Critical pressure in liquids due to dynamic choking

Jiří Vacula * and Pavel Novotný
Institute of Automotive Engineering, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2896/2, 616 69 Brno, Czech Republic

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The existence of a critical pressure ratio due to gas-dynamic choking is well known for an ideal gas. It is reasonable to assume that liquids whose compressibility is defined by the bulk modulus also have a critical pressure ratio. The problem discussed here is a fundamental one because it deals with the basic principles of the compressible flow of liquids. It has been shown that even though an ideal gas with a constant heat capacity ratio value has a critical pressure ratio, liquid with a constant bulk modulus value experiences a critical pressure difference. As the outlet pressure gradually decreases, the liquid reaches the local speed of sound, and further reduction of this pressure does not lead to an increase in mass flow. This phenomenon occurs in liquids without considering the change from a liquid to a gaseous phase. Behavior is confirmed analytically for different bulk modulus models, and for a constant bulk modulus value, the phenomenon is verified by numerical simulation using computational fluid dynamics. The conclusions published in this work point to striking analogies between the behavior of liquids and ideal gas. The equations governing the motion of liquids derived in this work, thus complete the fundamental description of the critical flow of fluids.

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I. INTRODUCTION

It is well known that the velocity of gas leaving the reservoir of pressure p_0 and velocity $v_0 \approx 0$ through a duct increases due to a gradual reduction of the outlet pressure until the local speed of sound is reached. Mass flow increases until the ratio p/p_0 acquires the so-called critical pressure ratio, where p is local pressure in the duct and further reduction of the outlet pressure no longer leads to an increase in mass flow.

This phenomenon is called gas-dynamic choking and can occur even in the case of flow with no energy losses. The critical pressure ratio represents a limitation of the mass flow, not in terms of energy losses in the gas but as a physical limitation resulting from the nature of the change in density and velocity of the expanding gas. Therefore, critical flow is a limiting factor that causes the flow to choke, even in the case of zero gas viscosity. To prevent flow choking when the pressure ratio is smaller than critical, a convergent-divergent pipe must be used. Detailed analyses of one-dimensional flow can be found in almost any literature dealing with compressible flows [1–4].

The fundamental steps of the derivation of critical pressure are shown here to develop the problem for liquids. To determine the critical pressure ratio of an ideal gas, it is necessary to take into account the equation of the adiabatic process of the ideal gas

$$p\rho^{-\kappa} = p_0\rho_0^{-\kappa} = \text{constant}, \tag{1}$$

where ρ is density, κ is heat capacity ratio, and subscript 0 indicates the location inside the reservoir. Conservation of momentum is regarded in the form

$$v\,dv = -\frac{dp}{\rho}. \tag{2}$$

Note that Eq. (2) assumes a one-dimensional stationary flow, neglecting the influence of body and friction forces. Substituting density ρ from Eq. (1) into the conservation of momentum Eq. (2) and its subsequent integration, the velocity of the gas leaving a convergent nozzle, taking the adiabatic process into account, can be determined as

$$v = \sqrt{\frac{2\kappa}{\kappa-1}\frac{p_0}{\rho_0}\left[1-\left(\frac{p}{p_0}\right)^{(\kappa-1)/\kappa}\right]} \tag{3}$$

and assumes negligible velocity of the gas in the reservoir, $v_0 = 0$. The integral form of the continuity equation for one-dimensional stationary flow has the form

$$\dot{m} = \rho v A = \text{constant}, \tag{4}$$

where ρ is determined by Eq. (1), v is determined by Eq. (3), and A denotes the size of the cross-sectional area through which the fluid flows. At this point, it would be appropriate to investigate the product, ρv; however, function ψ is sometimes introduced in the literature [5]. The critical pressure ratio can be derived both from the product ρv and the form of the continuity equation considering function ψ. For the purposes of this work, function ψ is introduced and shown here. The continuity equation with the introduction of function ψ takes the form

$$\dot{m} = A\psi\sqrt{2p_0\rho_0}, \tag{5}$$

*jiri.vacula@vutbr.cz

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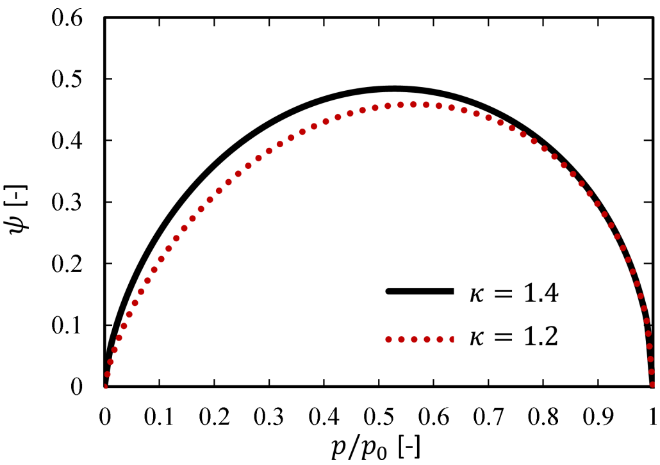

FIG. 1. Function ψ of an ideal gas for heat capacity ratio $\kappa = 1.2$ and $\kappa = 1.4$.

where the introduced function ψ is given as

$$\psi = \sqrt{\frac{\kappa}{\kappa - 1}} \sqrt{\left(\frac{p}{p_0}\right)^{2/\kappa} - \left(\frac{p}{p_0}\right)^{(\kappa+1)/\kappa}}. \tag{6}$$

It should be emphasized that when the inlet pressure p_0 increases, the mass flow increases even if the ratio p/p_0 acquires the maximum critical value because the mass flow $\dot{m}$ is linearly proportional to the term $\sqrt{2p_0\rho_0}$, according to Eq. (5). For this reason, it is more appropriate to state that critical pressure p exists in lieu of a critical pressure ratio p/p_0. The critical pressure ratio must be understood as the ratio p/p_0 for a fixed pressure in the reservoir p_0.

