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Semiclassical approach to S-matrix energy correlations and time delay in chaotic systems

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The M-dimensional scattering matrix $S(E)$ which connects incoming to outgoing waves in a chaotic systyem is always unitary, but shows complicated dependence on the energy. This is partly encoded in correlators constructed from traces of powers of $S(E+\epsilon)S^\dagger(E-\epsilon)$, averaged over E, and by the statistical properties of the time delay operator, $Q(E) = -i\hbar S^\dagger dS/dE$. Using a semiclassical approach for systems with broken time-reversal symmetry, we derive two kinds of expressions for the energy correlators: one as a power series in $1/M$ whose coefficients are rational functions of ϵ, and another as a power series in ϵ whose coefficients are rational functions of M. From the latter we extract an explicit formula for $\mathrm{Tr}(Q^m)$ which is valid for all m and is in agreement with random matrix theory predictions.

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I. INTRODUCTION

Scattering of waves of energy E can be described by the $S(E)$ matrix, which connects incoming to outgoing amplitudes. We consider a finite region with chaotic classical dynamics, characterized by a single timescale τ_D, the dwell time, the average amount of time spent inside the region by a classical particle injected at random. This chaotic region is connected to the outside world by means of M channels, so that S is M dimensional and always unitary as a consequence of the energy conservation.

If time-reversal symmetry is broken, one statistical approach, random matrix theory (RMT), assumes $S(E)$ to be uniformly distributed in the unitary group [1,2], according to the invariant Haar measure, for every E. To understand the correlations between S matrices at different energies has always been a challenge. One way to quantify this is to compute S at one energy and $S^\dagger$ at another, and take the trace of their product, $\mathrm{Tr}[S(E+\frac{\epsilon\hbar}{2\tau_D})S^\dagger(E-\frac{\epsilon\hbar}{2\tau_D})]$. This will be equal to M for $\epsilon = 0$, but in general a widely fluctuating function of E. Averaging within a local energy window produces a well-behaved function of ϵ. Such energy correlations have traditionally been studied by modeling the Hamiltonian of the system as a random Hermitian matrix coupled to scattering channels [3–9].

A more detailed characterization of energy correlations is the calculation of

$$C_n(M,\epsilon) = \langle \mathrm{Tr}[S(E+\epsilon')S^\dagger(E-\epsilon')]^n\rangle \tag{1}$$

for integer n, where $\epsilon' = \frac{\epsilon\hbar}{2\tau_D}$ is a classically small energy increment. The above quantity is expected to be universal, i.e., independent of the system's details, as long as the dynamics is fully chaotic, the particle spends enough time in the scattering region, i.e., τ_D is large compared to other classical timescales (also compared to the Ehrenfest time $\tau_E \sim |\log\hbar|$), and the system is in the semiclassical regime. Besides M and ϵ, it should depend only on whether time-reversal symmetry is present or not. In this work we focus our attention on systems where this symmetry is broken.

Related to energy dependence of the S matrix is the time delay matrix [10–13]

$$Q(E) = -i\hbar S^\dagger \frac{dS}{dE}. \tag{2}$$

Its real eigenvalues $\{\tau_1,\dots,\tau_M\}$ are commonly referred to as proper time delays and provide the lifetimes of metastable states. Its normalized trace $\tau_W = \frac{1}{M}\mathrm{Tr}(Q)$ is known as the Wigner time delay, which provides a measure of the density of states of the open system. Its average value equals the classical dwell time, $\langle\tau_W\rangle = \tau_D$. More detailed information is encoded in higher spectral moments such as

$$Q_n = \langle \mathrm{Tr}(Q^n)\rangle. \tag{3}$$

The statistical properties of time delay have been much studied. Within RMT, perhaps the main point of departure is the distribution of the inverse matrix Q^{-1}, which is known to conform to the Laguerre ensemble [14,15]. This led to the calculation of the distribution function of τ_W in different regimes and to expressions for the above spectral moments [16–23] (see the review [24]).

