Contour dynamics of two-dimensional dark solitons

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Equations for contour dynamics of trough-shaped dark solitons are obtained for the general form of the nonlinearity function. Their self-similar solution which describes the nonlinear stage of the bending instability of dark solitons is studied in detail.

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I. INTRODUCTION

Dynamics of dark solitons plays an important role in nonlinear optics and physics of Bose-Einstein condensates (BECs) (see, e.g., [1,2] and references therein). In particular, if the condensate is confined in a quasi-one-dimensional (quasi-1D) harmonic trap, such a soliton oscillates with a frequency different from the trap frequency, contrary to the behavior of bright solitons [3–6]. Dynamics of dark solitons becomes even more complicated for solitons with two spatial dimensions, which are typical, for example, in physics of polariton condensates formed in planar microresonators (see, e.g., [7]). Such solitons have the form of troughs or dips in the distributions of the polaritons' density. Such dips are localized along some curves in the plane with two space coordinates: the density changes slowly along such a curve and at each point the density has a minimum in the transverse (normal to the curve) direction. We shall call such a soliton a *2D dark soliton* and the curve of its localization will be called *the soliton's position*. Of course, such solitons should not be mixed with solitons or other nonlinear structures localized in both spatial directions. As was shown in Refs. [8–10], 2D dark solitons are unstable with respect to the bending ("snake") instability. As a result, a dark soliton breaks down with formation of vortices, and this phenomenon was observed experimentally in Refs. [11–14].

Theoretical description of the transition from the exponential growth of the unstable "snake" modes at the linear stage of their evolution to the nonlinear stage leading eventually to formation of vortices is a difficult task, and several possible scenarios were identified depending on the soliton's amplitude [15] (see also review article [16] and references therein). An interesting approach to the description of the nonlinear evolution of instability of deep enough dark solitons was suggested in Refs. [17,18] on the basis of the *contour dynamics* [19]. Mironov *et al* [17,18] assumed that the local radius of curvature of a dark soliton is much greater than its local width, so that the position of this soliton can be represented with high accuracy by a curved line: the soliton's "contour." Then the bending dynamics of such a contour is determined by two variables: the local velocity of the soliton and its local curvature. Mironov *et al* [17,18] derived the equations governing this dynamics in the framework of perturbation theory for the case of BEC dynamics obeying the standard Gross-Pitaevskii equation, and studied the nonlinear stage of development of instability of dark solitons. Later this theory was generalized in Ref. [20] to the instability dynamics of dark solitons in a polariton condensate. To avoid any confusion, we would like to stress that the contour dynamics of Mironov *et al* [17,18] differs from dynamics of contours around 2D vortex patches developed by Zabusky *et al.* [21] (see also review article [22] and references therein).

In this paper, we derive the equations of contour dynamics of dark solitons for media whose evolution obeys the generalized Gross-Pitaevskii equation [or generalized nonlinear Schrödinger (NLS) equation]

$$i\psi_t + \tfrac{1}{2}(\psi_{xx} + \psi_{yy}) - f(|\psi|^2)\psi = 0 \tag{1}$$

with general form of the positive nonlinearity function $f > 0$. Our derivation is based on physical reasoning rather than on the formal application of the perturbation theory. After that we study analytically in some detail the self-similar solutions of these equations. These solutions describe the nonlinear stage of the bending instability of dark solitons and considerably extend the results obtained in Refs. [17,18].

II. DARK SOLITON SOLUTION OF THE GENERALIZED NLS EQUATION

First, we shall present here the basic results of the dark soliton theory. For definiteness, we shall interpret Eq. (1) as the Gross-Pitaevskii equation for dynamics of BEC, so that $\rho = |\psi|^2$ has the meaning of the condensate's density and the gradient of the phase $\mathbf{u} = \nabla\phi$ has the meaning of the condensate's flow velocity. These definitions imply the representation of the condensate wave function ψ in the form

$$\psi = \sqrt{\rho(\mathbf{r}, t)}\exp[i\phi(\mathbf{r}, t) - i\mu t], \tag{2}$$

where

$$\mu = f(\rho_0) \tag{3}$$

is the chemical potential of the condensate with a uniform density ρ_0 far from the dark soliton. Substitution of Eq. (2) into Eq. (1) and standard calculations yield the equations of

