Long-term properties of finite-correlation-time isotropic stochastic systems

A. S. Il'yn,* A. V. Kopyev,† V. A. Sirota,‡ and K. P. Zybin§
P. N. Lebedev Physical Institute, RAS, Leninskij Prospekt 53, 119991 Moscow, Russia

(Received 14 March 2022; accepted 20 April 2022; published 20 May 2022)

We consider finite-dimensional systems of linear stochastic differential equations $\partial_t x_k(t) = A_{kp}(t)x_p(t)$, $\mathbf{A}(t)$ being a stationary continuous statistically isotropic stochastic process with values in real $d \times d$ matrices. We suppose that the laws of $\mathbf{A}(t)$ satisfy the large-deviation principle. For these systems, we find exact expressions for the Lyapunov and generalized Lyapunov exponents and show that they are determined in a precise way only by the rate function of the diagonal elements of $\mathbf{A}$.

DOI: 10.1103/PhysRevE.105.054130

I. INTRODUCTION

Finite-dimensional first-order systems of linear differential equations with stochastic coefficients appear in stochastic dynamics, the theory of turbulence and turbulent transport, kinematic dynamos, and many other physical, chemical, and biological problems (see, e.g., [1,2]). The long-term solutions of these systems have been investigated for the case of Gaussian and δ-correlated-in-time stochastic noise [3,4]. In this paper we study the case of isotropic non-Gaussian statistics and the finite correlation time of the noise.

Consider a system of linear equations

$$\partial_t x_k(t) = A_{kp}(t)x_p(t), \quad k = 1, \dots, d, \tag{1}$$

where $\mathbf{A}(t)$ is a stationary continuous real $d \times d$ matrix stochastic process with a known law. It is assumed to be statistically isotropic, i.e., $\mathbf{O}\mathbf{A}(t)\mathbf{O}^T$ and $\mathbf{A}(t)$ are identically distributed at any t and have the same time correlations for any $\mathbf{O} \in O(d)$.

The formal solution of (1) can be written as

$$x_k(t) = Q_{kp}(t)x_p(0),$$

where the evolution matrix $\mathbf{Q}$ satisfies the equation

$$\partial_t \mathbf{Q}(t) = \mathbf{A}(t)\mathbf{Q}(t), \quad \mathbf{Q}(0) = \mathbf{1}. \tag{2}$$

For each continuous realization of $\mathbf{A}(t)$ there exists a well-defined solution of (2) given by the Volterra product integral [5] (in quantum-mechanical terminology this is also called T exponential)

$$\mathbf{Q}(t) = \prod_{\tau=0}^{t}[\mathbf{1} + \mathbf{A}(\tau)d\tau]. \tag{3}$$

So the solutions of (2) can be interpreted as continual products of random matrices.

In many physical problems (e.g., stochastic dynamics and turbulent transport) one is basically interested in the infinite-time asymptotic behavior of the norms of the basis multivectors [6,7]

$$E_1 = \|\mathbf{Q}\vec{e}_1\|, \quad E_2 = \|\mathbf{Q}\vec{e}_1 \wedge \mathbf{Q}\vec{e}_2\|,$$
$$E_d = \|\mathbf{Q}\vec{e}_1 \wedge \cdots \wedge \mathbf{Q}\vec{e}_d\|.$$

For instance, the absolute value of a passive vector advected by a random smooth flow is E_1; the density of an advected passive scalar in the presence of weak molecular diffusion is $\prod_{k=1}^{d} \min\{E_{k-1}E_k^{-1}, 1\}$ [8,9]; the energy of the magnetic field generated by a turbulent magnetohydrodynamic flow with strong conductivity is $\min\{E_1^2/E_2, E_2^2/E_1\}$ [10–12].

From the multiplicative Oseledets theorem [13] it follows that almost surely there exist the limits

$$\lim_{T\to\infty}\left(\frac{1}{T}\ln E_k\right) = \lambda_1 + \cdots + \lambda_k, \quad k = 1, \dots, d. \tag{4}$$

The constants λ_k are called Lyapunov exponents (LEs). Complete information about the asymptotic behavior of the set $\{E_k\}$ is contained in the generalized Lyapunov exponents (GLEs) [14] defined by

$$w(\eta_1, \dots, \eta_d) = \lim_{T\to\infty}\frac{1}{T}\ln\left\langle E_1^{\eta_1-\eta_2}E_2^{\eta_2-\eta_3}\cdots E_{d-1}^{\eta_{d-1}-\eta_d}E_d^{\eta_d}\right\rangle, \tag{5}$$

where $\langle\cdots\rangle$ is the average over all realizations of $\mathbf{A}(t)$, $\eta_k \in \mathbb{R}$. For example, the LEs can be expressed in terms of GLEs:

$$\lambda_k = \frac{\partial}{\partial \eta_k} w(0). \tag{6}$$

The calculation of the values of (4) and (5) is the subject of this paper.

Most previous investigations considered the case of a Gaussian $\mathbf{A}(t)$ that is δ correlated in time. For this case, the exact expressions for LEs were found in [15–17]. The δ-correlated process typically has paths that have a break at

*Also at National Research University Higher School of Economics, Myasnitskaya Ulitsa 20, 101000 Moscow, Russia; asil72@mail.ru
†kopyev@lpi.ru
‡sirota@lpi.ru
§Also at National Research University Higher School of Economics, Myasnitskaya Ulitsa 20, 101000 Moscow, Russia; zybin@mail.ru

each of their points, so Eq. (2) requires a stochastic convention; the known results for that case refer to the Stratonovich convention [18,19].

However, in many physical applications random processes acting as multiplicative noise are generally non-Gaussian and have finite (nonzero) correlation time. For continuous processes the solutions (3) are well defined, so there is no need for stochastic conventions. It is well known [20,21] that for such systems, the central limit theorem *is not valid* in the sense that all connected correlators (cumulants) of $\mathbf{A}(t)$ contribute equivalently to the LEs and GLEs. So the result may differ essentially from the Gaussian case.

To deal with nonzero correlation time, the renovation model has been developed by different authors [22,23]: A nonstationary piecewise constant process is substituted for the stationary continuous process $\mathbf{A}(t)$. Alternatively, we stay in the frame of stationarity and use the large-deviation principle; in quantum mechanics and quantum field theory the corresponding technique is known as the low-frequency limit. For the processes $A_{ij}(t)$ that satisfy the large-deviation principle, this allows us to get exact expressions for LEs and GLEs; it appears that they are completely determined by the rate function of the diagonal elements A_{kk}.

Iwasawa decomposition

To get explicit expressions for the LEs and GLEs, one performs the Iwasawa decomposition of the evolution matrix

$$\mathbf{Q} = \mathbf{RDZ},$$

where $\mathbf{R}$ is the orthogonal matrix, $\mathbf{D}$ is positive diagonal, and $\mathbf{Z}$ is the upper triangular unipotent matrix:

$$\mathbf{RR}^T = \hat{\mathbf{I}}, \quad D_{ij} = \delta_{ij}D_i, \quad D_i > 0, \quad Z_{i<j} = 0, \quad Z_{jj} = 1.$$

Then $E_k = D_1 \dots D_k$ and the LEs can be expressed by

$$\lambda_k = \lim_{T\to\infty} \frac{1}{T} \ln D_k, \quad \lambda_1 \geqslant \cdots \geqslant \lambda_k. \tag{7}$$

The expression for the GLEs takes the form

$$w(\eta_1, \dots, \eta_d) = \lim_{T\to\infty} \frac{1}{T} \ln\langle D_1^{\eta_1} \cdots D_d^{\eta_d}\rangle. \tag{8}$$

The statistical properties of $\mathbf{R}$, $\mathbf{D}$, and $\mathbf{Z}$ behave differently during the evolution (2); we are interested only in the statistics of $\mathbf{D}$.

