1) = e^{H_B/2}Q(0, 1) = e^{H_B}$ using $H_B = 2H$. Note that Eq. (A10) is continuous at $t = 0$ since $Q(x, t = 0) = 1$. Hence P_B, Q_B constructed as above are the solution of the WNT for Brownian initial conditions.

In the previous paragraph we constructed $P_F(x,t), Q_F(x,t)$ using symmetries. It is not *a priori* obvious that these functions should coincide with $P(x,t), Q(x,t)$ extended to the interval $t \in [-1, 1]$ as constructed in the text. It turns out that this is the case and one has

$$Q(x,t) = Q_F(x,t), \quad P(x,t) = P_F(x,t), \quad t \in [-1,1]. \tag{A17}$$

This implies, from Eqs. (A10) and (A11), that the solutions obtained in the text for $P(x,t)$, $Q(x,t)$ should satisfy, for $t \in [-1,1]$,

$$Q(x,t)Q(-x,-t) = 1 \tag{A18}$$

as well as

$$2gP(x,t)Q(x,t) + \partial_x^2 \log Q(x,t) = 2gP(-x,-t)Q(-x,-t) + \partial_x^2 \log Q(-x,-t).$$

These conditions are highly nontrivial to check on the analytical form of the solutions provided in the text. Thus we have performed some numerical checks, e.g., we have checked numerically that the symmetry (A18) holds (see below in Appendix D).

Note that all the above construction is correct for each given branch of solutions. For $H = \frac{H_B}{2} \leqslant H_{c2}$ one thus inserts in Eqs. (A12) and (A13) the solution P, Q for the flat IC given in the text for the main and second branches, and one obtains the solution for the Brownian IC for $H_H \leqslant 2H_{c2}$. For $H \geqslant H_{c2}$ (third branch) there are three simultaneous solutions, as discussed in the text. One of these solutions (with the choice $\{\kappa_0, \kappa_0\}$ for the solitonic rapidities) is even in x and corresponds to the flat IC solution. This solution does not allow to obtain the solution for the Brownian IC (it corresponds to a subleading contribution to the dynamical action). The two other solutions (with the choice $\{\kappa_1, \kappa_0\}$ and $\{\kappa_0, \kappa_1\}$ for the solitonic rapidities), denoted as $P^{\pm}, Q^{\pm}$ in the text, break the $x \to -x$ symmetry and are mirror images of each other. These are the solutions which should be inserted in Eqs. (A12) and (A13) to obtain the solution for the Brownian IC in that regime. Note the symmetries (A10) and (A11) are never broken for any of these solutions, irrespective of whether $x \to -x$ is broken or not.

3. Rate functions: Relations between flat and Brownian

Let us recall our result in the text for the rate function $\Psi(z)$ for the flat IC in the main branch $z > 0$. It reads

$$\Psi(z) = \Psi_0(z) := -\int_{\mathbb{R}} \frac{dq}{4\pi} \mathrm{Li}_2\left(-\frac{z^2}{q^2} e^{-2q^2}\right). \tag{A19}$$

Comparing with the result for the rate function $\Psi_B(\tilde{z})$ for the Brownian initial condition (A2) in the main branch, we see that the following relation holds:

$$\Psi_0(z) = \frac{1}{2\sqrt{2}} \Psi_{B,0}(\tilde{z} = 2z^2). \tag{A20}$$

Let us recall that the rate functions Ψ_0 and $\Psi_{B,0}$ are related to the rate functions $\Phi(H)$ and $\Phi_B(H_B)$ in the main branch through the Legendre transform

$$\Psi_0(z) = \min_H (\Phi(H) + ze^H), \tag{A21}$$

$$\Psi_{B,0}(\tilde{z}) = \min_{H_B} (\Phi_B(H_B) + 2\sqrt{\tilde{z}} e^{H_B}). \tag{A22}$$

One can easily verify that this is compatible with the relation obtained in Ref. [16],

$$\Phi(H) = \frac{1}{2\sqrt{2}} \Phi_B(2H). \tag{A23}$$

This is easily checked inserting $\Phi(H)$ from this relation into the first equation in Eqs. (A21) and defining $z = \sqrt{\tilde{z}/2}$. In fact the relation

$$\Psi(z) = \frac{1}{2\sqrt{2}} \Psi_B(\tilde{z} = 2z^2) \tag{A24}$$

holds for each branch and each solution. As a consequence the jumps are also related. One has

$$\Delta_0(z) = \frac{1}{2\sqrt{2}} \Delta_{0,B}(\tilde{z} = 2z^2) \tag{A25}$$

as can be checked by comparing Eqs. (27) and (A5). The same relation holds between $\Delta_1(z)$ and $\Delta_{B,1}(z)$. Finally in the third branch the spatially asymmetric solutions discussed in the text associated to $\Psi(z) = \Psi_0(z) + \frac{\Delta_0(z)+\Delta_1(z)}{2}$ correspond to the result in the third line of Eqs. (A7) for the Brownian initial condition via the same relation.

