

Numerical test of the replica-symmetric Hamiltonian for correlations of the critical state of spin glasses in a field

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A growing body of evidence indicates that the sluggish low-temperature dynamics of glass formers (e.g., supercooled liquids, colloids, or spin glasses) is due to a growing correlation length. Which is the effective field theory that describes these correlations? The natural field theory was drastically simplified by Bray and Roberts in 1980. More than 40 years later, we confirm the tenets of Bray and Roberts's theory by studying the Ising spin glass in an externally applied magnetic field, both in four spatial dimensions (data obtained from the Janus collaboration) and on the Bethe lattice.

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I. INTRODUCTION

Spin glasses [1–3] in a magnetic field [but above the de Almeida-Thouless (dAT) line [4]], structural glasses close to (but above) their mode coupling temperature [5], or hard spheres above the Gardner transition [6] all display large correlation lengths and slow relaxations that are typical of a second-order phase transition. These features are predicted by mean field (MF) theory [1] and have been identified both in experiments and in numerical simulations [7–32]. However, the very existence of the phase transition has been long debated [33–36]. Indeed, it has been frequently suggested that these *critical* features might be connected to a crossover rather than to a true phase transition [37–43]: The corrections to MF theory would destroy the transition, or (in some cases) move it to zero temperature. In this paper we do not claim against, nor in favor, of the presence of a transition. Instead, our aim is understanding in detail the properties of the correlations in the region where the susceptibilities are large (e.g., 10^3 times their natural value).

Let us consider the framework of spin glasses in a magnetic field. The theory is complex [44]. Three different two-point correlators (and their associated susceptibilities) become critical. We also have eight nonlinear susceptibilities associated to the eight three-point correlators (there are eight different coupling constants) [45]. However, in an expansion at around MF, one finds a linear transformation such that only one of the three susceptibilities is divergent at the critical temperature T_c . Similarly, the divergence at T_c is more violent for two of the nonlinear susceptibilities: At first order in perturbation theory,

they scale as $1/(T - T_c)^3$, while two nonlinear susceptibilities diverge as $1/(T - T_c)^2$, another one as $1/(T - T_c)$, and the remaining three are finite at T_c . As expected, only the couplings that correspond to the most divergent nonlinear susceptibilities are relevant near the transition. The linear transformations that *diagonalize* the singularity structure are well known, and they have a physical meaning. Corrections to the MF could completely destroy this divergence structure (or they may just modify the values of the critical exponents). A systematic investigation of the correctness of the above picture has never been attempted using numerical simulations. This paper fills the lacuna in the particular case of spin glasses. We show that these qualitative predictions are satisfied in the region of large susceptibilities. It is quite possible that the same situation is present in other contexts, beyond spin glasses.

II. SUMMARY OF THE THEORETICAL FRAMEWORK

The standard tool to understand the fate of a transition in finite spatial dimension D is the Wilsonian renormalization group (RG) [46]. Unfortunately, the standard perturbative construction fails in these models. The most relevant corrections to MF theory are due to the presence of cubic terms in the effective Landau-Ginzburg theory (LGT), see Eq. (A1) in Appendix A, and two couplings (\tilde{w}_1 and \tilde{w}_2) are known to be relevant for $D \lesssim 6$. In fact, in spin glasses and also in models with the same LGT, the construction of the $D = 6 - \epsilon$ expansion fails because no fixed point is present in the weak-coupling region [33]. The action of the RG brings

the corrections to the Gaussian behavior in the region where the effective couplings are large. The fate of the parameter $\lambda_r \equiv w_{2,r}/w_{1,r}$ ($w_{i,r}$, $i = 1, 2$, are the renormalized couplings, see below) is of particular interest. Indeed, λ_r plays a crucial role in the mode coupling theory where it must be $0 \leq \lambda_r \leq 1$. Moreover, as discovered by Gross *et al.* [47] and recently stressed by Höller and Read [48], having $\lambda_r > 1$ would imply a peculiar first-order-like transition, like the calorimetric transition of glasses (see, e.g., [5]).

Unfortunately, in spite of the relevance of the renormalized parameters [49], they have not been obtained in simulations, partly because of the complexity of the computation. Here we show that such a computation is feasible: We present results for spin glasses in a magnetic field, both in the Bethe lattice and in the $D = 4$ hypercubic lattice. Our model choice is based on its relative simplicity, but our techniques can be straightforwardly extended to more complex models. The Bethe lattice computation is a test of the viability of the approach and of the formulas used. Indeed, corrections to MF disappear in an infinite Bethe lattice, and the value of λ_r , which is unaffected by fluctuations (i.e., loop corrections), is analytically known. On the other hand, the $D = 4$ Edwards-Anderson (EA) mode may be well thermalized in the region of very large susceptibilities, and we have some estimates of the position of the extrapolated dAT transition [24]. Our results are suggestive of the presence of a fixed-point value $\lambda_r \approx 0.5$ and clearly exclude a value of λ_r greater than 1.

