Stochastic telegrapher's approach for solving the random Boltzmann-Lorentz gas

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The 1D random Boltzmann-Lorentz equation has been connected with a set of stochastic hyperbolic equations. Therefore, the study of the Boltzmann-Lorentz gas with disordered scattering centers has been transformed into the analysis of a set of stochastic telegrapher's equations. For global binary disorder (Markovian and non-Markovian) exact analytical results for the second moment, the velocity autocorrelation function, and the self-diffusion coefficient are presented. We have demonstrated that time-fluctuations in the lost of energy in the telegrapher's equation, can delay the entrance to the diffusive regime, this issue has been characterized by a timescale t_c which is a function of disorder parameters. Indeed, producing a longer ballistic dynamics in the transport process. In addition, fluctuations of the space probability distribution have been studied, showing that the mean value of a stochastic telegrapher's Fourier mode is a good statistical object to characterize the solution of the random Boltzmann-Lorentz gas. In a different context, the stochastic telegrapher's equation has also been related to the run-and-tumble model in Biophysics. Then a discussion devoted to the potential applications when swimmers' speed and tumbling rate have time fluctuations has been pointed out.

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I. INTRODUCTION

A notable advance in understanding electromagnetic wave propagation, in a conducting medium, was achieved through the analysis of the telegrapher's equation (TE) [1,2]. In addition, this hyperbolic equation has been used in quite different areas of research: Cattaneo-Fick transport [3,4], neuroscience [5,6], biomedical optics [7], short electric transmission lines [8], tunneling in microwave experiments [9–11], electromagnetic fields in multilayered conductor planes [12], anisotropic diffusion from Boltzmann-Lorentz scattering [13–15], applied models in 2D and 3D for engineers problems [16,17], finite-velocity diffusion in heterogeneous media [18,19]; as well as, damping and propagation of surface gravity waves on a random bottom [20,21]. Recently, new sources of inspiration in the use of the TE have also emerged due to the practical interest for understanding transient anomalous diffusion [22,23], telegraphic processes with stochastic resetting [24], flagella motion [25], wavelike transport and delocalization with fluctuating absorption of energy [26,27].

In a previous paper, one of us, has shown that the 1D Boltzmann-Lorentz equation can be reduced to a set of hyperbolic partial differential equations (PDEs) of the second order, which can be solved in Fourier-Laplace representation, see appendices A and B in Ref. [28]. This hyperbolic PDE is the so called TE which describes—at short times—a ballistic transport, while—at long times—it is diffusive [1,3,14,29,30]. Here we have worked out the Boltzmann-Lorentz gas with global disorder, reducing the problem to a set of uncoupled stochastic hyperbolic PDEs, which, for the binary intermittent disorder, has been solved in an exact manner by algebraic methods.

For understanding the role of the fluctuations, for different models of disorder in the Lorentz scattering, we have introduced the connection of the random Botzmann-Lorentz gas and the hyperbolic diffusion approach. In the context of hyperbolic diffusion, one of the important point was to consider stochastic absorption of energy in the TE. In this case the mean value of the packet shows a nontrivial crossover in its entrance to the diffusion regime. This fact can be understood in terms of the analysis of the delayed dispersion of the propagating wavelike front [26]. In the present paper, we have studied this issue considering different model of fluctuations in the absorption of energy, this is one of the goal of our work. Therefore, it turns to be crucial to check whether the mean value of the solution of the stochastic TE is a good statistical object for describing the propagation, this is the second task of the paper. Exact analytical results for different noise correlation structures of (global) binary disorder are shown. Indeed, the second moment of the front propagation, the ballistic-diffusive transition point t_c, as well as the velocity autocorrelation function and the renormalized diffusion coefficient are calculated. In addition, the fluctuation dispersion of the space-distribution of the TE is studied for Markovian and non-Markovian noise in the rate of absorption of energy.

Our results are of interest in the analysis of stochastic Lorentz gas models [31,32], finite-velocity diffusion in heterogeneous media [33,34], as well as, in hyperbolic diffusion

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in the presence of external force [28]. In addition, here we discuss another point of interest in the stochastic TE which has not relation with the Boltzmann-Lorentz gas. This is motivated by biological applications in the study of bacteria motion [35,36], recently referred to as the run-and-tumble models [25]. The organization of the paper is as follows: in Sec. II we present the Boltzmann-Lorentz gas and the TE. In Sec. III the connection with the stochastic TE is established for the case when considering fluctuations in the scattering process. Section IV is devoted to introduce the type of binary fluctuations that we will use: Markovian, non-Markovian and its intermittent limit. In Sec. V we present exact mean-value results for different model of fluctuations (noise correlations). Section VI is devoted to the analysis of the fluctuation dispersion of the field. Conclusions and extensions of the present work are given in Sec. VII. General analysis of hyperbolic PDEs associated to the 1D Botzmann-Lorentz gas with *global* and *local* disorder are also presented in the Appendix A. The rest of the Appendices are devoted to mathematical details.

II. BOLTZMANN-LORENTZ GAS AND THE TELEGRAPHER'S EQUATION

The linear Boltzmann-Lorentz equation is very useful to describe noninteracting classical particles moving in an array of equal scatterers [31,37]. These scattering centers can be located at random positions in space (local disorder), or globally the intensity of the scattering function may fluctuate in time (global disorder) due to some environmental makeup (screening). All these possibilities can be considered in the context of the Botzmann-Lorentz model and therefore can be studied using suitable random scattering functions [32]. Nevertheless, in this case, the mathematical problem turns to be a stochastic integrodifferential equation which may be hard to be solved, depending on the structure of the disorder. All these possibilities can be put in the context of the pioneer Lorentz model [38].

In the present paper, we are concerned with the probability distribution function (PDF) in space $P(x,t)$. Then, the main focus has been on the marginal Boltzmann function: $\int f(x,v,t)dv = P(x,t)$, i.e., without taking into account the direction of the velocity of the particle. In general, to manage the Boltzmann-Lorenz' gas in the presence of random scattering processes we have reduced the problem to a set of coupled stochastic TEs, then we have studied the mean value $\langle P(x,t)\rangle$ characterizing the "telegrapher's front" propagation. In a point model if the scatterer centers are uniformly distributed in the line, we can write Lorentz scattering function in the form

$$W(x,v,v') = q_0\,\delta(v+v'),\qquad q_0 = \text{constant} > 0. \tag{1}$$

Introducing this scattering function in the linear free of force 1D Boltzmann's equation:

$$\begin{aligned}(\partial_t + v\partial_x)f(x,v,t) &= \int dv'\,W(x,v,v')f(x,v',t)\\ &\quad - f(x,v,t)\int dv'\,W(x,v,v') \equiv \left(\frac{df}{dt}\right)_{\text{Coll}},\end{aligned} \tag{2}$$

we get

$$(\partial_t \pm v\partial_x)f(x,\pm v,t) = q_0[f(x,\mp v,t) - f(x,\pm v,t)], \tag{3}$$

which is, as expected, a velocity-invariant $|v'| \to |v|$ and space nonstationary process. Equation (3) can conveniently be transformed into a PDE for the PDF in space [28,39]. Thus, we will be interested in the time evolution of the marginal distribution: $P(x,t) = f(x,v,t) + f(x,-v,t)$, where $f(x,v,t)$ is Boltzmann's distribution. Note that the dimension of q_0 is given in units of 1/ time, because we adopt $\int dx \int dv\, f(x,v,t) = 1$.

A. The telegrapher's equation

From Eq. (3) and after some algebra it is possible to prove that we recover the 1D homogeneous TE for the evolution of the PDF $P(x,t)$, that is (see Appendix A 1)

$$\left[\partial_t^2 + 2q_0\partial_t - v^2\partial_x^2\right]P(x,t) = 0, \tag{4}$$

where $2q_0 = \tau^{-1}$ measures the rate of absorption of energy in the TE (Cattaneo-Fick's equation) [3,4]. The TE can be derived in a number of ways, but always the parameter τ^{-1} is related to absorption of energy, for example, in electric circuits and electromagnetic waves [1,2], turbulence [40], transport [29,30], to mention a few.

We can also prove that the associated current: $vQ(x,t) = v[f(x,v,t) - f(x,-v,t)]$ fulfills the same TE. These equations must be solved with suitable initial conditions for $P(x,t)$ and $Q(x,t)$, for example, the symmetric initial condition in distribution:

$$P(x,t)\big|_{t=0} = \delta(x) \text{ and } Q(x,t)\big|_{t=0} = 0. \tag{5}$$

We note that in this "telegrapher's picture" the velocity v is a parameter characterizing the velocity of the propagation of the diffusionlike front. The fact that we are working in a 1D space is reflected in that Boltzmann's distribution $f(x,v,t)$ can only be two (possible) functions: $f(x,v,t) \equiv f_+(x,t)$ and $f(x,-v,t) \equiv f_-(x,t)$. This issue makes the connection between the persistent random walk and the anisotropic scattering approach [13,14,29,39,41].

B. Boltzmann-Lorentz gas in the presence of a force

The linear Boltzmann-Lorentz equation in the presence of a force $\mathcal{F}$ can also be worked out in a similar way. Working with functions: $P(x,v,t) = f(x,v,t) + f(x,-v,t)$ and $Q(x,v,t) = f(x,v,t) - f(x,-v,t)$, it is possible to find a set of equations to describe the time evolution of the system. We note however that in this case the PDF $P(x,v,t)$ and the current $vQ(x,v,t)$ may vary as a function of the magnitude $|v|$ due to the term $\frac{1}{m}\mathcal{F}\partial_v$ in the Liouville's operator. In this case the corresponding PDEs for $P(x,v,t)$ and $Q(x,v,t)$ are analogous to the phase-space Kramers-Fokker-Planck equation, but they explicitly show at short time the ballistic and at long time the Brownian behaviors, see Appendix B.2 in Ref. [28].

III. STOCHASTIC SCATTERING IN THE BOLTZMANN-LORENTZ GAS

A. Time fluctuations in the scattering centers

Now comes up the question of what happens if the intensity of scattering has time-fluctuations (global disorder); that is, we propose to study the generalized scattering model:

$$W(t,v,v')=q(t)\,\delta(v+v'),\quad q(t)\geqslant 0,\quad \forall t\geqslant 0, \tag{6}$$

here $q(t)$ may represent a time dependent function or a noise, see Eq. (A1) for its interpretation.

