# Efficiency at optimal performance: A unified perspective based on coupled autonomous thermal machines

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We show that coupled autonomous thermal machines, in the presence of three heat reservoirs and following a global linear-irreversible description, can have efficiency at maximum power (EMP) which is analogous in form to the EMP of models with two (hot and cold) reservoirs. In particular, the temperature dependence of EMP in the coupled model is via only the ratio of hot and cold temperatures if the intermediate reservoir temperature is expressed as an algebraic mean of these temperatures. Many popular expressions of EMP in the literature can be recovered by making a choice of some standard mean. Further, the universal properties of EMP near equilibrium can be explained in terms of the properties of symmetric means. For the case of broken time-reversal symmetry, a universal second-order coefficient of $6/49$ is predicted in the series expansion of EMP, analogous to the $1/8$ coefficient in the time-reversal symmetric case.

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## I. INTRODUCTION

We observe that engines in the real world involve fluxes of matter and energy and undergo processes with finite rates. Linear-irreversible thermodynamics is by far the simplest phenomenological theory that assumes the fluxes to be proportional to the small thermodynamic forces driving them [1]. Heat engines based on this premise and other auxiliary assumptions bound the efficiency at maximum power (EMP), e.g., as $\eta_C/2$ [2], where $\eta_C$ is the Carnot efficiency. Other irreversible models [3–14] may predict EMP that goes beyond the linear-response result. These expressions for EMP are usually model specific (see Table I for a few examples), although they fall within certain bounds, as for example

$$\frac{\eta_C}{2} \leqslant \eta_{MP} \leqslant \frac{\eta_C}{2-\eta_C}. \tag{1}$$

Invariably, expressions of $\eta_{MP}$ exhibit a dependence on the ratio of cold to hot reservoir temperatures ($T_c/T_h$), an important feature also of the Carnot efficiency, $\eta_C = 1 - T_c/T_h$. Other universal or model-independent features can be identified at small values of $\eta_C$ (near-equilibrium situations), whereby the EMP satisfies the series expansion $\eta_{MP} \approx \eta_C/2 + \eta_C^2/8 + \mathcal{O}[\eta_C^3]$. Here, the first-order coefficient ($1/2$) corresponds to the linear-response behavior, while the second-order coefficient ($1/8$) has been analyzed in terms of a certain left-right symmetry of the specific model [14–17]. The fact that many proposed models do show universal features in EMP suggests the possibility of a generic thermodynamic model that might incorporate the various expressions within a single framework [18]. However, a scheme for autonomous machines that may accommodate the myriad expressions for EMP in a unified framework while accounting for its universal features is lacking.

In this paper, we analyze the global performance of two autonomous heat engines which are tightly coupled via a third heat reservoir having a temperature intermediate between the hot and cold reservoirs (see Fig. 1). Within a linear-irreversible framework, we optimize the total power output and show that EMP is bounded as $\eta^* \leqslant \eta_C(1 + T_c/T_0)^{-1}$, where the upper bound is achieved under a strong-coupling (SC) condition. The previous bound of $\eta_C/2$ is recovered for $T_0 = T_c$, but can be breached for $T_0 > T_c$. Further, the requirement that EMP depends only on the ratio $T_c/T_h$, or equivalently upon $\eta_C$ [19], requires that $T_0$ be expressed as a mean value of the hot and cold temperatures. Interestingly, specific choices of some common means for $T_0$ rather lead to well-known expressions for the EMPs of two-reservoir heat engines (Table I). This also attributes the above-mentioned universal features to EMP, if the choice is restricted to the so-called symmetric means. We also derive EMP for suboptimal coupling and suggest a different universality class for EMP in the case of broken time-reversal symmetry (TRS). More precisely, in place of the universal $1/8$ coefficient in the series expansion of EMP, we derive a universal coefficient of $6/49$ for the case of broken TRS. Finally, apart from the engine, we are able to optimize the cooling power in the refrigerator