In this work we do not rely on random matrices, but instead employ a semiclassical approach, in which the elements of S are approximated, in the short-wavelength regime, as infinite sums over scattering rays [25,26]. It has been very successful in treating transport properties at fixed energy [27–33]. It was adapted by Kuipers and Sieber [34,35] in order to take into account the variable ϵ and handle correlators like (1). It has grown into an independent line of attack to this kind of problems [36–42].

We follow recent advances in the semiclassical theory and formulate correlation functions in terms of auxiliary matrix integrals [42–45]. These matrix integrals are related to concepts in representation theory and combinatorics and thereby the power of these fields can be brought to bear on the problem. This approach leads to two explicit formulas for $C_n(M,\epsilon)$:

one as a power series in $1/M$ whose coefficients are rational functions of ϵ, and another as a power series in ϵ whose coefficients are rational functions of M. From the latter we extract an explicit formula for $\mathrm{Tr}(Q^n)$ which is valid for arbitrary values of n and M and which is in agreement with random matrix theory predictions.

In Sec. II we present the semiclassical matrix integral which is the crux of the theory. In Secs. III and IV we use it to compute $C_n(M,\epsilon)$ in two different ways. In Sec. V we make the connection with Q_n. We conclude in Sec. VI.

II. SEMICLASSICAL MATRIX INTEGRALS

The semiclassical approximation to quantum scattering has been extensively discussed in previous works [27,28,31,42]. When correlations among scattering trajectories are taken into account, and the required integrations over phase space have been performed, the theory has a diagrammatic formulation which is a perturbative theory in the parameter M^{-1}. Kuipers and Sieber obtained the diagrammatic rules governing this theory when applied to (1). The contribution of any given diagram factorizes into the contributions of individual vertices and edges: a vertex of valence $2q$ gives rise to $-M(1-iq\epsilon)$; channels of any valence give rise to M; each edge gives rise to $[M(1-i\epsilon)]^{-1}$.

Recently, the semiclassical approach has been developed in terms of appropriate matrix integrals [42–45] into which the diagrammatic rules are built by design. For systems with broken time-reversal symmetry, which are our focus, the result is that C_n is given by

$$\lim_{N\to 0}\int e^{-\sum_{q=1}^{\infty}(M/q)(1-iq\epsilon)\mathrm{Tr}(ZZ^\dagger)^q}\mathrm{Tr}[ZPZ^\dagger P]^n\frac{dZ}{\mathcal{Z}}, \quad (4)$$

where Z is an N-dimensional complex matrix, P is an orthogonal projector from $\mathbb{R}^N$ to $\mathbb{R}^M$, and

$$\mathcal{Z}=\int e^{-M(1-i\epsilon)\mathrm{Tr}(ZZ^\dagger)}dZ \quad (5)$$

is a normalization.

The way this matrix model works is that the factor $e^{-M(1-i\epsilon)\mathrm{Tr}(ZZ^\dagger)}$ is kept as a Gaussian measure while the rest of the exponential is Taylor expanded. Each trace then becomes a vertex in a diagram, along with the correct factor $-M(1-iq\epsilon)$. Then the integration is performed by invoking Wick's rule, and edges are produced along with the correct factor $[M(1-i\epsilon)]^{-1}$. The term $\mathrm{Tr}[ZPZ^\dagger P]^n$ mimics the correlator we want to compute. Finally, the limit $N\to 0$ is necessary to remove spurious contributions coming from unwanted periodic orbits [43].