After some algebraic calculations, it is possible to prove that the set of uncoupled evolution equations is

$$\left[\partial_t^2+2q(t)\partial_t-v^2\partial_x^2\right]P(x,t)=0, \tag{7}$$

$$\left[\partial_t^2+2\partial_t q(t)-v^2\partial_x^2\right]Q(x,t)=0, \tag{8}$$

to be used with suitable initial conditions as before. Notably the structure for the time evolution of the PDF $P(x,t)$ remains the same TE but with a time dependent function $q(t)>0$. While the current evolves with a "generalized" time-dependent rate $q(t)$; see Appendix A 3.

1. *Evolution for the stochastic telegrapher's field $P(x,t)$*

Adopting the time-fluctuating model (global disorder):

$$2q(t)=2q_0+\theta\xi(t),\quad \langle\xi(t)\rangle=0,$$
$$\theta\xi(t)\geqslant -2q_0,\quad \forall t\geqslant 0. \tag{9}$$

It is simple to see from Eq. (7) that the *exact* evolution for the PDF turns to be:

$$\left[\partial_t^2+\frac{1}{\tau}\partial_t-v^2\partial_x^2\right]P(x,t)=-\theta\xi(t)\partial_t P(x,t),\quad \tau^{-1}=2q_0, \tag{10}$$

see also Eqs. (11) and (12). This is the stochastic TE that we will solve in the present paper, for different (bounded from below) noises $\xi(t)$. Model Eq. (10) represents a stochastic absorption of energy in the telegrapher's field. In particular, we will be interested in the response of the system for different noise correlation functions: $\langle\langle\xi(t)\xi(t')\rangle\rangle$. In Ref. [26] this problem was solved using a Terwiel's expansion series, this approach was very useful because for the Markovian binary noise the series cuts leading to an exact result, while if the binary noise has a nonexponential correlation, infinite higher order terms in the series must be taken into account.

In the present paper, we are interested in the exact analysis of Eq. (10) for different correlations of the noise $\xi(t)$, therefore we will present an alternative approach for this purpose: the enlarged master equation [39,42–44].

2. *The stochastic TE and the run-and-tumble biological model*

As we have mentioned before in Sec. II A, the TE can be connected with the persistent random walk approach [14,29]. In this context, the interesting situation when the propagation speed v may depend on space and time: $v\to v(x,t)$ has also been worked out to lead to a generalized hyperbolic diffusion [45]. In particular, if $v\to v(x)$ is a random function the associated TE is of the type of equation worked out in the problem of finite-velocity diffusion in random media [18]; while if $v\to v(t)$ is a time-random function the associated TE is of the type of equation worked out in the problem of propagation of electromagnetic waves with stochastic absorption of energy; therefore, showing delocalization of plane waves [27].

Recently, inspired in the flagella motion, it has been shown that starting from the generalized persistent random walk scheme [45], the run-and-tumble biological model can be generalized [25]. Actually the original concept of stick–slip process, corresponds to the "run-and-tumble" pattern long used to describe bacterial motion [36].

In addition, considering the situation when swimmers' speed $v(t)$ and tumbling rate $\lambda(t)$ can be different time-dependent quantities. Our Eq. (10) represents the case when the *effective* tumbling rate $\lambda_{\rm eff}(t)$ has time fluctuations for a constant swimmers' speed v_0 [25], and so our results on delayed ballistic-diffusion processes, mean-square dispersion, velocity correlation function, and fluctuation-dispersion relation are of interest in biological problems.

IV. GLOBAL DISORDER IN THE ABSORPTION OF ENERGY

To study stochastic realizations of the TE (10) we introduce auxiliary fields:

$$\psi(x,t)\equiv P(x,t)\quad\text{and}\quad \phi(x,t)\equiv\partial_t P(x,t),$$

therefore we can write in the Fourier representation, with $K^2\equiv v^2k^2$, a set of equations equivalent to the TE (10):

$$\partial_t\psi=\phi, \tag{11}$$

$$\partial_t\phi=-K^2\psi-\tau^{-1}\phi-\theta\xi(t)\phi, \tag{12}$$

we note that in this short notation the Fourier label k has been dropped because for this stochastic differential equation this label is superfluous. Before going on with this presentation we show in the next sections realizations representing a Markovian and a non-Markovian (biexponential) binary noise.

Simulations are necessary when the global disorder cannot be analytically solved, this is the case of Poisson's noise perturbations as it is commented in Appendix D.

A. Exponential-correlated binary noise

A symmetric Markovian binary process: $\xi(t)=\pm 1,\forall t\geqslant 0$ corresponds to be exponentially correlated; that is, $\langle\xi(t)\xi(t')\rangle=e^{-|t-t'|/T}$. We can generate one realization of a symmetric binary noise $\xi(t)$ by

$$\xi(t)=\sum_{j=1}^{n}(-1)^j\,W(t_j,t_{j+1}|t),\ t\geqslant 0,\ \xi(0)=1, \tag{13}$$

where $W(t_j,t_{j+1}|t)$ is the window function: $W(t_j,t_{j+1}|t)=\Theta(t-t_j)-\Theta(t-t_{j+1})$, with $\Theta(u)=$ step function, the number of dots n in Eq. (13) is Poisson distributed, and the random location of independent times t_j are uniformly distributed in $[0,\infty]$ with density a. That is, we generate the statistical independent time-increments $\Delta_{i,j}\equiv t_i-t_j$ with an exponential waiting time, $\varphi(\Delta_{ij})=a\exp(-a\Delta_{ij})$, where $a=1/2T$. Clearly the stationary PDF of this binary noise, $\xi(t)$, is independent of the correlation timescale T; that is,

$\Pi_{St}(\xi) = 1/2, \forall\xi = \pm 1$. Strong (global) disorder corresponds to the case when $\theta = \tau^{-1}$. In this case there can be a realization for the rate $2q(t) = \tau^{-1} + \theta\xi(t)$ [Lorentz scattering intensities Eq. (9)] that could be zero for a random interval of time $[t_i, t_{i+1}]$.

B. Intermittent binary noise

A symmetric intermittent binary process, $\xi(t) = \pm 1, \forall t \geqslant 0$, can be represented, for example, with a nonexponential correlation function having two characteristic timescales. The stationary distribution $\Pi_{St}(\xi)$ still is the same as for the Markovian binary case. The only difference, in the intermittent case, is the waiting-time function that generates the statistical independent time increments $\{t_j - t_{j-1}\}$ in the generic formula Eq. (13) [39]. We have proved that an intermittent binary noise can be obtained by introducing a biexponential waiting time for the random time increments Δ_{ij} in Eq. (13):

$$\varphi(\Delta_{ij}) = \alpha p \exp(-\alpha\Delta_{ij}) + q\beta \exp(-\beta\Delta_{ij}),$$
$$p + q = 1, \quad \forall\{\alpha, \beta\} > 0. \tag{14}$$

Here the important parameters to characterize intermittence are $\alpha \gg \beta$ (different timescales) and $1 > p \gg q$, that is, very different statistical weight for each waiting timescale [42]; if $\alpha = \beta$ or if $p = \{1, 0\}$, then we recover the Markovian case. Strong (global) disorder corresponds, as in the previous Markovian case, to the situation when $\theta = \tau^{-1}$.

V. ANALYTICAL SOLUTIONS OF THE STOCHASTIC TELEGRAPHER'S EQ. (10)

Equation (10) corresponds to the TE with stochastic absorption of energy [26]. In this section we are going to present the solution of Eq. (10) when the stochastic process $\xi(t)$ is a Markov binary noise. As we mention before this noise can be worked out using different techniques; in particular, here we are going to use the enlarged master equation [39,43,44]. This approach allows also to introduce in a straightforwardly way the possibility to tackle the case when the binary noise is also intermittent [42].

A. Enlarged master equation for Markovian binary absorption of energy in the TE

When the noise $\xi(t)$ is Markovian the stochastic evolution Eq. (10) can rigorously be written in the context of an *enlarged* semigroup for the probability $\mathcal{P}(P, \partial_t P, \xi, t)$; see Appendix B.

Using auxiliary functions: $\psi^\pm$ and $\phi^\pm = \partial_t\psi^\pm$ [see Eq. (17)] and the fact that the system Eq. (10) is linear we can go one step further and write a system for the *marginal* averages (in Fourier's representation):

$$\partial_t \begin{pmatrix} \psi^+(k,t) \\ \phi^+(k,t) \\ \psi^-(k,t) \\ \phi^-(k,t) \end{pmatrix} = \begin{pmatrix} -a & 1 & a & 0 \\ -K^2 & -(\tau^{-1}+\theta+a) & 0 & a \\ a & 0 & -a & 1 \\ 0 & a & -K^2 & -(\tau^{-1}-\theta+a) \end{pmatrix} \begin{pmatrix} \psi^+(k,t) \\ \phi^+(k,t) \\ \psi^-(k,t) \\ \phi^-(k,t) \end{pmatrix}, \tag{15}$$

where $K^2 \equiv v^2k^2$, θ is the intensity of the perturbation, and we have used that $\xi(t) = \pm 1, \forall t \geqslant 0$ with $\langle\xi(t)\rangle = 0$ is the symmetric binary noise with correlation-function:

$$\langle\xi(t)\xi(t')\rangle = \exp(-|t - t'|/T), \quad \text{with } a = 1/2T. \tag{16}$$

To write Eq. (15) we have used the 2×2 master Hamiltonian $\mathbf{H} = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$ associated to the Markovian (2-state) binary process. Defining a 4-dimension vector $\vec{\Psi}(t)$ and a matrix $\mathbf{A} \in \mathcal{R}_e^{4\times 4}$, the solution of Eq. (15) is $\vec{\Psi} = e^{\mathbf{A}t} \cdot \vec{\Psi}(0)$. Therefore, the mean value (over the noise) of the fields $P(k, t)$ and $\Phi(k, t) \equiv \partial_t P(k, t)$ are given by

$$\langle P(k,t)\rangle = \psi^+(k,t) + \psi^-(k,t),$$

and

$$\langle\Phi(k,t)\rangle = \phi^+(k,t) + \phi^-(k,t). \tag{17}$$

The system Eq. (15) can easily be solved in the Laplace representation,

$$\psi^\pm(k,s) = \int_0^\infty e^{-st}\, \psi^\pm(k,t)dt,$$

and

$$\phi^\pm(k,s) = \int_0^\infty e^{-st}\, \phi^\pm(k,t)dt, \tag{18}$$

that is, $\vec{\Psi}(s) = (s - \mathbf{A})^{-1} \cdot \vec{\Psi}(0)$. Thus, all the information is contained in the roots of the characteristic polynomial:

$$\begin{aligned} 0 = {} & s^4 + 2(a+y)s^3 + (-a^2 + 2K^2 - \theta^2 + 4ay + y^2)s^2 \\ & + (-2a^3 + 2aK^2 - 2a\theta^2 + 2K^2y + 2ay^2)s \\ & + (2a^2K^2 + K^4 + 2aK^2y), \end{aligned} \tag{19}$$

with $y \equiv \tau^{-1} + a$. These roots can be found analytically and they characterize the dispersive behavior of the mean value of the stochastic telegrapher's field, due to the presence of Markovian binary fluctuations in the absorption of energy, with correlation-time $T = 1/2a$.