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TABLE I. The intermediate temperature $T_0$ as some well-known symmetric means of $T_h$ and $T_c$, and the corresponding upper bound for EMP, $\eta_u^*$, obtained under strong coupling, where $\eta_C = 1 - T_c/T_h$. Various finite-time models derive these forms of EMP in the listed references.

| Mean | $T_0 \equiv \mathcal{M}(T_h, T_c)$ | $\eta_u^* = \eta_C\left(1+\frac{T_c}{T_0}\right)^{-1}$ | Physical model |
|---|---|---|---|
| Geometric | $\sqrt{T_h T_c}$ | $1-\sqrt{1-\eta_C}$ | Ref. [3] |
| Harmonic | $\frac{2T_hT_c}{T_h+T_c}$ | $\frac{2\eta_C}{4-\eta_C}$ | Refs. [4,5,11,20] |
| Arithmetic | $\frac{T_h+T_c}{2}$ | $\frac{(2-\eta_C)\eta_C}{4-3\eta_C}$ | Ref. [21] |
| Logarithmic | $\frac{T_h-T_c}{\ln T_h-\ln T_c}$ | $\frac{\eta_C^2}{\eta_C-(1-\eta_C)\ln(1-\eta_C)}$ | Refs. [22–25] |
| Lehmer | $\frac{T_h^\sigma+T_c^\sigma}{T_h^{\sigma-1}+T_c^{\sigma-1}}$ | $\frac{\eta_C}{2-\frac{\eta_C}{1+(1-\eta_C)^\sigma}}$ | $\sigma\in R$, Ref. [26] |

mode—a goal which proves to be elusive in some of the previously studied models.

The paper is organized as follows. In Sec. II, we describe the model of two tightly coupled autonomous engines within a linear-irreversible framework based on a weighted mean of hot and cold fluxes. In Sec. III A, we optimize the power output and derive a simple expression for EMP. Using $T_0$ in the form of an algebraic mean of hot and cold temperatures, we discuss, in Sec. III B, a few examples of EMP within our framework. This is followed by a discussion on the universal properties of EMP, again using basic properties of the so-called symmetric means. In Sec. IV, we extend our framework to the case of broken time-reversal symmetry (or a violation of the Onsager reciprocal relation) and predict a different universality class for EMP in this regime. Section V outlines our approach for the model of coupled refrigerators and the optimization of the cooling power is discussed. Section VI highlights the main conclusions of the paper.

## II. COUPLED ENGINE MODEL

Based on Fig. 1, let us now consider the performance of the subengines. The reservoirs $T_h$ and $T_0$ are coupled via an autonomous engine leading to power output $\dot{W}_1 = \dot{Q}_h - \dot{Q}_0$, and a rate of entropy generation, $\dot{S}_1 = -\dot{Q}_h/T_h + \dot{Q}_0/T_0$, which can be written as

$$\dot{S}_1 = -\frac{\dot{W}_1}{T_0} + \dot{Q}_h\left(\frac{1}{T_0} - \frac{1}{T_h}\right). \tag{2}$$

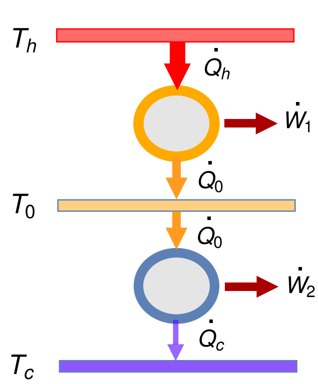


FIG. 1. Two autonomous heat engines tightly coupled via a third heat reservoir at temperature $T_0$, which satisfies $T_c \leqslant T_0 \leqslant T_h$. The total power output is $\dot{W} = \dot{W}_1 + \dot{W}_2$.