The traditional singular value decomposition

$$Z=UDV^\dagger, \quad (6)$$

where U and V are unitary matrices, leads to $\mathcal{Z}=\mathcal{G}\int e^{-M(1-i\epsilon)\mathrm{Tr}(X)}|\Delta(X)|^2dX$, where the Vandermonde

$$\Delta(X)=\prod_{1\leqslant i<j\leqslant N}(x_j-x_i) \quad (7)$$

is the Jacobian of the change of variables, $X=D^2$ has the same eigenvalues as $ZZ^\dagger$, and $\mathcal{G}=\int dU\,dV$ is the result of a double integration over the unitary group. This integral is the partition function of the Laguerre ensemble of random matrix theory [46] and a particular case of the Selberg integral [47]. It is well known that it gives

$$\mathcal{Z}=\mathcal{G}[M(1-i\epsilon)]^{-N^2}\prod_{j=1}^{N}j!(j-1)!. \quad (8)$$

Let $\chi_\lambda(\mu)$ be the characters of the irreducible representations (irreps) of the permutation group S_n (these representations are labeled by integer partitions, denoted by $\lambda\vdash n$ or $|\lambda|=n$). They are useful in expressing the trace of a power of a matrix in terms of Schur polynomials,

$$\mathrm{Tr}(A^n)=\sum_{\lambda\vdash n}\chi_\lambda(n)s_\lambda(A). \quad (9)$$

Schur polynomials are characters of irreducible representations of the unitary group [48]. It follows that they satisfy

$$\frac{1}{\int dU}\int dU\,s_\lambda(UAU^\dagger B)=\frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(1^N)}, \quad (10)$$

where 1^N is the identity matrix in N dimensions. The quantity [49,50]

$$s_\lambda(1^N)=\frac{d_\lambda}{n!}[N]^\lambda, \quad (11)$$

is the dimension of the irrep of the unitary group, where $d_\lambda=\chi_\lambda(1^n)$, given by

$$n!\prod_{i=1}^{\ell(\lambda)}\frac{1}{(\lambda_i-i+\ell)!}\prod_{j=i+1}^{\ell(\lambda)}(\lambda_j-j-\lambda+i), \quad (12)$$

is the dimension of the irrep of the permutation group, while $[N]^\lambda$ is a monic polynomial in N,

$$[N]^\lambda=\prod_{j=1}^{\ell(\lambda)}\frac{(N+\lambda_j-j)!}{(N-j)!}, \quad (13)$$

which is a generalization of the rising factorial. For future reference, let us also define a corresponding generalization of the falling factorial,

$$[N]_\lambda=\prod_{j=1}^{\ell(\lambda)}\frac{(N+j)!}{(N-\lambda_j+j)!}. \quad (14)$$

Using (9) and (10) to perform the angular integration of the term $\mathrm{Tr}[ZPZ^\dagger P]^n$ in Eq. (4), we get

$$C_n(M,\epsilon)=\lim_{N\to 0}\sum_{\lambda\vdash n}\left(\frac{[M]^\lambda}{[N]^\lambda}\right)^2\chi_\lambda(n)\mathcal{I}_\lambda, \quad (15)$$

with

$$\mathcal{I}_\lambda=\frac{\mathcal{G}}{\mathcal{Z}}\int e^{-\sum_{q=1}^{\infty}(M/q)(1-iq\epsilon)\mathrm{Tr}(X)^q}s_\lambda(X)dX. \quad (16)$$

It is known that $\chi_\lambda(n)$ is different from zero only if $\lambda=(n-k,1^k)$, a so-called hook partition [49,50]. In that case, $\chi_\lambda(n)=(-1)^k$ and $d_\lambda=\binom{n-1}{k}$. We denote by H_n the set of all hook partitions of n. For example, $H_4=\{(4),(3,1),(2,1,1),(1^4)\}$. We also define the quantity

$$t_\lambda=(n-k-1)!k!=\frac{(n-1)!}{d_\lambda}. \quad (17)$$

III. CORRELATOR AS POWER SERIES IN 1/M

Let b_β be the size of the conjugacy class of the permutation group containing permutations of cycle type β (i.e., the largest cycle has length β_1, the second largest has length β_2, and so on) and let us define the function

$$g_\beta(\epsilon) = \prod_{q\in\beta}(1 - iq\epsilon). \tag{18}$$

Then it is clear that we can write the Taylor series of $e^{-\sum_{q>1}(M/q)(1-iq\epsilon)\mathrm{Tr}(X^q)}$ as

$$\sum_m \sum_{\beta\vdash m} \frac{1}{m!} b_\beta (-M)^{\ell(\beta)} g_\beta(\epsilon) p_\beta(X), \tag{19}$$

where

$$p_\beta(X) = \prod_{j=1}^{\ell(\beta)} \mathrm{Tr}(X^{\beta_j}) \tag{20}$$

is a power sum symmetric polynomial. In the sum (19) the term $m = 1$ is excluded, and the partition β has no parts equal to 1.