BEC dynamics in the hydrodynamic-like form

$$\begin{aligned} &\rho_t + \nabla(\rho\mathbf{u}) = 0, \\ &\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} + \nabla f(\rho) + \nabla\left(\frac{(\nabla\rho)^2}{8\rho^2} - \frac{\Delta\rho}{4\rho}\right) = 0. \end{aligned} \tag{4}$$

Linearization of these equations with respect to a uniform quiescent BEC with $\rho = \rho_0$, $\mathbf{u} = \mathbf{u}_0 = 0$ gives the Bogoliubov dispersion relation

$$\omega = k\sqrt{c_0^2 + k^2/4} \tag{5}$$

for linear waves $\propto \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, where c_0 is the sound velocity

$$c_0 = \sqrt{\rho_0 f'(\rho_0)} \tag{6}$$

of waves in the long wavelength limit.

It is not difficult to find the soliton solution of Eqs. (4) for which the variables ρ and $\mathbf{u}$ depend only on the distance $\xi = (\mathbf{v}/v)\cdot(\mathbf{r} - \mathbf{v}t)$ from the straight line normal to vector $\mathbf{v}$ of the soliton velocity ($v = |\mathbf{v}|$). Substitution of the ansatz $\rho = \rho(\xi)$, $\mathbf{u} = (\mathbf{v}/v)u(\xi)$ gives with account of the boundary conditions $\rho \to \rho_0$, $u \to 0$ as $|\xi| \to \infty$ the relationship

$$u(\xi) = v\left(1 - \frac{\rho_0}{\rho}\right) \tag{7}$$

and the equation for $\rho(\xi)$ (see Ref. [6])

$$\rho_\xi^2 = Q(\rho), \tag{8}$$

where

$$Q(\rho) = 8\rho\int_\rho^{\rho_0}[f(\rho_0) - f(\rho')]d\rho' - 4v^2(\rho_0 - \rho)^2. \tag{9}$$

Integration of Eq. (8) gives at once

$$\xi = \int_{\rho_m}^{\rho}\frac{d\rho}{\sqrt{Q(\rho)}}, \tag{10}$$

where ρ_m is the minimal density at the center $\xi = 0$ of the soliton. The function $Q(\rho)$ has a double zero at $\rho = \rho_0$, hence $dQ/d\rho|_{\rho=\rho_0} = 0$, and this equation yields the relationship between ρ_m and the soliton velocity v,

$$v^2 = \frac{Q_0(\rho_m)}{4(\rho_0 - \rho_m)^2}, \tag{11}$$

where

$$Q_0(\rho) = 8\rho\int_\rho^{\rho_0}[f(\rho_0) - f(\rho')]d\rho'. \tag{12}$$

The inverse of the function $\xi = \xi(\rho)$ defined by Eq. (10) gives the profile $\rho = \rho(\xi)$ of density of the condensate with the soliton propagating through it, and substitution of this expression for $\rho(\xi)$ into Eq. (7) provides the profile of the corresponding flow velocity $u = u(\xi)$.

The soliton's energy per unit length can be calculated by the method of Ref. [5] and it is given by the expression (see Ref. [6])

$$\varepsilon = \frac{1}{2}\int_{\rho_m}^{\rho_0}\frac{Q_0(\rho)d\rho}{\rho\sqrt{Q(\rho)}}. \tag{13}$$

Here ρ_m is a function of v according to Eq. (11) and the same is true for functions $Q_0(\rho)$ and $Q(\rho)$, so we can consider the soliton's energy as a known function of its velocity v:

$$\varepsilon = \varepsilon(v). \tag{14}$$

For example, in case of standard "Kerr-like" nonlinearity $f(\rho) = \rho$ our formula reduces to the well-known expression

$$\varepsilon(v) = \tfrac{4}{3}(\rho_0 - v^2)^{3/2}. \tag{15}$$

In all above formulas the background condensate density ρ_0 is a constant parameter.

Now we can proceed to derivation of equations of the contour dynamics.