Remark. In Ref. [26] we have obtained the series expansion

$$\Psi_{B,0}(\tilde{z}) = \frac{1}{\sqrt{4\pi}} \sum_{n\geqslant 1} (-1)^{n-1} \frac{(4\tilde{z})^{n/2}}{n!} \Gamma\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^{\frac{n-3}{2}}. \tag{A26}$$

It is useful to note that this provides, using the relation (A20), the following series expansion for the rate function of the flat IC, for $z > 0$:

$$\Psi_0(z) = \frac{1}{\sqrt{4\pi}} \sum_{n\geqslant 1} (-1)^{n-1} \frac{(2z)^n}{n!} \Gamma\left(\frac{n}{2}\right) n^{\frac{n-3}{2}}. \tag{A27}$$

Remark. We can give an alternative interpretation of the rate function $\Psi(z)$ of the flat IC. Consider now the solution to the SHE (in rescaled variables) for the droplet IC considered in Ref. [9] and denote it by $Z_\delta(x,t)$. Then one has

$$\overline{\exp\left(-\frac{z}{\sqrt{T}} \int_{\mathbb{R}} dx\, Z_\delta(x,1)\right)} \sim \exp\left(-\frac{\Psi(z)}{\sqrt{T}}\right). \tag{A28}$$

This implies that the probability distribution function (PDF) of the rate function for the variable $\int_{\mathbb{R}} dx\, Z_\delta(x,1)$ is the same as $\Phi(H)$ for the flat IC.

Indeed, to compute the lhs of Eq. (A28) one performs the same manipulations as in Ref. [9] choosing $j(x,t) = -z\delta(t-1)$ in Eq. (6) there. This leads to the P, Q system with boundary conditions $P_\delta(x,1) = 1$ and $Q_\delta(x,0) = \delta(x)$. Upon the transformation

$$Q_\delta(x,t) = P(x,1-t), \quad P_\delta(x,t) = Q(x,1-t), \tag{A29}$$

which leaves invariant the $\{P, Q\}$ system, one reduces the problem to studying the flat IC and measuring the height field at time $t = 1$. Note that this relation is in fact more general and valid beyond the WNT as an identity in law between the partition function with flat IC and the integral over space of the partition function with droplet IC (both being the so-called point to line partition sum of directed polymers).

APPENDIX B: ADDITIONAL CONSERVATION LAW AND ORDER PARAMETER

In the case considered here where Q does not vanish at infinity, there is an additional nontrivial conservation law which was not discussed in Ref. [9]. Indeed it is easy to check, using the equations for the $\{P, Q\}$ system, that

$$\partial_t \frac{\partial_x Q(x,t)}{Q(x,t)} = \partial_x J_0(x,t),$$
$$J_0(x,t) = 2gP(x,t)Q(x,t) + \frac{\partial_x^2 Q(x,t)}{Q(x,t)}. \tag{B1}$$

Assuming that J_0 vanishes at $x \to \pm\infty$ this implies the conservation law

$$\frac{d}{dt}\int_{\mathbb{R}} dx \frac{\partial_x Q(x,t)}{Q(x,t)} = \frac{d}{dt}[\log Q(+\infty,t) - \log Q(-\infty,t)] = 0. \tag{B2}$$

Note that Eqs. (B1) can also be written in terms of the height field and the response (or noise) field (see definitions in Eqs. (S42) and (S43) in Sec. S-B of Ref. [9]),

$$\partial_t\partial_x h(x,t) = \partial_x\big(2\tilde{h}(x,t) + \partial_x^2 h(x,t) + (\partial_x h)^2\big), \tag{B3}$$

which in these variables is simply the time derivative of Eq. (S42) in Ref. [9].

It is interesting to note (although we will not use it here) that a similar conservation equation holds for P, i.e.,

$$\partial_t \frac{\partial_x P(x,t)}{P(x,t)} = \partial_x \tilde{J}_0(x,t),$$
$$\tilde{J}_0(x,t) = -2gP(x,t)Q(x,t) - \frac{\partial_x^2 P(x,t)}{P(x,t)}, \tag{B4}$$

which under similar assumptions implies the conservation of $\log P(+\infty,t) - \log P(-\infty,t)$.

Hence the order parameter defined in the text,

$$\Delta_h = h(+\infty,t) - h(-\infty,t) = \log Q(+\infty,t) - \log Q(-\infty,t), \tag{B5}$$

is time independent. If the solution is even by spatial parity one has $\Delta_h = 0$, as is the case for the flat IC and in the main and second branches for the Brownian IC. If the spatial parity is broken, as in the third branch for the Brownian IC, it is nonzero.

Although we have not attempted to prove it, we believe that this conserved quantity takes a "simple" value in our case. To provide a guess we have examined the value of this quantity in the case of a low-rank soliton. Let us consider as in Sec. S-D of Ref. [9] the case where $\mathcal{A}_{xt}$ and $\mathcal{B}_{xt}$ are rank n_1 and n_2 operators, respectively, i.e., $\mathcal{A}_{xt} = \sum_{j=1}^{n_1} q_{\kappa_j}|\kappa_j\rangle\langle\kappa_j|$ and $\mathcal{B}_{xt} = \sum_{i=1}^{n_2} p_{\mu_i}|\mu_i\rangle\langle\mu_i|$, and $q_{\kappa_j} = q_{\kappa_j}(x,t) = \tilde{q}_j e^{-\kappa_j x + \kappa_j^2 t}$ and $p_{\mu_i} = p_{\mu_i}(x,t) = \tilde{p}_i e^{-\mu_i x - \mu_i^2 t}$ are plane waves. In that case we obtained the formula

$$Q(x,t) = \sum_{i,j=1}^{n_1} q_{\kappa_i}(I + g\sigma\gamma)^{-1}_{ij},$$
$$\gamma_{ij} = \frac{p_{\mu_i} q_{\kappa_j}}{\mu_i + \kappa_j}, \quad \sigma_{ij} = \frac{1}{\kappa_i + \mu_j}. \tag{B6}$$

