Statistical physics of exchangeable sparse simple networks, multiplex networks, and simplicial complexes

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Exchangeability is a desired statistical property of network ensembles requiring their invariance upon relabeling of the nodes. However, combining sparsity of network ensembles with exchangeability is challenging. Here we propose a statistical physics framework and a Metropolis-Hastings algorithm defining exchangeable sparse network ensembles. The model generates networks with heterogeneous degree distributions by enforcing only *global constraints* while existing (nonexchangeable) exponential random graphs enforce an extensive number of *local constraints*. This very general theoretical framework to describe exchangeable networks is here first formulated for uncorrelated simple networks and then it is extended to treat simple networks with degree correlations, directed networks, bipartite networks, and generalized network structures including multiplex networks and simplicial complexes. In particular here we formulate and treat both uncorrelated and correlated exchangeable ensembles of simplicial complexes using statistical mechanics approaches.

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I. INTRODUCTION

Networks constitute the architecture of the vast majority of complex systems ranging from the brain to finance [1,2]. Maximum entropy network ensembles [3–18] and in general information theory and modeling frameworks [19–22] are key to analyze such realistic networks, and can be used for a wide variety of applications. Due to the deep relation between information theory and statistical mechanics [23], maximum entropy network ensembles can to a large extent be treated as traditional statistical mechanics ensembles. Indeed recently it was shown in Ref. [7] that network ensembles can be distinguished between canonical and microcanonical network ensembles enforcing respectively soft and hard constraints. For instance, Erdős-Rényi networks of N nodes can enforce either a given total number of links, L [giving rise to the $G(N, L)$ ensemble] or a given expected number of links [giving rise to the $G(N, p)$ ensemble where p is the probability that any two links are connected]. Erdős-Rényi random networks are certainly important; however, in many applications it is observed that nodes have heterogeneous degree distribution [24], typically deviating from the Poisson degree distribution of Erdős-Rényi networks in the sparse network regime. In order to treat network ensembles with heterogeneous degree distribution, exponential random graphs [3,4,25] are considered instead. Exponential random graphs are canonical network ensembles enforcing a given expected degree sequence. While the Erdős-Rényi ensembles impose a single global constraint such as the expected total number of links, exponential random graphs enforce an extensive number of local constraints each given by the expected degree of a single node of the network. This feature of exponential random graphs makes these ensembles significantly different from the Erdős-Rényi ensembles. The first main difference is that these ensembles are not any more equivalent to their conjugated microcanonical network ensemble [26] (the configuration model), which enforces a given degree sequence of the network. The second main difference is that these ensembles are not any more exchangeable. Exchangeability is a notion originally introduced by de Finetti [27] whose theorem states that a sequence of random variables $X_1, X_2, \dots$ is exchangeable if and only if there exists a random probability measure Θ such that the X_i are conditionally identically independent variables given Θ. This notion has been then extended to 2-arrays (i.e., networks) for which the Aldous-Hoover theorem applies [28–31].

Exchangeability is a desired statistical property of network ensembles that ensures invariance of the model upon relabeling of the nodes. In network sampling, when the labels of the nodes depend on the sampling order, exchangeability ensures that the marginal probability of a link is unchanged if nodes are sampled in a different order. Together with projectivity, implying that the marginal probability of a link does not change if the network size increases, or if a part of the network is hidden, exchangeability is a fundamental property of network models that allows their reliable use for sampling and for preserving privacy when processing real network data [32–34]. In the dense network regime, graphons [30,35] have been shown to be exchangeable and projective and are known to allow a well-defined infinite graph limit [28,29]. Graphons are dense in the sense that they have a number of links of the same order of the number of nodes to the power 2, i.e., $L = O(N^2)$. However, this regime is seldom encountered in real networks. The mathematics literature has recently proposed several approaches to face the challenge of modeling exchangeable networks with a number of links that scales like

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$L = O(N^{1+\hat{\alpha}})$ with $0 < \hat{\alpha} < 1$ [31,36–38] and to define ways to define the infinite network limit [37] for such models. All these approaches are based on point processes on $\mathbb{R}^2_+$.

In this paper we propose a statistical physics approach to model sparse exchangeable network ensembles of a given number of nodes, N, and a number of links that scales as $L = O(N)$. Therefore, these network ensembles cover the scaling regime $L = O(N)$ which is important for the vast majority of applications. The exchangeable network ensembles are Hamiltonian and are not based on point processes. These ensembles generate networks with given heterogeneous degree distribution $p(k)$ imposing only two global constraints: the energy (expressing the value of a global exchangeable Hamiltonian of the network ensemble) and the total number of links. The proposed exchangeable sparse network ensembles have the property that each link of the network has the same marginal probability, still the network display and heterogeneous degree distribution. Moreover, the probability that two nodes are connected, when conditioned on their degrees, reduces to the probability of the exponential random graph in the uncorrelated limit. Note that the model is not projective and in particular the marginal distribution of a link is actually dependent on the network size. Therefore, we do not consider the network generated in the limit $N \to \infty$; instead we take an equilibrium statistical mechanics approach and we consider N finite but large. Indeed our result does not contradict the Aldous-Hoover theorem [28–31] that states that infinitely exchangeable sparse networks are empty. In fact for our model in the limit $N \to \infty$ the marginal probability of any link goes to zero. Although this can be considered a shortcoming with respect to graphons, this does not limit the applicability of the model. Indeed many widely used network models are not projective, and have a vanishing marginal in the limit $N \to \infty$, including the Erdős-Rényi networks and the exponential random graphs. Here the sparse exchangeable ensemble of simple networks is simulated with a constructive Metropolis algorithm and is extended to network models with degree correlations, to directed, to bipartite networks and to generalized network structures such as multiplex networks [8,39,40] and simplicial complexes [41–46].