III. EQUATIONS OF CONTOUR DYNAMICS

As was indicated in the Introduction, the density of a 2D dark soliton is localized along a curve called the soliton's position. We assume that the instant position of such a soliton in the (x, y) plane is given in a parametric form,

$$\mathbf{r}(s) = (x(s), y(s)), \tag{16}$$

where s is the length of the curve's arc starting from its "zero" point to the point (16). Following the rules of elementary differential geometry (see, e.g., [23]), we introduce the tangent vector $\mathbf{t}(s) = d\mathbf{r}/ds$, $|\mathbf{t}| = 1$, and the unit normal vector $\mathbf{n}(s)$, $|\mathbf{n}| = 1$, which obey the Frenet-Serret equations

$$\frac{\partial\mathbf{t}}{\partial s} = \kappa\mathbf{n}, \qquad \frac{\partial\mathbf{n}}{\partial s} = -\kappa\mathbf{t} \tag{17}$$

for plane curves, where κ is the curvature of the curve at the point $\mathbf{r}$. We use the partial derivatives here to indicate that they are taken for an instant position of the curve (16) (the soliton's contour). Now we take into account that the soliton moves and deforms, so that its point with the coordinate s at the moment of time t has velocity

$$\frac{\partial\mathbf{r}}{\partial t} = v\mathbf{n} + w\mathbf{t}, \tag{18}$$

where the first term corresponds to the motion of the curve in the normal direction and the second term corresponds to its stretching with change of the length s. The condition $\mathbf{r}_{ts} = \mathbf{r}_{st}$ yields

$$w_s = \kappa v, \qquad \mathbf{t}_t = (v_s + \kappa w)\mathbf{n}; \tag{19}$$

that is,

$$w = \int_0^s \kappa v\, ds'. \tag{20}$$

If we differentiate the second equation (19) with respect to s and replace $\mathbf{t}_s$ and w with the use of (17) and (20), then we get

$$\kappa_t\mathbf{n} + \kappa\mathbf{n}_t = \left[v_{ss} + \left(\int_0^s \kappa v\, ds'\right)\kappa_s + \kappa^2 v\right]\mathbf{n} - \kappa(v_s + \kappa w)\mathbf{t};$$

that is, with account of $(\mathbf{n}\cdot\mathbf{t})_t = 0$ and the second Eq. (19), we obtain

$$\kappa_t - \left(\int_0^s \kappa v\, ds'\right)\kappa_s = v_{ss} + \kappa^2 v. \tag{21}$$

This is a kinematic equation of the contour dynamics which follows from purely geometric consideration (see also discussions of the contour dynamics in Refs. [17–19]). The second term in its left-hand side has the meaning of change of the curvature κ due to transfer of the soliton's points along the arc with velocity w: $ds = -w\,dt$. This means that if we mark some point on the soliton's location curve, then the length of the arc from the zero point on the curve to this marked point changes during the evolution of the curve, hence the parameter s is not attached to the marked point and changes with time. In other words, the contour's motion leads to the reparametrization of its points, and the derivative in the left-hand side of Eq. (21) is interpreted as a "substantial derivative" of the curvature at the "marked" points of the curve:

$$\frac{d\kappa}{dt} = \left(\frac{\partial}{\partial t} + \frac{ds}{dt}\frac{\partial}{\partial s}\right)\kappa = \left[\frac{\partial}{\partial t} - \left(\int_0^s \kappa v\,ds'\right)\frac{\partial}{\partial s}\right]\kappa. \quad (22)$$

Now we turn to derivation of the dynamical equation for the soliton's contour motion. The energy of a dark soliton decreases with increase of its velocity. Actually, this is the reason for its bending instability [24,25]. Indeed, if some segment of the curve bends forward by getting greater velocity, then it acquires greater length and, hence, smaller energy per unit of length. Consequently, the velocity of this segment increases further and it bends even more. This reasoning implies the trough-like shape of the dark soliton only and it cannot be applied to other two-dimensional structures where one can meet an opposite condition for their stability; see, e.g., Ref. [26]. If a straight soliton moving along the x axis with velocity v_0 undergoes a small bending disturbance $x'(y,t)$, then its instant position is described by the function $x = v_0 t + x'(y,t)$ and its local velocity

$$v = v_0 + x'_t(y,t) \quad (23)$$

becomes a function of the coordinate y along the soliton. For small x' the velocity v can still be considered as the velocity component v_n normal to the soliton's contour at the point y: $v_n \approx v$. Then the energy per unit length is equal to $\varepsilon(v)$ and it is different from $\varepsilon(v_0)$ due to the local disturbance, that is due to the growth of the length l with the rate $dl/dt = \kappa v_n \approx \kappa v$ (see, e.g., formula (61,2) in Ref. [27]). Consequently, we get