Next, we use the relation between power sums and Schur polynomials [49,50], $p_\beta(X) = \sum_\rho \chi_\rho(\beta) s_\rho(X)$, and then join $s_\rho(X)$ with the $s_\lambda(X)$ with the one already in the integrand, according to

$$s_\rho(X)s_\lambda(X) = \sum_\nu c^\nu_{\lambda\rho} s_\nu, \tag{21}$$

where $c^\nu_{\lambda,\rho}$ are the Littlewood-Richardson coefficients [50]. The integral to be done is then

$$\frac{\mathcal{G}}{\mathcal{Z}} \int e^{-M(1-i\epsilon)\mathrm{Tr}(X)} |\Delta(X)|^2 s_\nu(X) dX. \tag{22}$$

This is an integral of Selberg type [47], given by

$$\frac{d_\nu}{|\nu|!}([N]^\nu)^2 [M(1-i\epsilon)]^{N^2-|\nu|}. \tag{23}$$

The limit $N \to 0$ can be taken by noticing that, since λ is a hook, we have [45]

$$[N]^\lambda = N(-1)^{\ell(\lambda)-1} t_\lambda + O(N^2). \tag{24}$$

This means only partitions ν that are also hooks will contribute, because for more general partitions the quantity $[N]^\nu$ will be at least quadratic in N for small N. This observation leads to

$$C_n = \sum_{\lambda\in H_n} \frac{\chi_\lambda(n)}{t_\lambda^2} ([M]^\lambda)^2 B_\lambda, \tag{25}$$

where

$$B_\lambda = \sum_m \sum_{\beta\vdash m} \frac{b_\beta(-M)^{\ell(\beta)} g_\beta(\epsilon)(n+m-1)!}{m!(n+m)[M(1-i\epsilon)]^{n+m}} D_{\lambda\beta}, \tag{26}$$

with

$$D_{\lambda\beta} = \sum_{\rho\nu} \chi_\rho(\beta) \frac{c^\nu_{\lambda\rho}}{d_\nu}. \tag{27}$$

For a given pair of hooks, λ, ν, it follows from the so-called Murnaghan-Nakayama rule [50] that there are two different ρ for which $c^\nu_{\lambda\rho}$ is not zero. If $\lambda = (n-k, 1^k)$ and $\nu = (n+m-r, 1^r)$, then $\rho_1 = (m+k-r, 1^{r-k})$ and $\rho_2 = (m+k-r+1, 1^{r-k-1})$. We thus have the sum

$$\sum_{\rho\in H_m} \chi_\rho(\beta) c^\nu_{\lambda\rho} = \chi_{\rho_1}(\beta) + \chi_{\rho_2}(\beta). \tag{28}$$

It is a standard fact from representation theory that the restriction from S_{n+1} to S_n of the irreducible character χ_λ is the sum of irreducible characters χ_ρ over all partitions ρ that result from the Young diagram of λ by removing a box [50]. Hence, the above sum equals $\chi_\omega(\beta, 1)$ with $\omega = (m+k-r+1, 1^{r-k})$.