In the next section we consider the main strategy and the application of the large-deviation theory on a simple one-dimensional example. Then we proceed to the multidimensional case and make use of the isotropy to calculate the GLEs. In conclusion, we reformulate the results in terms of the effective δ process, which is a useful tool for physical applications.

II. ONE-DIMENSIONAL CASE

A. Rate function and GLEs

Consider one-dimensional differential equation with multiplicative noise

$$\partial_t x(t) = \xi(t)x(t), \quad x(0) = 1, \tag{9}$$

where $\xi(t)$ is a continuous stationary random process with a given law. We are interested in the moments $\langle x^\eta\rangle$, $\eta \in \mathbb{R}$.

For any realization of $\xi(t)$, the solution of (9) is

$$x(T) = \exp\left(\int_0^T \xi(t)dt\right).$$

We assume that $\xi(t)$ satisfies the large-deviation principle [24], i.e., that the probability density of its time average

$$\frac{1}{T}\int_0^T \xi(t)dt = \bar{\xi}$$

satisfies at large $T \to \infty$ the relation

$$\rho_{\bar{\xi}}(a) \sim e^{-TJ(a)},$$

where the sign $\sim$ means that there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \ln \rho_{\bar{\xi}}(a) = -J(a).$$

Here $J(a)$ is the rate function (Cramér function). Then

$$\langle x^\eta(T)\rangle = \int e^{T\eta a}\rho_{\bar{\xi}}(a)da \sim e^{Tw(\eta)}, \tag{10}$$

where $w(\eta)$ is the Legendre transform of the rate function

$$w(\eta) = \sup_a[\eta a - J(a)]. \tag{11}$$

This proves the existence of the limit

$$\lim_{T\to\infty} \frac{1}{T} \ln\langle x^\eta(T)\rangle = w(\eta), \tag{12}$$

where $w(\eta)$ is the GLE of the process $\xi(t)$. We note that, according to (10), the function $w(\eta)T$ at large T coincides with the cumulant-generating function of the integral $\int_0^T \xi(t)dt$.

Let the cumulants (or connected correlators)

$$\langle \xi(t_1)\cdots\xi(t_n)\rangle_c = W^{(n)}(t_1 - t_2, \dots, t_1 - t_n) \tag{13}$$

be regular fast-decaying functions (i.e., let $\int W^{(n)}dt_2\cdots dt_n$ converge). Then the cumulant-generating functional

$$W[\eta(t)] = \ln\left\langle \exp\left(\int dt\, \eta(t)\xi(t)\right)\right\rangle \tag{14}$$

can be expanded in the infinite series in $\eta(t)$:

$$W[\eta(t)] = \sum_n \frac{1}{n!}\int W^{(n)}(t_1 - t_2, \dots, t_1 - t_n) \times \eta(t_1)\cdots\eta(t_n)dt_1\cdots dt_n. \tag{15}$$

In accordance with (14),

$$\langle x^\eta(T)\rangle = \left\langle \exp\left(\eta\int_0^T \xi(t)dt\right)\right\rangle = e^{W[\eta I_{[0,1]}(t/T)]}, \tag{16}$$

where $I_{[0,1]}$ is the indicator function of the segment $[0, 1]$. Substituting (15), we find

$$\lim_{T\to\infty} \frac{1}{T} \ln\langle x^\eta(T)\rangle = \sum_n \frac{w^{(n)}}{n!}\eta^n, \tag{17}$$

where

$$w^{(n)} = \int W^{(n)}(t_1 - t_2, \dots, t_1 - t_n)dt_2\cdots dt_n. \tag{18}$$

Comparing (10) with (17), we get

$$w(\eta) = \sum_n \frac{w^{(n)}}{n!}\eta^n. \tag{19}$$

From (13) and (19) we see that the long-time asymptotics of the moments $\langle x^\eta \rangle$, as well as the GLEs, do not depend on the details of the cumulants and are determined only by their integrals (18).

B. Low-frequency limit

We now consider an alternative way to find the GLEs based on the Lagrangian formalism. The aim is to derive the relation between the Lagrangian density and the rate function.

We start from the probability density functional $\wp[\xi(t)]$, which can be presented in the form

$$\wp[\xi] \sim e^{-\mathcal{S}[\xi]}, \quad \mathcal{S}[\xi(t)] = \int dt\, \mathcal{L}(\xi, \partial_t\xi, \partial_t^2\xi, \dots).$$

Here $\mathcal{L}$ is the Lagrangian density. For example, the case $\mathcal{L} = \frac{1}{2}(\epsilon\partial_\tau\xi)^2 + \frac{1}{2}\xi^2$ corresponds to Ornstein-Uhlenbeck process with the only nonzero cumulant $W^{(2)}(t_1 - t_2) = \frac{1}{2\epsilon}e^{-|t_1-t_2|/\epsilon}$. Generally, $\mathcal{L}$ is nonlocal and contains derivatives of all orders. The generating functional can then be written in the form of the Feynman-Kac integral [25]

$$e^{W[\eta(t)]} \sim \int [d\xi] \exp\left(-\int dt[\mathcal{L}(t) - \eta(t)\xi(t)]\right). \tag{20}$$

In the Section II A we saw that the logarithm of the moments $\langle x^m \rangle$ is determined by the functions $\eta(t)$ that are constant in the time range $[0, T]$ [Eq. (16)]. For this reason, we are interested in the values of $W[\eta(t)]$ on slowly varying functions with characteristic timescale T: $\eta_T(t) = \eta(t/T)$ (the low-frequency limit). In (20) we rescale the time $t = \tau T$ and change accordingly the integration variable: $\xi(t) = \xi_T(\tau)$. Then from (20) we obtain

$$e^{W[\eta_T(t)]} \sim \int [d\xi_T] \exp\left(-T\int d\tau[\mathcal{L}_T(\tau) - \eta(\tau)\xi_T(\tau)]\right), \tag{21}$$

where

$$\mathcal{L}_T(\tau) = \mathcal{L}\left(\xi_T, \frac{1}{T}\partial_\tau\xi_T, \frac{1}{T^2}\partial_\tau^2\xi_T, \dots\right).$$

As $T \to \infty$, we can substitute the rate function

$$J(\xi_T) = \mathcal{L}(\xi_T, 0, 0, \dots) + C \tag{22}$$

for $\mathcal{L}_T$. Here C is a normalization constant. Then we estimate the integral by means of the saddle-point method

$$\int [d\xi_T] \exp\left(-T\int d\tau[J(\xi_T(\tau)) - \eta(\tau)\xi_T(\tau)]\right) \sim \exp\left(-T\int d\tau[J(\xi_s(\tau)) - \eta(\tau)\xi_s(\tau)]\right),$$

where $\xi_s(\tau)$ is defined by the minimum condition

$$\frac{\partial J(\xi_s)}{\partial \xi} = \eta(\tau). \tag{23}$$

Eventually, we get

$$e^{W[\eta_T(t)]} = e^{W[\eta(t/T)]} \sim \exp\left(T\int w(\eta(\tau))d\tau\right), \tag{24}$$

where

$$w(\eta) = \sup_\xi[\eta\xi - J(\xi)]. \tag{25}$$

The normalization condition $W[0] = 0$ results in the claim $w(0) = 0$. Now, substituting (24) in (16), we find

$$\langle x^\eta(T)\rangle \sim \exp\left(T\int_0^1 w(\eta)d\tau\right) = e^{Tw(\eta)}.$$

Comparing this with (10) and (13), we see that the function $w(\eta)$ defined in this section coincides with the GLE found in (11) and (19). From (11) and (25) it also follows that the function $J(\xi)$ defined in (22) coincides with the rate function.

So both ways to determine the rate function are equivalent. In the multidimensional case the second way appears to be more convenient.