Figure 1 shows that function ψ acquires a local maximum on the interval $(0; 1)$. By setting the derivative of function ψ equal to zero, that is, $d\psi/d(p/p_0) = 0$, it holds that function ψ reaches its maximum for the pressure ratio

$$\left(\frac{p}{p_0}\right)_{\psi\ \max} = \left(\frac{2}{\kappa + 1}\right)^{\kappa/(\kappa-1)}. \tag{7}$$

Equation (7) is a well-known result, which shows that the flow of an ideal gas reaches its maximum at a certain value of the pressure ratio p/p_0 for convergent pipes. Nonetheless, it must be noted that Eq. (7) is valid only for an ideal gas, not for liquids because Eq. (1) is valid for an ideal gas. The maximum gas velocity is equal to the local speed of sound for the critical pressure ratio

$$v_{\psi\ \max} = \sqrt{\kappa \frac{p}{\rho}} = \sqrt{\kappa r T}, \tag{8}$$

where r denotes the specific gas constant and T denotes thermodynamic temperature. The modifications shown below, as stated by Shapiro [1], make it possible to deduce the pressure gradient of the flow in an upstream and downstream direction depending on the pipe shape for subsonic and supersonic flow regimes. By deriving continuity Eq. (4), the equation

$$\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{dv}{v} = 0 \tag{9}$$

can be obtained. Substituting expression dv/v into the conservation of momentum Eq. (2), it is possible to derive the expression

$$\frac{dA}{A} = \frac{dp}{\rho}\left(\frac{1}{v^2} - \frac{d\rho}{dp}\right) = 0. \tag{10}$$

In isentropic processes, denoted by subscript s, the local speed of sound is defined by the relationship

$$v_{\text{sound}} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{\text{s}}}. \tag{11}$$

Supposing that $(\partial p/\partial \rho)_{\text{s}} = dp/d\rho$ and introducing Mach number

$$\text{Ma} = \frac{v}{\sqrt{\frac{dp}{d\rho}}}, \tag{12}$$

Eq. (13) can be obtained as

$$\frac{dA}{A} = \frac{dp}{\rho v^2}(1 - \text{Ma}^2). \tag{13}$$

As Shapiro [1] notes, Eq. (13) is generally valid for fluids, i.e., for both gases and liquids. Equation (13) is derived based on continuity Eq. (4) and conservation of momentum Eq. (2) as they are valid laws for both gases and liquids. Equation (13) makes it possible to determine whether the pressure increases or decreases in the flow direction depending on the ratio of the cross sections of the pipe, A/A_0, in the case of subsonic and supersonic flow. The conservation of momentum Eq. (2) indicates that the streamwise velocity gradient is equal to the negative streamwise pressure gradient, i.e., if the pressure increases in the flow direction, the velocity decreases and vice versa. Four familiar cases can occur, as listed in Table I.

However, the aforementioned equations do not provide any information about the value of prospective critical pressure ratio for liquid. Although the conclusions made about Table I, which is valid for both gases and liquids, suggest that a critical pressure ratio must also exist for liquids, Eq. (7) cannot be used for liquids as it is only valid for an ideal gas. This equation was derived based on Eq. (1). The conclusions summarized in Table I are drawn from Eq. (13) and provide a qualitative idea of the behavior of liquids in terms of the existence of liquid-dynamic choking, but it is not possible to quantify the value of critical pressure for liquids.

Researchers from various branches of fluid mechanics are generally aware that even liquids behave analogously to gases with respect to the existence of a critical pressure ratio, which certainly exists in liquids. Therefore, it is interesting that determining the value of critical pressure in liquids, as is the case for ideal gases, has so far escaped the attention of theoretical considerations.

Unlike gases, liquids do not have an equation of state expressed in an elegant form that combines pressure, density, and thermodynamic temperature that allows a description of the selected processes. For liquids, processes are expressed by the type of bulk modulus, the value of which is experimentally determined for the selected process. This work aims to demonstrate that the value of critical pressure exists in liquids, and it can be clearly defined for selected models describing

TABLE I. Overview of change in pressure and fluid velocity depending on the cross section for subsonic and supersonic regimes.

	Ratio $\frac{A}{A_0}$	Pressure gradient	Velocity gradient
Subsonic divergent nozzle	$\frac{A}{A_0} > 1$	$p > p_0$	$v < v_0$
Subsonic convergent nozzle	$\frac{A}{A_0} < 1$	$p < p_0$	$v > v_0$
Supersonic divergent nozzle	$\frac{A}{A_0} > 1$	$p < p_0$	$v > v_0$
Supersonic convergent nozzle	$\frac{A}{A_0} < 1$	$p > p_0$	$v < v_0$

the bulk modulus of a liquid. The theory is initiated analytically using fundamental equations that govern the motion of liquids.

II. REVIEW

In the literature, the terms critical flow and choked flow have different meanings and applications. Therefore, it should be emphasized that this work will present an unfinished analogy between gas-dynamic choking and liquid-dynamic choking. Knowledge of the critical pressure ratio for an ideal gas has been widely used in engineering and science, especially in the aerospace industry, for example, in the construction of engines and nozzles, and this phenomenon has been studied for a long time [6–8]. Kluwick and Scheichl [9] investigated a supersonic nozzle for the nonstationary flow of dense gases. Drikakis and Tsangris [10] examined the supersonic flow of real gas in a convergent-divergent nozzle. Sirignano [11] more recently studied the differences in conditions leading to choking between an ideal gas and real gas using fundamental equations to describe the flow of an ideal gas while introducing compressibility factors. The results showed the variations in speed, density, enthalpy, or speed of sound between an ideal gas and a real gas.

Research has also been conducted on choking in liquids; however, it is not caused by so-called dynamic choking but other physical phenomena. Birkhoff *et al.* [12] and Brennen [13] investigated choked flow and cavity flow when the maximum liquid velocity is limited for a given stream and cavity pressure. In their work, liquid choking was a consequence of cavitation. Richardson [14] referred to critical flow in liquids as a state of choking due to an increase in viscosity as a result of a rise in liquid pressure. He investigated choking as a result of pressure losses, although he noted that an increase in pressure led to the liquid being heated, therefore decreasing its viscosity.

Although LeMartelot *et al.* [15] studied critical flow in a mixture of gas and liquid, the equations for the critical pressure ratio published in this work are based on equations for an ideal gas. Similarly, Ros [16] examined a mixture of gas and liquid to analyze the flowmeter formula. However, the equations for the liquid-gas mixture are also based on those valid for an ideal gas using polytropic process equations.

Moncalvo and Friedel [17] examined flow choking at a critical pressure ratio considering a homogeneous mixture of water. The determination of critical pressure was based on the equations valid for an ideal gas. Hardekopf and Mewes [5] also studied the critical pressure ratio for two-phase flow and demonstrated that the critical pressure ratio of water in the state of saturated liquid is $(p/p_0)_{\psi\ \max} = 0.85$ as determined by the Henry-Fauske model. However, even in the case of saturated liquid flow, this model is based on the assumption of a two-phase mixture flow because it is assumed that the vaporization of the liquid may occur locally at the nozzle inlet.

The Henry-Fauske model was also used by Kim [18–20] and Geng *et al.* [21], where critical flow in the case of liquids was considered to be a flow that, under certain conditions, could choke because of gaseous or vapor components present in the liquid. Moody [22] determined the maximum flow rate in the flow of a one-component water vapor mixture using the term critical flow. Schrock *et al.* [23] studied nonstationary steam flow considering condensation and used equations for an ideal gas. Arina [24] also dealt with the numerical simulation of supercritical flow, which is understood to be a liquid flow that is close to the liquid-vapor critical point.

Indeed, an abundance of literature highlighted the existence of maximum flow in the nozzle due to rapid decompression, in which flashing inception occurs [25–28]. However, these works used saturated or subcooled water, and maximum mass flow was determined for the nucleation of the vapor component in the liquid flowing through the nozzle. Therefore, flow choking was understood as a consequence of the local phase change from liquid to gas.