Let us summarize the theoretical understanding for spin glasses in a magnetic field h . The effective action can be written using the replica formalism (we recall in Appendix A the main results, that are well described in the literature). We aim to express all our results in terms of correlation functions than can be computed in a numerical simulation. Let us start from the two-point correlation functions. As usual in disordered systems, we need to distinguish between the thermal average, $\langle(\dots)\rangle$, and the average over disorder, $\overline{(\dots)}$. For a system of linear size L , with $N = L^D$ spins $S_i = \pm 1$, we have three relevant susceptibilities:

$$\chi_1 \equiv \frac{1}{N} \sum_{ij} \overline{\langle S_i S_j \rangle^2} - q^2, \quad (1)$$

$$\chi_2 \equiv \frac{1}{N} \sum_{ij} \overline{\langle S_i S_j \rangle \langle S_i \rangle \langle S_j \rangle} - q^2, \quad (2)$$

$$\chi_3 \equiv \frac{1}{N} \sum_{ij} \overline{\langle S_i \rangle^2 \langle S_j \rangle^2} - q^2, \quad (3)$$

where $q \equiv \overline{\langle S_i \rangle^2}$ is the average overlap. If we expand around the MF solution, we find at all orders of the perturbation theory that the so-called replicon susceptibility is divergent near the transition,

$$\chi_R \equiv \chi_{SG} \equiv \frac{1}{N} \sum_{ij} \overline{\langle S_i S_j \rangle_c^2} = \chi_1 - 2\chi_2 + \chi_3, \quad (4)$$

where by $\langle(\dots)\rangle_c$ we denote the connected correlation function (e.g., $\langle S_i S_j \rangle_c = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$, see, for instance, [50]). For later use we introduce the longitudinal and anomalous susceptibilities, χ_L and χ_A , respectively. The two are degenerated

in the presence of a magnetic field,

$$\chi_L = \chi_A = \chi_1 - 4\chi_2 + 3\chi_3. \quad (5)$$

If we consider Gaussian-distributed random magnetic fields, χ_L is proportional to the staggered magnetic susceptibility (see Appendix C for a detailed discussion). Then the physically motivated assumption that the magnetic susceptibility is not critical implies that χ_L is not critical either. Only the average of the (squared) connected correlator becomes critical. This is in sharp contrast with the $h = 0$ case where $\chi_2 = \chi_3 = 0$ and $\chi_A = \chi_L = \chi_R$. We expect a crossover region for small L and h , where χ_L and χ_A seem critical (because χ_L and χ_A are critical at the $h = 0$ transition).

The renormalized coupling $w_{1,r}$ and $w_{2,r}$ are defined in terms of the exact vertices w_1 and w_2 , i.e., the coefficients of the Gibbs free energy. The exact vertices can be expressed as $w_i = \omega_i/\chi_R^3$ ($i = 1, 2$) in terms of connected correlations at zero external momentum [44]:

$$\omega_1 \equiv \frac{1}{N} \sum_{ijk} \overline{\langle S_i S_j \rangle_c \langle S_j S_k \rangle_c \langle S_k S_i \rangle_c}, \quad (6)$$

$$\omega_2 \equiv \frac{1}{2N} \sum_{ijk} \overline{\langle S_i S_j S_k \rangle_c^2}. \quad (7)$$

The coupling constants w_1, w_2 diverge at the transition while the renormalized coupling constants remain finite. They are obtained by renormalizing the lengths and the overlap fields, leading to

$$w_{1,r} = \frac{\omega_1}{\chi_R^{3/2} \xi_2^{D/2}}, \quad w_{2,r} = \frac{\omega_2}{\chi_R^{3/2} \xi_2^{D/2}}, \quad (8)$$

where ξ_2 is the second-moment correlation length. It follows that

$$\lambda_r = \frac{w_{1,r}}{w_{2,r}} = \frac{\omega_1}{\omega_2}. \quad (9)$$

Note that $\lambda_r = w_1/w_2$; hence, λ does not renormalize and we will drop thereafter the subindex r .

Finite-volume corrections are very strong, so we do not consider here the computation of the renormalized couplings $w_{i,r}$. However, we can introduce the dimensionless quantities

$$\Lambda_1 = \frac{\omega_1}{\chi_R^{3/2} L^{D/2}}, \quad \Lambda_2 = \frac{\omega_2}{\chi_R^{3/2} L^{D/2}}, \quad (10)$$

which should scale with L as Binder's cumulant [51]. Notice that, at the critical point, $\Lambda_i \propto w_{i,r}$.

Before discussing our numerical findings for λ , it is important to stress that there are nonequivalent ways of taking the relevant limit for $\lambda(L, T)$ in the onset of a second-order phase transition at T_c :

$$\lambda^* = \lim_{L \rightarrow \infty} \lim_{T \rightarrow T_c} \lambda(L, T), \quad \lambda(T_c^+) = \lim_{T \rightarrow T_c^+} \lim_{L \rightarrow \infty} \lambda(L, T). \quad (11)$$

The fact that $\lambda^* \neq \lambda(T_c^+)$ is hardly surprising [52]. Similarly, the corresponding limits for renormalized coupling $w_{1,r}$ and $w_{2,r}$ do not commute. $\lambda(T_c^+)$ is in general more difficult to estimate than λ^* , but the former could be more desirable given that the RG β functions (see, e.g., [50,53,54]) are typically expressed in terms of the thermodynamic quantities in analytical computations.

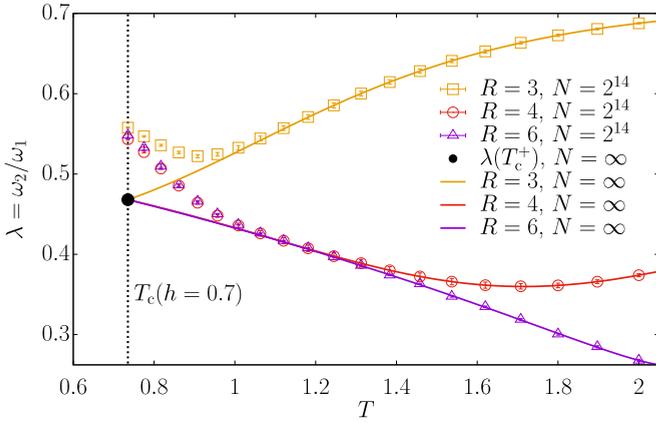


FIG. 1. Temperature dependence of the ratio of renormalized couplings λ , see Eq. (9), computed with a magnetic field $h = 0.7$ on a Bethe lattice. The critical temperature is marked with a vertical line. We plot the data obtained with the three-, four-, and six-replica estimators. The black dot reports the value of $\lambda(T_c^+) \simeq 0.47$, see Eq. (11), that has been computed analytically in [55]. All three estimators take the same value $\lambda^* \simeq 0.55$ at the critical temperature. The continuous lines, marked with $N = \infty$, are the extrapolations of the data considering scaling corrections [56] and are compatible with the analytical computation $\lambda(T_c^+)$.