III. MULTIDIMENSIONAL EQUATION

A. Equation for the Iwasawa components

Now we return to the matrix equation (2), where $A(t)$ is a stationary continuous stochastic process with regular fast-decaying connected correlations

$$\langle A_{i_1j_1}(t_1)\cdots A_{i_nj_n}(t_n)\rangle_c = W^{(n)}_{i_1j_1\cdots i_nj_n}(t_1 - t_2, \dots, t_1 - t_n) \tag{26}$$

and nonlocal Lagrangian density $\mathcal{L}_A(\mathbf{A}, \partial_t\mathbf{A}, \partial_t^2\mathbf{A}, \dots)$. The well-defined solution (3) exists for any continuous realization of $\mathbf{A}(t)$, but noncommutativity makes it difficult to use: There is a T exponential instead of a usual exponential and it seems impossible to apply the large-deviation approach for $\int \mathbf{A}\,dt$. However, we will see in what follows that this is possible in the case of isotropic law of $\mathbf{A}$ at least for the important *diagonal* (in the sense of Iwasawa decomposition) part of the evolution matrix.

To separate the Iwasawa components, we rewrite Eq. (2) in the form

$$\mathbf{A} = \partial_t\mathbf{Q}\,\mathbf{Q}^{-1}$$

and substitute the Iwasawa decomposition for $\mathbf{Q}$. We obtain

$$\mathbf{A} = \mathbf{R}\mathbf{X}\mathbf{R}^{-1}, \quad \mathbf{X} = \boldsymbol{\xi} + \boldsymbol{\zeta} + \boldsymbol{\theta}, \tag{27}$$

where

$$\boldsymbol{\xi} = (\partial_t\mathbf{D})\mathbf{D}^{-1}, \quad \boldsymbol{\zeta} = \mathbf{D}(\partial_t\mathbf{Z})\mathbf{Z}^{-1}\mathbf{D}^{-1}, \quad \boldsymbol{\theta} = \mathbf{R}^{-1}(\partial_t\mathbf{R}). \tag{28}$$

The matrices $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\theta}$ are diagonal, nilpotent upper triangular, and antisymmetric, respectively. Eq. (28) can be rewritten as

$$\partial_t\mathbf{D} = \boldsymbol{\xi}\mathbf{D}, \tag{29}$$

$$\partial_t\mathbf{Z} = \mathbf{D}^{-1}\boldsymbol{\zeta}\mathbf{D}\mathbf{Z}, \tag{30}$$

$$\partial_t\mathbf{R} = \mathbf{R}\boldsymbol{\theta}. \tag{31}$$

Thus, treating $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\theta}$ as independent variables, we could separate the equations for $\mathbf{D}$ and for $\mathbf{R}$. Moreover, the elements D_i satisfy one-dimensional Eq. (29), similarly to Eq. (9). So (8) takes the form

$$w(\eta_1,\dots,\eta_d)\equiv w_\xi(\eta_1,\dots,\eta_d)=\lim_{T\to\infty}\frac{1}{T}\times\ln\left\langle\exp\left(\int_0^T(\xi_1\eta_1+\dots+\xi_d\eta_d)dt\right)\right\rangle. \tag{32}$$

To calculate this, it would be enough to know the rate function of $\xi=\{\xi_1,\dots,\xi_d\}$ and make use of (25). So in the next section we discuss the relation between the Lagrangian densities of $\mathbf{X}$ and $\mathbf{A}$.

B. Change of variables

One can consider (27) as a functional transformation from A to X variables,

$$\mathbf{A}=\mathbf{R}[\mathbf{X}]\mathbf{X}\mathbf{R}^{-1}[\mathbf{X}], \tag{33}$$

where the dependence $\mathbf{R}(\mathbf{X})$ is determined by (31). To find the probability density of $\mathbf{X}(t)$, one has to calculate the Jacobian

$$\mathcal{J}[X]=\mathrm{Det}\left(\frac{\delta A_{ij}(t)}{\delta X_{kp}(t')}\right), \tag{34}$$

which was calculated, e.g., in [21] as

$$\mathcal{J}[X]=\exp\left(\int\mathrm{tr}(\boldsymbol{\eta}_0\mathbf{X}(t))dt\right), \tag{35}$$

where

$$(\eta_0)_{kp}=\left(\frac{d+1}{2}-k\right)\delta_{kp}. \tag{36}$$

In Appendix A to this paper we derive this result by taking the continuous limit of a stochastic difference equation.

Taking into account (34) and (33), from the condition $\mathcal{P}(\mathbf{X})[d\mathbf{X}]=\mathcal{P}(\mathbf{A})[d\mathbf{A}]$ we get the expression for the Lagrangian density of $\mathbf{X}$,

$$\mathcal{L}_X=-\mathrm{tr}(\boldsymbol{\eta}_0\mathbf{X})+\mathcal{L}_A(\mathbf{A},\partial_t\mathbf{A},\partial_t^2\mathbf{A},\dots),$$

where $\mathbf{A}$ is a function of $\mathbf{X}$ in accordance with (33). From (31) it follows that for any $\mathbf{F}(t)$,

$$\partial_t(\mathbf{R}(t)\mathbf{F}(t)\mathbf{R}^{-1}(t))=\mathbf{R}(t)(D_t\mathbf{F}(t))\mathbf{R}^{-1}(t), \tag{37}$$

where

$$D_t\mathbf{F}=\partial_t\mathbf{F}+[\boldsymbol{\theta},\mathbf{F}], \tag{38}$$

with $[a,b]=ab-ba$. Substituting this for the arguments of $\mathcal{L}_A$, we obtain

$$\mathcal{L}_X=-\mathrm{tr}(\eta_{\mathbf{0}}\mathbf{X})+\mathcal{L}_A\big(\mathbf{R}\mathbf{X}\mathbf{R}^{-1},\ \mathbf{R}(D_t\mathbf{X})\mathbf{R}^{-1},\mathbf{R}\big(D_t^2\mathbf{X}\big)\mathbf{R}^{-1},\dots\big). \tag{39}$$

This expression contains $\mathbf{R}$, which is the Volterra product integral of the components of $\mathbf{X}$, $\mathbf{R}[\mathbf{X}]=\prod_{\tau=0}^{t}(\mathbf{1}+\boldsymbol{\theta}(\tau)d\tau)$. However, we will see below that in the case of isotropic processes $\mathbf{R}$ vanishes.

IV. ISOTROPIC SYSTEMS

We now make use of the claim that $\mathbf{A}$ is isotropic, in particular, $\mathbf{A}(t)$ has the same probability density as $\mathbf{O}\mathbf{A}(t)\mathbf{O}^T$ for any $\mathbf{O}\in O(d)$. For such processes, the Lagrangian density $\mathcal{L}_A$ can be presented as a sum of different combinations of traces with arguments containing products of A, A^T, and their derivatives. As one substitutes $\mathbf{R}\mathbf{X}\mathbf{R}^{-1}$ for $\mathbf{A}$, the plates $\mathbf{R}$ and $\mathbf{R}^{-1}$ in the expressions vanish; however, in accordance with (37) and (39), all time derivatives in the expression for the Lagrangian density change to long derivatives (38),

$$\mathbf{A}\mapsto\mathbf{X},\quad \partial_t^p\mathbf{A}\mapsto D_t^p\mathbf{X}.$$

As we proceed from $\mathcal{L}_X$ to the rate function as in (22), we have to set all the time derivatives of $\mathbf{X}$ zero. However, the commutators stay in their places, so the rate function takes the form

$$J_X(\mathbf{X})=\mathcal{L}_X(\mathbf{X},0,0,\dots)+C=-\mathrm{tr}(\boldsymbol{\eta}_0\mathbf{X})+\mathcal{L}_A(\mathbf{X},[\boldsymbol{\theta},\mathbf{X}],[\boldsymbol{\theta},[\boldsymbol{\theta},\mathbf{X}]],\dots)+C. \tag{40}$$