The aforementioned works assume the existence of a gaseous or vapor component present in the liquid to determine the critical pressure ratio. However, the value of the liquid's critical pressure ratio, without considering a phase change similar to an ideal gas, is still unknown. This work demonstrates that liquids, without considering the phase change, experience a certain pressure drop value at which choking in the convergent pipe occurs, and a further increase in mass flow is possible only by using a convergent-divergent pipe. The equations presented in this work have not yet been investigated, perhaps due to no need to deal with the supersonic flow of liquids without considering the liquid phase change or the presence of the gaseous component.

Generally, due to the high value of the ratio between the bulk modulus and the density of liquids, the achievement of the sonic velocity of liquids is associated with the creation of extremely high pressure drop values. In any case, the following equations represent the theoretical limit of the critical pressure drop leading to choking in the flow of liquids. This is not because of the change of particles from liquid to vaporous form or another gas but for equivalent reasons, as in the case of gas-dynamic choking.

III. CRITICAL PRESSURE IN LIQUIDS

As stated by Lighthill [29], the most accurate form that expresses the law of conservation of momentum for a compressible fluid without considering external forces, written using Einstein's summation notation, is governed by the equation

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) - \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{14}$$

where $\partial/\partial t$ is the time derivative, v_i is the velocity vector with index i and v_j is the velocity vector with index j, σ_{ij} is the Cauchy stress tensor, and x_j is a spatial coordinate with index j; indices $i,\ j = 1,\ 2,\ 3$. The Cauchy stress tensor takes the form $\sigma_{ij} = -\delta_{ij} p + \Pi_{ij}$, where $-\delta_{ij}$ is Kronecker delta and Π_{ij} represents the tensor of irreversible changes, that is, the deviatoric stress tensor. The continuity equation for compressible fluid may be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0. \tag{15}$$

The following assumptions are used for

(1) stationary flow $\partial/\partial t = 0$,

(2) inviscid flow $\Pi_{ij} = 0$,

(3) one-dimensional flow $v_2 = v_3 = 0$, $\partial/\partial x_2 = \partial/\partial x_3 = 0$,

(4) the insignificant effect of density change, and

(5) no phase change.

Note that the conservation of momentum Eq. (2) in the form $v\ dv = -dp/\rho$, can be obtained from Eq. (14), and continuity Eq. (15) assumes that density ρ is constant, i.e., $\rho \neq \rho(t, x_i)$. Despite these assumptions, the conservation of momentum Eq. (2) is used for a perfectly compressible ideal gas. Concerning one-dimensional flow conditions, velocity v_1 is further written without an index, $v_1 = v$.

To derive function ψ for liquids, the compressibility of the fluid must be considered. Equation (1) cannot simulate the behavior of a liquid under adiabatic flow. For the sake of clarity, it is worth noting that considering $\kappa = 1$ represents Boyle's law, from which it follows that a double increase in absolute static pressure at a constant temperature will lead to a double increase in the density of a liquid. Alternatively, considering a high value of heat capacity ratio, e.g., $\kappa = 1000$, results in doubling the pressure ratio p/p_0, leading to a very slight increase in density ratio ρ/ρ_0, according to Eq. (1), as is normally expected for most liquids. Nevertheless, based on the heat capacity ratio definition, $\kappa = 1000$ means that $c_\mathrm{p} = 1000 c_\mathrm{v}$, which is obviously not correct. Hence, it is apparent that in the case of liquids, it is not physically correct to assign artificial values of heat capacity ratio κ to liquids in Eq. (1) and apply such an artificial κ in Eq. (7) as κ is defined as $\kappa = c_\mathrm{p}/c_\mathrm{v}$, where c_p and c_v denote isobaric and isochoric specific heat, respectively.

The quantity that defines the relationship between pressure and density of liquids is the bulk modulus. Gholizadeh *et al.* [30] and Hayward [31] summarized several definitions found in the literature, such as the secant or tangent bulk modulus. For the analytical purposes in this work, it is appropriate to use the definition of tangent bulk modulus K_τ because it expresses the instantaneous change in the density of the liquid and is defined by a differential equation. Furthermore, it is possible to distinguish whether it is a tangent isothermal bulk modulus $K_{\tau,\mathrm{T}}$ or a tangent isoentropic bulk modulus $K_{\tau,\mathrm{S}}$. The difference between these types of bulk moduli lies in how the volume of a liquid changes, whether it changes at a constant temperature or without heat exchange. For this work, the type of bulk modulus is not that important because the formal definition of the tangent bulk modulus is the same for both isothermal and isentropic modules. Therefore, it is not distinguished and is shown without an index. Equations and conclusions are applicable for $K_{\tau,\mathrm{T}}$ and $K_{\tau,\mathrm{S}}$.

A. Function ψ for constant bulk modulus

Bulk modulus K can be written in the form

$$K = \rho \frac{\partial p}{\partial \rho}. \tag{16}$$

It is assumed that density is a function of pressure, $\rho = \rho(p)$. In Eq. (16), partial derivatives can be exchanged for total derivatives and Eq. (16) may then be rearranged to take the form

$$\frac{d\rho}{\rho} = \frac{dp}{K}. \tag{17}$$

Bulk modulus is considered to be a constant value, $K = K_\mathrm{const}$. Subsequent integration satisfying boundary condition $p = p_0$: $\rho = \rho_0$ yields Eq. (18). This boundary condition expresses the known values of the density and pressure of the liquid in the reservoir. Density ρ is then equal to

$$\rho = \rho_0 e^{(p-p_0)/K_\mathrm{const}}. \tag{18}$$

Furthermore, the conservation of momentum Eq. (2) considering Eq. (17) has the form

$$v\ dv = -\frac{dp}{\rho} = -\frac{d\rho}{\rho^2} K_\mathrm{const} \tag{19}$$

and its integration satisfying boundary condition $v = 0$: $\rho = \rho_0$ gives Eq. (20). This boundary condition assumes a negligibly low velocity inside the reservoir and a known density ρ_0. It holds that

$$v = \sqrt{2K_\mathrm{const}\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)}, \tag{20}$$

where ρ is defined by Eq. (18). Substituting Eq. (18) for Eq. (20), the velocity of the liquid obtains the form

$$v = \sqrt{\frac{2K_\mathrm{const}}{\rho_0}(e^{(p_0-p)/K_\mathrm{const}} - 1)}. \tag{21}$$

Naturally, Eq. (21) can also be obtained by substituting $\rho = \rho(p)$ from Eq. (18) into Eq. (19) and subsequent integration with respect to p and satisfying boundary condition $v = 0$: $p = p_0$. The continuity equation is then rewritten

$$\dot{m} = A\rho_0 e^{(p-p_0)/K_\mathrm{const}} \sqrt{\frac{2K_\mathrm{const}}{\rho_0}(e^{(p_0-p)/K_\mathrm{const}} - 1)}$$
$$= A\psi_{K_\mathrm{const}}\sqrt{2K_\mathrm{const}\rho_0}. \tag{22}$$