Similarly, reservoirs $T_0$ and $T_c$ are coupled via another such engine that leads to the power output $\dot{W}_2 = \dot{Q}_0 - \dot{Q}_c$, and a rate of entropy generation, $\dot{S}_2 = -\dot{Q}_0/T_0 + \dot{Q}_c/T_c$, which can be written as

$$\dot{S}_2 = -\frac{\dot{W}_2}{T_0} + \dot{Q}_c\left(\frac{1}{T_c} - \frac{1}{T_0}\right). \tag{3}$$

Since the two subengines are tightly coupled with each other, the net heat flux exchanged with the intermediate reservoir is zero. Then, $\dot{W}_1 + \dot{W}_2 = \dot{Q}_h - \dot{Q}_c = \dot{W}$, and $\dot{S}_1 + \dot{S}_2 = \dot{S}$ is written as

$$\dot{S} = -\frac{\dot{W}}{T_0} + \dot{Q}_h\left(\frac{1}{T_0} - \frac{1}{T_h}\right) + \dot{Q}_c\left(\frac{1}{T_c} - \frac{1}{T_0}\right). \tag{4}$$

Let us define $X_h = 1/T_0 - 1/T_h \geqslant 0$ and $X_c = 1/T_c - 1/T_0 \geqslant 0$, so that $X_h + X_c = 1/T_c - 1/T_h$. Then, we can write Eq. (4) as

$$\begin{aligned}\dot{S} &= -\frac{\dot{W}}{T_0} + \frac{\dot{Q}_h X_h + \dot{Q}_c X_c}{X_h + X_c}(X_h + X_c)\\ &= -\frac{\dot{W}}{T_0} + \dot{Q}_{\rm av}\left(\frac{1}{T_c} - \frac{1}{T_h}\right),\end{aligned} \tag{5}$$

where the average or effective thermal flux is given by

$$\dot{Q}_{\rm av} = (1-\omega)\dot{Q}_h + \omega\dot{Q}_c, \tag{6}$$

with $\omega = X_c/(X_h + X_c)$ satisfying $0 \leqslant \omega \leqslant 1$. In standard approaches, the reference reservoir is usually chosen to be the coldest reservoir available, and so $T_0 = T_c$. Within the present framework, the reference reservoir is an additional resource at $T_0$ and the relevant thermal flux is the average value $\dot{Q}_{\rm av}$. Finally, the total power flux is given as $\dot{W} = F\dot{x}$, where $F$ is the load and $\dot{x} \equiv \dot{x}_1 + \dot{x}_2$ is the total rate of displacement generated.

### A. Linear-irreversible framework

Now, assuming a linear-irreversible description at the level of global performance of the coupled engines, we identify the following flux-force pairs,

$$J_1 = \dot{x}, \quad X_1 = -\frac{F}{T_0}, \tag{7}$$

$$J_2 = \dot{Q}_{\rm av}, \quad X_2 = \frac{1}{T_c} - \frac{1}{T_h}, \tag{8}$$

so that the rate of entropy production is cast in a bilinear form $\dot{S} = \sum_{i=1}^{2} J_i X_i$. Second, the linear regime implies the flux-force relations of the form $J_i = \sum_{j=1}^{2} L_{ij} X_j$, where $i = 1, 2$. Here, the phenomenological coefficients $L_{ij}$ are assumed fixed due to the small magnitudes of the forces. Then, the second-law inequality imposes the following conditions:

$$L_{11}, L_{22} \geqslant 0, \quad 4L_{11}L_{22} \geqslant (L_{12} + L_{21})^2. \tag{9}$$

We first assume the principle of microscopic time-reversal symmetry (TRS) which allows the use of the Onsager reciprocity relation $L_{21} = L_{12}$. In this case, the third inequality above reduces to $L_{11}L_{22} \geqslant L_{12}^2$. This makes it convenient to define a measure, $q = L_{12}/\sqrt{L_{11}L_{22}}$, for the coupling strength between thermodynamic forces, which satisfies $-1 \leqslant q \leqslant +1$.