One can separate the rate function of the matrix $\mathbf{A}$,

$$J_A(\mathbf{A})=\mathcal{L}_A(\mathbf{A},0,\dots),$$

and present J_X in the form

$$J_X(\mathbf{X})=-\mathrm{tr}(\boldsymbol{\eta}_0\mathbf{X})+J_A(\mathbf{X})+\delta J(\mathbf{X})+C, \tag{41}$$

$$\delta J(\mathbf{X})=\mathcal{L}_A(\mathbf{X},[\boldsymbol{\theta},\mathbf{X}],[\boldsymbol{\theta},[\boldsymbol{\theta},\mathbf{X}]],\dots)-\mathcal{L}_A(\mathbf{X},0,\dots).$$

We define $w_X(\boldsymbol{\mu})$ for the matrix process $\mathbf{X}$ in the same way as in (11) and (25),

$$w_X(\boldsymbol{\mu})=\sup_{\mathbf{X}}[\mathrm{tr}(\boldsymbol{\mu}\mathbf{X})-J_X(\mathbf{X})]=\lim_{T\to\infty}\frac{1}{T}\times\ln\left\langle\exp\left(\int_0^T\mathrm{tr}(\boldsymbol{\mu}\mathbf{X})dt\right)\right\rangle. \tag{42}$$

To this purpose, we have to find the extremum point $\mathbf{X}_s$ analogous to ξ_s in (23):

$$\frac{\partial J_X(\mathbf{X}_s)}{\partial X_{qr}}=\mu_{qr}. \tag{43}$$

This equation system is very complicated. However, it can be simplified significantly as we are interested only in the GLEs of the diagonal elements $X_{kk}=\xi_k$. From (32) it follows that

$$w_\xi(\eta_1,\dots,\eta_d)=w_X(\boldsymbol{\eta}), \tag{44}$$

where $\boldsymbol{\eta}=\mathrm{diag}(\eta_1,\eta_2,\dots,\eta_d)$. So, to find w_ξ we can restrict ourselves to the diagonal matrices $\boldsymbol{\mu}=\boldsymbol{\eta}$ in (43), so the extremum condition takes the form

$$\frac{\partial J_X(\mathbf{X}_s)}{\partial X_{qr}}=0,\quad q\neq r$$
$$\frac{\partial J_X(\mathbf{X}_s)}{\partial X_{qq}}=\eta_q,\quad q=1,\dots,d. \tag{45}$$

This system has the diagonal solution

$$\mathbf{X}_s=\boldsymbol{\xi}_s, \tag{46}$$

where $\boldsymbol{\xi}_s$ satisfies the relation

$$\frac{\partial J_A(\boldsymbol{\xi}_s)}{\partial X_{qq}} - (\eta_0)_{qq} = \eta_q. \quad (47)$$

Indeed, we recall that $\mathbf{X} = \boldsymbol{\xi} + \boldsymbol{\theta} + \boldsymbol{\zeta}$ and $J_X(\mathbf{X})$ is a combination of traces $\mathrm{tr}(f(\boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{\zeta}))$, where f is some product of the matrices. Since $\boldsymbol{\xi}$ is diagonal and both $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ have zero diagonal elements, each summand in J_X contains either zero or more than one of the matrices $\boldsymbol{\theta}$ or $\boldsymbol{\zeta}$. Taking the derivative with respect to X_{qr}, $q \neq r$, leaves the summands with at least one multiplier $\boldsymbol{\theta}$ or $\boldsymbol{\zeta}$, and subsequent setting $\mathbf{X} = \boldsymbol{\xi}_s$ makes them zero; so the first equation in (45) holds automatically. On the other hand, each summand in $\delta J(\mathbf{X})$ contains at least one $\boldsymbol{\theta}$ as a multiplier, so $\partial \delta J/\partial X_{qq} = 0$.

We note that $\boldsymbol{\theta}_s = 0$ and thus $\delta J(\mathbf{X}_s) = 0$. From (44), (42), and (41) it then follows that

$$w_\xi(\eta_1, \eta_2, \ldots, \eta_d) = \mathrm{tr}((\boldsymbol{\eta} + \boldsymbol{\eta}_0)\boldsymbol{\xi}_s) - J_A(\boldsymbol{\xi}_s) - C. \quad (48)$$

The statistical isotropy of $\mathbf{A}$ implies $\langle \mathbf{A}_{i\neq j}\rangle = 0$. Thus, $\partial J_A/\partial A_{qr}(\boldsymbol{\xi}_s) = 0$ for $q \neq r$, and $J_A(\boldsymbol{\xi})$ coincides with the rate function of the diagonal elements $\alpha_q = A_{qq}$:

$$J_\alpha(\alpha_1, \ldots, \alpha_d) = J_A(\boldsymbol{\alpha}),$$

$$\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1, \ldots, \alpha_d).$$

So (48) proves that the GLEs and, as a consequence, also the Lyapunov exponents are completely determined by the rate function of the diagonal elements of $\mathbf{A}$.

By analogy with (42), we define the local cumulant-generating function of these diagonal elements:

$$w_\alpha(\boldsymbol{\eta}) = \sup_{\boldsymbol{\alpha}}[\mathrm{tr}(\boldsymbol{\eta}\boldsymbol{\alpha}) - J_\alpha(\alpha_1, \ldots, \alpha_d)] = \lim_{T\to\infty} \frac{1}{T} \ln\left\langle \exp\left(\int_0^T \mathrm{tr}(\boldsymbol{\eta}\boldsymbol{\alpha})dt\right)\right\rangle.$$

It is related to the cumulants of α_i by

$$w_\alpha(\eta_1, \ldots, \eta_d) = \sum_{n=1}^{\infty} \sum_{i_1\ldots i_n} \frac{w^{(n)}_{i_1\cdots i_n}}{n!} \eta_{i_1} \cdots \eta_{i_n}, \quad (49)$$

where

$$w^{(n)}_{i_1\cdots i_n} = \int dt_2 \cdots dt_n \langle \alpha_{i_1}(t_1) \cdots \alpha_{i_n}(t_n)\rangle_c.$$

From (48) we see that $w_\xi(\eta_1, \ldots, \eta_d)$ can be reduced to $w_\alpha(\boldsymbol{\eta} + \boldsymbol{\eta}_0)$; taking into account the normalization condition $w_\xi(0) = \ln\langle 1\rangle = 0$, we obtain GLEs and the Lyapunov exponents

$$w_\xi(\eta_1, \eta_2, \ldots, \eta_d) = w_\alpha(\boldsymbol{\eta} + \boldsymbol{\eta}_0) - w_\alpha(\boldsymbol{\eta}_0), \quad (50)$$

$$\lambda_k = \frac{\partial}{\partial \eta_k} w_\alpha(\boldsymbol{\eta}_0), \quad (51)$$

$$(\eta_0)_{kp} = \left(\frac{d+1}{2} - k\right)\delta_{kp}.$$

Gaussian process

Consider now the following important particular case: Let $\mathbf{A}(t)$ be Gaussian continuous process with zero mean and the given second-order correlator

$$\langle A_{ij}(t)\rangle = 0, \quad \langle A_{ij}(t_1)A_{kp}(t_2)\rangle_c = K_{(ij)(kp)}\Phi(t_1 - t_2),$$

where

$$K_{(ij)(kp)} = -a\delta_{ij}\delta_{kp} + b\delta_{ik}\delta_{jp} + c\delta_{ip}\delta_{jk}, \quad (52)$$

with a, b, and c constants, and $\Phi(t)$ is some regular even fast decaying function $\int \Phi(t)dt = 1$. The form (52) of $K_{(ij)(kp)}$ is determined by the isotropy.