III. NUMERICAL SIMULATION RESULTS

In a simulation, the above quantities are computed from real *replicas* (i.e., systems that evolve independently under the same coupling constants). It is well known that one needs *two* real replicas to compute q , *four* replicas for the three susceptibilities, and *six* replicas for the $\omega_{i=1,2}$ in Eqs. (6) and (7). In spite of this and *only at the critical point* is it possible to compute both ω_i using only three and four replicas. We shall denote the estimate obtained with R replicas by $\omega_i^{(R=3,4)}$. Away from the critical point, one has for the differences $\omega_i - \omega_i^{(R)} = O(|T - T_c|^{\rho(R)})$, where $\rho(R)$ is a suitable exponent (see Appendix B for a more complete discussion).

a. Numerical results in the Bethe lattice. To study the behavior of the three- and four-replica estimators in a controlled setting, we have simulated an Ising spin glass in a magnetic field on a Bethe lattice (random regular graph with fixed degree 4). In this case there is little doubt that a true dAT transition is present. Furthermore, the divergence of the susceptibilities (both linear and nonlinear) closely matches our description above.

In Fig. 1 we plot the parameter λ for the Bethe lattice, as obtained from the exact expression together with the three- and four-replica estimators $\lambda^{(3)} \equiv \omega_2^{(3)}/\omega_1^{(3)}$ and $\lambda^{(4)} \equiv \omega_2^{(4)}/\omega_1^{(4)}$. In this case T_c and $\lambda(T_c^+)$ are known analytically [55], and we see that the estimators extrapolate to the correct value at the critical temperature, although close to the critical point there are finite-size corrections. Note as well that the finite-size corrections of the true λ (i.e., the six-replica estimator) and the four-replica estimator coincide in the critical region. The same effect is expected for the three-replica estimator, but it is masked by preasymptotic effects at the sizes considered. At any rate, we find that the deviations are

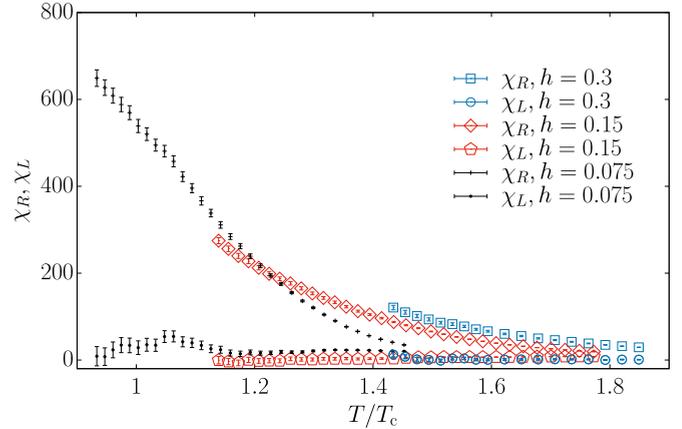


FIG. 2. Replicon (χ_R) and longitudinal (χ_L) susceptibilities, Eqs. (4) and (5) vs temperature, as computed for the $D = 4$ Edwards-Anderson model at magnetic fields $h = 0.075, 0.15$, and 0.3 (for each h , the temperature is plotted rescaled by the corresponding best estimate for T_c [24]). To avoid cluttering the plot, we only show data for our largest system, $L = 16$.

consistent with the predicted MF values $\omega_i - \omega_i^{(3)} = O(|T - T_c|)$ and $\omega_i - \omega_i^{(4)} = O(|T - T_c|^3)$ [56].

b. Numerical results in four dimensions. The discussion of the three- and four-replica estimators is of great practical and theoretical importance in this case. The theoretical importance relies on the fact that, at variance with the Bethe lattice case, one cannot take for granted that the transition is described by the theory outlined above. For instance, we could have a continuous transition described by a different theory and therefore the three- and four-replica estimators would yield conflicting results, thus indicating a wrong choice for the starting field theory. Furthermore, due to the lack of a perturbative RG fixed point below six dimensions, one could even question the very existence of such a theory for $D < 6$. Thus the fact that the three- and four-replica expressions yield consistent estimates provides a nontrivial indication that the region of large susceptibilities is actually described by the replica-symmetric field theory of Bray and Roberts [33].

The practical importance of the three- and four-replica estimators lies in that, in the present study, we have reanalyzed equilibrium configurations obtained by the Janus Collaboration [24] using the Janus-I supercomputer [57], where the four-dimensional Ising spin glass in the presence of a constant magnetic field was simulated (see Appendix D). Those equilibrium configurations were obtained for only four real replicas. Therefore λ can be computed only through the three- and four-replica estimators (although the computation will not be exact away from the dAT line).

In [24] the critical temperatures and the critical exponents were estimated for three different magnetic fields ($h = 0.075, 0.15$, and 0.3) by looking only to one of the two-point correlators, namely, the replicon. We study the same magnetic fields considered in [24] for temperatures near (but above) their estimated critical temperatures.