1. *Second moment of $\langle P(x, t)\rangle$ in the presence of binary fluctuations*

The exact second-moment of $\langle P(x, t)\rangle$ in the presence of binary fluctuations in the absorption of energy can be calculated from the Fourier-Laplace representation: $\langle P(k, s)\rangle$. That is, in the Laplace representation the second-moment of the averaged telegrapher's field is

$$\overline{x(s)^2} = \int x^2\, \langle P(x,s)\rangle dx = -\partial_k^2(\psi^+(k,s) + \psi^-(k,s))\big|_{k=0}. \tag{20}$$

This second moment can be obtained from the Laplace representation of Eq. (15), that is,

$$\begin{pmatrix}\psi^+(k,s)\\ \phi^+(k,s)\\ \psi^-(k,s)\\ \phi^-(k,s)\end{pmatrix} = (s-\mathbf{A})^{-1}\cdot\begin{pmatrix}1/2\\0\\1/2\\0\end{pmatrix}, \tag{21}$$

where we have used the initial condition $\vec{\Psi}(0) = (1/2, 0, 1/2, 0)$, in accordance with $P(x,t)|_{t=0} = \delta(x)$ and $\partial_t P(x,t)|_{t=0} = 0$. Then we get

$$\begin{aligned}\overline{x(s)^2} &= -\partial_k^2\langle P(k,s)\rangle\big|_{k=0}\\ &= \frac{2\tau v^2(1+sT+T/\tau)}{s^2[(s\tau+1)(sT+1)+(s\tau+1-\tau^2\theta^2)T/\tau]}.\end{aligned} \tag{22}$$

This result shows the crossover between the ballistic to the Brownian motion, for the dynamics of the mean-value solution of the Boltzmann-Lorentz gas with stochastic scattering intensities; that is, the behavior of the field mean value $\langle P(x,t)\rangle$ when the absorption of energy has binary fluctuations. This result was also predicted using Terwiel's series expansion [26]. The interesting point in using the present approach is that a similar calculation can be done for the case when the binary noise $\xi(t)$ is intermittent.

By using the Tauberian theorem [39], from Eq. (22) it is possible to check the short and long-time behavior. From the analysis of $\overline{x(s)^2}$, for $s\to\infty$ and $s\to 0$, we get both time-dependent limits; that is, for $s\to 0$ we get

$$\begin{aligned}\lim_{s\to 0}\overline{x(s)^2} &\simeq \frac{2\tau v^2(T+\tau)}{T+\tau-T\tau^2\theta^2}\frac{1}{s^2}\\ &\quad+\frac{2\tau^2v^2(T^2+2T\tau+\tau^2+T^2\tau^2\theta^2)}{T+\tau-T\tau^2\theta^2}\frac{1}{s}+\cdots,\end{aligned}$$

which means that $\overline{x(t\to\infty)^2}\to C_\infty\, t+\cdots$. While in the opposite limit $s\to\infty$ we get

$$\lim_{s\to\infty}\overline{x(s)^2}\simeq\frac{v^2}{s^3}-\frac{2v^2}{\tau s^4}+\cdots,$$

which means that $\overline{x(t\to 0)^2}\to C_0\, t^2+\cdots$.

For strong disorder the full analysis of Eq. (22) has been presented in Table 1 of Ref. [26].

2. *The coefficient of self-diffusion*

In this section we will calculate the renormalized transport coefficient using the well-known Green-Kubo expression for the self-diffusion coefficient. First using Eq. (22) we can write the velocity autocorrelation function (VAF) $C(t)\equiv\langle V(0)V(t)\rangle$, in the Laplace representation as

$$C(s)=\frac{s^2}{2}\,\overline{x(s)^2}. \tag{23}$$

Then the renormalized diffusion coefficient is given by

$$D=\int_0^\infty C(t)dt=\lim_{s\to 0}C(s), \tag{24}$$

this formula is nothing but the famous Einstein relation [39]. Therefore, from Eq. (24) we get

$$D=\frac{\tau v^2(1+T/\tau)}{[1+(1-\tau^2\theta^2)T/\tau]}, \tag{25}$$

as can be seen, by taking $\theta=0$ in Eq. (25), the *ordered* limit is recovered and it is in accordance with the TE (4); that is, $D_{\rm TE}=\tau v^2$. Another interesting limit to take is the case of strong disorder: $\theta=\tau^{-1}$, in this case we get $D_{\theta=\tau^{-1}}=\tau v^2(1+T/\tau)$, saying that the renormalized diffusion coefficient is larger than in the ordered case. This result can be understood as follows: when the "light particle" is moving freely in space (ballistically—before a collision) there are realizations for the (global) disorder where the *particle* does not find any scattering center with a nonzero total intensity: $2q(t)=\tau^{-1}+\theta\xi(t)$; see Eq. (6). However, for weak Markovian disorder $0<\theta<\tau^{-1}$, the diffusion coefficient shows a monotonic behavior with the correlation timescale parameter T; see Eq. (16).

B. Enlarged master equation for intermittent absorption of energy in the TE

If the noise $\xi(t)$ in Eq. (10) is binary with a biexponential waiting time, see Eq. (14), then the stochastic evolution for $P(k,t)$ can also be studied in the context of an *enlarged* master equation (Appendix B). Now the dimension of the matrix system is 8×8 and the auxiliary variables are: $\psi^\pm_{\alpha,\beta}$ and $\phi^\pm_{\alpha,\beta}=\partial_t\psi^\pm_{\alpha,\beta}$. Thus, *marginal* averages are characterized by the coupled system:

$$\partial_t\begin{pmatrix}\psi_\alpha^+\\ \phi_\alpha^+\\ \psi_\alpha^-\\ \phi_\alpha^-\\ \psi_\beta^+\\ \phi_\beta^+\\ \psi_\beta^-\\ \phi_\beta^-\end{pmatrix}=\begin{pmatrix}-\alpha & 1 & \alpha p & 0 & 0 & 0 & \beta p & 0\\ -K^2 & L(+,\alpha) & 0 & \alpha p & 0 & 0 & 0 & \beta p\\ \alpha p & 0 & -\alpha & 1 & \beta p & 0 & 0 & 0\\ 0 & \alpha p & -K^2 & L(-,\alpha) & 0 & \beta p & 0 & 0\\ 0 & 0 & \alpha q & 0 & -\beta & 1 & \beta q & 0\\ 0 & 0 & 0 & \alpha q & -K^2 & L(+,\beta) & 0 & \beta q\\ \alpha q & 0 & 0 & 0 & \beta q & 0 & -\beta & 1\\ 0 & \alpha q & 0 & 0 & 0 & \beta q & -K^2 & L(-,\beta)\end{pmatrix}\begin{pmatrix}\psi_\alpha^+\\ \phi_\alpha^+\\ \psi_\alpha^-\\ \phi_\alpha^-\\ \psi_\beta^+\\ \phi_\beta^+\\ \psi_\beta^-\\ \phi_\beta^-\end{pmatrix}, \tag{26}$$

which can be written in the form $\partial_t \vec{\Psi} = e^{\mathbf{A}t} \cdot \vec{\Psi}(0)$, where $\mathbf{A} = \mathbf{L} + \mathbf{H}$, and $\mathbf{L}$ explicitly depends on $K^2 \equiv v^2 k^2$, $L(\pm, \alpha) \equiv -(\tau^{-1} \pm \theta + \alpha)$, and $L(\pm, \beta) \equiv -(\tau^{-1} \pm \theta + \beta)$. Note that here we have used the 4×4 master Hamiltonian $\mathbf{H}$ of the Markovian four-state binary process [42], this allows to represent an intermittent binary noise with statistical weights (p, q) for each timescales $\{\alpha^{-1}, \beta^{-1}\}$. Then, ordering the set of values of ξ as $\{\xi_\alpha^+, \xi_\beta^+, \xi_\alpha^-, \xi_\beta^-\}$ we get

$$\mathbf{H} = \begin{pmatrix} -\alpha & 0 & \alpha p & \beta p \\ 0 & -\beta & \alpha q & \beta q \\ \alpha p & \beta p & -\alpha & 0 \\ \alpha q & \beta q & 0 & -\beta \end{pmatrix}. \tag{27}$$

In Eq. (14) we give the physical interpretation of $\mathbf{H}$ in terms of a renewal (biexponential) waiting-time function $\varphi(\Delta_{ij})$, in this case it is possible to see that the correlation $\langle \xi(t)\xi(t') \rangle$ is biexponential too [42]. As in the previous case Eq. (15), but now using a matrix $\mathbf{A} \in \mathcal{R}_e^{8\times 8}$ the general solution follows.

Second moment of $\langle P(x, t) \rangle$ in the presence of intermittent fluctuations

In the Laplace representation the solution of Eq. (26) is $\vec{\Psi}(s) = (s - \mathbf{A})^{-1} \cdot \vec{\Psi}(0)$, where we used, in accordance with the initial conditions Eq. (5), the vector $\vec{\Psi}(0) = (1/4, 0, 1/4, 0, 1/4, 0, 1/4, 0)$. The space dispersion (second-moment) of $\langle P(x, s) \rangle$, when the energy absorption has intermittent fluctuations is

$$\begin{aligned} \overline{x(s)^2} &= -\partial_k^2 \langle P(k, s) \rangle\big|_{k=0} = -\partial_k^2 \big(\psi_\alpha^+(k, s) + \psi_\beta^+(k, s) \\ &\quad + \psi_\alpha^-(k, s) + \psi_\beta^-(k, s)\big)\big|_{k=0} \\ &= 2(s^4 + f_3 s^3 + f_2 s^2 + f_1 s^1 + f_0)/ \\ &\quad \times (s^7 + g_6 s^6 + g_5 s^5 + g_4 s^4 + g_3 s^3 + g_2 s^2); \end{aligned} \tag{28}$$

constants f_n and g_n are given in Appendix C. From Eq. (28) a novel crossover from short to long-time behaviors can analytically be studied.