So, the constitutive relations for the fluxes in Eqs. (7) and (8) can be written in the following form:

$$\dot{x} = -L_{11}\frac{F}{T_0} + L_{12}X_2, \tag{10}$$

$$\dot{Q}_{\rm av} = -L_{12}\frac{F}{T_0} + L_{22}X_2. \tag{11}$$

Using Eqs. (6), (11), and $\dot{Q}_h - \dot{Q}_c = \dot{W}$, we can derive the following relations:

$$\dot{Q}_h = -L_{12}\frac{F}{T_0} + L_{22}X_2 + \omega\dot{W}, \tag{12}$$

$$\dot{Q}_c = -L_{12}\frac{F}{T_0} + L_{22}X_2 - (1-\omega)\dot{W}. \tag{13}$$

## III. OPTIMIZATION OF POWER OUTPUT

### A. Efficiency at maximum power (EMP)

By using Eq. (10), we optimize the power output, $\dot{W} = F\dot{x}$, with respect to the load $F$. The optimal load is obtained at $F^* = L_{12}T_0X_2/2L_{11}$. The optimal power, $\dot{W}^* \equiv \dot{W}(F^*)$, is given by

$$\dot{W}^* = \frac{L_{12}^2 T_0 X_2^2}{4L_{11}}. \tag{14}$$

Similarly, the hot flux, $\dot{Q}_h^* \equiv \dot{Q}_h(F^*)$, is obtained from Eq. (12) as

$$\dot{Q}_h^* = \left[1 + \frac{q^2}{4}\left(\frac{T_0}{T_c} - 3\right)\right]L_{22}X_2. \tag{15}$$

Then, the efficiency at maximum power (EMP), $\eta^* = \dot{W}^*/\dot{Q}_h^*$, is evaluated to be

$$\eta^* = \eta_{\rm C}\left[1 + \left(\frac{4-2q^2}{q^2} - 1\right)\frac{T_c}{T_0}\right]^{-1}. \tag{16}$$

For given reservoir temperatures, the EMP can be varied by tuning the coupling strength $q$, but it remains bounded as

$$0 \leqslant \eta^* \leqslant \eta_{\rm C}\left(1 + \frac{T_c}{T_0}\right)^{-1} \equiv \eta_u^*, \tag{17}$$

where the upper bound is saturated for strong coupling ($q^2 = 1$). Furthermore, to discuss the two reservoirs set up at hot and cold temperatures, we may set $T_0 = T_c$. Then, the EMP of Eq. (16) reduces to $\eta^* = q^2\eta_{\rm C}/(4 - 2q^2)$, as derived in Ref. [2]. This EMP is upper bounded by $\eta_C/2$. Thus, the presence of a third reservoir at $T_0 > T_c$, helps to go beyond this linear-response result, so that $\eta_{\rm C}/2$ now becomes the lower bound. In other words, if we consider $\eta_u^*$ of Eq. (17) as a function of $T_0$, then $\eta_u^*$ is bounded as in Eq. (1).

### B. Examples

The previous studies on the form of EMP were mostly carried out on two-reservoir setups, where the EMP obtained depends upon the ratio $T_c/T_h$. In the present model, with three reservoirs, the EMP depends on two ratios involving the three temperatures, as in Eq. (16). Now, $T_0$ may be assigned some numerical value in the interval $[T_c, T_h]$. However, as we show in the following, when $T_0$ is expressed as an algebraic mean of $T_h$ and $T_c$, then the EMP depends only on $T_c/T_h$ and we can establish a comparison with the EMP of two-reservoir models. Interestingly, many known expressions for EMP can be derived by assigning a specific mean to $T_0$. The few examples of Table I pertain to the scenario $q^2 = 1$, for which Eq. (16) yields $\eta^* = \eta_{\rm C}(1 + T_c/T_0)^{-1}$. Upon comparison between this formula and a known expression for EMP, the corresponding $T_0$ may be inferred.