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon}{dv}\frac{dv}{dt}$$

on one hand and

$$\frac{d\varepsilon}{dt} = \varepsilon\frac{dl}{dt} = \varepsilon\kappa v$$

on the other hand, so that equality of these two expressions yields

$$\frac{dv}{dt} = \frac{v\varepsilon}{d\varepsilon/dv}\kappa = \frac{\varepsilon}{m_*}\kappa, \quad (24)$$

where $m_* = 2d\varepsilon/dv^2 < 0$ is an "effective soliton mass" per unit length. Now we take into account the stretching of the contour with the local velocity $-w$ and replace dv/dt by the substantial derivative (22):

$$v_t - \left(\int_0^s \kappa v\,ds'\right)v_s = \frac{\varepsilon}{m_*}\kappa. \quad (25)$$

Equations (21) and (25) comprise the system of the contour dynamics equations. For the case of the nonlinearity $f(\rho) = \rho$ they were derived in Ref. [17,18] from the Gross-Pitaevskii equation (1) by means of the regular perturbation theory.

As a simple application of these equations, let us consider a linear approximation when a straight soliton ($\kappa_0 = 0$) moving with velocity v_0 is slightly disturbed and the above equations reduce to ($\kappa = \kappa_0 + \kappa'$, $v = v_0 + v'$)

$$\kappa'_t \approx v'_{yy}, \qquad v'_t \approx \frac{\varepsilon}{m_*}\kappa'. \quad (26)$$

Looking for the solution in the form $\kappa', v' \propto \exp(iky + \Gamma t)$ we find

$$\Gamma = -\frac{\varepsilon}{m_*}k^2; \quad (27)$$

that is, we have reproduced the result of Ref. [25].

Now we can turn to more interesting self-similar solutions of the obtained equations.

IV. SELF-SIMILAR SOLUTION

As was noticed in Ref. [17,18], Eqs. (21) and (25) are invariant with respect to the scaling transformation $s = \alpha\tilde{s}$, $t = \alpha\tilde{t}$, $\kappa = \tilde{\kappa}/\alpha$, $v = \tilde{v}$. Therefore this system has the solution in the form

$$v = V(\zeta), \qquad \kappa = \frac{K(\zeta)}{t}, \qquad \zeta = \frac{s}{t}, \quad (28)$$

where ζ is a self-similar variable. Such a form of the solution implies that in the limit $t \to -0$ the solution becomes singular; that is, the contour dynamics approach loses its applicability when the radius of curvature t/K becomes smaller than the soliton's width. At the same time, the solution describes curved moving solitons which can greatly deviate from their standard straight-line form.

Substitution of (28) into (21) and (25) yields

$$\begin{aligned} &\frac{d(\zeta K)}{d\zeta} + \frac{d}{d\zeta}\left(K\int_0^\zeta VK\,d\zeta'\right) = -\frac{d^2V}{d\zeta^2}, \\ &\left(\zeta + \int_0^\zeta VK\,d\zeta'\right)\frac{dV}{d\zeta} = -\frac{\varepsilon}{m_*}\kappa. \end{aligned} \quad (29)$$

Following [17,18], we introduce the function

$$\Phi(\zeta) = \zeta + \int_0^\zeta VK\,d\zeta', \quad K = \frac{1}{V}\left(\frac{d\Phi}{d\zeta} - 1\right), \quad (30)$$

and integrate the first equation (29) to obtain

$$K\Phi = -\frac{dV}{d\zeta} + A, \quad (31)$$

where the integration constant A is determined by the condition

$$A = \left.\frac{dV}{d\zeta}\right|_{\zeta=0}. \quad (32)$$

The second equation (29) and (31) give the system of ordinary differential equations

$$\frac{dV}{d\zeta} = \frac{(\varepsilon/m_*)A}{\varepsilon/m_* - \Phi^2}, \qquad \frac{d\Phi}{d\zeta} = 1 - \frac{AV\Phi}{\varepsilon/m_* - \Phi^2}. \quad (33)$$