In Fig. 1 we have shown the transition from ballistic to Brownian transport. In this figure we have compared the crossover in the second-moment $\overline{x(t)^2}$ when the stochastic perturbation is a Markovian binary noise against the case when the binary noise is intermittent. Both plots have been obtained by Laplace inversion of the exact result Eq. (28). We note that taking the limit $\alpha \to \beta = 1/2T$ in Eq. (28) we get Eq. (22); that is, the exponential-correlated Markovian binary case [26].

In Fig. 2 we have plotted (for a fixed $\alpha = 1$) a characteristic transition point between the ballistic to diffusive regime, as a function of p and for different values of β. This timescale t_c is a possible way to characterize the delay of the entrance to the diffusive regime. We have calculated t_c numerically as the intersection of the ballistic and diffusive lines; that is the function $\overline{x(t)^2}$ in logarithm scale, see Fig. 1. As expected, values $p = 1$ and $p = 0$ $(q = 1)$ correspond to Markovian cases; however it is possible to see that t_c increases for small values of β. To fully understand the behavior of the entrance to the diffusive regime, it is convenient to calculate also the proportional constant in the temporal dispersion: $\overline{x(t)^2} \sim C_\infty\, t$; that is, the coefficient of diffusion. In the next sections a generic calculus of $\overline{x(t)^2}$, the VAF and the explicit calculation of the diffusion coefficient will be presented.

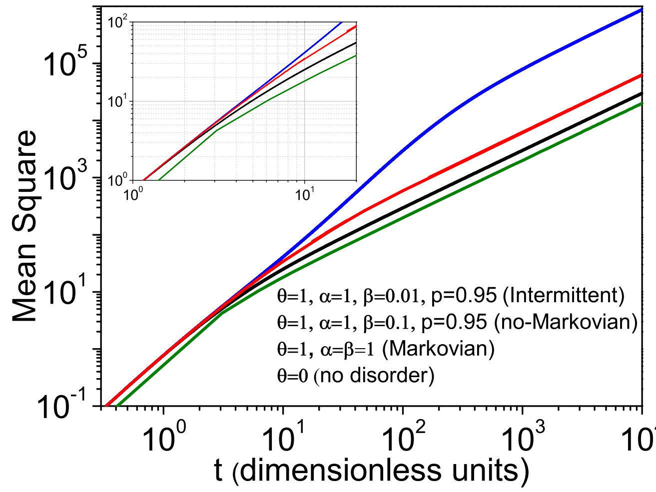

FIG. 1. Mean square $\overline{x(t)^2}$ of the PDF $\langle P(x, t) \rangle$ in dimensionless time $t \to t/\tau$ and space $x \to x/v\tau$ variables for Markovian and non-Markovian fluctuations $\xi(t)$ in Eq. (10). From the top to the bottom: blue line for the intermittent binary noise $(\alpha = 1, \beta = 0.01, p = 0.95, \theta = 1)$, red line for $(\alpha = 1, \beta = 0.1, p = 0.95, \theta = 1)$, black line for Markovian binary noise $(\alpha = \beta = 1, \theta = 1)$, and the nonperturbed case $\theta = 0$ (zero noise) in olive line. The inset shows a zoom at the beginning of the crossover around telegrapher's characteristic timescale $\tau = 1$. In general, a typical timescale t_c for the crossover can be seen from these plots; see Fig. 2.

C. Delayed Brownian dispersion in the mean value $\langle P(x, t) \rangle$

In general, if we are interested in the time-dependent second-moment of the mean-value solution, then we need to

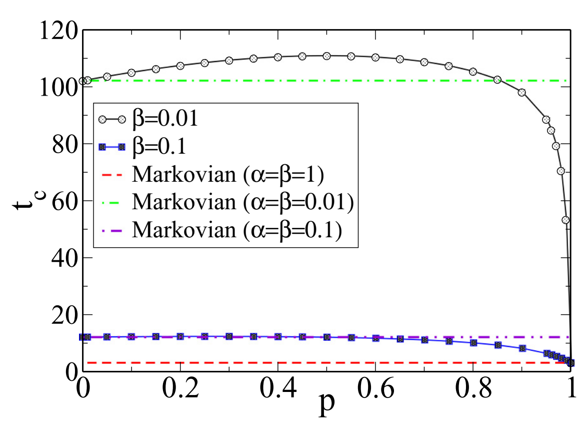

FIG. 2. The timescale t_c (in dimensionless time $t \to t/\tau$) as a function of p characterizing the transition point from ballistic to diffusive regime. Black circles correspond to the intermittent case (the dashed green line is the Markovian case with $\beta = 0.01$). Blue squares correspond to a typical non-Markovian case (the dashed magenta line is the Markovian case with $\beta = 0.1$). The red dashed line, in the bottom, corresponds to the Markovian case with $\alpha = \beta = 1$.

calculate

$$\overline{x(t)^2} = \int x^2 \langle\psi(x,t)\rangle dx = -\partial_k^2 \langle\psi(k,t)\rangle\big|_{k=0},$$
$$\langle\psi(x,t)\rangle \equiv \langle P(x,t)\rangle. \quad (29)$$

Denoting $\partial_k\phi(k,t) \equiv \phi'(t)$, $\partial_k^2\phi(k,t) \equiv \phi''(t)$, and $\partial_k\psi(k,t) \equiv \psi'(t)$, etc., and using that for each realization of $\xi(t)$ we know that $\psi(k,t)|_{k=0} = 1$, the second-moment (for a fixed $\xi(t)$) can be calculated by the following trick:

$$\partial_t\phi''(t) = -2v^2 - \tau^{-1}\phi''(t) - \theta\xi(t)\phi''(t), \quad \phi''(t=0) = 0, \quad (30)$$

then averaging $\phi''(t)$ over many realizations of the noise $\xi(t)$ we can numerically simulate the second-moment from the evolution equation:

$$\partial_t \overline{x(t)^2} = -\partial_t\big\langle \partial_k^2\psi(k,t)\big|_{k=0}\big\rangle \equiv -\partial_t\langle\psi''(t)\rangle = -\langle\phi''(t)\rangle. \quad (31)$$

The short- and long-time asymptotic solution shows the ballistic and Brownian behaviors, respectively; in Appendix D a scheme for tackling different noise's structures is also presented.

In addition to this numerical approach, we have plotted the *exact* second moment $\overline{x(t)^2}$, $\forall t \geqslant 0$ from Eqs. (22) and (28) after Laplace inversion [in dimensionless time and space: $t \to t/\tau$ and $x \to x/(\tau v)$]. In Fig. 1 we show the second moment of $\langle P(x,t)\rangle$. For the Markovian binary case the theoretical prediction is given by Eq. (22) when $\xi(t)$ is characterized by Eqs. (13) and (16). As can be seen, due to stochastic absorption of energy the timescale τ in the TE does not control any more the transition between the ballistic to the diffusive regime. Now there is a nontrivial crossover controlled by noise parameters: T, θ. In Fig. 1 we have used $\theta = \tau^{-1}(=1)$ corresponding to the strong disorder case. The theoretical prediction given in Eq. (22) shows, for example, that for timescales: $T \gg 1$ and $s \ll 1$ there can be a ballistic or a diffusive behavior depending on the following inequalities: if $2T \gg s^{-1} \Rightarrow (t \ll 2T)$, then it gets $\overline{x(t)^2} \simeq t^2/2$, while for $2T \ll s^{-1} \Rightarrow (t \gg 2T)$ it gets $\overline{x(t)^2} \simeq 2Tt$. However (using dimensional units $\tau = 1$) for timescales $T \ll 2^{-1}$ and $s \ll 1$ the behavior is just diffusive; that is, if $2T \ll 1$ with $s^2 \ll T^{-1} \Rightarrow (\sqrt{T} \ll t)$ we get $\overline{x(t)^2} \simeq 2t$. All these regimes are referred in Table 1 of Ref. [26].

In Fig. 1 we also have shown the second moment $\overline{x(t)^2}$, $\forall t \geqslant 0$ when the perturbation $\xi(t)$ is a non-Markovian binary noise. In this case, the global disorder $\xi(t)$ is characterized by Eqs. (13) and (14) with a characteristic mean waiting time, $\tau_{\rm WT} = \frac{p}{\alpha} + \frac{q}{\beta}$, and the theoretical prediction for $\overline{x(t)^2}$ is given by Eq. (28). In Fig. 1 we have also plotted the intermittent case corresponding to $\alpha \gg \beta$ (very different times-scales) with $1 > p \gg q$ (very different statistical weights). In addition, in the same Fig. 1 we have shown the ordered case (the rate of absorption of energy has not fluctuations).

Concerning the crossover between the ballistic to the diffusive regimes in the mean value $\langle P(x,t)\rangle$ (which is characterized by the timescale t_c, see Fig. 2) several conclusions can be drawn:

First, using non-Markovian binary fluctuations the noise produces an important delay in the entrance to the diffusive regime; that is, the ballistic behavior in the front $\langle P(x,t)\rangle$ is maintained for longer times. We recall that the intensity of the binary noise is $\theta \leqslant \tau^{-1}$, and the limit $\theta \to \tau^{-1}$ corresponds to strong disorder, which is the case plotted in Figs. 1 and 2.