From (49) the expression for the cumulant-generating function of the diagonal elements of $\mathbf{A}$ follows:

$$w_{\boldsymbol{\alpha}}(\boldsymbol{\eta}) = -\frac{a}{2}[\mathrm{tr}(\boldsymbol{\eta})]^2 + \frac{b+c}{2}\mathrm{tr}(\boldsymbol{\eta}^2).$$

From (50) we get

$$w_\xi(\eta_1, \ldots, \eta_d) = (b+c)\sum_{k=1}^{d}\left(\frac{d+1}{2} - k\right)\eta_k + \frac{1}{2}\sum_{k,p=1}^{d}[(b+c)\delta_{kp} - a]\eta_k\eta_p, \quad (53)$$

$$\lambda_k = (b+c)(d+1-2k). \quad (54)$$

In many applications one considers traceless matrices $\mathrm{tr}(\mathbf{A}) = 0$. With this additional requirement, the coefficients a, b, and c are associated by the relation $b + c - ad = 0$, which can be taken into account in (53).

V. CONCLUSION: EFFECTIVE δ PROCESS

In the paper we have considered a system of linear stochastic equations (1) with a statistically isotropic matrix random process $\mathbf{A}(t)$ that has regular fast-decaying connected correlations (26). We found explicit expressions (50) for the generalized Lyapunov exponents in terms of rate functions of the diagonal elements of $\mathbf{A}$.

Now we reformulate the results in a form that is useful for physical applications. We find that the correlations of the diagonal elements of $\mathbf{A}$ contribute to GLEs only via their integrals:

$$w^{(n)}_{i_1\cdots i_n} = \int dt_2 \cdots dt_n W^{(n)}_{i_1 i_1\cdots i_n i_n}(t_1 - t_2, \ldots, t_1 - t_n) \quad \text{(no summation)}.$$

In (50), the GLEs are expressed in terms of the cumulant-generating function of the diagonal elements of the matrix $\mathbf{A}$,

$$w_\alpha(\eta_1, \ldots, \eta_d) = \sum_{n=1}^{\infty}\sum_{i_1\cdots i_n} \frac{w^{(n)}_{i_1\cdots i_n}}{n!}\eta_{i_1}\cdots\eta_{i_n}.$$

So there exists the sequence of formal random processes $\mathbf{A}_\epsilon$ with connected correlations

$$\frac{1}{\epsilon^{n-1}} W^{(n)}_{i_1 j_1\cdots i_n j_n}\left(\frac{t_1 - t_2}{\epsilon}, \ldots, \frac{t_1 - t_n}{\epsilon}\right),$$

which produce identical GLEs. Going to the formal limit $\epsilon \to 0$, we can define the effective δ process $\mathbf{A}_0$ with singular correlation functions

$$\Delta^{(n)}_{ij\cdots kp} = w^{(n)}_{ij\cdots kp}\delta(t_1 - t_2)\cdots\delta(t_1 - t_n). \tag{55}$$

This formal process provides the same GLEs and allows us to split correlations and get closed equations for different averages (see Appendix B). Despite its formal nature, it is a handy instrument for calculations for the problems that appear in the theory of turbulence, turbulent transport, kinematic dynamos in turbulent flows, etc. [26–28].

In the particular case of the Gaussian isotropic processes, the result (54) coincides with the well-known expressions [18,19] obtained for the differential equation $\dot{\mathbf{Q}} = d\mathbf{W}(t)/dt\mathbf{Q}$ in the frame of Stratonovich stochastic convention, $d\mathbf{Q}(t) = d\mathbf{W}(t)\circ\mathbf{Q}(t)$. Thus, for these processes the effective δ process not only has a formal sense but can also be expressed in terms of the derivative of the Wiener process. This corresponds to the Wong-Zakai theorem [29].

We also make some comments on the relation between our approach and the renovation model [23]. In our approach, the noise is stationary for any correlation time, while in the renovation model it becomes stationary only as $\tau \to 0$. Actually, the results of both approaches coincide as $\tau \to 0$. Possibly, the nonstationarity can be taken into account in our approach by means of corrections to $w^{(n)}_{i_1\cdots i_n}$. However, this is a subject for future study.

It is also important to note that even in the isotropic case, one can substitute the effective δ process for the real matrix process only when calculating the long-term asymptotics of E_k and their combinations. For the quantities that depend on the nondiagonal elements θ_{ij} and ζ_{ij} (e.g., the coordinates x_k), the asymptotic behavior is determined not only by the rate function of the matrix elements A_{ij} but also by the shape of their correlation functions. This illustrates the fact that the possibility to introduce the effective δ process is a nontrivial feature of multidimensional isotropic stochastic systems with multiplicative noise.

ACKNOWLEDGMENTS

The authors are grateful to Professor A. V. Gurevich for his constant attention to their work.

APPENDIX A: CALCULATION OF THE JACOBIAN (34)

The Jacobian was calculated in [21] by means of operations with continuous stochastic processes and functional integrals. The notion of the Jacobian is difficult to define for a continuous process. It is natural to define it for a discrete process and then take the continuous limit. The result must not depend on the method of discretization. Here we calculate the Jacobian for one particular discretization of the random process $\mathbf{Q}$ and the corresponding stochastic difference equation. Then we show that for another choice of discretization with the same continuous limit, the result is the same. In an analogous way, one can check that any difference equation with the same continuous limit would lead to the same result.

1. Stratonovich-type discretization

Again we start with Eq. (2). Although $A(t)$ is a finite-correlation-time process and thus the differential equation is well defined, we still consider its discrete analog in order to substitute ordinary derivatives for variational derivatives. Thus, we split T into N discrete intervals, Δt being much smaller than the correlation time of $\mathbf{A}$, and consider the discrete equation

$$\Delta\mathbf{Q}_t \equiv \mathbf{Q}_{t+1} - \mathbf{Q}_t = \mathbf{A}_t\frac{\mathbf{Q}_t + \mathbf{Q}_{t+1}}{2}\Delta t. \tag{A1}$$

We use the Stratonovich-type discretization because to calculate the Jacobian, we need the second-order accuracy in Δt and the Stratonovich choice allows to get this accuracy for $\Delta\mathbf{Q}$ without writing the second-order derivative.

Multiplying (A1) by $\mathbf{Q}_t^{-1}$ and by $\mathbf{Q}_{t+1}^{-1}$ and taking the sum, we get

$$\mathbf{A}_t\Delta t = \tfrac{1}{2}\big(\mathbf{Q}_{t+1}\mathbf{Q}_t^{-1} - \mathbf{Q}_t\mathbf{Q}_{t+1}^{-1}\big) + O(\Delta t^3).$$

Now we make use of the Iwasawa decomposition and substitute $\mathbf{Q}_t = \mathbf{R}_t\mathbf{D}_t\mathbf{Z}_t$ and $\mathbf{Q}_{t+1} = \mathbf{R}_{t+1}\mathbf{D}_{t+1}\mathbf{Z}_{t+1}$. With unified notation $\Delta\mathbf{F}_t = \mathbf{F}_{t+1} - \mathbf{F}_t$ and taking into account $\Delta(\mathbf{F}^{-1})_t = -\mathbf{F}_t^{-1}\Delta\mathbf{F}_t\mathbf{F}_t^{-1} + \mathbf{F}_t^{-1}\Delta\mathbf{F}_t\mathbf{F}_t^{-1}\Delta\mathbf{F}_t\mathbf{F}_t^{-1} + O(\Delta t^3)$, we obtain