In the present case we take $n_1 = 2$ and $n_2 = 1$ and choose $\kappa_2 = 0$ and $\tilde{q}_2 \neq 0$ corresponding to $A_t(x)$ being constant and equal to $\tilde{q}_2$ as $x \to +\infty$:

$$Q(x,t) = \frac{\tilde{q}_1 e^{-\kappa_1 x}\big(g\kappa_1^2 \tilde{p}_1\tilde{q}_2 e^{-\mu_1 x} + \mu_1^2(\kappa_1+\mu_1)^2\big) + \mu_1^2\tilde{q}_2(\kappa_1+\mu_1)^2}{g\mu_1^2\tilde{p}_1\tilde{q}_1 e^{-\kappa_1 x - \mu_1 x} + (\kappa_1+\mu_1)^2\big(g\tilde{p}_1\tilde{q}_2 e^{-\mu_1 x} + \mu_1^2\big)}\Bigg|_{\tilde{q}_1 \to \tilde{q}_1 e^{\kappa_1^2 t}, \tilde{p}_1 \to \tilde{p}_1 e^{-\mu_1^2 t}}. \tag{B7}$$

It is easy to check that

$$Q(+\infty,t) = \tilde{q}_2, \quad Q(-\infty,t) = \tilde{q}_2\frac{\kappa_1^2}{\mu_1^2}, \tag{B8}$$

hence we find that the order parameter in that case is

$$\Delta_h = 2\log\frac{\mu_1}{\kappa_1}. \tag{B9}$$

We believe that this result extends to our case (the asymmetric branches for the Brownian IC) with $\mu_1 \to \kappa_0$ and $\kappa_1 \to \kappa_1$ where κ_0 and κ_1 are defined in the text. This conjecture is supported by the data in Fig. 2 in the text.

Remark. Note that in the case of purely solitonic solutions, the standard conserved quantities are equal to

$$C_n = \frac{\mu_1^n - (-\kappa_1)^n}{n}. \tag{B10}$$

Interestingly, the additional conservation law presented here and Eq. (B9), although it does not belong to the standard family of conserved quantities, correspond to (twice) the limit ΔC_n for $n \to 0$.

Remark. In a recent work [43], a similar-looking additional conservation law, previously missed in the literature, was identified in a discretized integrable version of the nonlinear Schrodinger equation.

Remark. The formula for the order parameter Δ_h as a function of H_B indicated in the text,

$$\Delta_h = 2\log\frac{\kappa_1}{\kappa_0}\,|_{g=g(H=H_B/2)}, \tag{B11}$$

is evaluated there explicitly (see Fig. 2) from the parametric system

$$\Delta_h = 2\log\frac{\sqrt{-W_{-1}(-2g^2)}}{\sqrt{-W_0(-2g^2)}}, \tag{B12}$$
$$\Psi'(-g) = e^{H_B/2}. \tag{B13}$$

APPENDIX C: MORE DETAILS ON THE SCATTERING PROBLEM

We give some details on the determination of the scattering amplitudes mentioned in the text.

Equation for $\bar{\phi}$ at $t = 1$. Consider the ∂_x equation of the Lax pair for $\bar{\phi}$ at $t = 1$. Using that $P(x,1) = \delta(x)$ it reads in components

$$\partial_x\bar\phi_1 = -\mathbf{i}\frac{k}{2}\bar\phi_1 - g\delta(x)\bar\phi_2, \quad \partial_x\bar\phi_2 = \mathbf{i}\frac{k}{2}\bar\phi_2 + Q(x,1)\bar\phi_1. \tag{C1}$$

Let us integrate the first equation. Since $\bar\phi_1$ vanishes at $-\infty$ it gives

$$\bar\phi_1(x,1) = -ge^{-\mathbf{i}\frac{k}{2}x}\Theta(x)\bar\phi_2(0,1). \tag{C2}$$

Taking the limit $x\to+\infty$, we obtain from the asymptotics (11) that

$$\tilde b(k,t=1) = -g\bar\phi_2(0,1). \tag{C3}$$

To determine $\bar\phi_2(0,1)$ we can integrate the second of Eqs. (C1), which gives, using Eqs. (C2) and (C3),

$$\begin{array}{ll} e^{-\mathbf{i}\frac{k}{2}x}\bar\phi_2(x,1) = \bar\phi_2(0,1) + \tilde b(k,1)\int_0^x dx' Q(x',1)e^{-\mathbf{i}kx'}, & x>0 \\ \bar\phi_2(x,1) = -e^{\mathbf{i}\frac{k}{2}x}, & x<0, \end{array} \tag{C4}$$

where in the second equation we have used that $\bar\phi_2(x,1)\simeq -e^{\mathbf{i}\frac{k}{2}x}$ for $x\to-\infty$. Assuming continuity of $\bar\phi_2(x,1)$ at $x=0$, this leads to $\bar\phi_2(0,1)=-1$ and to

$$\tilde b(k,t=1) = g \quad \Rightarrow \quad \tilde b(k) = ge^{-k^2} \tag{C5}$$

since we recall that $\tilde b(k,t) = \tilde b(k)e^{k^2 t}$.