Function ψ_{K_const} can be defined in a manner analogous to that of an ideal gas. By introducing substitution $\Delta p = p_0 - p$, function ψ_{K_const} can be written as

$$\psi_{K_\mathrm{const}} = e^{-\Delta p/K_\mathrm{const}}\sqrt{(e^{\Delta p/K_\mathrm{const}} - 1)}. \tag{23}$$

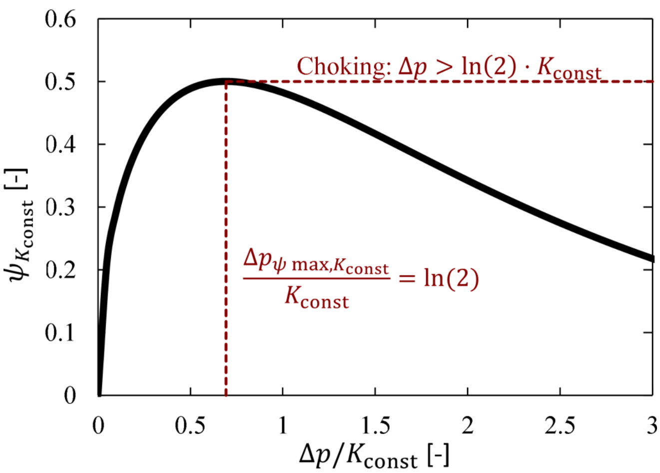

FIG. 2. The course of function $\psi_{K_{const}}$ of a liquid with a constant bulk modulus K_{const}.

The graph of function $\psi_{K_{const}}$ as a dependence of the dimensionless variable $\Delta p/K_{const}$ is shown in Fig. 2.

The graph in Fig. 2 demonstrates that function $\psi_{K_{const}}$ has a local maximum value that may be found by $d\psi_{K_{const}}/d(\Delta p/K_{const}) = 0$. At the maximum of function $\psi_{K_{const}}$, the ratio $\Delta p/K_{const}$ is equal to

$$\frac{\Delta p_{\psi\ \max,\ K_{const}}}{K_{const}} = \ln(2) \doteq 0.69. \tag{24}$$

Equation (24) shows that liquids whose compressibility is defined by the bulk modulus show a certain pressure drop value, similar to the case of a perfectly compressible ideal gas. During the gradual decrease in outlet pressure for a fixed pressure in the reservoir p_0, the mass flow reaches its maximum. Unlike an ideal gas, there is no critical pressure ratio in liquid with a constant bulk modulus K_{const}, but there is a critical pressure difference. Equation (24) states that, even for liquids, there is a certain limit at which the mass flow reaches its maximum, and the value of this limit is a pressure difference proportional to 0.69 times the bulk modulus of the liquid. Additionally, analogously to an ideal gas and in terms of Eq. (22), liquid-dynamic choking must be understood as a phenomenon that occurs under the condition of constant pressure p_0 in the reservoir. As the pressure in the reservoir increases, the density ρ_0 increases as well. For this reason, it is also appropriate to understand the maximum of function $\psi_{K_{const}}$ as a consequence of the reduction of the outlet pressure.

Nonetheless, the question also arises as to what value the velocity of the flowing liquid takes at the maximum of function $\psi_{K_{const}}$. Substituting expression ln(2) from Eq. (24) into Eq. (21) and using the relationship $\rho_0 = \rho e^{\ln(2)}$ from Eq. (18), which is valid for the maximum of function $\psi_{K_{const}}$, it follows that the velocity in the maximum of function $\psi_{K_{const}}$ is obtained as

$$v\ \psi\ \max,\ K_{const} = \sqrt{\frac{K_{const}}{\rho}} = v_{sound}. \tag{25}$$

Equation (25) proves that the velocity of a liquid is equal to the local speed of sound at the maximum of function $\psi_{K_{const}}$. Therefore, Eq. (24) can be interpreted that when the pressure difference reaches 0.69 times the bulk modulus of the liquid, the liquid reaches the local speed of sound.

As with an ideal gas, the Mach number $Ma = v/\sqrt{K_{const}/\rho}$ can be introduced for further consideration because when the pressure difference increases, the velocity in the critical cross section acquires the local velocity of sound. Using Eq. (17) and the definition of the Mach number, it follows that $v^2 = Ma^2 K_{const}/\rho$. The differential form of continuity Eq. (9) can thus be converted into

$$\frac{dA}{A} = \frac{1 - Ma^2}{K_{const} Ma^2} dp, \tag{26}$$

and after its integration that includes boundary condition $A = A_0$: $p = p_0$, the pressure p is equal to

$$p = \frac{K_{const} Ma^2}{1 - Ma^2} \ln\left(\frac{A}{A_0}\right) + p_0. \tag{27}$$

By analyzing Eq. (27) and considering the conservation of momentum Eq. (2), four known cases can occur that are analogous to an ideal gas; these are summarized in Table I.

B. Function ψ for linear dependence of the bulk modulus on pressure

The above derivation of function ψ is based on assuming a constant value for the bulk modulus. In general, the bulk modulus is a function of pressure and temperature $K = K(p, T)$. The following derivation assumes a linear dependence of the bulk modulus on pressure in the form

$$K_{lin} = ap + b, \tag{28}$$

where a and b are real coefficients. Using Eq. (17), it can be written

$$\frac{d\rho}{\rho} = \frac{dp}{ap + b}. \tag{29}$$

After integrating Eq. (29), assuming that $a \neq 0$ and considering boundary condition $p = p_0$: $\rho = \rho_0$, density ρ takes the form

$$\rho = \rho_0 \left(\frac{ap + b}{ap_0 + b}\right)^{1/a}. \tag{30}$$

By substituting density ρ from Eq. (30), the conservation of momentum Eq. (19) is

$$v\ dv = -\frac{dp}{\rho(p)} = -\frac{dp}{\rho_0 \left(\frac{ap+b}{ap_0+b}\right)^{1/a}}. \tag{31}$$

Integrating Eq. (31), assuming that $a \neq 1$ and taking into account the boundary condition $v = 0$: $p = p_0$, velocity v may be written as

$$v = \sqrt{\frac{2(ap_0 + b)^{1/a}}{\rho_0(a - 1)}[(ap_0 + b)^{(a-1)/a} - (ap + b)^{(a-1)/a}]}. \tag{32}$$

The continuity equation takes the form

$$\dot{m} = A\rho v$$
$$= A\sqrt{\frac{2\rho_0}{(a-1)}(ap_0+b)\left[\left(\frac{ap+b}{ap_0+b}\right)^{2/a} - \left(\frac{ap+b}{ap_0+b}\right)^{(a+1)/a}\right]}. \tag{33}$$