We start by studying in Fig. 2 the replicon and longitudinal susceptibilities, recall Eqs. (4) and (5). We clearly see that χ_R increases and becomes very large as the temperature is lowered, while χ_L saturates at a much smaller plateau value

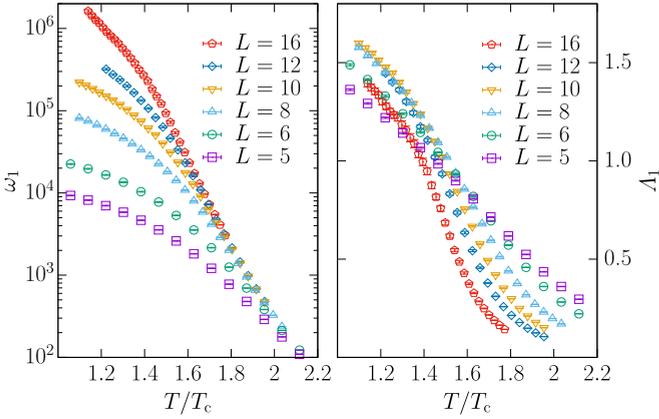


FIG. 3. $D = 4$ Edwards-Anderson model with magnetic field $h = 0.15$. Left: Four-replica estimate, $\omega_1^{(4)}$, for the nonlinear susceptibility in Eq. (6) vs temperature. Right: Dimensionless quantity Λ_1 , recall Eq. (10), vs temperature. In both panels, T is in units of the estimated critical temperature T_c [24].

[58]. We conclude, in agreement with our MF-based expectations and with previous dynamic investigations in $D = 3$ [22], that correlations extend to much larger distances for the replicon mode than for the longitudinal one, thus excluding the possibility that the critical behavior in χ_R is due to the $h = 0$ fixed point.

We have considered also the nonlinear susceptibilities, the most divergent ones being ω_1 and ω_2 , see Eqs. (6) and (7). We find that ω_1 grows significantly upon decreasing T and (at a fixed, low T) upon increasing L , see Fig. 3 (left). The suggested divergence in ω_1 makes it advisable to consider the dimensionless $\Lambda_1(L, T)$, see Eq. (10). At a critical point, the curves of Λ_1 as function of T , computed for different sizes L , should cross or merge at T_c . Our data for $L = 10, 12$, and 16 in Fig. 3 (right) do not clearly cross nor merge, making it difficult to compute T_c from these data (indeed, the authors of Ref. [24] could locate T_c only by considering quantities at nonzero external momentum). The crucial point, however, is the absence of any evidence in Fig. 3 (right) for a runaway

trajectory where Λ_1 becomes bigger and bigger upon increasing L . This observation makes unlikely the scenario with a first-order transition [48].

Once we know that $\omega_{1,2}$ behaves as expected, we can consider their ratio λ , which is the main quantity of interest. Figure 4 shows the three- and four-replica estimators for magnetic fields $h = 0.075, 0.15$, and 0.30 . At variance with our findings for the Bethe lattice [where the difference between $\lambda(T_c^+)$ and λ^* is very clear, recall Eq. (11) and Fig. 1], our data for the $4D$ case shown in Fig. 4 do not manifest large finite-size effects approaching the critical point: Data barely depend on temperature for $T < T_c(h = 0)$, thus suggesting λ^* and $\lambda(T_c^+)$ should be very close. The only visible finite-size effect in $4D$ data is a monotonic in L decrease for $R = 3$ and increase for $R = 4$, which actually helps in bracketing λ^* between the values measured on the largest lattice $L = 16$. Indeed, our data are consistent with a universal value $\lambda^* \approx 0.55$ at the critical temperature. We remark as well that both the $R = 3$ and the $R = 4$ estimates verify $\lambda(L, T) < 1$. Hence we conclude $\lambda(T_c^+) < 1$ in $4D$ spin glasses in a field, which is the main result of this paper.

IV. CONCLUSIONS

Irrespective of the ongoing debate about whether the glass transition is a true phase transition or a crossover, it is undeniable that glass formers display slow dynamics and large correlations. When the length scale for fluctuations becomes large, the natural tool to study the problem is a field theory. Unfortunately, symmetry considerations do not constrain the Hamiltonian much. In the particular case of spin glasses in a magnetic field, we end with a extremely complex theory containing eight different coupling constants. Bray and Roberts [33] drastically simplified the theory. Their so-called replica-symmetric Hamiltonian has been the basis for many analyses. In spite of this, up to now it was not possible to test in a nontrivial problem the basic hypothesis underlying the theory. We have overcome this challenge thanks to two crucial ingredients: (i) a detailed scaling description for the many linear and nonlinear susceptibilities in the problem [44], and