We now have to compute

$$\sum_{\nu\in H_{n+m}} \frac{\chi_\omega(\beta,1)}{d_\nu} = \sum_{r=0}^{n+m-1} \chi_\omega(\beta,1) \frac{(n+m-r-1)!r!}{(n+m-1)!}. \tag{29}$$

Using that

$$\frac{(n+m-r-1)!r!}{(n+m)!} = \int_0^1 u^r(1-u)^{n+m-r-1} du \tag{30}$$

we end up having to compute the sum

$$\sum_{r=0}^{n+m-1} \chi_\omega(\beta,1) x^r, \tag{31}$$

where $x = u/(1-u)$. Fortunately, the characters χ_ω with ω a hook have already been studied in connection with the problem of factorizing permutations [51], and it turns out that

$$\sum_{r=0}^{n+m-1} \chi_\omega(\beta,1) x^{\ell(\omega)} = x f_\beta(x), \tag{32}$$

where

$$f_\beta(x) = \prod_{q\in\beta} [1 - (-x)^q]. \tag{33}$$

Hence, we arrive at an integral representation for the sum in Eq. (29):

$$F_{n,m,k}(\beta) = \sum_\nu \frac{\chi_\omega(\beta,1)}{d_\nu(n+m)} = \int_0^1 u^k(1-u)^{n+m-k-1} f_\beta\left(\frac{u}{1-u}\right) du. \tag{34}$$

Notice that the integral above can be done exactly for any partition β, as it is always a beta function.

The quantity B_λ is then given by

$$\sum_m \sum_{\beta\vdash m} \frac{b_\beta(-M)^{\ell(\beta)} g_\beta(\epsilon)(n+m-1)!}{m![M(1-i\epsilon)]^{n+m}} F_{n,m,k}(\beta) \tag{35}$$

and we arrive at

$$C_n = \sum_{k=0}^{n-1} \frac{(-1)^k [(M-k)^{(n)}]^2}{(n-k-1)!^2 k!^2} \sum_m \sum_{\beta\vdash m} \times \frac{b_\beta(-M)^{\ell(\beta)} g_\beta(\epsilon)(n+m-1)!}{m![M(1-i\epsilon)]^{n+m}} F_{n,m,k}(\beta), \tag{36}$$

where we have used that $[M]^\lambda = (M-k)^{(n)}$.

This expression is very explicit and easy to implement in the computer. Even indeed, we can compute many orders in $1/M$ and they all agree with the generating functions presented in [37].

IV. CORRELATOR AS POWER SERIES IN ϵ

Alternatively, we may express $C_n(M,\epsilon)$ as a power series in ϵ. Such a series is not convergent: its radius of convergence cannot be finite because the integral in Eq. (16) clearly does not exist if ϵ has a negative imaginary part. But the series can still be asymptotic and therefore useful, in the sense that its first d terms give an accurate representation of the function for small ϵ, up to an error of order ϵ^d.

After we expand

$$e^{Mi\epsilon\mathrm{Tr}(X/1-X)}=\sum_{m=0}^{\infty}\frac{(iM\epsilon)^m}{m!}\sum_{\mu\vdash m}d_\mu s_\mu\Big(\frac{X}{1-X}\Big), \tag{37}$$

we express C_n as

$$\lim_{N\to 0}\sum_{\lambda\in H_n}\chi_\lambda(n)\bigg(\frac{[M]^\lambda}{[N]^\lambda}\bigg)^2\sum_{m=0}^{\infty}\frac{(iM\epsilon)^m}{m!}\sum_{\mu\vdash m}d_\mu I_{\lambda\mu}, \tag{38}$$

where $I_{\lambda\mu}$ is given by

$$\frac{\mathcal{G}}{\mathcal{Z}}\int\det(1-X)^M|\Delta(X)|^2 s_\mu\Big(\frac{X}{1-X}\Big)s_\lambda(X)dX. \tag{39}$$

Schur polynomials can be expressed as a ratio of determinants,

$$s_\lambda(X)=\frac{\det\big(x_j^{N+\lambda_i-i}\big)}{\det\big(x_j^{N-i}\big)}=\frac{\det\big(x_j^{N+\lambda_i-i}\big)}{\Delta(X)}. \tag{40}$$