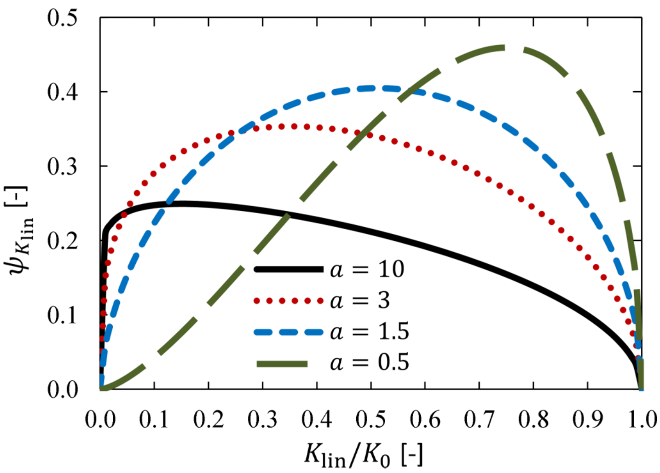

FIG. 3. The course of function $\psi_{K_{\text{lin}}}$ of a liquid having linear dependence on the bulk modulus K_{lin} of pressure depends on the ratio of the bulk moduli K_{lin}/K_0 for selected coefficients a.

By introducing a substitution based on the assumption expressed by Eq. (28), $K_{\text{lin}} = ap + b$, and a substitution expressing the value of the bulk modulus in the reservoir for pressure p_0 as $K_0 = ap_0 + b$, the continuity equation can be written as

$$\dot{m} = A\rho v = A\sqrt{\frac{2\rho_0}{(a-1)}K_0\left[\left(\frac{K_{\text{lin}}}{K_0}\right)^{2/a} - \left(\frac{K_{\text{lin}}}{K_0}\right)^{(a+1)/a}\right]}$$
$$= A\psi_{K_{\text{lin}}}\sqrt{2\rho_0 K_0}. \quad (34)$$

Function $\psi_{K_{\text{lin}}}$ for the linear dependence of the bulk modulus on pressure is introduced in Eq. (34) as

$$\psi_{K_{\text{lin}}} = \sqrt{\frac{1}{a-1}\left[\left(\frac{K_{\text{lin}}}{K_0}\right)^{2/a} - \left(\frac{K_{\text{lin}}}{K_0}\right)^{(a+1)/a}\right]}. \quad (35)$$

Function $\psi_{K_{\text{lin}}}$ introduced in this way acquires values from the field of real numbers only under the assumption $a > 0 \wedge a \neq 1$. Function $\psi_{K_{\text{lin}}}$ is shown in Fig. 3.

Based on the course of function $\psi_{K_{\text{lin}}}$, it is apparent that it acquires a maximum found by $d\psi_{K_{\text{lin}}}/d(K_{\text{lin}}/K_0) = 0$. The solution to this condition is ratio K_{lin}/K_0, for which function $\psi_{K_{\text{lin}}}$ acquires a maximum given by the relationship

$$\left(\frac{K_{\text{lin}}}{K_0}\right)_{\psi\ \max,\ K_{\text{lin}}} = \left(\frac{2}{a+1}\right)^{a/(a-1)}. \quad (36)$$

Based on Eq. (36), it is possible to determine the pressure for which function $\psi_{K_{\text{lin}}}$ reaches a maximum using the expression

$$p_{\psi\ \max,\ K_{\text{lin}}} = \frac{1}{a}\left(\frac{a+1}{2}\right)^{a/(1-a)}(ap_0 + b) - \frac{b}{a}. \quad (37)$$

One can see from Eq. (37) that the pressure at the maximum of function $\psi_{K_{\text{lin}}}$ increases linearly with increasing pressure p_0. Substituting Eq. (37) for Eq. (32), expressing pressure p_0 from Eq. (37) with formal modifications, the expression for the velocity in the maximum of function $\psi_{K_{\text{lin}}}$ is obtained as

$$v_{\psi\ \max,\ K_{\text{lin}}} = \sqrt{\frac{ap+b}{\rho}} = \sqrt{\frac{K_{\text{lin}}}{\rho}} = v_{\text{sound}}. \quad (38)$$

As in the case of constant bulk modulus K_{const} and the linear dependence of K_{lin} on pressure p, the velocity at the maximum of function $\psi_{K_{\text{lin}}}$ is equal to the local speed of sound, v_{sound}. Therefore, Eq. (36) can be interpreted as if the ratio of the local value of bulk modulus in duct K_{lin} and bulk modulus K_0 in the reservoir reaches the value of the expression $[2/(a+1)]^{a/(a-1)}$ during the gradual decrease in outlet pressure, the liquid reaches the local speed of sound, and mass flow also reaches its maximum. One cannot fail to notice the interesting formal analogy between Eq. (36) and Eq. (7), which is a consequence of the analogy between Eq. (35) and Eq. (6).

Quite analogously, Eq. (26) can be considered; however, instead of K_{const}, the relationship $K_{\text{lin}} = ap + b$ is valid, giving rise to the equation

$$\frac{dA}{A} = \frac{1 - \text{Ma}^2}{(ap+b)\text{Ma}^2}dp. \quad (39)$$

By integrating Eq. (39) satisfying boundary condition $A = A_0$: $p = p_0$, the following equation is obtained:

$$\frac{ap+b}{ap_0+b} = \left(\frac{A}{A_0}\right)^{a\text{Ma}^2/(1-\text{Ma}^2)}. \quad (40)$$

Analogously, four cases occur, which are summarized in Table I. The limit transition $a \to 0$ in Eq. (37) yields Eq. (24):

$$p_{\substack{\psi\ \max,\ K_{\text{lin}} \\ a\to 0}} = \lim_{a\to 0}\left\{\frac{1}{a}\left(\frac{a+1}{2}\right)^{a/(1-a)}(ap_0+b) - \frac{b}{a}\right\}$$
$$= p_0 - b\ln(2) = p_{\psi\ \max,\ K_{\text{const}}}. \quad (41)$$

C. A generalization of the existence of critical pressure

In general, $K = K(p)$. Based on Eq. (17), density ρ can be determined as

$$\rho(p) = \rho_0 e^{\int_{p_0}^{p}[1/K(p)]dp}. \quad (42)$$

The conservation of momentum Eq. (2) then has the form

$$v\,dv = -\frac{dp}{\rho(p)} = -\frac{dp}{\rho_0 e^{\int_{p_0}^{p}[1/K(p)]dp}}. \quad (43)$$

By integrating Eq. (43), assuming a negligibly small mean velocity inside the reservoir, a general form for the velocity of the flowing liquid is obtained as

$$v(p) = \sqrt{-\frac{2}{\rho_0}\int_{p_0}^{p} e^{-\int_{p_0}^{p}[1/K(p)]dp}dp}. \quad (44)$$

Function ψ can then be introduced as the product of density $\rho(p)$ and velocity $v(p)$; that is, the product of Eqs. (42)

FIG. 4. Computational domain representing the flow of liquid from the reservoir to the nozzle.