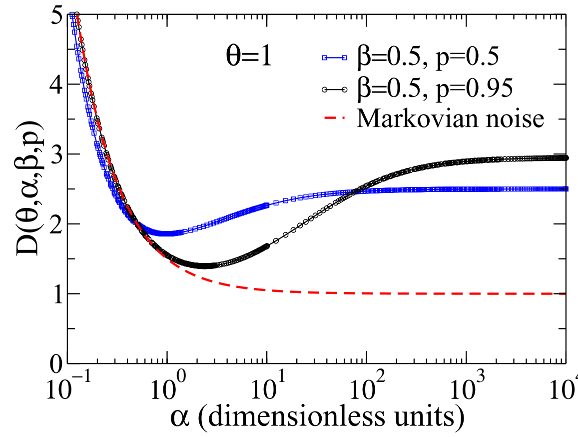

FIG. 3. Log-plot of the selfdiffusion coefficient $D(\theta,\alpha,\beta,p)$ in dimensionless units, when the intensity of the scattering has time fluctuations (strong disorder $\theta = \tau^{-1}$); see Eqs. (6) and (9). In the context of the TE approach this situation corresponds to consider strong global disorder in the absorption of energy, see Eq. (10). For: $\beta = 0.5$, $p = 0.95$ Circles (black line) and $\beta = 0.5$, $p = 0.5$ Squares (blue line), as a function of α. The bottom dashed line (red) corresponds to the Markovian binary noise in the strong disorder case, see Eqs. (34) and (35). The limit $\alpha \to 0$ corresponds to take an infinite correlation in the binary noise, or what is equivalent infinite mean waiting time: $\tau_{WT} = \frac{p}{\alpha} + \frac{q}{\beta}$; see Eq. (14).

Second, in the very short-time regime [such that $\overline{x(t)^2} \propto \mathcal{O}(t^2)$], the behavior does not show any difference when changing the structure of the noise, see Appendix D. While the long-time behavior strongly depends on the noise correlation.

Thus, in general we can conclude that any non-Markovian noise structure tends to preserve the ballistic regime in the front propagation, and for long time the renormalized diffusion coefficient strongly depends on the features of the noise, see Fig. 3.

D. The velocity autocorrelation function

In general, the VAF can be calculated from the Green-Kubo Eqs. (23) using Eq. (28). This is a formulation which is valid even when the global disorder shows intermittence. Then, the VAF can be obtained from an inverse Laplace transform. From this result, it is simple to see that the long-time behavior is exponential and its relaxation is controlled by the pole: g_2/g_3, which can be read from Appendix C. That is, in the long-time limit we can write

$$C(t\to\infty) = \lim_{t\to\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} C(s)e^{-st}ds$$
$$\sim \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f_0\, e^{-st}}{(g_3 s + g_2)}ds \propto \frac{f_0}{g_3}\, e^{\frac{-g_2}{g_3}t}. \quad (32)$$

The asymptotic result $\sim e^{-t/\tau_{\rm VAF}}$ is well understood because the disorder is global; this would not be the case if the disorder in the Lorentz scattering were local. We can see that there is a slowing down in the relaxation of the VAF if the timescale $\tau_{\rm VAF} \equiv g_3/g_2 \to \infty$, but this can only happens in the *extreme* limit: $\alpha \to 0$ with $\beta \neq 0$ (or equivalently for $\beta \to 0$ with $\alpha \neq 0$), we recall that the mean waiting time for the non-Markovian noise is $\tau_{\rm WT} = \frac{p}{\alpha} + \frac{q}{\beta}$, and the intermittent limit corresponds to take $\{\alpha \gg \beta, 1 > p \gg q = 1 - p\}$; see Eq. (14).

The renormalized diffusion coefficient follows from the $\lim_{s\to 0} C(s)$; that is, using Eqs. (23) and (28) we get

$$D = \frac{f_0}{g_2}. \tag{33}$$

In general, quantities f_0 and g_2 can analytically be read from the mean-value Green function $\langle P(k, s)\rangle$; see Eq. (B10) for the Markovian and Eq. (B13) for the non-Markovian (biexponential) stochastic absorption of energy.

To simplify the analysis we adopt dimensionless units [time: $t \to t/\tau$ and space: $x \to x/(\tau v)$]; therefore, $D = D(\theta, \alpha, \beta, p)$. In Appendix C the expressions f_0 and g_2 are given (for $\tau = v = 1$), also note that $q = 1 - p$. In addition, the following limits can be readily checked: the ordered case $D(\theta = 0, \alpha, \beta, p) = D_{\rm TE} = 1$ and Markovian disordered limits; see Eq. (25) in dimensionless units with $T = 1/2\alpha = 1/2\beta$:

$$D(\theta, \alpha, \beta, 1) = D(\theta, \alpha, \alpha, p) = \frac{1 + 2\alpha}{1 + 2\alpha - \theta^2}, \tag{34}$$

$$D(\theta, \alpha, \beta, 0) = D(\theta, \beta, \beta, p) = \frac{1 + 2\beta}{1 + 2\beta - \theta^2}. \tag{35}$$

Intermittent absorption of energy corresponds to take $\{\alpha \gg \beta, 1 > p \gg q\}$ or $\{\alpha \ll \beta, 0 < p \ll q\}$. In Fig. 3 we have plotted $D(\theta, \alpha, \beta, p)$, for strong disorder and for different values of parameters: $\{\alpha, \beta, p\}$.

VI. FLUCTUATIONS OF THE STOCHASTIC TELEGRAPHER'S FIELD $P(x, t)$

To study the fluctuations in the stochastic field $P(x, t)$, governed by Eq. (10) or by Eqs. (11) and (12), here we propose to analyze the dispersion of the stochastic Fourier mode $P(k, t)$; that is, $\sigma_P^2(k, t) = \langle P(k, t)^2\rangle - \langle P(k, t)\rangle^2$. This quantity teaches us, over time, how good of a statistical object is the mean value $\langle P(k, t)\rangle$ to characterize the field $P(k, t)$. For strong (global) disorder ($\theta = \tau^{-1}$) we have plotted in Figs. 4 and 5 the fluctuations $\sigma_P^2(k, t)$ as a function of time for several values of the Fourier wave number k, for both Markovian and non-Markovian binary noises $\xi(t)$. These fluctuations have been calculated by solving, numerically, the coupled equations presented in Appendix B 2 for the Markovian and non-Markovian binary noise cases. As expected, wild oscillations can be seen for large Fourier numbers. Comparing these dispersions we see that a non-Markovian noise induces prolonged fluctuation dispersions when compared with the Markovian case. For both binary noises the exact calculation follows straightforwardly using the enlarged master equation approach; see Appendix B 2.

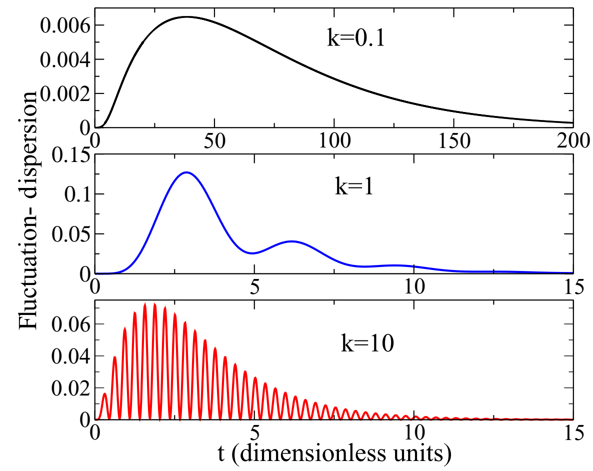

FIG. 4. Fluctuation dispersion $\sigma_P^2(k, t)$ for three values of k, as a function of dimensionless time and space ($t \to t/\tau, k \to v\tau k$) when the stochastic absorption of energy in Eq. (10) is the (Markov) binary noise Eq. (13) with intensity $\theta = 1$ and mean waiting time $a = 1$.

Fluctuation dispersion $\sigma_P^2(k, t)$ in the presence of (global) binary disorder

For the Markovian binary disorder, solving Eqs. (B4)–(B9) and putting these results in Eq. (B12) we get the fluctuation dispersion of the stochastic telegrapher's field. As can be seen for any value of the Fourier wave number k the dispersion goes to zero for $t \to \infty$. Similarly, using Eqs. (B13) and (B14) we can work out the case when the binary disorder has two timescales (leading to intermittence), see Eq. (14). In Fig. 4 we have shown $\sigma_P^2(k, t)$ when the stochastic absorption of energy has exponential-correlated fluctuations, while in Fig. 5 we have shown $\sigma_P^2(k, t)$ when $\xi(t)$ has nonexponential

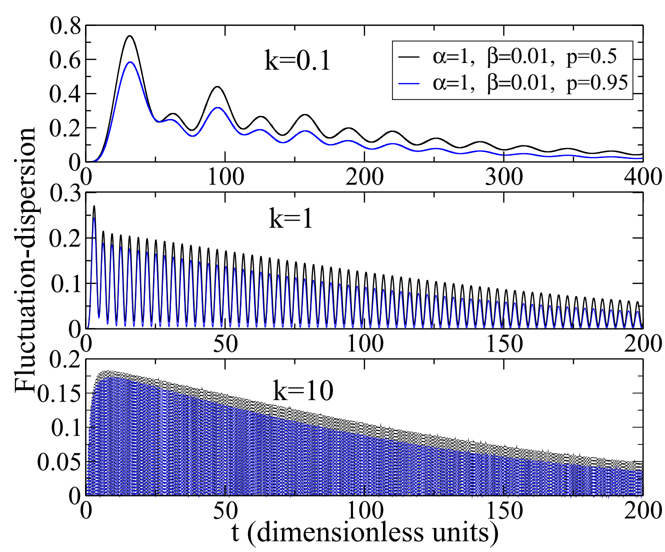

FIG. 5. Fluctuation dispersion $\sigma_P^2(k, t)$ for three values of k, as a function of dimensionless time and space ($t \to t/\tau, k \to v\tau k$), when the stochastic absorption of energy in Eq. (10) is the non-Markovian binary noise Eq. (13) with intensity $\theta = 1$ and waiting-time function Eq. (14), with parameters: from top to the bottom, black-line ($\alpha = 1, \beta = 0.01, p = 0.5$), blue-line ($\alpha = 1, \beta = 0.01, p = 0.95$).

correlated fluctuations. The intermittent case corresponds to use parameters: $\alpha \gg \beta$ and $1 > p \gg q$, for comparison we also plot the case $\alpha \gg \beta$ with $p = q$. From these figures it is possible to see that the scenario is similar for all cases of binary disorder, while for the Markovian binary noise these fluctuations are 100 times smaller than for the intermittent case. In general, the dispersion starts at zero value (the initial condition is certain) has a maximum (at some k^*, t^*) and decays to zero at long time when the distribution reaches the unique zero distribution $P(x, t \to \infty) \to 0$.