In the present case this gives

$$s_\lambda\bigg(\frac{X}{(1-X)}\bigg)=\det\bigg[\bigg(\frac{x_k}{(1-x_k)}\bigg)^{N+\lambda_i-i}\bigg]\frac{1}{\Delta\big(\frac{X}{(1-X)}\big)}. \tag{41}$$

It is easy to express the above Vandermonde as

$$\Delta\bigg(\frac{X}{(1-X)}\bigg)=\frac{\Delta(X)}{\det(1-X)^{N-1}}. \tag{42}$$

The integral can then be computed by means of the Andreief identity,

$$\int\det[f_i(x_k)]\det[g_j(x_k)]dX=N!\det\bigg[\int f_i(x)g_j(x)dx\bigg], \tag{43}$$

which gives

$$I_{\lambda\mu}=N!\frac{\mathcal{G}}{\mathcal{Z}}\det\bigg[\int_0^1(1-x)^{M-\mu_j+j-1}x^{2N+\mu_j-j+\lambda_i-i}dx\bigg], \tag{44}$$

or

$$I_{\lambda\mu}=N!\frac{\mathcal{G}}{\mathcal{Z}}\prod_{j=1}^{N}\frac{(M-\mu_j+j-1)!}{(M+2N+\lambda_j-j)!}\times\det[(2N+\mu_j-j+\lambda_i-i)!]. \tag{45}$$

The above determinant can be computed as follows. First, we write $\det[(2N+\mu_j-j+\lambda_i-i)!]$ as

$$\det\bigg[\int_0^\infty x^{2N+\mu_j-j+\lambda_i-i}e^{-x}dx\bigg]. \tag{46}$$

Next, we use the Andreief identity in reverse to express this as a multiple integral,

$$\begin{aligned}&\frac{1}{N!}\int_0^\infty\det\big[x_j^{N+\lambda_i-i}\big]\det\big[x_i^{N+\mu_j-j}\big]e^{-\mathrm{Tr}(X)}dX\\&=\frac{1}{N!}\int_0^\infty s_\mu(X)s_\lambda(X)|\Delta(X)|^2e^{-\mathrm{Tr}(X)}dX.\end{aligned} \tag{47}$$

The Schur polynomials can be combined by using the Littlewood-Richardson coefficients and the resulting integral is of Selberg type [47], leading to

$$\frac{1}{N!}\sum_\nu c^\nu_{\lambda\mu}\frac{d_\nu}{|\nu|!}([N]^\nu)^2\prod_{j=1}^{N}j!(j-1)!. \tag{48}$$

After the product over j cancels with an equal term coming from the normalization $\mathcal{Z}$ [see (8)], we can take $N\to 0$. From this limit it results that all contributions come from those ν that are hooks. Moreover, in that case $c^\nu_{\lambda\mu}$ is different from zero, and equal to 1, if and only if μ is a hook as well. We also recognize in (45) an expression for the generalized falling factorial, so that

$$C_n=\sum_{\lambda\in H_n}\frac{\chi_\lambda(n)}{t_\lambda^2}[M]^\lambda\widetilde{B}_\lambda, \tag{49}$$

with $\widetilde{B}_\lambda$ given by

$$\sum_{m=0}^{\infty}\frac{(iM\epsilon)^m}{m!}\sum_{\mu\in H_m}\frac{d_\mu}{[M]_\mu}\frac{(n+m-1)!}{(n+m)}\sum_\nu\frac{c^\nu_{\lambda\mu}}{d_\nu}. \tag{50}$$

When $\lambda=(n-k,1^k)$ and $\mu=(m-l,1^l)$, it again follows from the so-called Murnaghan-Nakayama rule [50] that the coefficient $c^\nu_{\lambda\mu}=1$ if and only if $\nu=(n+m-k-l,1^{k+l})$ or $\nu=(n+m-k-l-1,1^{k+l+1})$. Hence,

$$\frac{1}{n+m}\sum_{\nu\in H_{n+m}}\frac{c^\nu_{\lambda\mu}}{d_\nu}=\frac{(n+m-k-l-2)!(k+l)!}{(n+m-1)!} \tag{51}$$

or

$$\frac{1}{n+m}\sum_{\nu\in H_{n+m}}\frac{c^\nu_{\lambda\mu}}{d_\nu}=\frac{t_{\lambda\circ\mu}}{(n+m-1)!}, \tag{52}$$

where $\lambda\circ\mu=(n+m-k-l-1,1^{k+l})$.