and (44) can be written as

$$\psi(p) = \rho(p)\, v(p)$$
$$= \rho_0 e^{\int_{p_0}^{p} [1/K(p)]\mathrm{d}p} \sqrt{-\frac{2}{\rho_0}\int_{p_0}^{p} e^{-\int_{p_0}^{p}[1/K(p)]dp} dp}. \quad (45)$$

From the condition of the local extreme of function $\psi(p)$, i.e., $d\psi(p)/dp = 0$, it is possible to obtain the expression

$$\frac{1}{K(p)} e^{\int_{p_0}^{p}[1/K(p)]dp} \int_{p_0}^{p} e^{-\int_{p_0}^{p}[1/K(p)]dp} dp + \frac{1}{2} = 0. \quad (46)$$

It should be noted that functions $v(p) > 0$ and $\rho(p) > 0$ are under the assumption that $\rho_0 > 0$. Function $\psi(p)$ is positive and function $\psi(p = p_0) = 0$ because of $v(p = p_0) = 0$, so it is unnecessary to introduce the condition $d^2\psi(p)/dp^2 < 0$ for the maximum. Thus, Eq. (46) determines the class of functions $K(p)$ satisfying this equation, for which there is a maximum of function $\psi(p)$; therefore, critical pressure exists. Equation (24) can easily be obtained by substituting the expression $K(p) = K_{\mathrm{const}}$, which assumes a constant bulk modulus value, into Eq. (46).

IV. NUMERICAL SIMULATION

It is reasonable to verify if the derived equations above really predict flow choking. Therefore, a three-dimensional geometric model representing a liquid reservoir is created, including a nozzle with a constant cross section opening into the free space (Fig. 4). This geometric model is spatially discretized and using computational fluid dynamics (CFD), finite volume method numerical analysis is performed. To partly confirm the equations derived above, the CFD analysis is performed so that the parameters are as close as possible to the simplifying assumptions under which the equations are derived. The analysis is an inviscid steady-state type ($\partial/\partial t = 0$, $\Pi_{ij} = 0$) considering a single-phase liquid neglecting heat transfer. The external body forces are not considered either. The liquid is considered to be barotropic and the bulk modulus is considered to be constant K_{const}. Therefore, the density of a liquid is a function of the pressure according to Eq. (18). The theory presented in this paper is new; for this reason, the dimensions of the geometric model have no connection with any test device. The length of the nozzle is long enough to ensure numerical stability.

The boundary conditions consist of total inlet pressure and static outlet pressure. The total pressure is set at the inlet because the combination of static pressure at the inlet and outlet is numerically unstable due to the fact that mass flow is an implicit result. As the velocity at the inlet to the reservoir is very low due to a significantly larger inlet area compared to the outlet area, the inlet total pressure is close to the inlet static pressure. Several cases are analyzed in which the outlet pressure is gradually reduced and the inlet pressure is fixed. Therefore, constant value Δp is set for each analyzed case by these boundary conditions.

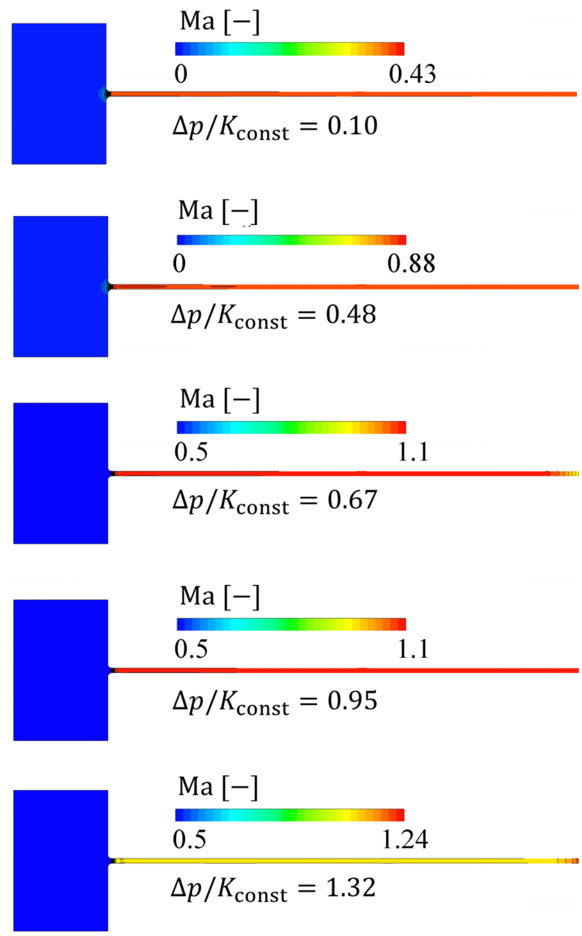

FIG. 5. Distribution of Mach number in the computational domain for selected values of ratio $\Delta p/K_{\mathrm{const}}$ obtained by CFD analysis performed for a constant bulk modulus.

Figure 5 shows that values $\Delta p/K_{\mathrm{const}} = 0.67$ and 0.95 have almost the same Mach number distribution in the outlet pipe. Case $\Delta p/K_{\mathrm{const}} = 1.32$ shows an increase in Mach number near the outlet boundary condition. In Fig. 5 the maximum of the Mach number in the legend corresponds to the maximum of the Mach number obtained by the CFD simulation for the given case. Figure 6 shows mass flow through the pipe as per CFD analyses, and it is obvious that at $\Delta p/K_{\mathrm{const}} = \ln(2)$, maximum flow through the nozzle occurs and a further decrease of outlet pressure no longer leads to an increase in mass flow. The results obtained by CFD simulation confirm the validity of Eq. (24).

V. DISCUSSION

The question arises whether it is possible to achieve dynamic choking in liquids under the conditions described in Sec. III. Knowledge of the equations that allow the value of the critical pressure difference to be quantified is important.

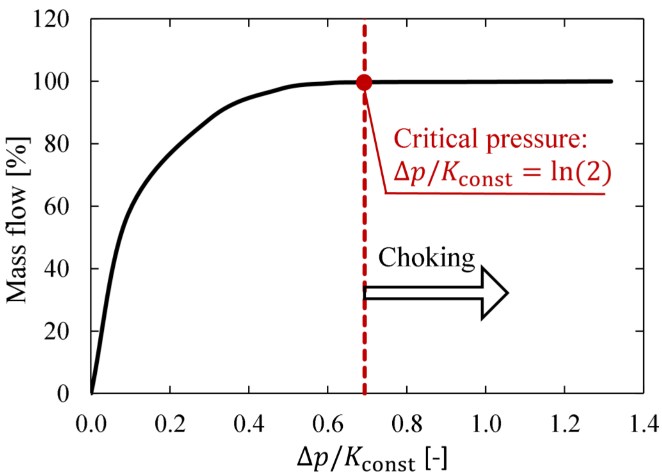

FIG. 6. The flow through the nozzle during an increase in the pressure difference between the inlet and outlet based on CFD analysis for the constant value of bulk modulus K_{const}.