We suppose that at the origin $s = \zeta = 0$ the soliton is black, that is $v(0) = 0$, but the gradient of velocity A is not equal to zero here. Then the system (33) must be solved with the initial conditions

$$V(0) = 0, \qquad \Phi(0) = 0. \tag{34}$$

For small $\zeta \ll 1$ we get $\Phi \approx \zeta$, $V \approx A\zeta$, $K \approx -\frac{A}{\varepsilon/m_*}\zeta$, whereas for large $\zeta \gg 1$ we obtain the estimates $\Phi \sim \zeta$, $V \approx V_m = \text{const}$, $K \sim A/\zeta$. Consequently, the transition from one asymptotic regime to the other one occurs at $\zeta \sim (|\varepsilon/m_*|)^{1/2}$ and for small A the solution has the order of magnitude $V \sim (|\varepsilon/m_*|)^{1/2}A$ and $K \sim A/(|\varepsilon/m_*|)^{1/2}$. Hence, in case of small $A \ll 1$ in the leading approximation with respect to this small parameter we can put $V = 0$ in the function ε/m_* and consider this function as a constant parameter. Then the first equation (33) with $\Phi \approx \zeta$ becomes

$$\frac{dV}{d\zeta} = \frac{A}{1 + \zeta^2/(\varepsilon/|m_*|)} \tag{35}$$

with the obvious solution

$$V(\zeta) = A\sqrt{\frac{\varepsilon}{|m_*|}}\,\arctan\left(\frac{\zeta}{\sqrt{\varepsilon/|m_*|}}\right). \tag{36}$$

With the same accuracy we obtain from (31)

$$K(\zeta) = \frac{A\zeta}{\varepsilon/|m_*| + \zeta^2}. \tag{37}$$

In the limit $\zeta \to \infty$ we find

$$V \approx V_m = \frac{\pi}{2}\sqrt{\frac{\varepsilon}{|m_*|}}A, \quad K \approx \frac{A}{\zeta}, \quad A \ll 1. \tag{38}$$

In case of large A the system (33) is to be solved numerically. For example, if we take in Eq. (1) the Kerr-like nonlinearity $f(\rho) = \rho$, then we get

$$\frac{\varepsilon}{m_*} = -\frac{1}{3}(1 - V^2), \tag{39}$$

where we have assumed $\rho_0 = 1$, and the system (33) takes the form

$$\frac{dV}{d\zeta} = \frac{A(1 - V^2)}{1 - V^2 + 3\Phi^2}, \quad \frac{d\Phi}{d\zeta} = 1 + \frac{3AV\Phi}{1 - V^2 + 3\Phi^2}. \tag{40}$$

Plots of its solutions for two different values of A are depicted in Fig. 1 (see also [17,18]). These solutions confirm the above estimates. The dependence of the limiting value V_m on A is shown in Fig. 2, where the red dashed line corresponds to the formula $V_m \approx (\pi/2\sqrt{3})A$ which is a particular case of Eq. (38) for the Kerr-like nonlinearity. As we see, the agreement with the limit of small A is good enough for $A \lesssim 0.5$. In the asymptotic region $\zeta \gg 1$ the first equation (40) reduces to

$$\frac{dV}{d\zeta} = \frac{A(1 - V^2)}{3\zeta^2} \tag{41}$$

and it can be easily integrated to give

$$V(\zeta) \approx V_m - \frac{A(1 + V_m)^2}{3\zeta}, \qquad \zeta \gg 1, \tag{42}$$