Taking the $x\to+\infty$ limit of Eqs. (C4) and adding and subtracting c we see that it is compatible with the asymptotics (11) and gives in addition a relation between $\tilde a(k)$ and $Q(x,1)$:

$$\tilde a(k) = \tilde a(k,1) = 1 - g\lim_{x\to+\infty}\left(\int_0^x dx'(Q(x',1)-c)e^{-\mathbf{i}kx'} - \frac{c}{-\mathbf{i}k}\right) \tag{C6}$$

$$= 1 - g\int_0^{+\infty} dx'(Q(x',1)-c)e^{-\mathbf{i}kx'} + g\frac{c}{-\mathbf{i}k}. \tag{C7}$$

Equation for ϕ at $t=1$. Consider the ∂_x equation of the Lax pair for ϕ at $t=1$. Using that $P(x,1)=\delta(x)$ it reads in components

$$\partial_x\phi_1 = -\mathbf{i}\frac{k}{2}\phi_1 - g\delta(x)\phi_2, \quad \partial_x\phi_2 = \mathbf{i}\frac{k}{2}\phi_2 + Q(x,1)\phi_1, \tag{C8}$$

which can be rewritten as

$$[e^{\mathbf{i}\frac{k}{2}x}\phi_1(x,1)]' = -g\delta(x)\phi_2(x,1)e^{\mathbf{i}\frac{k}{2}x}, \quad [e^{-\mathbf{i}\frac{k}{2}x}\phi_2(x,1)]' = Q(x,1)\phi_1(x,1)e^{-\mathbf{i}\frac{k}{2}x}. \tag{C9}$$

Integrating these two equations, and using the asymptotics (11) at $x\to+\infty$ and $\phi_1(x,1)\to e^{-\mathbf{i}kx/2}$ and $\phi_2(x,1)\to\frac{c}{-\mathbf{i}k}e^{-\mathbf{i}kx/2}$ at $x\to-\infty$, we obtain

$$\begin{aligned} &\phi_1(x,1) = e^{-\mathbf{i}\frac{k}{2}x}(\Theta(-x)+a(k)\Theta(x)), \quad a(k)-1 = -g\phi_2(0,1), \\ &\phi_2(x,1) = e^{\mathbf{i}\frac{k}{2}x}\lim_{X\to-\infty}\left(\int_X^x dx' Q(x',1)e^{-\mathbf{i}kx'}(\Theta(-x')+a(k)\Theta(x')) + e^{-\mathbf{i}kX}\frac{c}{-\mathbf{i}k}\right), \end{aligned} \tag{C10}$$

where we used that $a(k,t)=a(k)$. The last equation can be rewritten as

$$\phi_2(x,1) = e^{\mathbf{i}\frac{k}{2}x}\lim_{X\to-\infty}\left(\int_X^x dx' Q(x',1)e^{-\mathbf{i}kx'}(\Theta(-x')+a(k)\Theta(x')) - \int_X^x dx' e^{-\mathbf{i}kx'}c + e^{-\mathbf{i}kx}\frac{c}{-\mathbf{i}k}\right). \tag{C11}$$

Setting $x=0$ we obtain a relation between $\tilde a(k)$ and $Q(x,1)$:

$$\begin{aligned} a(k) &= 1 - g\phi_2(0,1) \\ &= 1 - g\int_{-\infty}^0 dx'(Q(x',1)-c)e^{-\mathbf{i}kx'} - g\frac{c}{-\mathbf{i}k}. \end{aligned} \tag{C12}$$

Note that integrating the second of Eqs. (C9) for $\phi_2(x,1)e^{-\mathbf{i}\frac{k}{2}x}$ between zero and $+\infty$ and using the asymptotics (11) leads to an expression for $b(k)$; however, this expression is equivalent to the one obtained from the relation $a(k)\tilde a(k)+b(k)\tilde b(k)=1$ obtained from the Wronskian (see the main text) together with the above results for $\tilde b(k)$, $\tilde a(k)$, $a(k)$.

From the above results we see that if $Q(x,1)$ is even one has $\tilde a(k)=a(-k)=a^*(k)$ (for real k). From the Wronskian relation and Eq. (C5) one thus gets $b(k)ge^{-k^2} = 1 - a(k)a(-k) = 1-|a(k)|^2$; hence $b(k)$ is real and even in k. Alternatively one sees that $|a(k)|$ is fixed by $b(k)$ so one can write

$$a(k) = e^{-\mathbf{i}\varphi(k)}\sqrt{1-gb(k)e^{-k^2}}, \tag{C13}$$

where $\varphi(k)$ is a real and odd function $\varphi(k)=-\varphi(-k)$, as discussed in the text.

It is important to note that the analysis of the scattering equation was performed here assuming that the parity is not broken, which holds for the flat IC.