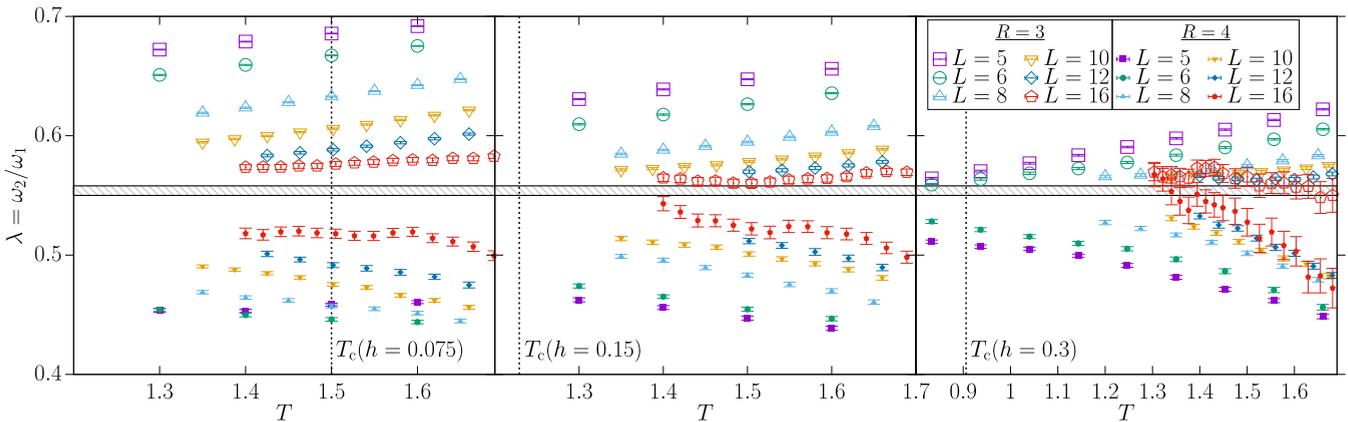


FIG. 4. Three- and four-replica estimators for λ as a function of the temperature in the $D = 4$ Ising spin glass (the value of the magnetic field is indicated above each panel). Vertical lines report the three critical temperatures taken from [24]. The band around $\lambda^* \approx 0.55$ is our best $L \rightarrow \infty$ extrapolation, assuming three- and four-replica estimators converge to a common value for all the three simulated values of the magnetic field (the width of the band represents the uncertainty in our extrapolation for $h = 0.075$).

(ii) a reanalysis of the equilibrated configurations obtained with the Janus I supercomputer [24]. We have found that the crucial scaling relations are fulfilled beyond the mean field approximation, close to (but above) the de Almeida-Thouless line. Furthermore, it is quite probable that our approach will be relevant for the study of other physical systems as well (e.g., glass-forming liquids). In addition, our results for the renormalized coupling λ seem to exclude the suggested scenario of a first-order transition [48].

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APPENDIX A: THE REPLICA-SYMMETRIC FIELD THEORY

Standard arguments [33,34] tell us that the D -dimensional Ising spin glass in the presence of a magnetic field is described at criticality by the following Replica Symmetric (RS) Hamiltonian for the replicated overlap $\phi_{ab}(x)$ [$\phi_{aa}(x) = 0$]:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \int d^D x \left[m_1 \sum_{ab} \phi_{ab}^2 + \frac{1}{2} \sum_{ab} (\nabla \phi_{ab})^2 + \right. \\ & + m_2 \sum_{abc} \phi_{ab} \phi_{ac} + m_3 \sum_{abcd} \phi_{ab} \phi_{cd} + \\ & \left. - \frac{1}{6} \tilde{w}_1 \sum_{abc} \phi_{ab} \phi_{bc} \phi_{ca} - \frac{1}{6} \tilde{w}_2 \sum_{ab} \phi_{ab}^3 \right]. \quad (\text{A1}) \end{aligned}$$

Note that the cubic couplings in the Hamiltonian are written as \tilde{w}_1, \tilde{w}_2 . In general they are different from the corresponding coefficients w_1, w_2 of the Gibbs free energy discussed in Ref. [44] (the vertices in field theoretical language). The Gibbs free energy as usual is the Legendre transform of the free energy, and the corresponding coefficients of the free energy are ω_1, ω_2 introduced previously. The coefficients \tilde{w}_i and w_i are respectively bare and dressed couplings. They coincide only at the level of the tree approximation in field theory; in general they are different.

At the MF level (where $\tilde{w}_i = w_i$), m_1 vanishes linearly on the dAT line and, in the spin glass (SG) phase, the solution displays Replica Symmetry Breaking (RSB) with a breaking point at a value equal to w_2/w_1 [47,59]. It follows that $\lambda \equiv w_2/w_1$ must be smaller than 1 for consistency. It should

be also noted that the parameter λ controls the MF values of equilibrium and off-equilibrium dynamical exponents in a variety of contexts [44,60,61].

The idea of Höller and Read [48] (that started from [62]) is to apply the RG to the above replicated Hamiltonian until the mass term m_1 (which is initially small because we start close to the dAT line) becomes equal to 1; then the RG flow is stopped and the new Hamiltonian is analyzed at the MF level. Note that they actually follow Bray and Roberts [33] and project on the replicon subspace effectively sending the longitudinal and anomalous masses to infinity. To obtain subcritical behavior, one must keep the massive modes finite; see Refs. [63,64] for a thorough comparative discussion of the two approaches. Höller and Read suggest that below the upper critical dimension, λ becomes larger than 1 under the RG flow on the whole dAT line and therefore the transition becomes first order. One should note that treating a Wilson Hamiltonian at the MF level is always an approximation, although it may be accurate close to the upper critical dimension. Essentially, one is approximating the true Gibbs free energy with the Wilson's Hamiltonian, i.e., fluctuations are neglected. While the coefficients of the Wilson's Hamiltonian are bare parameters that cannot be measured, the coefficients of the Gibbs free energy (proportional of the renormalized couplings) can be expressed in terms of physical observables and thus are directly accessible to measurements [50,53,54].

The renormalized couplings $w_{1,r}$ and $w_{2,r}$ have finite and model-dependent values except at the critical temperature where, if scaling holds, they have *finite* universal values $w_{1,r}^*$ and $w_{2,r}^*$. The spin-glass susceptibility and correlation length diverge as $\chi_R \propto |T - T_c|^{-\gamma}$ and $\xi_2 \propto |T - T_c|^{-\nu}$, respectively, and consistently ω_1 and ω_2 diverge as

$$\omega_{1,2} \propto |T - T_c|^{-\gamma_3}, \quad \gamma_3 = 3\nu - \frac{3}{2}\nu\eta + \frac{\nu D}{2}. \quad (\text{A2})$$

Notice that renormalized coupling constants $w_{1,r}^*$ and $w_{2,r}^*$ are universal quantities at criticality and play a key role in computations of critical exponents [50,53,54], being the zeros of the β functions.

Note that Eqs. (8) in the main text follow from Eqs. (91) in Ref. [44], noticing that when the RG flow is stopped, the overlaps are effectively rescaled by a factor $\chi_R^{1/2}$ and the length is rescaled by a factor ξ_2 , since the coefficient of the term $(\nabla \phi_{ab})^2$ is fixed to 1 in the RG flow.