Interestingly, these analysis show that apart from the neighborhood of space $\sim 1/k^*$ at time $\sim t^*$ scales (which depends on noise's correlation) the mean value $\langle P(k, t)\rangle$ is a good statistical object to represent the stochastic field $P(k, t)$; which in addition represents the mean-value propagation of the random Boltzmann-Lorenz' gas. To show the difference between an intermittent and a nonintermittent noise [42] in Fig. 5 the fluctuation dispersion $\sigma_P^2(k, t)$ is shown for different values of waiting-time parameters α, β and weight p.

In a different context, we note that a dissipative system driven by this type of biexponential binary noise can show an unusual behavior like *dynamical locking of the realizations*, which is not related to the intermittence of the noise, while it is related to the interplay between the stochastic realization and the nature of the discrete noise having two characteristic timescales [46]. Thus, in the present context of hyperbolic diffusion, it is important to account for if the mean value of the field is, indeed, a good statistical object. Our results support the conclusion that the mean value of the stochastic field $P(k, t)$ is a good statistical object to characterize a transport problem in the presence of stochastic absorption of energy.

VII. CONCLUSIONS

From a mathematical viewpoint it seems of interest to have some soluble strongly non-Markovian transport process, this is the case of Lorentz gas with disorder. We have calculated the *exact* mean-value space distribution of the 1D random Boltzmann-Lorentz gas considering different models of fluctuations in the scattering processes. To tackle this problem first we have reduced the random integrodifferential equation for the Boltzmann's distribution $f(x, v, t)$, to a set of coupled stochastic hyperbolic PDEs for the space distribution $P(x, t) = f(x, v, t) + f(x, -v, t)$ and the associated current $vQ(x, t) = v[f(x, v, t) - f(x, -v, t)]$. Then, we have shown that for *local* disorder (random location of scatterer centers) the set of PDEs is coupled, while for *global* disorder (equal time-fluctuating intensity in any site x) the set of PDEs is decoupled. These PDEs can be called stochastic telegrapher's equations.

Using different correlated binary noises (exponential and biexponential), exact mean-value results for global disorder have been found from the space distribution of the stochastic $P(x, t)$. In this manner we have disentangled the *form* of the exponential-correlation from the intrinsic structure of the Markovian binary noise. Thus, we have studied the mean value of the telegrapher's profile $\langle P(x, t)\rangle$ when it is perturbed with Markovian and non-Markovian noises in the absorption of energy. From the mean value $\langle P(x, t)\rangle$ we have calculated, analytically, the dispersion $\overline{x(t)^2}$, showing a large crossover between the ballistic to the Brownian regime. We have shown that the timescale of separation t_c in this crossover is controlled by the correlation structure of the global disorder. In particular, intermittent absorption of energy produces a delay in converging to the diffusive regime, this has been shown from the analysis of the second-moment of $\langle P(x, t)\rangle$. Then we have proved that the 1D Boltzmann-Lorentz gas with global disorder has convergence to Brownian motion after the characteristic timescale t_c. In the ordered case this timescale is $t_c = \tau \equiv 1/2q_0$, which is controlled by the intensity of the scattering function Eq. (1), while in the presence of intermittent fluctuations it can be $t_c \gg \tau$; see Figs. 1 and 2. Then, for short timescales $t \ll t_c$ the transport is ballistic, while for long timescales $t \gg t_c$ it is diffusive. All these conclusions have been proved from the exact mean-value solution of a stochastic hyperbolic PDE (our TE picture). Thus, our approach gives an interpretation of the stochastic Botzmann-Lorentz gas in terms of a diffusionlike process with a finite-velocity restriction for the evolution of the PDF $P(x, t)$ and the current $vQ(x, t)$.

Using the Green-Kubo formula, the velocity autocorrelation function (long-time behavior) and the coefficient of selfdiffusion have analytically been calculated, showing its dependence with noise parameters; see Fig. 3. In particular the renormalized diffusion coefficient is shown to have a nonmonotonic behavior as a function of α for different values of p; as well as the relaxation timescale τ_{VAF} of the VAF is shown to have a slowing-down in the extreme limit when the waiting time of the noise diverges $\tau_{\mathrm{WT}} \to \infty$.

When the rate $q = q(t)$ has time fluctuations (global disorder), we have shown that the fluctuation dispersion of the field $\sigma_P^2(k, t)$ may be a function both of k and time t, but except for space scale near $\sim 1/k^*$ and timescale near t^* we need not concern ourselves with notable fluctuations of the stochastic Fourier mode $P(k, t)$; see Figs. 4 and 5. That is, $\sigma_P^2(k, t)$ has a small global maximum at $\{k^*, t^*\}$ and these values strongly depend on the noise correlation-function. We can conclude that $\langle P(k, t)\rangle$ is a good statistical object.

In recent literature, generalizations of the TE have attracted interest in describing flagellalike motion in Biophysics [36]; in particular, when the dynamics of the swimmers is described by run-and-tumble models [25]. Nevertheless, a general relationship between tumbling rate and swim speed, for different microorganisms, is still an open question. Our results on delayed diffusion are concerned with fluctuations in the effective tumbling rate (absorption of energy in the TE); see Sec. III A 2. Therefore, we expect that mapping our results to Biophysics can be of interest with generic applications.

Extending the present approach to tackle other models of global fluctuations, for example, using a Poisson' noise [47], is an open question that merits study to account for the relevance of the noise structure in the response of the system. Local disorder in the Lorentz's model can also be worked out using our approach; nevertheless, in this case we have to solve the coupled set of hyperbolic Eqs. (A13) and (A14) with some effective medium approximation [18,39,48].

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ACKNOWLEDGMENTS

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M.O.C. conceived and designed the study, performed the analytical calculations, and wrote the paper. M.N. contributed analysis tools and did the numerical calculations. M.O.C. and M.N. had final approval of the version of the paper to be published.

APPENDIX A: RANDOM SCATTERER CENTERS IN THE 1D LORENTZ MODEL

Consider the generalized 1D Lorentz scattering function:

$$W(x,t,v,v') = q(ax+bt)\,\delta(v+v'),$$
$$q(ax+bt) \geqslant 0, t \geqslant 0, x \in \mathcal{R}_e, \forall\{a,b\} \in \mathcal{R}_e. \quad \text{(A1)}$$

In this model, "point" scatterer centers are located at any site x where Boltzmann collision operator is applied; nevertheless, the scattering intensity may depend on space and time. Therefore, $q(ax+bt)$ is a generic function [in the stochastic case the following analysis applied for a fixed realization of the process $q(ax+bt)$]. In 1D, a function $q=q(x)$ can represent a space-dependent *intensity* (emulating local disorder), while in the case $q=q(t)$ there is a time-variable *intensity* (equal) for all scatterer centers located at any x (global disorder), for example, due to some screening. Note that $q=q(t)$ does not emulate any (thermal) time-varying border of a finite-size scatterer center [32,49], this is so because Lorentz scattering Eq. (A1) is a point model.

Introducing this scattering function in Eq. (2), the collision operator turns out to be

$$\left(\frac{df}{dt}\right)_{\text{Coll}} = q(ax+bt)f(x,-v,t) - q(ax+bt)f(x,v,t)$$
$$\equiv q\cdot(f_- - f_+), \quad \text{(A2)}$$

where we have used a short notation in the last equality (note that $f=f_+$).

Because v appears as a free parameter, we can write the Boltzmann-Lorentz Eq. (2) as

$$(\partial_t + v\partial_x)f = q\cdot(f_- - f_+). \quad \text{(A3)}$$

Taking the time derivative in this equation we get

$$\left(\partial_t^2 + v\partial_t\partial_x\right)f = bq'\cdot(f_- - f_+) + q\cdot(\dot{f}_- - \dot{f}_+), \quad \text{(A4)}$$

where $bq' = \partial_t q(ax+bt)$ and $\dot{f}_\pm \equiv \partial_t f_\pm$. From Eq. (A3) we can solve $\partial_t f_\pm$ as

$$\partial_t f = -v\partial_x f + q\cdot(f_- - f_+).$$

Therefore, Eq. (A4) can be written in the form:

$$\begin{aligned}\partial_t^2 f &= -v\partial_t\partial_x f + bq'(f_- - f_+) + q(\dot{f}_- - \dot{f}_+)\\ &= -v\partial_x[-v\partial_x f + q(f_- - f_+)] + bq'(f_- - f_+) + q(\dot{f}_- - \dot{f}_+)\\ &= v^2\partial_x^2 f - v\partial_x[q(f_- - f_+)] + bq'(f_- - f_+) + q(\dot{f}_- - \dot{f}_+)\\ &= v^2\partial_x^2 f - v[aq'(f_- - f_+) + q(f'_- - f'_+)] + bq'(f_- - f_+) + q(\dot{f}_- - \dot{f}_+),\end{aligned} \quad \text{(A5)}$$

where $aq' = \partial_x q(ax+bt)$ and $f'_\pm \equiv \partial_x f_\pm$. Collecting terms we can write

$$\left(\partial_t^2 - v^2\partial_x^2\right)f_+ = (-vaq' + bq')(f_- - f_+)$$
$$- vq(f'_- - f'_+) + q\left(\dot{f}_- - \dot{f}_+\right); \quad \text{(A6)}$$

introducing the replacement $v \to -v$ in this equation we get

$$\left(\partial_t^2 - v^2\partial_x^2\right)f_- = -(vaq' + bq')(f_- - f_+) - vq(f'_- - f'_+)$$
$$- q(\dot{f}_- - \dot{f}_+). \quad \text{(A7)}$$

Using the notation: $P(x,t) = (f_+ + f_-)$ and $Q(x,t) = (f_+ - f_-)$ from Eqs. (A6) and (A7) we get the set of equations

$$\left(\partial_t^2 - v^2\partial_x^2\right)P(x,t) = 2vaq'Q(x,t) + 2vqQ'(x,t), \quad \text{(A8)}$$

$$\left(\partial_t^2 - v^2\partial_x^2\right)Q(x,t) = -2bq'Q(x,t) - 2q\dot{Q}(x,t), \quad \text{(A9)}$$

where $Q' \equiv \partial_x Q$ and $\dot{Q} \equiv \partial_t Q$. Noting from the conservation law (in 1D) that $\partial_t P = -v\partial_x Q(x,t)$ we can write

$$\left(\partial_t^2 + 2q\partial_t - v^2\partial_x^2\right)P(x,t) = 2vaq'Q(x,t), \quad \text{(A10)}$$

$$\left(\partial_t^2 + 2q\partial_t - v^2\partial_x^2\right)Q(x,t) = -2bq'Q(x,t), \quad \text{(A11)}$$

to be solved with suitable initial conditions, where we use the notation $q \equiv q(ax+bt)$ and $q' = dq(z)/dz$. From this general result we can analyze different cases.