Remark: Small k behavior. Since we expect that $Q(x,1)$ is smooth and decays fast towards c as $x\to\pm\infty$ we can extract from the relations obtained above the behavior of the scattering amplitudes as $k\to 0$,

$$a(k)\simeq g\frac{c}{\mathbf{i}k},\quad \tilde{a}(k)\simeq g\frac{c}{-\mathbf{i}k}, \tag{C14}$$

which implies

$$b(k)=\frac{1}{\tilde{b}(k)}(1-a(k)\tilde{a}(k))\simeq -gc^2\frac{1}{k^2}, \tag{C15}$$

which is consistent with Eq. (16) in the text. The integrands in the functions $A_t(x)$ and $B_t(x)$ in Eqs. (20) and (21), i.e., the reflection amplitudes $r(k)$ and $\tilde{r}(k)$, thus behave respectively for small k as

$$r(k)=b(k)/a(k)\simeq -\tilde{a}(k)/g\simeq \frac{c}{\mathbf{i}k},$$
$$\tilde{r}(k)=\tilde{b}(k)/(g\tilde{a}(k))\simeq \frac{-\mathbf{i}k}{cg}. \tag{C16}$$

Remark: Schrödinger equation. It is interesting to note that the ∂_x equation of the Lax pair can always be written as a Schrödinger equation, albeit with a complex potential in the general case. One has

$$\partial_x\phi_1(x)=-\mathbf{i}\frac{k}{2}\phi_1(x)-gP(x)\phi_2(x),$$
$$\partial_x\phi_2(x)=\mathbf{i}\frac{k}{2}\phi_2(x)+Q(x)\phi_1(x), \tag{C17}$$

where here $Q(x)=Q(x,t)$, $P(x)=P(x,t)$, and t can be arbitrary and fixed, so we suppress the time variable. One can eliminate ϕ_1 and one obtains that ϕ_2 satisfies

$$\phi_2''(x)-\frac{\phi_2'(x)Q'(x)}{Q(x)}+\left(gP(x)Q(x)+\frac{k^2}{4}+\frac{\mathbf{i}kQ'(x)}{2Q(x)}\right)\times\phi_2(x)=0. \tag{C18}$$

The first derivative term can be eliminated by writing

$$\phi_2(x)=\sqrt{Q(x)}f_2(x), \tag{C19}$$

where now $f_2(x)$ satisfies a Schrödinger equation

$$f_2''(x)+\frac{1}{4}f_2(x)\left(4gP(x)Q(x)+k^2+\frac{2(Q''(x)+\mathbf{i}kQ'(x))}{Q(x)}-\frac{3Q'(x)^2}{Q(x)^2}\right)=0. \tag{C20}$$

In the general case the potential is complex, and the problem is non-Hermitian. However, for the flat IC, $Q(x)=c$, it simplifies and one obtains the simple result given in the text.

APPENDIX D: NUMERICAL EVALUATIONS

In this section we present some additional numerical evaluations which support the results presented in the text.

1. Functions φ, A_t, and B_t

First we have plotted in Fig. 3 the function $\varphi(k)$ defined in Eq. (22) as a function of k. It clearly shows that it has a discontinuity at $k=0$ with $\varphi(0^\pm)=\mp\frac{\pi}{2}$ as stated in the text.

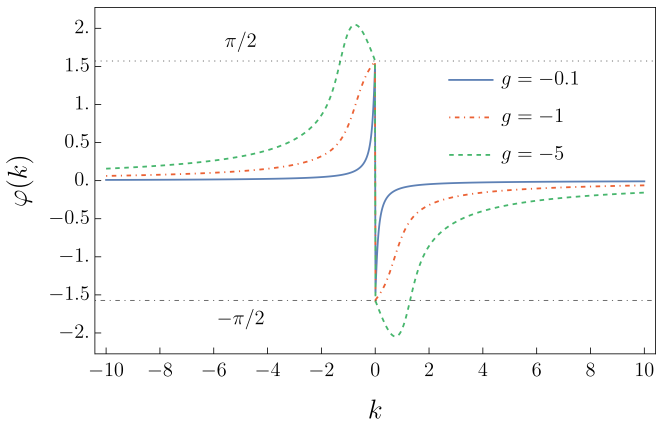

FIG. 3. The phase $\varphi(k)$ defined in Eq. (22) plotted versus k for various values of g.

Next we have plotted in Fig. 4 the function $A_t(x)$ for several values of positive time t and g corresponding to the main branch (20) and to the second branch (26), as well as at the critical point $g=g_{c2}$. We recall that the function $g(H)$ is plotted in Fig. 2 in the text.

In Fig. 5 we have plotted the function $B_t(x)$ for several values of positive time t and g corresponding to the main branch (21) and second branch (26), as well as at the critical point $g=g_{c2}$.

We have also plotted these functions for negative times (as is of interest for the Brownian IC; see text) for the same values of g for the main and second branches. These are shown in Figs. 6 and 7. Note the relation (see text) $A_t''(x)=gB_{-t}(x)$ valid in all the symmetric branches.

2. Optimal height and noise, evaluation of P, Q

From the above exact solutions for $A_t(x)$ and $B_t(x)$ we obtain the solutions to the $\{P,Q\}_g$ system through the Fredholm operator inversion formula (18) for various values of H and g. We use the numerical method developed in Sec. S-l of Ref. [9].

For the solution for the flat IC, we have performed several numerical checks of some highly nontrivial consequences of the formulas, which validate our conjecture:

(i) The functions P, Q are even in x.
(ii) $Q(x,t=1)=A_1(|x|)$.
(iii) $Q(0,t=1)=e^H=\Psi'(-g)$.
(iv) $Q(\pm\infty,t)=1$.

We found them to hold in all three branches in the case of the flat IC. The results for the optimal height $h_{\rm opt}(x,t)=\log Q(x,t)$ are plotted in Fig. 2. Concerning the extension of the flat IC solution to negative times, of interest for the Brownian initial condition, we have also performed a numerical check of the symmetry (A18) in the main branch, the second branch, and the symmetric third branch.