As another example, a tandem construction of linear-irreversible engines [2] leads to the EMP, $\eta^* = 1 - (1 - \eta_C)^\beta$. Comparing this expression for EMP with Eq. (16), we obtain

$$T_0 = \frac{\beta - 1}{\beta}\frac{T_h^\beta - T_c^\beta}{T_h^{\beta-1} - T_c^{\beta-1}}, \tag{18}$$

a special case of the generalized mean [27,28]. Due to $0 \leqslant \beta \leqslant 1/2$, $T_0$ is bounded as $T_hT_c/T_{\rm L} \leqslant T_0 \leqslant \sqrt{T_hT_c}$, with $T_{\rm L} = (T_h - T_c)/\log(T_h/T_c)$ as the logarithmic mean. Here, Curzon-Ahlborn (CA) efficiency [3] is obtained with $\beta = 1/2$, for which $T_0 = \sqrt{T_hT_c}$.

Further, it is not hard to find examples of asymmetric means, $\mathcal{M}(T_h, T_c) \neq \mathcal{M}(T_c, T_h)$, that can parametrize more general expressions of EMP. Thus, the use of weighted harmonic mean $T_0 = T_hT_c/[(1-\alpha)T_h + \alpha T_c]$ in Eq. (16) yields $\eta^* = \eta_{\rm C}/(2 - \alpha\eta_{\rm C})$, where $0 \leqslant \alpha \leqslant 1$. The symmetric case of $\alpha = 1/2$ has been already mentioned in Table I. The above expression has been derived in various models, where, for instance, the parameter $\alpha$ may quantify the ratio of heat transfer coefficients [4] or dissipation constants [11,17] on the hot and cold sides of the engine.

### C. Universal properties of EMP

Next, we address the universal properties of EMP in the context of our coupled model. Let $\mathcal{M}(a, b)$ define an algebraic mean of two real numbers $a, b > 0$, which satisfies $\min[a, b] < \mathcal{M}(a, b) < \max[a, b]$. So, we define $\mathcal{M}(a, a) = a$. Further, $\mathcal{M}$ is a homogeneous function of its arguments, satisfying $\mathcal{M}(\lambda a, \lambda b) = \lambda\mathcal{M}(a, b)$, for all real $\lambda$. Thus, we can write $\mathcal{M}(a, b) = a\mathcal{M}(1, b/a)$. Assuming $T_0$ to be such a mean of hot and cold temperatures, i.e., $T_0 \equiv \mathcal{M}(T_h, T_c)$, we can write $T_0 \equiv T_h\mathcal{M}(1, T_c/T_h) = T_h\mathcal{M}(1, 1 - \eta_{\rm C})$. In other words, $\eta^*$ of Eq. (16) becomes a function only of $\eta_{\rm C}$, or the ratio of cold to hot temperatures.

Then, for a small difference between the hot and cold temperatures ($\eta_{\rm C}$ as a small parameter), we may develop $\mathcal{M}$

as a Taylor series in $(-\eta_C)$,

$$\mathcal{M}(1, 1-\eta_C) = 1 + a_1(-\eta_C) + a_2(-\eta_C)^2 + \mathcal{O}[\eta_C^3], \quad (19)$$

where the coefficients $a_1, a_2, \ldots$ are determined by the form of the given mean [29]. The corresponding series expansion of Eq. (16) is then given by

$$\eta^* = \frac{q^2}{4-2q^2}\eta_C + (1-a_1)\frac{(4-3q^2)q^2}{(4-2q^2)^2}\eta_C^2 + \mathcal{O}[\eta_C^3]. \quad (20)$$