1. The ordered case

In this limit there is not local disorder nor time-fluctuations in the intensity of the scattering, thus $q = q_0 =$ constant; therefore, from Eqs. (A10) and (A11) we get Eq. (4) and the same equation for $Q(x,t)$.

2. Local disorder in scatterer centers

In the case when the intensity is modified at random positions we propose the scattering function to be

$$W(x,v,v') = q(x)\,\delta(v+v'),$$
$$q(x) \geqslant 0, \quad \forall x \in (-\infty,+\infty), \quad \text{(A12)}$$

where $q(x)$ may represent a heterogeneous medium or a random function. This case corresponds to take $a=1, b=0$ in the collision operator Eq. (A2). Thus, from Eqs. (A10) and (A11) we get the coupled set of equations

$$\left[\partial_t^2 + 2q(x)\partial_t - v^2\partial_x^2\right]P(x,t) = 2vq'(x)Q(x,t), \quad \text{(A13)}$$

$$\left[\partial_t^2 + 2q(x)\partial_t - v^2\partial_x^2\right]Q(x,t) = 0, \quad \text{(A14)}$$

to be solved with initial conditions Eq. (5), where we use notation $q'(x) \equiv dq(x)/dx$. These expressions generalize the previous homogeneous case of Sec. II A.

We noted that Eq. (A13) is different to the one that is gotten when the persistent random walk or the TE, has a *random velocity* of propagation $v(x)$ [18,45]. This means that local disorder in $v(x)$ at the level of the phenomenological Cattaneo-Fick equation is not equivalent to a mesoscopic random scattering model at the level of the Boltzmann-Lorentz equation. Using an effective medium approximation the analysis of Eqs. (A13) and (A14) will be presented in a forthcoming paper.

3. Global disorder: Time-fluctuating scatterer centers

In the case when the intensity of the scattering centers depends on time we take $q = q(t)$, this corresponds to take $a = 0, b = 1$ in the collision operator Eq. (A2). Thus, from Eqs. (A10) and (A11) we get an independent set of equations

$$\left(\partial_t^2 + 2q\partial_t - v^2\partial_x^2\right)P(x,t) = 0, \quad \text{(A15)}$$

$$\left(\partial_t^2 + 2q\partial_t + 2q' - v^2\partial_x^2\right)Q(x,t) = 0, \quad \text{(A16)}$$

from which Eqs. (7) and (8) follow.

APPENDIX B: ENLARGED MASTER EQUATION FOR MARKOV GLOBAL DISORDER

1. General approach

The second order differential Eq. (10) can be written as a linear system for $X_1 = P \equiv \psi$ and $X_2 = \partial_t P \equiv \phi$, see also Eqs. (11) and (12); that is, in the Fourier representation it gets

$$\frac{d}{dt}X_i = \sum_l \mathbf{M}_{il}[\xi(t)]X_l, \quad \text{where} \quad \mathbf{M}[\xi(t)] \in \mathcal{R}_e^{2\times 2},\ t \geqslant t_0, \quad \text{(B1)}$$

with $\{X_i(t_0)\}$ = initial conditions $(i = 1, 2)$.

If the stochastic process $\xi(t)$ is Markovian, then there exists a semigroup associated to its conditional probability $\Pi_t(\xi|\xi')$; therefore, there is a master Hamiltonian $\mathbf{H}$ such that $\frac{d}{dt}\Pi_t(\xi|\xi') = \sum_{\xi''}\mathbf{H}_{\xi\xi''}\Pi_t(\xi''|\xi')$ (if $\xi(t)$ is continuous $\mathbf{H}$ is the Fokker-Planck operator). Although the process $\vec{X}(t)$ defined by Eq. (B1) is itself not Markovian, it can be considered as a projection of the Markovian process $[\vec{X}(t), \xi(t)]$; in this section the process $\xi(t)$ will be considered discrete with r states.

In general, for a vector $\vec{X}$ of dimension n, the Markovian character of a $n+1$ dimensional process is due to the Markovian character of process $\xi(t)$ and to the fact that for a given realization the solution of Eq. (B1); that is, $\vec{X}(t|t_0)$ depends on the values of $\xi(\tau)$ only in the time interval $t_0 \leqslant \tau \leqslant t$. As the enlarged process $[\vec{X}(t), \xi(t)]$ is Markovian, one can write a master equation for the joint probability $\mathcal{P}(\{X_i\}, \xi, t)$, which varies in time owing to the flow in X -space and jumps of ξ in an enlarged Liouville's phase-space [39,43,44]:

$$\partial_t \mathcal{P}(\{X_i\},\xi,t) = -\sum_{jl}\partial_{X_j}\mathbf{L}_{jl}(\xi)X_l\mathcal{P}(\{X_i\},\xi,t) + \sum_{\xi'}\mathbf{H}_{\xi\xi'}\mathcal{P}(\{X_i\},\xi',t),\ t \geqslant t_0. \quad \text{(B2)}$$

The initial condition in Eq. (B2) is taken in the stationary ensemble of process $\xi(t)$; that is: $\mathcal{P}(\{X_i\},\xi,t_0) = \Pi_{t_0}(\xi)\delta[X_i - X_i(t_0)]$.

The embedding matrices in the binary case are: $\mathbf{L}(\xi)$ and master Hamiltonian $\mathbf{H}$, these can be read from Eq. (15); that is, $\mathbf{A} = [\mathbf{L}(\xi) + \mathbf{H})] \in \mathcal{R}_e^{4\times 4}$.

If only the mean value of X_m is wanted, then we can go one step further and calculate the *marginal* average:

$$X_m(\xi,t) = \iint dX_1\cdots dX_n\ \mathcal{P}(\{X_i\},\xi,t)X_m. \quad \text{(B3)}$$

Thus, if Eq. (B2) is multiplied by X_m and integrated over all the variables X_j, then we get

$$\partial_t X_m(\xi,t) = \sum_j \mathbf{L}_{mj}(\xi)X_j(\xi,t) + \sum_{\xi'}\mathbf{H}_{\xi\xi'}X_m(\xi',t). \quad \text{(B4)}$$

Identifying $X_1(+,t) = \psi^+, X_2(+,t) = \phi^+, X_1(-,t) = \psi^-, X_2(-,t) = \phi^-$ we immediately get Eq. (15) for the case when $\xi(t)$ is the binary process characterized by the exponential correlation function Eq. (16). For example, the mean value of $\psi \equiv P$ over all noise realizations is just

$$\langle\psi\rangle \equiv \langle X_1(t)\rangle = \sum_\xi X_1(\xi,t) = \psi^+ + \psi^-, \quad \text{(B5)}$$

from which Eq. (17) follows.

Higher-order moments can also be worked out in a similar way. For example, defining the matrix

$$X_{ml}(\xi,t) = \iint dX_1\cdots dX_n\ \mathcal{P}(\{X_i\},\xi,t)\,X_mX_l, \quad \text{(B6)}$$

note that $X_{ml}(\xi,t) = X_{lm}(\xi,t)$. A cross moment can be studied multiplying Eq. (B2) by X_mX_l; then, after integrating by part and some algebra we get

$$\partial_t[X_{ml}(\xi,t)] = \sum_j \mathbf{L}_{mj}(\xi)X_{lj}(\xi,t) + \sum_j \mathbf{L}_{lj}(\xi)X_{mj}(\xi,t) + \sum_{\xi'}\mathbf{H}_{\xi\xi'}X_{ml}(\xi',t). \quad \text{(B7)}$$

In general, for $\{\vec{X}\} \equiv \{X_1\cdots X_n\}$ and the possible set of outcomes $\xi \in \{\xi_1\ldots\xi_r\}$, after solving Eq. (B7) we can write the mean value as

$$\langle X_{ml}(t)\rangle = \iint dX_1\cdots dX_n \sum_\xi \mathcal{P}(\{X_i\},\xi,t)\,X_mX_l = \sum_\xi X_{ml}(\xi,t); \quad \text{(B8)}$$

this is the key formula that will be used to calculate the fluctuation dispersion of the field $P(k,t)$.

2. Fluctuations in the stochastic telegrapher's field

a. Markovian binary noise

Following Sec. IV A, for the Markovian binary noise case, fluctuations of the Fourier mode $P(k,t)$ can be characterized by moments [using Eqs. (B7) and (B8) and $\xi=\pm 1$]

$$\partial_t[X_{mm}(\xi,t)] = 2\sum_j \mathbf{L}_{mj}(\xi)X_{mj}(\xi,t) + \sum_{\xi'}\mathbf{H}_{\xi\xi'}X_{mm}(\xi',t); \tag{B9}$$

these linear equations can be solved for $X_{mm}(\xi,t)$.

Noting that $\langle P(k,t)\rangle = X_1(+,t)+X_1(-,t)$ and $\langle\Phi(k,t)\rangle = X_2(+,t)+X_2(-,t)$ we write from Eq. (B5)

$$\langle P(k,t)\rangle = \sum_{\xi'} X_1(\xi',t), \quad \text{and} \quad \langle\Phi(k,t)\rangle = \sum_{\xi'} X_2(\xi',t), \tag{B10}$$

and from Eq. (B8) we get

$$\langle P(k,t)^2\rangle = \sum_{\xi'} X_{11}(\xi',t), \text{ and } \langle\Phi(k,t)^2\rangle = \sum_{\xi'} X_{22}(\xi',t). \tag{B11}$$

Thus, for the fluctuations of the field $P(k,t)$ we finally get

$$\sigma_P^2(k,t) = \sum_{\xi'} X_{11}(\xi',t) - \sum_{\xi'} X_1(\xi',t)\sum_{\xi''} X_1(\xi'',t). \tag{B12}$$

b. Intermittent binary noise

When the binary noise $\xi(t)$ in Eq. (10) has a biexponential waiting time [42], the mean value $\langle P(k,t)\rangle$ can also be studied in the context of an enlarged master equation. Thus, Eqs. (B9) and (B4) can also be used for the *intermittent* case. Embedding matrices are now $\mathbf{L}(\xi)\in\mathcal{R}_e^{8\times 8}$ and $\mathbf{H}\in\mathcal{R}_e^{8\times 8}$, which can be read from Eq. (26), in accordance with Eqs. (27) and (14) (the intermittent limit corresponds to $\alpha\ll\beta$ with $1>p\gg q$). In this case the noise $\xi(t)$ has four states, $\xi\in\{\xi_\alpha^+,\xi_\beta^+,\xi_\alpha^-,\xi_\beta^-\}$, and the fluctuations are characterized by the mean value

$$\langle P(k,t)\rangle = \sum_{\xi'} X_1(\xi',t) = \psi_\alpha^+(k,t)+\psi_\beta^+(k,t) + \psi_\alpha^-(k,t)+\psi_\beta^-(k,t), \tag{B13}$$

and the square fluctuation

$$\langle P(k,t)^2\rangle = \sum_{\xi'} X_{11}(\xi',t) = \psi_\alpha^+(k,t)^2+\psi_\beta^+(k,t)^2 + \psi_\alpha^-(k,t)^2+\psi_\beta^-(k,t)^2. \tag{B14}$$

Thus, for both type of binary noises exact expressions to calculate $\sigma_P^2(k,t)\equiv\langle P(k,t)^2\rangle - \langle P(k,t)\rangle^2$ are available.