APPENDIX B: COMPUTING ω_1 AND ω_2 USING THREE, FOUR, AND SIX REPLICAS

In order to compute ω_1 and ω_2 , we need to evaluate numerical quantities like

$$m_i^2 \equiv \overline{\langle \sigma_i \rangle^2}, \quad m_i^4 \equiv \overline{\langle \sigma_i \rangle^4}, \quad m_i^6 \equiv \overline{\langle \sigma_i \rangle^6}. \quad (\text{B1})$$

The standard approach consists in introducing K independent replicas of the system sharing the same disorder

$(\sigma^{(i)}, i = 1, \dots, K)$, obtaining

$$\begin{aligned} m_i^2 &= \overline{\langle \sigma_i^{(1)} \sigma_i^{(2)} \rangle}, \quad m_i^4 = \overline{\langle \sigma_i^{(1)} \sigma_i^{(2)} \sigma_i^{(3)} \sigma_i^{(4)} \rangle}, \\ m_i^6 &= \overline{\langle \sigma_i^{(1)} \sigma_i^{(2)} \sigma_i^{(3)} \sigma_i^{(4)} \sigma_i^{(5)} \sigma_i^{(6)} \rangle}. \end{aligned} \quad (\text{B2})$$

Both nonlinear susceptibilities, ω_1 and ω_2 , are suitable for numerical evaluation once expressed as [44]

$$\begin{aligned} \omega_1 &= \mathcal{W}_1 - 3\mathcal{W}_5 + 3\mathcal{W}_7 - \mathcal{W}_8, \\ \omega_2 &= \frac{1}{2}\mathcal{W}_2 - 3\mathcal{W}_3 + \frac{3}{2}\mathcal{W}_4 + 3\mathcal{W}_5 + 2\mathcal{W}_6 - 6\mathcal{W}_7 + 2\mathcal{W}_8, \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}_1 &\equiv N^2 \overline{\langle \delta Q_{12} \delta Q_{23} \delta Q_{31} \rangle}, \\ \mathcal{W}_2 &\equiv N^2 \overline{\langle \delta Q_{12}^3 \rangle}, \\ \mathcal{W}_3 &\equiv N^2 \overline{\langle \delta Q_{12}^2 \delta Q_{13} \rangle}, \\ \mathcal{W}_4 &\equiv N^2 \overline{\langle \delta Q_{12}^2 \delta Q_{34} \rangle}, \\ \mathcal{W}_5 &\equiv N^2 \overline{\langle \delta Q_{12} \delta Q_{13} \delta Q_{24} \rangle}, \\ \mathcal{W}_6 &\equiv N^2 \overline{\langle \delta Q_{12} \delta Q_{13} \delta Q_{14} \rangle}, \\ \mathcal{W}_7 &\equiv N^2 \overline{\langle \delta Q_{12} \delta Q_{13} \delta Q_{45} \rangle}, \\ \mathcal{W}_8 &\equiv N^2 \overline{\langle \delta Q_{12} \delta Q_{34} \delta Q_{56} \rangle}, \end{aligned}$$

where overlap fluctuations can be written in terms of independent real replicas with the same quenched disorder:

$$\delta Q_{ab} \equiv \frac{1}{N} \sum_i s_i^a s_i^b - \frac{1}{N} \sum_i \overline{\langle s_i \rangle^2}. \quad (\text{B3})$$

Each correlator \mathcal{W}_i requires a number of different real replicas equal to the largest index in its expression (right-hand side). Hence, we recall, we need two replicas to compute the overlap—four for the susceptibilities and six for ω_1 and ω_2 .

Can we use a smaller number of replicas? The theory predicts that there are six linear combinations of the \mathcal{W}_i 's that diverge less than the \mathcal{W}_i separately. By using these linear relationships one can express the eight coefficients in terms of only the three-replica estimators [44]:

$$\omega_1^{(3)} \equiv \frac{11}{30}\mathcal{W}_1 - \frac{2}{15}\mathcal{W}_2, \quad (\text{B4})$$

$$\omega_2^{(3)} \equiv \frac{4}{15}\mathcal{W}_1 - \frac{1}{15}\mathcal{W}_2. \quad (\text{B5})$$

Alternatively, the theory predicts that there are three linear combinations of the \mathcal{W} 's that remain finite at the critical temperature. Therefore one can express \mathcal{W}_7 and \mathcal{W}_8 as a function of the remaining cumulants obtaining the four-replica estimators [56]:

$$\begin{aligned} \omega_1^{(4)} &\equiv \frac{23\mathcal{W}_1}{30} + \frac{\mathcal{W}_2}{20} - \frac{3\mathcal{W}_3}{5} + \frac{9\mathcal{W}_4}{20} - \frac{6\mathcal{W}_5}{5} + \frac{\mathcal{W}_6}{2}, \\ \omega_2^{(4)} &\equiv \frac{7\mathcal{W}_1}{15} + \frac{2\mathcal{W}_2}{5} - \frac{9\mathcal{W}_3}{5} + \frac{3\mathcal{W}_4}{5} - \frac{3\mathcal{W}_5}{5} + \mathcal{W}_6. \end{aligned}$$

within the RS theory, the three- and four-replica estimators are different from the true ω_1 and ω_2 at any given temperature but *coincide* with them at the critical temperature. At a generic temperature $w_{1,r}$, $w_{2,r}$ and λ have model-dependent values and we are interested in the universal values they take only at the critical temperature. More precisely, one can show that close to the critical point,

$$\omega_i - \omega_i^{(3)} = O(|T - T_c|^{\gamma_\Delta}), \quad \omega_i - \omega_i^{(4)} = O(|T - T_c|^{\gamma_3}),$$

where the exponent γ_Δ is expected to be smaller than γ_3 (e.g., in MF one finds $\gamma_\Delta = 1$ and $\gamma_3 = 3$).