The first-order term above is the same as for a two-reservoir (hot and cold) setup [2], being independent of the intermediate temperature $T_0$. For $q^2 = 1$, this term yields the half-Carnot value. The coefficient of the second-order term depends on $q^2$ as well as on $a_1$ which is a characteristic of the mean $T_0$ [see Eq. (19)]. Remarkably, if $T_0$ is a *symmetric* mean, i.e., having the property $\mathcal{M}(T_h, T_c) = \mathcal{M}(T_c, T_h)$, then $a_1 = 1/2$ [30], and we may rewrite Eq. (20) as

$$\eta^* = \beta\eta_C + \frac{\beta(1-\beta)}{2}\eta_C^2 + \mathcal{O}[\eta_C^3], \quad (21)$$

where $\beta \equiv q^2/(4-2q^2)$ and $0 \leqslant \beta \leqslant 1/2$. Thus, we have a universal relation between the first- and second-order coefficients, which is valid for any choice of the symmetric mean $T_0$. In particular, for models with SC, $\beta = 1/2$, and thus we obtain $1/8$ as the second-order coefficient, analogous to the two-reservoir case [15].

## IV. BROKEN TIME-REVERSAL SYMMETRY (TRS)

The basic framework of Sec. II A can be easily generalized to scenarios with a broken TRS, for which the reciprocity relation is no longer true, i.e., $L_{21} \neq L_{12}$. Then, the second flux-force relation, Eq. (11), reads as $\dot{Q}_{\rm av} = -L_{21}F/T_0 + L_{22}X_2$. Following an analogous derivation as for the time-symmetric case, the EMP is given as

$$\eta^*_{\cancel{\rm TRS}} = \eta_C\left[1 + \left(\frac{1-\gamma}{\gamma}\right)\frac{T_c}{T_0}\right]^{-1}. \quad (22)$$

Here, $\gamma \equiv xy/(4+2y)$, with $x = L_{12}/L_{21}$ and $y = L_{12}L_{21}/(L_{11}L_{22} - L_{12}L_{21})$ [8]. For $x = 1$, we can write $y = q^2/(1-q^2)$ or $\gamma = \beta$, and so Eq. (22) reduces to Eq. (16), thus recovering the results of the model satisfying TRS. Second, note that for $T_0 = T_c$, results of the previous studies [8,9] are recovered, by which $\eta^*_{\cancel{\rm TRS}} = \gamma\eta_C$. Thus, the presence of an additional reservoir at $T_0 > T_c$ raises the EMP beyond $\gamma\eta_C$.

As noted in Refs. [8,9], for a given value of $x$, the parameter $\gamma$ lies in the range $0 \leqslant \gamma \leqslant x^2/(4x^2 - 6x + 4) \equiv \hat{\gamma}$. Since the EMP of Eq. (22) is a monotonic increasing function of $\gamma$, so the optimal EMP is given by

$$\eta^*_{\cancel{\rm TRS}} = \eta_C\left[1 + \frac{1-\hat{\gamma}}{\hat{\gamma}}\frac{T_c}{T_0}\right]^{-1}. \quad (23)$$

Clearly, for $x = 1$, we obtain $\hat{\gamma} = 1/2$, recovering the results of the strong-coupling case, $\eta^* = \eta_C/(1 + T_c/T_0)$. Figure 2 plots Eq. (23) for different special cases. As argued in Ref. [9], the upper bound of EMP can be breached in the case of broken TRS, yielding the optimal EMP as $4\eta_C/7(>\eta_C/2)$. It is apparent from Fig. 2 that the intermediate temperature $T_0$ helps to go beyond this result too and so the bound $4\eta_C/7$ is rendered just as the *lower* bound.