APPENDIX C: SECOND MOMENT OF THE MEAN-VALUE $\langle P(x,t)\rangle$

When the noise $\xi(t)$ in Eq. (10) is biexponential correlated, the second-moment $\overline{x(s)^2}$ of $\langle P(x,s)\rangle$ is given in Eq. (28). Using dimensionless time and space variables [$t\to t/\tau$ and $x\to x/(\tau v)$], the constants in the numerator are

$$\begin{aligned}
f_0 &= \alpha+2\alpha\beta+2\alpha^2q+4\alpha^2\beta q+\alpha^3q^2+2\alpha^3\beta q^2-\alpha p+\beta p+2\beta^2p+6\alpha\beta^2p-\alpha^2\beta qp+4\alpha^2\beta^2qp\\
&\quad+\alpha^3q^2p-5\alpha\beta^2p^2+2\alpha\beta^3p^2+\beta^3(1+q)p^2+3\alpha^2\beta qp^2+3\alpha\beta^2p^3-\alpha\theta^2-2\alpha\beta\theta^2+(\alpha-\beta)p\theta^2,\\
f_1 &= 1+5\alpha+2\beta+\alpha^2\beta q(5+q)+\alpha^3q^2(1+p)+\alpha\beta^2p(3+5q)+\beta^2(5+q)p+(\alpha-\beta)p(\theta^2-3)\\
&\quad+\beta^3(1+q)p^2+3\alpha^2\beta qp^2+3\alpha\beta^2p^3+2\alpha\beta(3+q+p^2)-(1+2\alpha+\beta)\theta^2+\alpha^2q(5+p),\\
f_2 &= 3+7\alpha+4\beta-3\alpha p+3\beta p+\beta^2(3+q)p+\alpha^2q(3+p)+2\alpha\beta(2+q+p^2)-\theta^2,\\
f_3 &= 3+3\alpha+2\beta-\alpha p+\beta p,
\end{aligned}$$

and in the denominator they are

$$\begin{aligned}
g_2 &= \alpha q+2\alpha^2q+2\alpha\beta q+4\alpha^2\beta q+\alpha^3q^2+2\alpha^3\beta q^2+\beta p+2\alpha\beta p+2\beta^2p+4\alpha\beta^2p-\alpha^2\beta qp+4\alpha^2\beta^2qp\\
&\quad+\alpha\beta^2(1+q)qp+\alpha^3q^2p-2\alpha\beta^2p^2+2\alpha\beta^3p^2+\beta^3(1+q)p^2+3\alpha^2\beta qp^2+2\alpha\beta^2p^3-2\alpha q\theta^2\\
&\quad-2\alpha^2q\theta^2-2\alpha\beta q\theta^2-2\beta p\theta^2-2\alpha\beta p\theta^2-2\beta^2p\theta^2+2\alpha^2\beta qp\theta^2-\alpha\beta^2(1+q)qp\theta^2+2\alpha\beta^2p^2\theta^2\\
&\quad-\beta^3(1+q)p^2\theta^2-2\alpha^2\beta qp^2\theta^2-2\alpha\beta^2p^3\theta^2-\alpha^3q^2(1+p)\theta^2-\alpha^2\beta qp(1+p)\theta^2+\alpha q\theta^4+\beta p\theta^4,\\
g_3 &= 1+2\alpha+2\beta+4\alpha\beta+4\alpha q+7\alpha^2q+6\alpha\beta q+10\alpha^2\beta q+2\alpha^3\beta q^2+4\beta p+4\alpha\beta p+6\beta^2p+10\alpha\beta^2p\\
&\quad+\beta^2(1+q)p+\alpha^2qp-4\alpha^2\beta qp+4\alpha^2\beta^2qp+2\alpha\beta^2(1+q)qp+2\alpha\beta p^2-4\alpha\beta^2p^2+2\alpha\beta^3p^2\\
&\quad+2\beta^3(1+q)p^2+4\alpha^2\beta qp^2+4\alpha\beta^2p^3+2\alpha^3q^2(1+p)+2\alpha^2\beta qp(1+p)-2\theta^2-2\alpha(1+2q)\theta^2\\
&\quad-2\alpha^2q\theta^2-2\alpha\beta q\theta^2-2\beta(1+2p)\theta^2-2\beta^2p\theta^2-\beta^2(1+q)p\theta^2-2\alpha\beta p^2\theta^2-\alpha^2q(1+p)\theta^2+\theta^4,\\
g_4 &= 4+6\alpha+6\beta+8\alpha\beta+6\alpha q+6\alpha^2q+6\alpha\beta q+6\alpha^2\beta q+6\beta p+2\alpha\beta p+6\beta^2p+6\alpha\beta^2p+2\beta^2(1+q)p\\
&\quad-2\alpha^2\beta qp+\alpha\beta^2(1+q)qp+4\alpha\beta p^2-2\alpha\beta^2p^2+\beta^3(1+q)p^2+2\alpha^2\beta qp^2+2\alpha\beta^2p^3+2\alpha^2q(1+p)\\
&\quad+\alpha^3q^2(1+p)+\alpha^2\beta qp(1+p)-4\theta^2-2\alpha\theta^2-2\beta\theta^2-2\alpha q\theta^2-2\beta p\theta^2,\\
g_5 &= 6(1+\alpha+\beta)+4\alpha\beta+4\alpha q+2\alpha^2q+2\alpha\beta q+4\beta p+2\beta^2p+\beta^2(1+q)p+2\alpha\beta p^2+\alpha^2q(1+p)-2\theta^2,\\
g_6 &= 4+2\alpha+2\beta+\alpha q+\beta p,
\end{aligned}$$

APPENDIX D: STOCHASTIC REPRESENTATION OF THE SECOND MOMENT

For a fixed realization of $\xi(t)$ the general solution of Eq. (30) is

$$\phi''(t) = \phi''(0)e^{-\int_0^t(\tau^{-1}+\theta\xi(z))dz} - 2v^2\int_0^t e^{-\int_{t'}^t[\tau^{-1}+\theta\xi(z)]dz}dt',\ t \geqslant 0; \quad \text{(D1)}$$

here we are interested in the initial conditions $P(x,t)|_{t=0} = \delta(x)$ and $\partial_t P(x,t)|_{t=0} = 0$, thus $\phi''(0) \equiv \phi''(k, t=0)|_{k=0} = \partial_k^2\partial_t\psi(k, t=0)|_{k=0} = 0$. The short-time behavior of $\phi''(t)$ follows from Taylor's series around $t \sim 0$, and the long-time behavior from the limit $t \to \infty$. Thus, from Eq. (D1) we get

$$\phi''(t \sim 0) = -2v^2 t + \mathcal{O}(t^2) + \cdots,$$

and

$$\phi''(t \to \infty) \sim -2v^2\tau(1+\cdots) = \text{ const.} \quad \text{(D2)}$$

Therefore, using Eq. (31) we get for the asymptotic second moments $\overline{x(t\to 0)^2} \propto t^2$ and $\overline{x(t\to\infty)^2} \propto t$. Higher corrections can be calculated using the statistics of the integrated noise [39,47]:

$$dY(t) = \xi(t)dt,\ t \geqslant 0; \quad \text{(D3)}$$

note that $Y(t)$ is a nonstationary process.

Taking the mean value in Eq. (D1) and invoking an expansion in cumulants we get

$$\begin{aligned}\langle\phi''(t)\rangle &= -2v^2\int_0^t e^{-(t-t')/\tau}\langle e^{-\theta[Y(t)-Y(t')]}\rangle\, dt' \\ &= -2v^2\int_0^t e^{-(t-t')/\tau}\exp\sum_{n=1}^{\infty}(-\theta)^n \\ &\quad\times\frac{\langle\langle[Y(t)-Y(t')]^n\rangle\rangle}{n!}\, dt'. \end{aligned}\quad \text{(D4)}$$

Thus, we need to calculate cumulants of the form $\langle\langle[Y(t) - Y(t')]^n\rangle\rangle$. For example, the first and second cumulants are

$$\langle\langle Y(t) - Y(t')\rangle\rangle = \langle Y(t)\rangle - \langle Y(t')\rangle \quad \text{(D5)}$$

$$\begin{aligned}\langle\langle[Y(t)-Y(t')]^2\rangle\rangle &= \langle[Y(t)-Y(t')]^2\rangle - \langle[Y(t)-Y(t')]\rangle^2 \\ &= \langle\langle Y(t)^2\rangle\rangle + \langle\langle Y(t')^2\rangle\rangle - 2\langle\langle Y(t)Y(t')\rangle\rangle.\end{aligned} \quad \text{(D6)}$$

From Eq. (D4) the short- and long-time behavior of $\langle\phi''(t)\rangle$ can be calculated in terms of the statistics of the integrated noise $Y(t)$. Therefore, using that $\partial_t\,\overline{x(t)^2} = -\langle\phi''(t)\rangle$, we can calculate the asymptotic behavior of the second moment $\overline{x(t)^2}$ when the TE is perturbed by any arbitrary noise $\xi(t)$ in the absorption of energy, for example, when $\xi(t)$ is a Poisson's noise [47] and we cannot apply the present enlarged master equation technique.

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