$$\begin{aligned}\mathbf{A}\Delta t &= \tfrac{1}{2}\big(\Delta\mathbf{R}_t\mathbf{D}_{t+1}\mathbf{Z}_{t+1} + \mathbf{R}_t\Delta\mathbf{D}_t\mathbf{Z}_{t+1} + \mathbf{R}_t\mathbf{D}_t\Delta\mathbf{Z}_t\big)\mathbf{Z}_t^{-1}\mathbf{D}_t^{-1}\mathbf{R}_t^{-1}\\ &\quad - \tfrac{1}{2}\mathbf{R}_t\mathbf{D}_t\mathbf{Z}_1\big[\Delta\big(\mathbf{Z}_t^{-1}\big)\mathbf{D}_{t+1}^{-1}\mathbf{R}_{t+1}^{-1} + \mathbf{Z}_t^{-1}\Delta\big(\mathbf{D}_t^{-1}\big)\mathbf{R}_{t+1}^{-1} + \mathbf{Z}_t^{-1}\mathbf{D}_t^{-1}\Delta\big(\mathbf{R}_t^{-1}\big)\big]\\ &= \mathbf{R}_t\big\{\tfrac{1}{2}\big(\mathbf{R}_t^{-1}\mathbf{R}_{t+1} - \mathbf{R}_{t+1}^{-1}\mathbf{R}_t\big) + \tfrac{1}{2}\big[\mathbf{R}_t^{-1}\mathbf{R}_{t+1}\Delta\mathbf{D}_t\mathbf{D}_t^{-1} + \Delta\mathbf{D}_t\mathbf{D}_t^{-1}\mathbf{R}_{t+1}^{-1}\mathbf{R}_t - \big(\Delta\mathbf{D}_t\mathbf{D}_t^{-1}\big)^2\big]\\ &\quad + \tfrac{1}{2}\big[\mathbf{R}_t^{-1}\mathbf{R}_{t+1}\mathbf{D}_{t+1}\Delta\mathbf{Z}_t\mathbf{Z}_t^{-1}\mathbf{D}_t^{-1} + \mathbf{D}_t\Delta\mathbf{Z}_t\mathbf{Z}_t^{-1}\mathbf{D}_{t+1}^{-1}\mathbf{R}_{t+1}^{-1}\mathbf{R}_t - \big(\mathbf{D}_t\Delta\mathbf{Z}_t\mathbf{Z}_t^{-1}\mathbf{D}_t^{-1}\big)^2\big]\big\}\mathbf{R}_t^{-1} + O(\Delta t^3).\end{aligned} \tag{A2}$$

We define

$$\boldsymbol{\theta}_t\Delta t \equiv \tfrac{1}{2}\big(\mathbf{R}_t^{-1}\mathbf{R}_{t+1} - \mathbf{R}_{t+1}^{-1}\mathbf{R}_t\big) = \tfrac{1}{2}\big(\mathbf{R}_t^{-1} + \mathbf{R}_{t+1}^{-1}\big)\Delta\mathbf{R}_t. \tag{A3}$$

This is an antisymmetric matrix; from (A3) it follows that, with accuracy $O(\Delta t^3)$,

$$\Delta\mathbf{R}_t = \frac{\mathbf{R}_t + \mathbf{R}_{t+1}}{2}\boldsymbol{\theta}_t\Delta t, \tag{A4}$$

which is the discrete analog to (31) in accord with the Stratonovich approach. Analogously, we claim that

$$\Delta \mathbf{D}_t = \boldsymbol{\xi}_t \frac{\mathbf{D}_t + \mathbf{D}_{t+1}}{2} \Delta t$$

and arrive at

$$\boldsymbol{\xi}_t \Delta t = \Delta \mathbf{D}_t \mathbf{D}_t^{-1} - \tfrac{1}{2}\left(\Delta \mathbf{D}_t \mathbf{D}_t^{-1}\right)^2 + O(\Delta t^3).$$

Requiring

$$\Delta \mathbf{Z}_t = \left(\frac{\mathbf{D}_t + \mathbf{D}_{t+1}}{2}\right)^{-1} \boldsymbol{\zeta}_t \left(\frac{\mathbf{D}_t + \mathbf{D}_{t+1}}{2}\right)\left(\frac{\mathbf{Z}_t + \mathbf{Z}_{t+1}}{2}\right)\Delta t,$$

we also get [up to $O(\Delta t^3)$ accuracy]

$$\begin{aligned}\boldsymbol{\zeta}_t \Delta t &= \left(\mathbf{D} + \frac{\Delta \mathbf{D}}{2}\right)\Delta \mathbf{Z}\left(\frac{\mathbf{Z}_t + \mathbf{Z}_{t+1}}{2}\right)^{-1}\left(\mathbf{D} + \frac{\Delta \mathbf{D}}{2}\right)^{-1}\\ &= \left(\mathbf{1} + \frac{\Delta \mathbf{D}_t \mathbf{D}_t^{-1}}{2}\right)\mathbf{D}_t \Delta \mathbf{Z}_t \mathbf{Z}_t^{-1} \mathbf{D}_t^{-1}\left(\mathbf{1} - \frac{\mathbf{D}_t \Delta \mathbf{Z}_t \mathbf{Z}_t^{-1} \mathbf{D}_t^{-1}}{2}\right)\left(\mathbf{1} - \frac{\Delta \mathbf{D}_t \mathbf{D}_t^{-1}}{2}\right) + O(\Delta t^3).\end{aligned}$$

The summand in the first parentheses in (A2) is $\boldsymbol{\theta}_t$; now we note that the summand in the first square brackets is

$$\mathbf{R}_t^{-1}\mathbf{R}_{t+1}\boldsymbol{\xi}_t \Delta t + \boldsymbol{\xi}_t \Delta t \mathbf{R}_t^{-1}\mathbf{R}_{t+1} + O(\Delta t^3) = (\mathbf{1} + \boldsymbol{\theta}_t \Delta t)\boldsymbol{\xi}_t \Delta t + \boldsymbol{\xi}_t \Delta t(\mathbf{1} + \boldsymbol{\theta}_t \Delta t) + O(\Delta t^3).$$

Finally, the summand in the second square brackets can be written as $\mathbf{R}_t^{-1}\mathbf{R}_{t+1}\boldsymbol{\zeta}_t \Delta t + \boldsymbol{\zeta}_t \Delta t \mathbf{R}_t^{-1}\mathbf{R}_{t+1} + O(\Delta t^3)$.

Summarizing, we rewrite (A2) as

$$\mathbf{A}_t = \mathbf{R}_t\left(\mathbf{X}_t + \tfrac{1}{2}[\boldsymbol{\theta}_t, \mathbf{X}_t]\Delta t\right)\mathbf{R}_t^{-1} + O(\Delta t^2), \qquad \text{(A5)}$$

where

$$\mathbf{X}_t = \boldsymbol{\theta}_t + \boldsymbol{\xi}_t + \boldsymbol{\zeta}_t.$$

Equation (A5) is the discrete analog to Eq. (27).

2. Calculation of the determinant

Now we have to calculate the Jacobian $\mathcal{J} = |\partial A_{ijt}/\partial X_{kmt'}|$. First, we note that from (A5) and (A4) it follows that

$$\partial A_{ijt}/\partial X_{kmt'} = 0 \quad \text{for any } t < t'.$$

(This is the manifestation of causality.) Thus, the matrix $(\partial \mathbf{A}/\partial \mathbf{X})_{ij,t;km,t'}$ is a block triangular matrix and its determinant is equal to the product of the determinants of the diagonal blocks, $t = t'$:

$$\mathcal{J} = \prod_t \left|\frac{\partial A_{ijt}}{\partial X_{kmt}}\right|. \qquad \text{(A6)}$$

Second, we note that, in accordance with the causality principle, from (A4) it follows that the value of the rotation matrix $\mathbf{R}_t$ depends only on the previous-time values of $\boldsymbol{\theta}_{t'<t}$ and does not depend on the simultaneous value $\boldsymbol{\theta}_t$,

$$\frac{\partial \mathbf{R}_t}{\partial \boldsymbol{\theta}_{t' \geqslant t}} = 0.$$