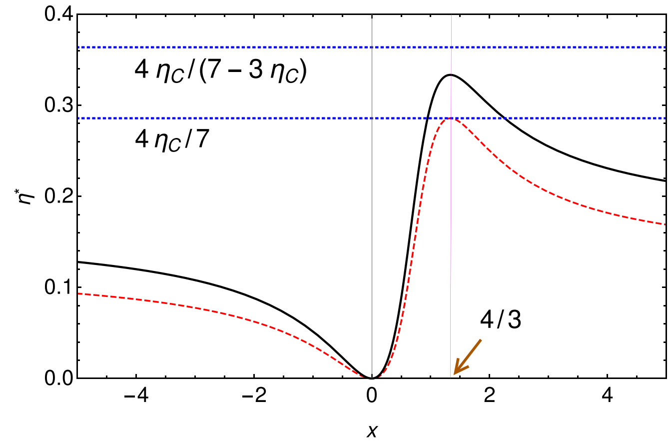


FIG. 2. The (red) dashed curve denotes EMP for $T_0 = T_c$ whose optimal value is $4\eta_C/7$, obtained at $x = 4/3$ [9]. For $T_c \leqslant T_0 \leqslant T_h$, the optimal EMP is bounded between the two dashed horizontal lines, and is able to breach the $4\eta_C/7$ value. The thick (black) curve is the EMP for $T_0 = (T_h + T_c)/2$, which is also optimal at $x = 4/3$.

Although the exact expression for EMP depends on the specific form of $T_0$, we can inquire into the universal features just as for the case with TRS. For an arbitrary symmetric mean $T_0$, and in proximity to equilibrium, we get

$$\eta^*_{\cancel{\rm TRS}} = \gamma\eta_C + \frac{\gamma(1-\gamma)}{2}\eta_C^2 + \mathcal{O}[\eta_C^3]. \quad (24)$$

The above series generalizes Eq. (21) which is obtained with $x = 1$, for which $\gamma = \beta$. For the case of optimal EMP where $\hat{\gamma} = 4/7$ ($x = 4/3$ [9]), the series expansion (24) is given by

$$\eta^*_{\cancel{\rm TRS}} = \frac{4}{7}\eta_C + \frac{6}{49}\eta_C^2 + \mathcal{O}[\eta_C^3]. \quad (25)$$

Thus, corresponding to a $\{1/2, 1/8\} \equiv \{4/8, 6/48\}$ pair of universal coefficients for optimal EMP in the time-symmetric case, we obtain $\{4/7, 6/49\}$ as the corresponding universal pair in the case of broken TRS.

## V. MODEL FOR COUPLED REFRIGERATORS

By reversing the energy flows in Fig. 1, we can study two tightly coupled refrigerators in a similar manner. In this case, it is possible to optimize the cooling power of the total machine, as we show below.

We can write the total rate of entropy generation as

$$\dot{S} = \frac{\dot{W}}{T_0} - \dot{Q}_{\rm av}\left(\frac{1}{T_c} - \frac{1}{T_h}\right). \quad (26)$$

Then, we identify the following flux-force pairs,

$$J_1 = \dot{x}, \quad X_1 = \frac{F}{T_0}, \quad (27)$$

$$J_2 = \dot{Q}_{\rm av}, \quad X_2 = -\left(\frac{1}{T_c} - \frac{1}{T_h}\right), \quad (28)$$

so that $\dot{S}$ is cast in a bilinear form, $\dot{S} \equiv J_1X_1 + J_2X_2$.

Within the linear-irreversible framework, the fluxes in Eqs. (27) and (28) take the following form:

$$\dot{x} = L_{11}\frac{F}{T_0} + L_{12}X_2, \tag{29}$$

$$\dot{Q}_{\rm av} = L_{12}\frac{F}{T_0} + L_{22}X_2. \tag{30}$$

Then, we can derive the following relations:

$$\dot{Q}_h = L_{12}\frac{F}{T_0} + L_{22}X_2 + \omega\dot{W}, \tag{31}$$

$$\dot{Q}_c = L_{12}\frac{F}{T_0} + L_{22}X_2 - (1-\omega)\dot{W}. \tag{32}$$

### A. Maximum cooling power

We optimize the cooling power by setting

$$\frac{\partial \dot{Q}_c}{\partial F} = 0. \tag{33}$$

The optimal value of $F \equiv \hat{F}$ is given by

$$\hat{F} = \frac{L_{12}X_2T_0(2T_h - T_0)}{2L_{11}(T_0 - T_h)}. \tag{34}$$

The coefficient of performance (COP) of the refrigerator at maximum cooling power is defined as