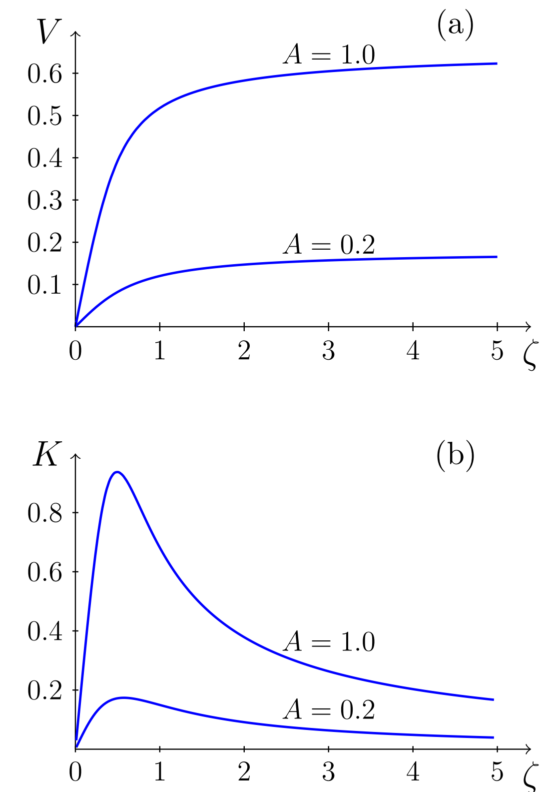

FIG. 1. (a) Plot of the function $V(\zeta)$ for $f(\rho) = \rho$ and two values of the parameter A. (b) Plot of the function $K(\zeta)$ for $f(\rho) = \rho$ and two values of the parameter A.

where the integration constant is chosen according to the condition $V(\zeta) \to V_m$ as $\zeta \to \infty$. This formula agrees with the asymptotic expression

$$V(\zeta) \approx \frac{\pi}{2\sqrt{3}}A - \frac{A}{3\zeta}, \qquad A \ll 1, \tag{43}$$

obtained from (36) in the limit $\zeta \to \infty$.

To find the form of the soliton at the moment t, we choose for definiteness (x, y) coordinates in such a way that the tangent vector $\mathbf{t}$ can be written in the form

$$\mathbf{t} = (\cos\theta, \sin\theta) \quad \text{or} \quad \frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta, \tag{44}$$

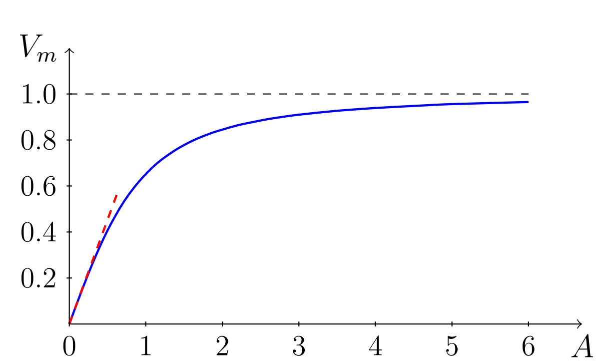

FIG. 2. The dependence of the limiting velocity V_m on the parameter A. The red dashed line corresponds to Eq. (38) applicable for $A \ll 1$.

and $\theta = 0$ at $s = 0$. Then from the first equation (17) we find at once

$$\kappa = \left|\frac{d\mathbf{t}}{ds}\right| = \frac{d\theta}{ds}. \quad (45)$$

In case of small A we obtain with the use of (37) the expression for the curvature,

$$\kappa = \frac{K}{t} = \frac{As}{(\varepsilon/m_*)t^2 + s^2}. \quad (46)$$

Consequently, integration of Eq. (45) gives

$$\theta(s) = \frac{A}{2}\ln\left[1 + \frac{s^2}{(\varepsilon/|m_*|)t^2}\right]. \quad (47)$$

At last, integration of Eqs. (44) yields the soliton's contour in a parametric form,

$$\begin{aligned} x(s,t) &= \int_0^s \cos\left\{\frac{A}{2}\ln\left[1 + \frac{s^2}{(\varepsilon/|m_*|)t^2}\right]\right\}ds, \\ y(s,t) &= \int_0^s \sin\left\{\frac{A}{2}\ln\left[1 + \frac{s^2}{(\varepsilon/|m_*|)t^2}\right]\right\}ds. \end{aligned} \quad (48)$$