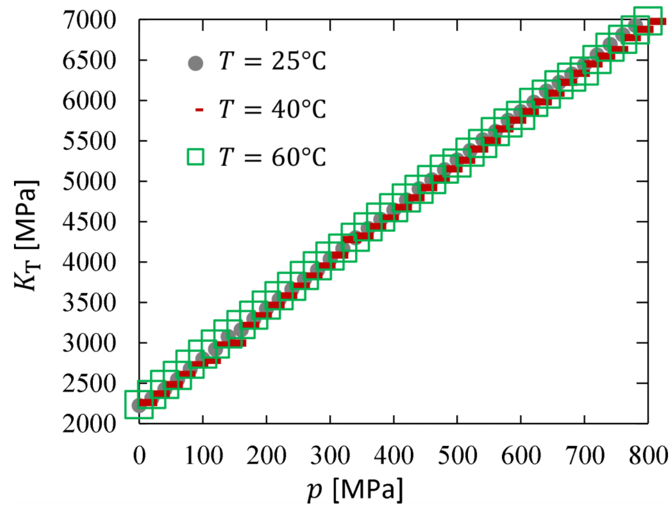

FIG. 7. Dependence of water bulk modulus on pressure at constant temperatures 25 °C, 40 °C, and 60 °C.

On the basis of Eqs. (24) and (37) it is possible to estimate the pressure at which choking of liquids occurs, not only due to phase changes but clearly due to the pipe geometry, quite analogous to an ideal gas. Most liquids, such as water, oil, or gasoline, have a bulk modulus greater than 1000 MPa and therefore, according to Eq. (24), pressure greater than 690 MPa is required to achieve the critical pressure difference. The nonzero viscosity of the liquids increases the required experimental pressure even further to cover pressure losses.

For real liquids subjected to enormous pressure, the transformation of their state of matter will become more important, which thus represents a significant complication for the experimental determination of the critical pressure difference in accordance with the assumptions described in Sec. III. This involves both high reservoir pressure leading to icing of the liquid as well as the risk of cavitation in the outlet pipe. Therefore, for a real liquid, the occurrence of dynamic choking is mainly conditional on the critical pressure difference existing exclusively in the region of liquid phase of the fluid. The limiting effect of cavitation could be experimentally suppressed by a sufficiently high outlet pressure, since Eq. (24) expresses the existence of a critical pressure difference regardless of the magnitude of the absolute pressure.

It is clear that the restrictive conditions for achieving the liquid-dynamic choking are an obstacle for realistic application and probably for this reason this phenomenon has not been thoroughly investigated yet. The derivation might be thereby rather considered as theoretical in nature, as it pertains to an idealized situation, in which the liquid does not undergo a phase change in the choked regime; however, the above equations cannot be considered as unusable. It is only a matter of time before an application using liquid transfer without occurrence of a phase change at a pressure difference higher than the critical pressure difference will be required.

A. Estimation of water critical pressure

In spite of the assumptions made in Sec. III, it is interesting to perform an analysis to approximate the critical pressure difference for water. With increasing pressure, the water bulk modulus increases approximately linearly, as shown in Fig. 7, which is compiled from data reported by Brostow *et al.* [32].

The coefficients a and b of the linear dependence of water bulk modulus on pressure according to Eq. (28) are determined from the interpolation of data shown in Fig. 7 by the least-squares method. They are shown in Table II.

The equations for the dependence of bulk modulus for temperatures 25 °C, 40 °C, and 60 °C show an insignificant influence of temperature on the course of bulk modulus in the pressure range 0.1–800 MPa. The estimations of critical pressure value for temperatures in the range of 25 °C–60 °C may therefore be considered almost identical. Figure 8 shows the dependence of function $\psi_{K_{\mathrm{lin}}}$ on pressure p determined by Eq. (35), considering coefficients a and b at 25 °C.

The graph in Fig. 8 highlights the shift to the right of the maxima of the curves of function $\psi_{K_{\mathrm{lin}}}$ as the pressure in reservoir p_0 increases. The course of the critical value of pressure $p_{\psi\ \max}$, which is the maxima of function $\psi_{K_{\mathrm{lin}}}$ in Fig. 8 computed according to Eq. (37), is shown in Fig. 9 (solid black line). The value of pressure $p_{\psi\ \max}$ determines how much pressure remains unused to reach maximum mass flow when pressurizing the water tank to p_0, unless a convergent-divergent pipe is used.

B. Notes to linear and constant bulk modulus

It should be noted that the coefficients a and b of the equation for the linear course of the bulk modulus of water are based on pressure values less than 800 MPa. Thus, the resulting graphs use a significantly extrapolated course of bulk

TABLE II. Coefficients a and b of the linear dependence of water bulk modulus on pressure for selected constant temperatures.

T (°C)	a (−)	b (Pa)
25	6.08	2.203×10^9
40	5.98	2.260×10^9
60	5.92	2.272×10^9

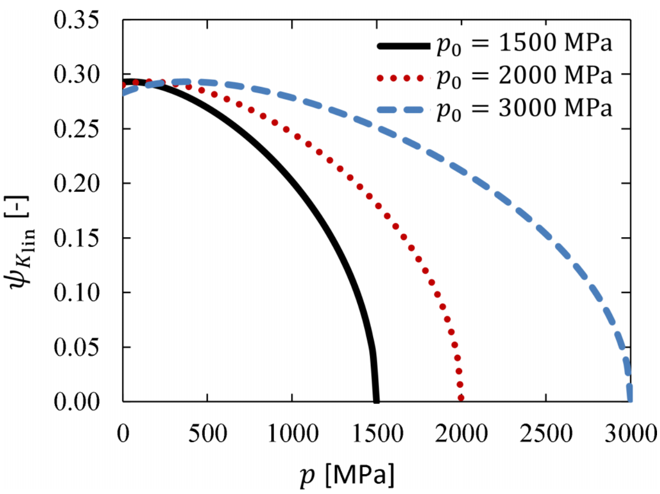

FIG. 8. The course of function $\psi_{K_{\rm lin}}$ on pressure p for water at 25 °C for three values of reservoir pressure $p_0 = 1500,\ 2000,$ and 3000 MPa.