APPENDIX E: THE LAMBERT W FUNCTION

We introduce the Lambert W function [40] which we use extensively throughout this work. Consider the function defined on $\mathbb{C}$ by $f(z)=ze^z$; the W function is composed of all inverse branches of f so that $W(ze^z)=z$. It does have two real

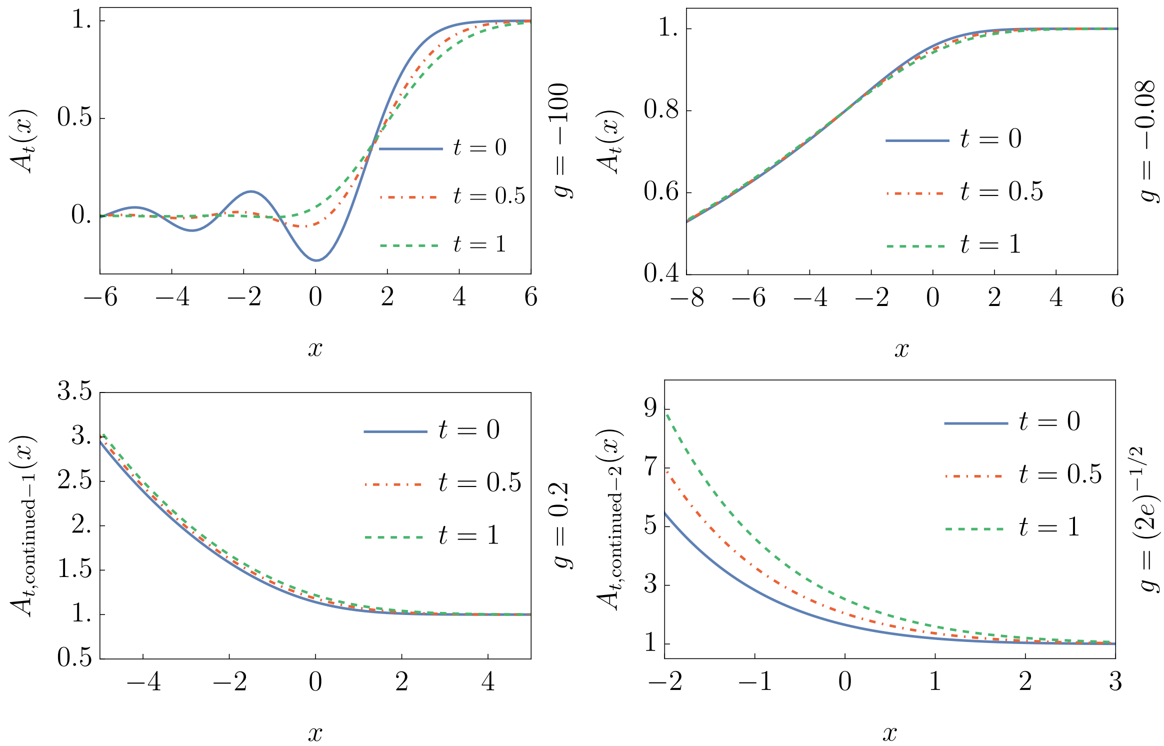

FIG. 4. Plot of the function $A_t(x)$ for various positive times t and coupling constants g for the main and second branches.

branches, W_0 and W_{-1}, defined respectively on $[-e^{-1}, +\infty[$ and $[-e^{-1}, 0[$. On their respective domains, W_0 is strictly increasing and W_{-1} is strictly decreasing. By differentiation of $W(z)e^{W(z)} = z$, one obtains a differential equation valid for all branches of $W(z)$:

$$\frac{dW}{dz}(z) = \frac{W(z)}{z(1 + W(z))}. \tag{E1}$$

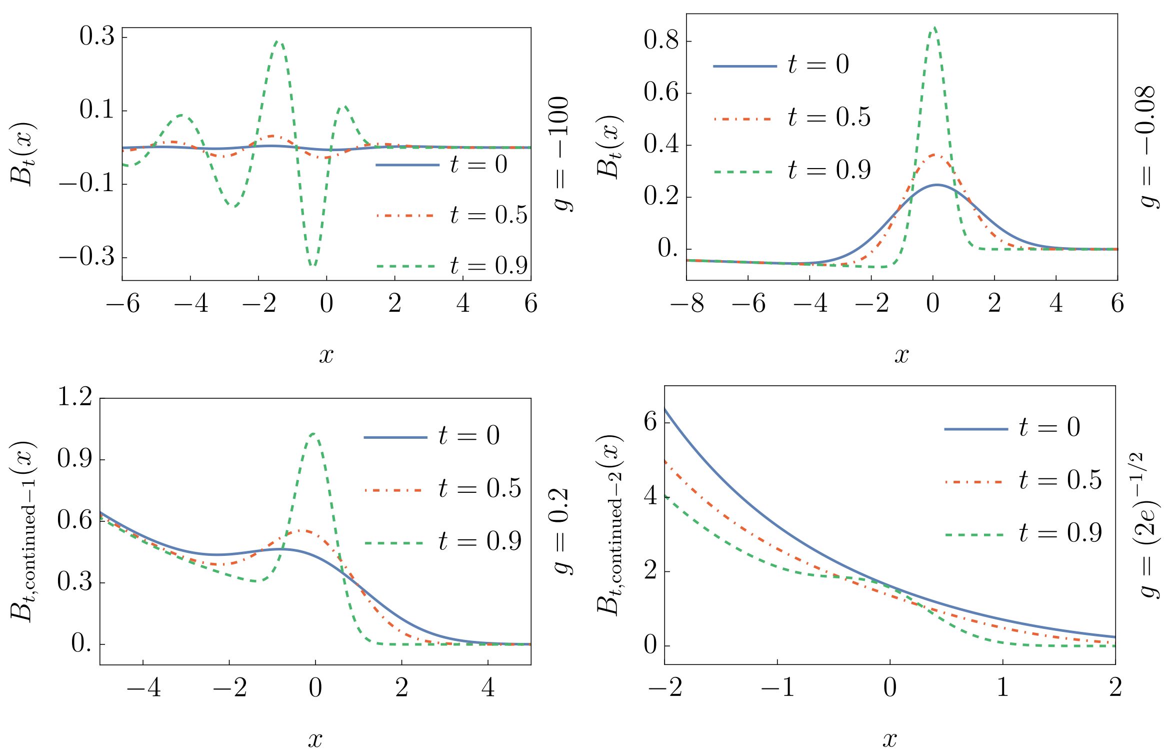

FIG. 5. Plot of the function $B_t(x)$ for various positive times t and coupling constants g for the main and second branches.

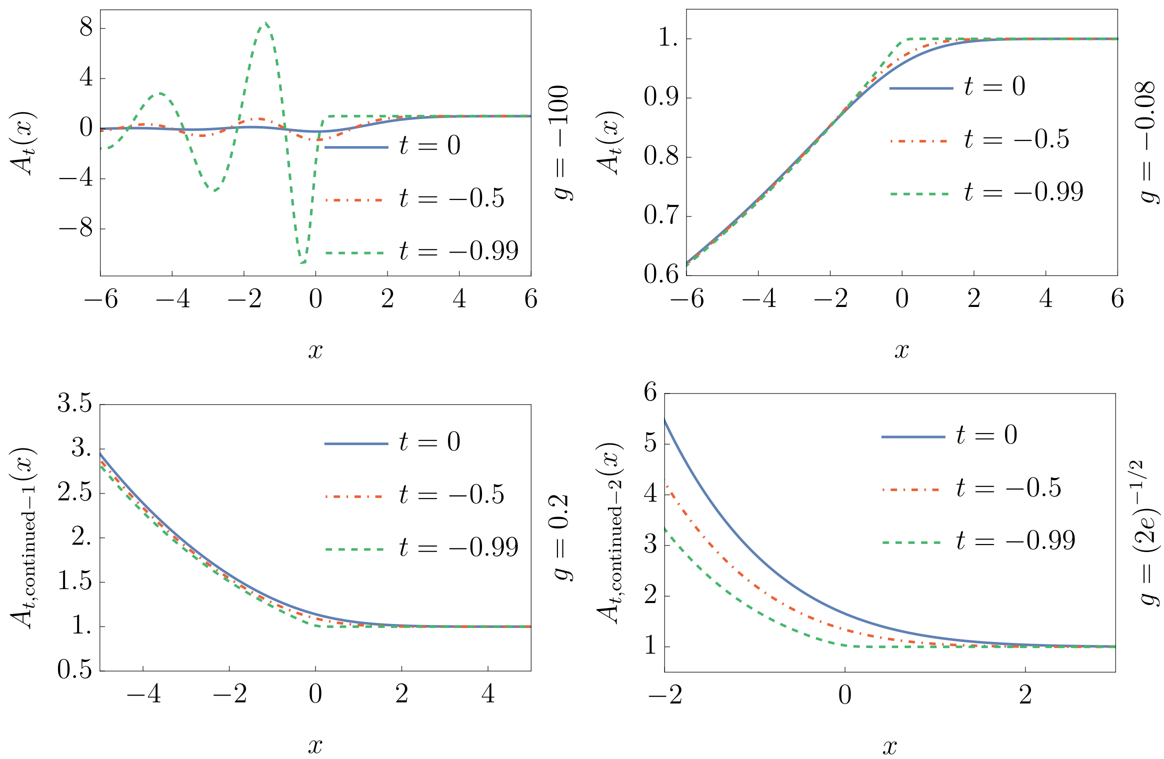

FIG. 6. Plot of the function $A_t(x)$ for various positive times t nad coupling constants g for the main and second branches.

Concerning their asymptotics, W_0 behaves logarithmically for large argument $W_0(z) \simeq_{z\to+\infty} \log(z) - \log\log(z)$ and is linear for small argument $W_0(z) =_{z\to 0} z - z^2 + \mathcal{O}(z^3)$. W_{-1} behaves logarithmically for small argument $W_{-1}(z) \simeq_{z\to 0^-} \log(-z) - \log(-\log(-z))$. Both branches join smoothly at the point $z = -e^{-1}$ and have the value $W(-e^{-1}) = -1$. These remarks are summarized in Fig. 8. More details on the other branches, W_k for integer k, can be found in Ref. [40].