Thus, in (A5) the derivative must be taken only over the multiplier in the square brackets. We now introduce the multi indices $\alpha = \{ij\}$ and the $d^2 \times d^2$ matrix

$$\mathcal{R}_{ij,km} = R_{ik}R_{jm}.$$

Then (A5) can be written in the form

$$A_\alpha = \mathcal{R}_{\alpha\beta}(X_\beta + M_\beta \Delta t),$$

where

$$\mathbf{M} = \tfrac{1}{2}[\boldsymbol{\theta}, \mathbf{X}].$$

So

$$\left|\frac{\partial A_\alpha}{\partial X_\gamma}\right| = |\mathcal{R}_{\alpha\beta}| \times \left|\delta_{\beta\gamma} + \frac{\partial M_\beta}{\partial X_\gamma}\Delta t\right|.$$

Since the matrices $\mathbf{R}$ are orthogonal, $\mathcal{R}$ is also orthogonal, i.e., $\mathcal{R}\mathcal{R}^T = \mathbf{1}$. Thus, $\det \mathcal{R} = 1$. The second determinant, to an accuracy of approximately $O(\Delta t)$, can be reduced to the trace of $\partial \mathbf{M}/\partial \mathbf{X}$:

$$\left|\frac{\partial A_\alpha}{\partial X_\gamma}\right| = 1 + \left(\frac{\partial M_\beta}{\partial X_\gamma}\right)\delta_{\beta\gamma}\Delta t + O(\Delta t^2). \qquad \text{(A7)}$$

Finally, we make use of the fact that only the lower triangular part of $\mathbf{X}$ determines the values of $\boldsymbol{\theta}$, while the diagonal and upper triangular components are responsible for $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, correspondingly. So

$$\theta_{ij} = \begin{cases} X_{ij} & \text{if } i > j \\ 0 & \text{if } i = j \\ -X_{ji} & \text{if } i < j. \end{cases}$$

In particular, $\partial \theta_{ij}/\partial X_{km} = 0$ if $k \leqslant m$. Thus, taking the derivative of $\mathbf{M}$, we obtain

$$\begin{aligned}\mathrm{tr}\left(\frac{\partial \mathbf{M}}{\partial \mathbf{X}}\right) &= \sum_{i,j}\left(\frac{\partial M_{ij}}{\partial X_{ij}}\right) = \frac{1}{2}\sum_{i,j}(\theta_{ii} - \theta_{jj})\\ &\quad + \sum_{i>j}(X_{jj} - X_{ii}) = \sum_{j=1}^{d}(d - 2j + 1)X_{jj}.\end{aligned}$$

Combining this with (A7) and (A6), we eventually get

$$\mathcal{J} = \prod_t \left(1 + \sum_{j=1}^{d} \eta_j X_{jj} \Delta t\right) + O(\Delta t^2)$$
$$= \exp\left(\sum_t \sum_{j=1}^{d} \eta_j X_{jj} \Delta t\right), \quad \eta_j = \frac{d - 2j + 1}{2}.$$

Taking the continuous limit $\Delta t \to 0$, we arrive at the integral

$$\mathcal{J} = \exp\left(\int \sum_{j=1}^{d} \eta_j X_{jj} dt\right),$$

which coincides with Eqs. (35) and (36).

3. Itô-type discretization

The same result can be obtained in other discretization settings; however, in the general case one has to keep the terms up to the second order in Δt in the initial difference equation for $\mathbf{Q}$. Here we consider the Itô-type discretization.

We rewrite the initial continuous differential Eq. (2), $\partial_t \mathbf{Q} = \mathbf{AQ}$, in the integral form

$$\mathbf{Q}(t) = \mathbf{Q}_0 + \int_{t_0}^{t} \mathbf{AQ}\, dt.$$

Applying this equation to the time range from t to $t + \Delta t$ and solving it by means of iterations, after two iterations we get

$$\mathbf{Q}(t + \Delta t) = \mathbf{Q}(t) + \bar{\mathbf{A}}\mathbf{Q}(t)\Delta t + \tfrac{1}{2}\bar{\mathbf{A}}^2\mathbf{Q}(t)\Delta t^2,$$

where $\bar{\mathbf{A}}$ is the time average of $\mathbf{A}(t)$ over the range Δt. Based on this equation, we write the difference equation

$$\mathbf{Q}_{t+1} = \mathbf{Q}_t + \left(\mathbf{A}_t \Delta t + \tfrac{1}{2}{\mathbf{A}_t}^2 \Delta t^2\right)\mathbf{Q}_t. \tag{A8}$$

Multiplying (A8) by $\mathbf{Q}_t^{-1}$ and making use of the Iwasawa decomposition, we present the left-hand side in the form

$$\mathbf{Q}_{t+1}\mathbf{Q}_t^{-1} = \mathbf{R}_t\left(\mathbf{1} + \mathbf{R}_t^{-1}\Delta\mathbf{R}_t\right)\left[\mathbf{1} + \Delta(\mathbf{DZ})_t(\mathbf{DZ})_t^{-1}\right]\mathbf{R}_t^{-1}, \tag{A9}$$

where $\Delta F_t \equiv F_{t+1} - F_t$.

Now we formally define $\boldsymbol{\xi}_t$ by

$$\boldsymbol{\xi}_t \Delta t = \Delta\mathbf{D}_t\mathbf{D}_t^{-1} - \tfrac{1}{2}\left(\Delta\mathbf{D}_t\mathbf{D}_t^{-1}\right)^2,$$

which coincides, up to the second order in Δt, with

$$\Delta\mathbf{D}_t = \mathbf{D}_{t+1} - \mathbf{D}_t = \left(\boldsymbol{\xi}_t \Delta t + \tfrac{1}{2}\boldsymbol{\xi}_t^2 \Delta t^2\right)\mathbf{D}_t.$$

In accordance with the chosen prescription, this difference equation corresponds to the differential Eq. (29). Accordingly, Eq. (31) corresponds to

$$\Delta\mathbf{R}_t = \mathbf{R}_{t+1} - \mathbf{R}_t = \mathbf{R}_t\left(\boldsymbol{\theta}_t \Delta t + \tfrac{1}{2}\boldsymbol{\theta}_t^2 \Delta t^2\right).$$

Taking into account (29), we rewrite the Eq. (30) for the upper triangular part in the form $\partial_t(\mathbf{DZ}) = (\boldsymbol{\xi} + \boldsymbol{\zeta})\mathbf{DZ}$. Then the corresponding difference equation takes the form

$$\Delta(\mathbf{DZ})_t = \mathbf{D}_{t+1}\mathbf{Z}_{t+1} - \mathbf{D}_t\mathbf{Z}_t$$
$$= \left[(\boldsymbol{\xi}_t + \boldsymbol{\zeta}_t)\Delta t + \tfrac{1}{2}(\boldsymbol{\xi}_t + \boldsymbol{\zeta}_t)^2\Delta t^2\right]\mathbf{D}_t\mathbf{Z}_t.$$

Substituting these expressions in (A9) and keeping the terms of the order of Δt^2, we obtain

$$\mathbf{Q}_{t+1}\mathbf{Q}_t^{-1} = 1 + \mathbf{R}_t\Bigg((\boldsymbol{\theta}_t + \boldsymbol{\xi}_t + \boldsymbol{\zeta}_t)\Delta t + \frac{[\boldsymbol{\theta}_t^2 + (\boldsymbol{\xi}_t + \boldsymbol{\zeta}_t)^2]\Delta t^2}{2} + \boldsymbol{\theta}_t(\boldsymbol{\xi}_t + \boldsymbol{\zeta}_t)\Delta t^2\Bigg)\mathbf{R}_t^{-1}.$$

Combining this with (A8), we get

$$\mathbf{A}_t = \mathbf{R}_t\left(\mathbf{X}_t + \tfrac{1}{2}[\boldsymbol{\theta}_t, \mathbf{X}_t]\Delta t\right)\mathbf{R}_t^{-1}, \quad \mathbf{X}_t = \boldsymbol{\xi}_t + \boldsymbol{\zeta}_t + \boldsymbol{\theta}_t.$$

This result coincides with (A5) obtained in the Stratonovich convention. The rest of the derivation is the same as in the Stratonovich case.