$$\xi^* = \frac{\dot{Q}_c(\hat{F})}{\dot{W}(\hat{F})}, \tag{35}$$

and is evaluated to be

$$\xi^* = \frac{\xi_{\rm C}(1-t_0)}{t_0^2(2-t_0)}\left[(2-t_0)^2 - \frac{4(1-t_0)}{q^2}\right], \tag{36}$$

where $t_0 = T_0/T_h$ and $\xi_{\rm C} = T_c/(T_h - T_c)$ is the Carnot bound for the COP. Now, for a given $q$ value, $\xi^*$ is a monotonic decreasing function of $t_0$. So, the bounds of $\xi^*$ are given as

$$0 \leqslant \xi^* \leqslant \frac{1}{\xi_{\rm C}(2+\xi_{\rm C})}\left[(2+\xi_{\rm C})^2 - \frac{4(1+\xi_{\rm C})}{q^2}\right]. \tag{37}$$

For models with SC, Eq. (36) gets simplified to $\xi^* = \xi_{\rm C}(1-t_0)/(2-t_0)$, which interpolates as $0 \leqslant \xi^* \leqslant \xi_{\rm C}/(2+\xi_{\rm C})$, with the lower and upper bounds obtained with $t_0 = 1$ and $t_0 = T_c/T_h \equiv \xi_{\rm C}/(1+\xi_{\rm C})$, respectively.

## VI. CONCLUSIONS

Concluding, we have studied the global performance of two tightly coupled engines within a three-reservoir setup. Assuming a linear-irreversible description where the total rate of entropy generation is defined in terms of a weighted average of the hot and cold fluxes, we have optimized the total power and analyzed the properties of the corresponding efficiency at maximum power. The EMP, in general, depends on two ratios involving the three reservoir temperatures. However, an interesting simplification occurs if the third temperature is chosen as an algebraic mean between the hot and cold temperatures. In this situation, the EMP can be expressed in terms of the Carnot efficiency of the total setup, or equivalently, the ratio of cold to hot temperatures. Further, the choice of this mean in the form of some common means (such as geometric mean, harmonic mean, and so on) yields well-known expressions for EMP found in previous studies on two-reservoir setups. Similarly, the universal properties of EMP found in the latter case can also be identified in the three-reservoir scenario, when the third temperature is a symmetric mean of hot and cold temperatures.

Further, universal features of EMP, surprising as they are, may be looked upon as a signature of the universality of thermodynamic approach. The present framework for the global performance of coupled machines provides an effective parameter in $T_0$ which may be tuned to obtain EMP in a desired form, thus bringing various mathematical forms of EMP under one formalism. Such an approach, apart from providing a unified viewpoint, can be instrumental in predicting novel features such as the 6/49 second-order coefficient for EMP in the case of broken TRS. The generality of thermodynamics deems it feasible that these features may be observed in systems with broken TRS, such as thermoelectric machines placed in an external magnetic field.

Finally, we have discussed the case of coupled refrigerators only briefly, mainly showing that the cooling power may be optimized within the present framework and deriving the corresponding COP. A more detailed analysis of the model for refrigerators [31,32] and a possible comparison with the observed COPs of refrigeration plants is left for future work.

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[29] For example, if $T_0 = [(1-p_1)/T_h + p_1/T_c]^{-1}$, a weighted harmonic mean ($0 \leqslant p_1 \leqslant 1$), then $\mathcal{M}(1, 1-\eta_C) = 1 - p_1\eta_C + p_1(p_1-1)\eta_C^2 + O[\eta_C^3]$, so that $a_1 = p_1$, $a_2 = p_1(p_1-1)$, and so on.
[30] Many well-known means are symmetric means, such as arithmetic mean $(a+b)/2$, geometric mean $\sqrt{ab}$, and harmonic mean $2ab/(a+b)$. Further, these means satisfy the following property, $\frac{\partial\mathcal{M}}{\partial a}|_{a=b} = \frac{\partial\mathcal{M}}{\partial b}|_{b=a} = \frac{1}{2}$, which implies $a_1 = 1/2$.
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