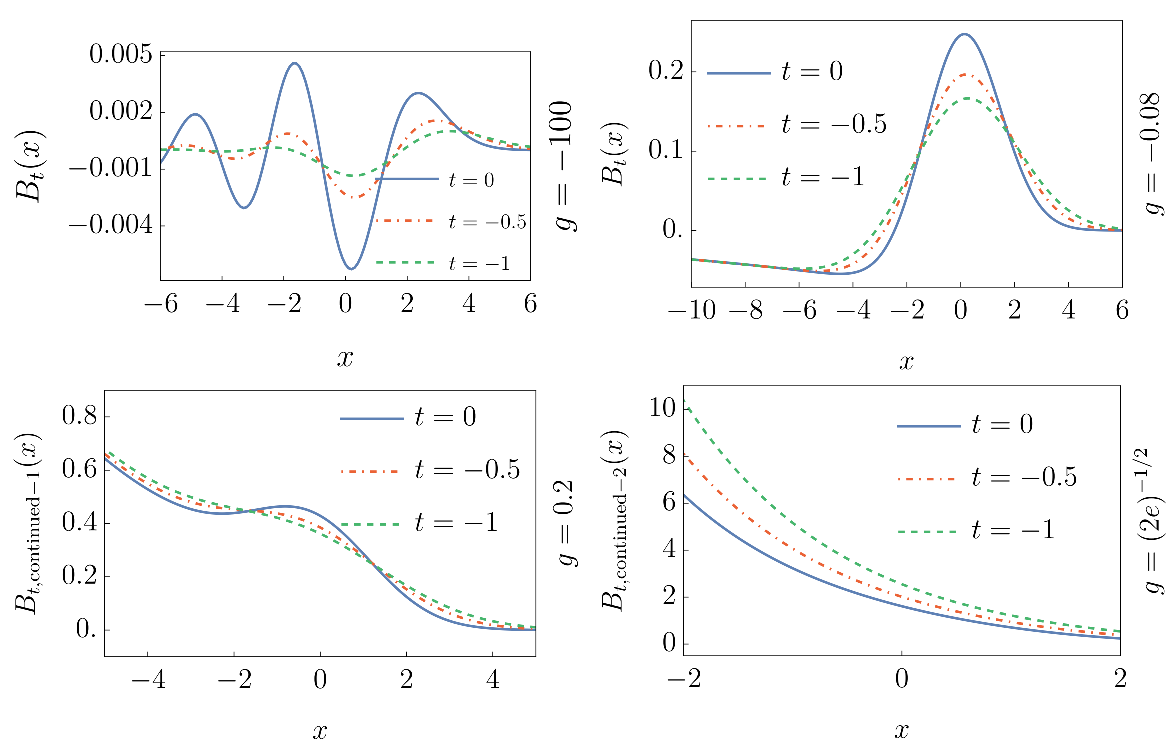

FIG. 7. Plot of the function $A_t(x)$ for various positive times t and coupling constants g for the main and second branches.

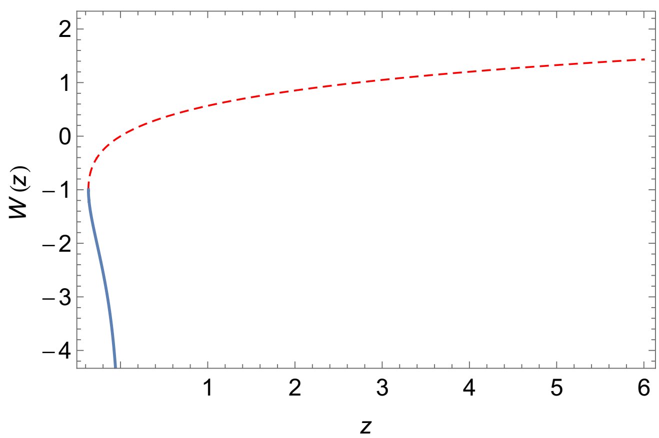

FIG. 8. The Lambert function W. The dashed red line corresponds to the branch W_0 whereas the blue line corresponds to the branch W_{-1}.

APPENDIX F: PHASE $\varphi(k)$ FROM THE AMPLITUDE $a(k)$

Let us recall briefly how the phase $\varphi(k)$ can be obtained from the analytic properties of the amplitude $a(k)$ from the Kramers-Kronig relations. Let us recall that $a(k)$ is assumed to be analytic in the upper half plane. By definition $a^*(k) = (a(k^*))^*$. Hence $a^*(k)$ is analytic in the lower half plane. Consider the contour C which passes just above the real axis, i.e., $p = \mathbf{i}0^+ + \mathbb{R}$, and which closes at infinity along a large half circle the upper half plane. One has

$$\int_C dp \frac{1}{p + \mathbf{i}0^+ - k} \log a(p) = 0. \tag{F1}$$

Since we know that $\log a(k) \simeq C_1/(\mathbf{i}k)$ for large $|k|$ the contribution of the large circle vanishes. In the contribution near the real axis we can replace $\frac{1}{p+\mathbf{i}0^+-k} = PV\frac{1}{p-k} - \mathbf{i}\pi\delta(p-k)$ (using the Sokhotski-Plemelj theorem) and one finds

$$\log a(k) = \fint_C \frac{dp}{\mathbf{i}\pi} \frac{1}{p-k} \log a(p). \tag{F2}$$

Taking the imaginary part one obtains

$$\mathrm{Im} \log a(k) = -\fint_C \frac{dp}{2\pi} \frac{1}{p-k} \log a(p)a^*(p). \tag{F3}$$

We have defined $a(k) = |a(k)|e^{-i\varphi(k)}$; hence we get

$$\varphi(k) = \fint_C \frac{dp}{2\pi} \frac{1}{p-k} \log a(p)a^*(p), \tag{F4}$$

which agrees with Eq. (22) in the text.

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