APPENDIX C: FINITENESS OF THE LONGITUDINAL SUSCEPTIBILITY

Let us consider the model in the presence of a Gaussian magnetic field which generates a new term in the Hamiltonian: $+h_o \sum_i h_i S_i$, where h_i are independent Gaussian variables with zero mean and unit variance. The staggered magnetization is defined as

$$m_{\text{st}} \equiv \overline{\langle h_i \sigma_i \rangle}, \quad (\text{C1})$$

where $\overline{\langle \dots \rangle}$ is the joint average over the couplings and the Gaussian magnetic field. Its susceptibility is

$$\chi_{\text{st}} = \frac{\partial m_{\text{st}}}{\partial h_o} = -\beta \sum_l \overline{\langle h_i S_i h_l S_l \rangle - \langle h_i S_i \rangle \langle h_l S_l \rangle}. \quad (\text{C2})$$

Integrating by parts Eq. (C2), one can finally obtain that $\chi_{\text{st}} = 2\beta \chi_L$. Therefore if the magnetic susceptibility does not diverge, neither does the longitudinal susceptibility.

APPENDIX D: THE MODELS

We study the 4D-dimensional EA model in a field h where $N = L^4$ Ising spins interact via

$$\mathcal{H} = - \sum_{(xy)} J_{xy} S_x S_y + h \sum_x S_x, \quad (\text{D1})$$

where the first sum is over nearest-neighbor pairs and $J_{xy} = \pm 1$ with 50% probability. In our 4D computation, the spins are located in the nodes of a hypercubic lattice with periodic boundary conditions. We have also simulated the model on a Bethe lattice where the spins occupy the vertices of a random regular graph with connectivity 4.

- [1] M. Mézard, G. Parisi, and M. Virasoro, *Spin-Glass Theory and Beyond* (World Scientific, Singapore, 1987).
- [2] J. A. Mydosh, *Spin Glasses: An Experimental Introduction* (Taylor and Francis, London, 1993).
- [3] A. P. Young, *Spin Glasses and Random Fields* (World Scientific, Singapore, 1998).
- [4] J. R. L. de Almeida and D. J. Thouless, *J. Phys. A: Math. Gen.* **11**, 983 (1978).

- [5] A. Cavagna, *Phys. Rep.* **476**, 51 (2009).
- [6] E. Gardner, *Nucl. Phys. B* **257**, 747 (1985).
- [7] S. Caracciolo, G. Parisi, S. Patarnello, and N. Sourlas, *J. Phys. France* **51**, 1877 (1990).
- [8] David A. Huse and Daniel S. Fisher, *J. Phys. I: France* **1**, 621 (1991).
- [9] S. Caracciolo, G. Parisi, S. Patarnello, and N. Sourlas, *J. Phys. I: France* **1**, 627 (1991).