The integrals here can be expressed in terms of the hypergeometric function (see, e.g., [28])

$$x(s,t) + iy(s,t) = sF\left(\frac{1}{2}, -\frac{iA}{2}, \frac{3}{2}, -\frac{s^2}{(\varepsilon/|m_*|)t^2}\right). \quad (49)$$

For small $|s| \ll |t|$ we get

$$x(s,t) \approx s, \qquad y(s,t) \approx \frac{As^3}{6(\varepsilon/|m_*|)t^2}; \quad (50)$$

that is, the soliton has the form of a cubic parabola here,

$$y(x,t) \approx \frac{Ax^3}{6(\varepsilon/|m_*|)t^2}, \qquad |x| \ll |t|. \quad (51)$$

The entire contour has the form of a spiral shown in Fig. 3 for different values of t. These curves have the maximal curvature

$$|\kappa_{\max}| = \frac{\sqrt{\varepsilon/|m_*|}A}{2|t|} \quad (52)$$

at $s = \pm\sqrt{\varepsilon/|m_*|}\,t$ and the coordinates of the points with the maximal curvature are to be found from the equation

$$x + iy = \pm\sqrt{\frac{\varepsilon}{|m_*|}}\,tF\left(\frac{1}{2}, -\frac{iA}{2}, \frac{3}{2}, -1\right). \quad (53)$$

The minimal radius of the curvature $1/|\kappa_{\max}| \propto |t|$ decreases as $t \to -0$ and, when it reaches the order of magnitude of the soliton's width, the contour dynamics approach loses its applicability. Numerical solution of the Gross-Pitaevskii equation performed in Refs. [17,18,29] shows that at this stage of evolution the dark soliton breaks with formation of vortex-antivortex pairs. The theory developed here describes the soliton's evolution before this breaking moment.

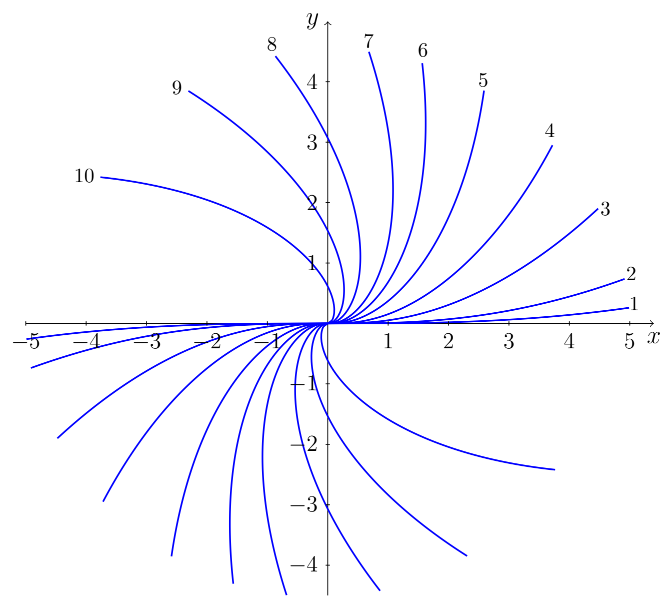

FIG. 3. Dark soliton contours at different moments of time: (*1*) $t = -10$; (*2*) $t = -5$; (*3*) $t = -2$; (*4*) $t = -1$; (*5*) $t = -0.5$; (*6*) $t = -0.3$; (*7*) $t = -0.2$; (*8*) $t = -0.1$; (*9*) $t = -0.05$; (*10*) $t = -0.02$. All of the curves correspond to $A = 0.5$.

V. CONCLUSION

We developed further the method of contour dynamics of dark solitons suggested first in Refs. [17,18]. A simplified derivation of equations of contour dynamics is given for the general form of the nonlinearity function in the Gross-Pitaevskii equation. The self-similar solution of the obtained equations is studied in detail. The results of this paper provide estimates for typical characteristics of dark solitons and the time of their breaking to vortex-antivortex pairs. We have considered evolution of solitons evolving in a uniform quiescent background, but the simple method used here can be applied to situations with nonuniform flowing condensates.

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