Finally,

$$C_n=\sum_{m=0}^{\infty}\frac{(iM\epsilon)^m}{m!}E_{nm}, \tag{53}$$

where

$$E_{nm}=\sum_{\lambda\in H_n}\sum_{\mu\in H_m}\chi_\lambda(n)d_\mu\frac{[M]^\lambda}{[M]_\mu}\frac{t_{\lambda\circ\mu}}{t_\lambda^2}. \tag{54}$$

This expression is of a different nature than the one obtained in the previous section, but it is also very explicit and easy to implement.

V. STATISTICS OF TIME DELAY

As discussed by Berkolaiko and Kuipers [36], the time delay moments $Q_m = \langle \mathrm{Tr}(Q^m)\rangle$ can be obtained from appropriate derivatives of the energy correlators,

$$Q_m = \frac{M\tau_D^m}{i^m m!}\left[\frac{d^m}{d\epsilon^m}\sum_{n=1}^{m}(-1)^{m-n}\binom{m}{n}C_n(\epsilon)\right]_{\epsilon=0}. \quad (55)$$

Using the expression we have just derived for C_n as a power series in ϵ, Eq. (53), it is easy to see that

$$\frac{M}{i^m}\frac{d^m C_n}{d\epsilon^m} = M^m \sum_{\lambda\in H_n}\sum_{\mu\in H_m}\chi_\lambda(n)d_\mu\frac{[M]^\lambda}{[M]_\mu}\frac{t_{\lambda\circ\mu}}{t_\lambda^2}. \quad (56)$$

From this we can derive

$$Q_m = (M\tau_D)^m\sum_{n=1}^{m}\binom{m}{n}\frac{(-1)^{n+m}}{m!}E_{nm}. \quad (57)$$

When $\lambda = (n-k, 1^k)$ we have $[M]^\lambda = (M-k)^{(n)}$. The sum over n and the sum over λ then become [52]

$$\sum_{n=1}^{m}\sum_{k=0}^{n-1}\binom{m}{n}(-1)^{n-k}\frac{(n+m-k-l-2)!(k+l)!}{(n-k-1)!^2 k!^2}(M-k)^{(n)}$$
$$= (-1)^{m-l}(M-l)^{(m)}, \quad (58)$$

with $\mu = (m-l, 1^l)$. Therefore, we arrive at a very simple expression,

$$Q_m = \frac{(M\tau_D)^m}{m!}\sum_{\mu\in H_m}\chi_\mu(m)d_\mu\frac{[M]^\mu}{[M]_\mu}. \quad (59)$$

This coincides exactly, for any m, with the result derived from random matrix theory [22], proving the exact equivalence between random matrix theory and semiclassical approximation for this problem. Previously, this equivalence had only been verified up to $m = 8$.

VI. CONCLUSION

Using a powerful semiclassical approach, based on matrix integrals, we investigated energy correlations in the scattering matrices of chaotic systems with broken time-reversal symmetry. We expressed the basic correlator $C_n(M, \epsilon)$, Eq. (1), in two different ways: as a power series in $1/M$ and as a power series in ϵ. From the latter we were then able to extract average spectral moments of the time delay operator. We found complete agreement with RMT predictions, thereby microscopically justifying that approach.

A natural extension of this work would be to perform analogous calculations for systems with intact time-reversal symmetry. That remains a challenge. Moreover, nonlinear statistics of time delay, like $\langle[\mathrm{Tr}(Q)]^n\rangle$, have been computed within RMT, but are not accessible to the present approach. We believe the alternative semiclassical treatment introduced in [53] is promising in that respect.

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