- [10] J. C. Ciria, G. Parisi, F. Ritort, and J. J. Ruiz-Lorenzo, *J. Phys. I: France* **3**, 2207 (1993).
- [11] G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Phys. Rev. B* **57**, 13617 (1998).
- [12] E. Marinari, G. Parisi, and F. Zuliani, *J. Phys. A: Math. Gen.* **31**, 1181 (1998).
- [13] E. Marinari, C. Naitza, and G. Parisi, *J. Phys. A: Math. Gen.* **31**, 6355 (1998).
- [14] E. Marinari, C. Naitza, F. Zuliani, G. Parisi, M. Picco, and F. Ritort, *Phys. Rev. Lett.* **81**, 1698 (1998).
- [15] J. Houdayer and O. C. Martin, *Phys. Rev. Lett.* **82**, 4934 (1999).
- [16] E. Marinari, G. Parisi, and F. Zuliani, *Phys. Rev. Lett.* **84**, 1056 (2000).
- [17] J. Houdayer and O. C. Martin, *Phys. Rev. Lett.* **84**, 1057 (2000).
- [18] A. Cruz, L. A. Fernández, S. Jiménez, J. J. Ruiz-Lorenzo, and A. Tarancón, *Phys. Rev. B* **67**, 214425 (2003).
- [19] A. P. Young and H. G. Katzgraber, *Phys. Rev. Lett.* **93**, 207203 (2004).
- [20] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Phys. Rev. Lett.* **101**, 107203 (2008).
- [21] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, *Phys. Rev. B* **87**, 024414 (2013).
- [22] M. Baity-Jesi, R. A. Baños, A. Cruz, L. A. Fernandez, J. M. Gil-Narvion, A. Gordillo-Guerrero, D. Iniguez, A. Maiorano, F. Mantovani, E. Marinari, V. Martín-Mayor, J. Monforte-Garcia, A. Muñoz Sudupe, D. Navarro, G. Parisi, S. Perez-Gaviro, M. Pivanti, F. Ricci-Tersenghi, J. J. Ruiz-Lorenzo, S. F. Schifano *et al.*, *Phys. Rev. E* **89**, 032140 (2014).
- [23] M. Baity-Jesi, R. A. Baños, A. Cruz, L. A. Fernandez, J. M. Gil-Narvion, A. Gordillo-Guerrero, D. Iniguez, A. Maiorano, M. F. E. Marinari, V. Martín-Mayor, J. Monforte-Garcia, A. Muñoz Sudupe, D. Navarro, G. Parisi, S. Perez-Gaviro, M. Pivanti, F. Ricci-Tersenghi, J. J. Ruiz-Lorenzo, S. F. Schifano *et al.*, *J. Stat. Mech.* (2014) P05014.
- [24] R. A. Baños, A. Cruz, L. A. Fernandez, J. M. Gil-Narvion, A. Gordillo-Guerrero, M. Guidetti, D. Iniguez, A. Maiorano, E. Marinari, V. Martín-Mayor, J. Monforte-Garcia, A. Muñoz Sudupe, D. Navarro, G. Parisi, S. Perez-Gaviro, J. J. Ruiz-Lorenzo, S. F. Schifano, B. Seoane, A. Tarancon, P. Tellez *et al.*, *Proc. Natl. Acad. Sci. USA* **109**, 6452 (2012).
- [25] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Phys. Rev. Lett.* **103**, 267201 (2009).
- [26] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. Ruiz-Lorenzo, *Philos. Mag.* **91**, 1917 (2011).
- [27] K. Geirhos, P. Lunkenheimer, and A. Loidl, *Phys. Rev. Lett.* **120**, 085705 (2018).
- [28] L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti, D. El Masri, D. L'Hôte, F. Ladieu, and M. Pierno, *Science* **310**, 1797 (2005).
- [29] S. Albert, T. Bauer, M. Michl, G. Biroli, J.-P. Bouchaud, A. Loidl, P. Lunkenheimer, R. Tourbot, C. Wiertel-Gasquet, and F. Ladieu, *Science* **352**, 1308 (2016).
- [30] A. P. Hammond and E. I. Corwin, *Proc. Natl. Acad. Sci. USA* **117**, 5714 (2020).
- [31] M. Dilucca, L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *Entropy* **22**, 250 (2020).
- [32] S. Albert, G. Biroli, F. Ladieu, R. Tourbot, and P. Urbani, *Phys. Rev. Lett.* **126**, 028001 (2021).
- [33] A. J. Bray and S. A. Roberts, *J. Phys. C: Solid State Phys.* **13**, 5405 (1980).
- [34] T. Temesvári and C. De Dominicis, *Phys. Rev. Lett.* **89**, 097204 (2002).
- [35] P. Charbonneau and S. Yaida, *Phys. Rev. Lett.* **118**, 215701 (2017).
- [36] M. C. Angelini, C. Lucibello, G. Parisi, G. Perrupato, F. Ricci-Tersenghi, and T. Rizzo, *Phys. Rev. Lett.* **128**, 075702 (2022).
- [37] A. J. Bray and M. A. Moore, in *Heidelberg Colloquium on Glassy Dynamics*, edited by J. L. van Hemmen and I. Morgenstern, Lecture Notes in Physics No. 275 (Springer, Berlin, 1987).
- [38] A. J. Bray and M. A. Moore, *Phys. Rev. Lett.* **58**, 57 (1987).
- [39] D. S. Fisher and H. Sompolinsky, *Phys. Rev. Lett.* **54**, 1063 (1985).
- [40] D. S. Fisher and D. A. Huse, *Phys. Rev. Lett.* **56**, 1601 (1986).
- [41] D. S. Fisher and D. A. Huse, *Phys. Rev. B* **38**, 373 (1988).
- [42] D. S. Fisher and D. A. Huse, *Phys. Rev. B* **38**, 386 (1988).
- [43] K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1993).
- [44] G. Parisi and T. Rizzo, *Phys. Rev. E* **87**, 012101 (2013).
- [45] We call susceptibilities (nonlinear susceptibilities), to the two-point (three-point) correlators in Fourier space, at zero momentum.
- [46] K. G. Wilson and J. Kogut, *Phys. Rep.* **12**, 75 (1974).
- [47] D. J. Gross, I. Kanter, and H. Sompolinsky, *Phys. Rev. Lett.* **55**, 304 (1985).
- [48] J. Höller and N. Read, *Phys. Rev. E* **101**, 042114 (2020).
- [49] As usual, the renormalized couplings are defined in terms of the renormalized correlation functions at zero momentum.
- [50] G. Parisi, *Statistical Field Theory* (Addison-Wesley, Reading, MA, 1988).
- [51] K. Binder, *Z. Phys. B: Condens. Matter* **43**, 119 (1981).
- [52] J. Salas and A. D. Sokal, *J. Stat. Phys.* **98**, 551 (2000).
- [53] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Clarendon Press, Oxford, 2005).
- [54] D. J. Amit and V. Martín-Mayor, *Field Theory, the Renormalization Group and Critical Phenomena*, 3rd ed. (World Scientific, Singapore, 2005).
- [55] G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, *J. Stat. Mech.: Theory Exp.* (2014) P04013.
- [56] M. Veca, Master's thesis, Universit La Sapienza-Roma, 2021.
- [57] F. Belletti, M. Cotallo, A. Cruz, L. A. Fernandez, A. Gordillo, A. Maiorano, F. Mantovani, E. Marinari, V. Martín-Mayor, A. Muñoz Sudupe, D. Navarro, S. Perez-Gaviro, J. J. Ruiz-Lorenzo, S. F. Schifano, D. Sciretti, A. Tarancon, R. Tripiccione, and J. L. Velasco (Janus Collaboration), *Comput. Phys. Commun.* **178**, 208 (2008).
- [58] The plateau value of χ_L in Fig. 2 approximately scales as h^{-x} , with x between 2 and 3. In MF, χ_L near T_c is proportional to $h^{-2/3}$ [64].
- [59] T. Rizzo, *Phys. Rev. E* **88**, 032135 (2013).
- [60] F. Caltagirone, U. Ferrari, L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, *Phys. Rev. Lett.* **108**, 085702 (2012).
- [61] F. Caltagirone, G. Parisi, and T. Rizzo, *Phys. Rev. E* **87**, 032134 (2013).
- [62] M. A. Moore and N. Read, *Phys. Rev. Lett.* **120**, 130602 (2018).
- [63] I. R. Pimentel, T. Temesvári, and C. De Dominicis, *Phys. Rev. B* **65**, 224420 (2002).
- [64] T. Temesvári, C. De Dominicis, and I. R. Pimentel, *Eur. Phys. J. B* **25**, 361 (2002).