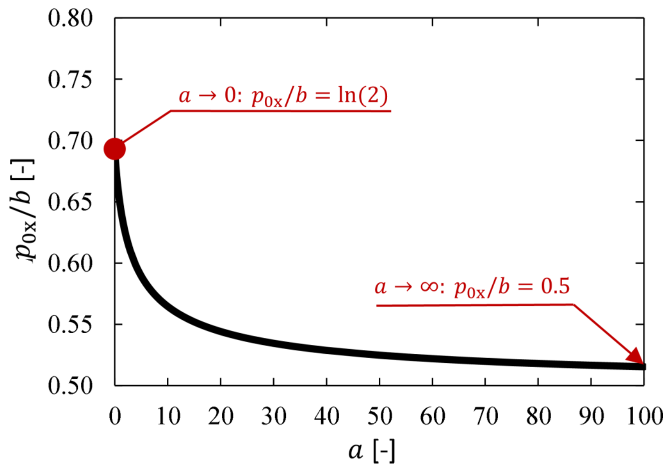

FIG. 10. The influence of parameter a in equation $K_{\rm lin} = ap + b$ on the value of the minimum pressure in the liquid reservoir when liquid-dynamic choking may occur.

modulus determined from a pressure range of 0.1–800 MPa up to 3000 MPa. Figure 9 also shows the course of $p_{\psi\ \max}$ for a constant bulk modulus value (dotted red line). The dependence of bulk modulus on pressure may fundamentally influence the value of critical pressure. As the linear dependence of the bulk modulus of water is based on extrapolated quantities, the course of critical pressure $p_{\psi\ \max}$ for $K_{\rm lin}$ (solid black line) must be understood as a rule of thumb. It is interesting that the intersection with axis p_0 is, in the case of $p_{\psi\ \max,\ K_{\rm lin}}$, smaller than $p_{\psi\ \max,\ K_{\rm const}}$. The dependence of intersection p_{0x} on the line $p_{\psi\ \max,\ K_{\rm lin}}$ with axis p_0 is normalized by parameter b and can be determined from Eq. (37) as

$$\frac{p_{0x}}{b} = \frac{1 - \left(\frac{a+1}{2}\right)^{a/(1-a)}}{a\left(\frac{a+1}{2}\right)^{a/(1-a)}}. \tag{47}$$

The value of the intersection of the curve at critical pressure $p_{\psi\ \max}$ with axis p_0 determines the minimum value of pressure p_0 in the liquid reservoir, for which there is a nonnegative absolute static pressure at pipe outlet $p_{\psi\ \max}$, at which liquid-dynamic choking may occur. In other words, value p_{0x} indicates the minimum value of the pressure in the liquid reservoir when it may be appropriate to consider the occurrence of liquid-dynamic choking. This dependence, given by Eq. (47), is shown in Fig. 10, which demonstrates that for $a \to 0$, i.e., the assumption of a constant bulk modulus $K_{\rm const}$, the ratio $p_{0x}/b = \ln(2)$. If $a \to \infty$, then the ratio $p_{0x}/b = 0.5$. If the liquid shows a significant increase in bulk modulus with increasing pressure, it can be roughly determined that the minimum value of the pressure in the reservoir—when it makes sense to consider liquid-dynamic choking—is 0.5 times the value of b, which is a value approximately equal to the bulk modulus at atmospheric pressure. An important conclusion from the graph in Fig. 9 is that the assessment of the possible formation of liquid-dynamic choking can be very misleading when using a constant bulk modulus $K_{\rm const}$.

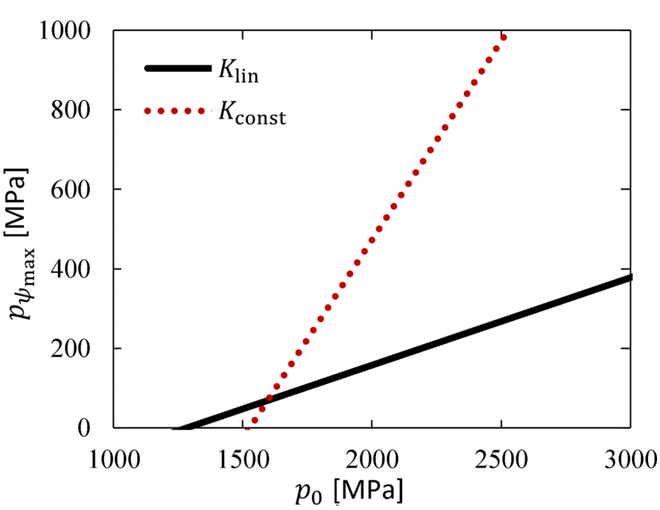

FIG. 9. Water: the course of critical pressure $p_{\psi\ \max}$ for different values of inlet pressure p_0 at 25 °C when considering $K_{\rm lin}$ and $K_{\rm const}$.

It is known from the phase diagram of water that at room temperature, 20 °C, and pressure close to 900 MPa, water changes from liquid to solid state. Therefore, the possibility of the transformation of liquid into ice must be taken into account. Nevertheless, according to Mohamed [33], the adiabatic heating of water during its compression in the reservoir reduces the risk of ice formation. Water that is isothermally compressed at 20 °C would freeze when reaching pressure $p_0 = 888$ MPa. However, during adiabatic compression, it heats up and freezes at $p_0 = 1322$ MPa [33]. According to the course shown in Fig. 9, critical pressure $p_{\psi\ \max} = 8.6$ MPa for the pressure in reservoir $p_0 = 1322$ MPa.

At a temperature of 40 °C, isothermally compressed water freezes at a pressure of approximately $p_0 = 1200$ MPa [34]. During adiabatic compression of water at an initial temperature of 40 °C, freezing can be expected at about $p_0 = 1730$ MPa [33]. At pressure $p_0 = 1730$ MPa, the value of critical pressure is $p_{\psi\ \max} = 99$ MPa.

As temperature has a significant effect on the pressure at which water freezes, the effect of critical pressure due to liquid-dynamic choking will be significantly dependent on the water temperature in the reservoir. However, based on an approximate estimate from the critical pressure shown in Fig. 9, in the case of water, this phenomenon will become more important when the pressure in the reservoir achieves several GPa.

VI. SUMMARY AND CONCLUSION

The results presented in this work show that liquids have a maximum mass flow in a convergent pipe even if there is no phase change and the flow is inviscid, which is quite analogous to gases. However, unlike gas-dynamic choking, liquid-dynamic choking shows a critical value of the pressure difference $\Delta p_{\psi\ \max}$ in the case of a constant bulk modulus. This critical pressure difference is equal to approximately $0.69K_{\text{const}}$. With the linear dependence of bulk modulus on pressure, the critical pressure is given by Eq. (37), and concerning Eq. (36), it can be stated that a critical ratio of bulk moduli exists.

In real applications, reaching $\Delta p_{\psi\ \max}$ is prevented by several influences. In viscous flow, the achievement of a critical pressure difference is limited by pressure losses that are roughly proportional to the product of the density and the square of the velocity. Thus, the resulting critical pressure difference is increased by these pressure losses in the case of viscous flow. Flow restriction due to phase change is also a significant barrier. However, if the outlet pressure is significantly higher than the saturated vapor pressure of the liquid, the probability of a change from the liquid phase to the gaseous phase can be minimized.

Nonetheless, the equations presented in this work are general and apply to any liquid whose change in density is defined by bulk modulus. These equations provide a description of the existence of critical pressure in liquids and thus allow one to quantify the phenomenon causing the limitation of the maximum flow rate of liquids in a duct when phase change of the liquid does not occur.

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