APPENDIX B: CORRELATION SPLITTING FOR δ PROCESSES

In this Appendix we derive the analog of the Furutsu-Novikov formula for the δ processes. The Furutsu-Novikov relation for regular processes has the form [30,31]

$$\langle A_{ij}(t)g[\mathbf{A}]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{R}^n}\langle A_{ij}(t)A_{i_1j_1}(t_1)\cdots A_{i_nj_n}(t_n)\rangle_c \times \left\langle\frac{\delta^n g[\mathbf{A}]}{\delta A_{i_1j_1}(t_1)\cdots\delta A_{i_nj_n}(t_n)}\right\rangle dt_1\cdots dt_n \tag{B1}$$

for any regular functional $g[\mathbf{A}]$. For the δ process with correlation functions (55),

$$\Delta^{(n)}_{ij\cdots kp} = w^{(n)}_{ij\cdots kp}\delta(t_1 - t_2)\cdots\delta(t_1 - t_n), \tag{B2}$$

it takes the form

$$\langle A_{ij}(t)g[\mathbf{A}]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}w^{(n+1)}_{iji_1j_1\cdots i_nj_n}\left\langle\frac{\delta^n g[\mathbf{A}]}{\delta A_{i_1j_1}(t_1)\cdots\delta A_{i_nj_n}(t_n)}\right\rangle. \tag{B3}$$

Here all the variational derivatives are taken at the same moment t.

However, in physical applications one often has to deal with causal functionals, i.e., the functionals that depend explicitly on time and satisfy the causality principle

$$\frac{\delta g[t,\mathbf{A}]}{\delta A_{ij}(t')}=0 \quad \text{if } t'>t.$$

For these functionals,

$$\frac{\delta^n g[t,\mathbf{A}]}{\delta A_{i_1j_1}(t_1)\cdots\delta A_{i_nj_n}(t_n)}=I_{[0,\infty)}(t-t_1)\cdots I_{[0,\infty)}(t-t_n)\times G^{(n)}_{iji_1j_1\cdots i_nj_n}[t,t_1,\dots,t_n;\mathbf{A}]$$

and

$$\langle A_{ij}(t')g[t,\mathbf{A}]\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}w^{(n+1)}_{iji_1j_1\cdots i_nj_n}\times\langle G^{(n)}_{i_1j_1\cdots i_nj_n}[t,t',\dots,t';\mathbf{A}]\rangle,\quad t'<t \tag{B4}$$

$$\langle A_{ij}(t')g[t,\mathbf{A}]\rangle=\langle A_{ij}(t')\rangle\langle g[t,\mathbf{A}]\rangle,\quad t'>t. \tag{B5}$$

However, Eq. (B3) is inapplicable for $t=t'$ because it contains undefined values $I_{[0,\infty)}(0)$. So, to calculate the simultaneous correlator, we have to return to (B1) and consider some sequence of (formal) processes with cumulants converging to (B2). Thus, we choose the sequence of cumulants

$$\frac{w^{(n)}_{i_1j_1\cdots i_nj_n}}{n}\sum_{k=1}^{n}\prod_{\substack{l=1\\ l\neq k}}^{n}\delta_\epsilon(t_k-t_l),$$

where $\delta_\epsilon(t)$ are even regular functions $\delta_\epsilon(t)\overset{\epsilon\to 0}{\to}\delta(t)$ and $\int\delta_\epsilon dt=1$. Taking into account

$$\int^t dt_1\cdots\int^t dt_n\prod_{l=1}^{n}\delta_\epsilon(t-t_l)=\frac{1}{2^n},$$

$$\int^t dt_1\cdots\int^t dt_n\delta_\epsilon(t-t_k)\prod_{\substack{l=1\\ l\neq k}}^{n}\delta_\epsilon(t_k-t_l)=\frac{1}{n}\left(1-\frac{1}{2^n}\right),$$

we arrive at

$$\langle A_{ij}(t)g[t,\mathbf{A}]\rangle=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}w^{(n+1)}_{iji_1j_1\cdots i_nj_n}\times\left\langle G^{(n)}_{i_1j_1\cdots i_nj_n}[t,\dots,t;\mathbf{A}]\right\rangle. \tag{B6}$$

This relation was presented without derivation in [32]; here we derive it by taking the formal limit. Comparing (B6) with (B4), we see that the correlations of causal functionals are discontinuous at $t'=t$ for δ processes, and the naive convention that $I_{[0,\infty)}(0)=\frac{1}{2}$ is valid only for the Gaussian processes.

[1] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[2] V. Sabel'nikov and O. Soulard, Phys. Rev. E **72**, 016301 (2005).
[3] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[4] A. P. Kazantsev, Zh. Eksp. Tear. Fiz. **53**, 180 (1968) [Sov. Phys. JETP **26**, 1031 (1968)].
[5] F. R. Gantmacher, *Matrix Theory*, 2nd ed. (AMS, Providence, 1990).
[6] J. M. Ottino, W. E. Ranz, and C. W. Macosko, AIChE J. **27**, 565 (1981).
[7] S. B. Pope, Int. J. Eng. Sci. **26**, 445 (1988).
[8] E. Balkovsky and A. Fouxon, Phys. Rev. E **60**, 4164 (1999).
[9] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, Phys. Rev. E **96**, 013117 (2017).
[10] Y. B. Zeldovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokoloff, J. Fluid Mech. **144**, 1 (1984).
[11] M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, Phys. Rev. Lett. **83**, 4065 (1999).
[12] A. S. Il'yn, A. V. Kopyev, V. A. Sirota, and K. P. Zybin, Phys. Fluids **33**, 075105 (2021).
[13] V. I. Oseledets, Trans. Moscow Math. Soc. **19**, 197 (1968).
[14] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices in Statistical Physics* (Springer, Berlin, 1993).
[15] Y. Le Jan, Z. Warsch. **70**, 609 (1985).
[16] P. H. Baxendale, in *Lyapunov Exponents*, edited by L. Arnold and V. Wihstutz, Lecture Notes in Mathematics Vol. 1186 (Springer, Berlin, 1986), p. 322.
[17] N. Newman, Commun. Math. Phys. **103**, 121 (1986).
[18] A. Gamba, J. Stat. Phys. **112**, 193 (2003).
[19] M. Liao, Ann. Probab. **25**, 1241 (1997).
[20] N. G. Van Kampen, Physica **74**, 215 (1974).
[21] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, J. Stat. Phys. **163**, 765 (2016).
[22] P. Dittrich, S. A. Molchanov, D. D. Sokolov, and A. A. Ruzmaikin, Astron. Nachr. **305**, 119 (1984).
[23] E. A. Illarionov and D. D. Sokoloff, Phys. Rev. E **104**, 015214 (2021).
[24] S. R. S. Varadhan, Ann. Probab. **36**, 397 (2008).
[25] J. Zinn-Justin, *Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
[26] A. V. Kopyev, A. M. Kiselev, A. S. Il'yn, V. A. Sirota, and K. P. Zybin, Astrophys. J. **927**, 172 (2022).
[27] A. V. Kopyev, A. S. Il'yn, V. A. Sirota, and K. P. Zybin, Phys. Fluids **34**, 035126 (2022).
[28] A. S. Il'yn, A. V. Kopyev, V. A. Sirota, and K. P. Zybin, Phys. Fluids **32**, 125114 (2020).
[29] E. Wong and M. Zakai, Ann Math. Stat. **36**, 1560 (1965).
[30] K. Furutsu, J. Res. Natl. Bur. Std. **67D**, 303 (1963).
[31] E. A. Novikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **47**, 1919 (1965) [Sov. Phys. JETP **20**, 1290 (1965)].
[32] V. I. Klyatskin, *Dynamics of Stochastic Systems* (Elsevier, Amsterdam, 2005).