II. FUNDAMENTAL PROPERTIES OF EXCHANGEABLE NETWORK ENSEMBLES

A network ensemble is exchangeable if the probability $\mathbb{P}(G)$ of a network $G = (V, E)$ is independent of the node labels, i.e.,

$$\mathbb{P}(G) = \mathbb{P}(\tilde{G}), \tag{1}$$

for any labeled network $\tilde{G}$ obtained from the network G by permuting the node labels; in particular this includes all labeled networks $\tilde{G}$ isomorphic to G.

Assuming that each labeled network G is uniquely determined by the adjacency matrix $\mathbf{a}$, and that the ensemble is determined by the probability $P(G) = P(\mathbf{a})$, we define the marginal probability of a link (i, j) as

$$p_{ij} = \sum_{\mathbf{a}} a_{ij} P(\mathbf{a}). \tag{2}$$

From the definition of an exchangeable network ensemble it follows that in an exchangeable network ensemble the marginal probability p_{ij} of a link between node i and node j must be invariant under any permutation σ of the node labels, i.e.,

$$p_{ij} = p_{\sigma(i),\sigma(j)}. \tag{3}$$

Note, however, that this is a necessary condition for exchangeable network ensembles and not a sufficient condition. Indeed knowing the marginal probabilities of the link of a network might not be enough to determine general network ensembles for which $P(G)$ does not factorize into independent link probabilities (see, for instance, the two-star or the Strauss model [2]). In an exponential random graph ensemble with given expected degree sequence $\mathbf{k} = \{k_1, k_2, \dots, k_N\}$ with $k_i \leqslant K \ll K_S = \sqrt{\langle k\rangle N}$ the marginal distribution p_{ij} takes the well-celebrated expression

$$p_{ij} = \frac{k_i k_j}{\langle k\rangle N}, \tag{4}$$

where $\langle k\rangle N = \sum_{i=1}^N k_i = 2L$ is twice the expected total number of links of the network. This network ensemble is not exchangeable, unless the expected degree of each node is the same. Indeed the marginal probability p_{ij} is not invariant upon permutation of the node labels if the expected degree distribution is heterogeneous. Only in the case in which the expected degree of each node is the same, $k_i = \langle k\rangle$, do we recover the exchangeable expression of the marginal probability of a sparse Poisson-Erdős-Rényi network $p_{ij} = \langle k\rangle/N$. Note that both the Erdős-Rényi network and the exponential random graph are not projective. This is an immediate consequence of the fact that the marginal probability p_{ij} of the link (i, j) depends on the network size N. Therefore, upon addition of new nodes, leading to an increase of N, the marginal probability between two previously existent nodes changes.

In the following section we will propose an exchangeable sparse network ensemble that imposes a heterogeneous expected degree distribution $p(k)$ enforcing only two global constraints: the energy of the ensemble and the total number of links. In this ensemble the marginal probability of a link between node i and node j is given by the exchangeable expression

$$p_{ij} = \sum_{k,k'} p(k)p(k')\frac{kk'}{\langle k\rangle N} = \frac{\langle k\rangle}{N}. \tag{5}$$

In other words, the marginal probability of any link is the same for every link, but when it is conditioned to the degree of the two linked nodes it is given by the uncorrelated network expression

$$p_{ij|k_i=k,k_j=k'} = p(k,k') = \frac{kk'}{\langle k\rangle N}. \tag{6}$$

Therefore, this ensemble has the same marginal of the Erdős-Rényi network but it can generate uncorrelated networks with heterogeneous degree distribution $p(k)$.

Note that as $N \to \infty$ the marginal probability given by Eq. (5) vanishes. This implies that our exchangeable sparse network ensemble does not contradict the Aldous-Hoover theorem, according to which any sparse infinite exchangeable network should vanish. However, our approach provides finite exchangeable network models with given degree distribution for any large but finite value of N.

As we will see in subsequent sections this network ensemble can be extended to sparse simple networks with degree correlations, to directed, bipartite networks, and to generalized network structures. In all these cases the marginal probability of an interaction is the same for any possible interaction of the network, yet the exchangeable ensembles can give rise to networks with very heterogeneous topology.

III. EXCHANGEABLE SPARSE SIMPLE NETWORK ENSEMBLES

In this section our goal is to construct ensembles of sparse simple networks of N nodes with degree distribution $p(k)$. Here by *sparse* we imply that these networks display a minimum degree $\hat{m}$ and a maximum degree (cutoff) K much smaller than the structural cutoff K_S, i.e., $k_i \leqslant K \ll K_S = \sqrt{\langle k\rangle N}$ for all $i \in \{1, 2, 3, \dots, N\}$ with $\langle k\rangle$ indicating the expected value over the $p(k)$ distribution, $\langle k\rangle = \sum_k kp(k)$. We assume that the nodes of the network can change their degree and we assign to each possible degree sequence $\mathbf{k} = \{k_1, k_2, \dots, k_N\}$ of the network the probability

$$P(\mathbf{k}) = \prod_{i=1}^{N} [p(k_i)\theta(K - k_i)\theta(k_i - \hat{m})], \tag{7}$$

where $\theta(x)$ indicates the Heaviside function with $\theta(x) = 1$ for $x \geqslant 0$ and $\theta(x) = 0$ otherwise. Therefore, the probability of a degree sequence results from the product of the probability that each node has the observed degree. For keeping the model general we assume that the minimum degree of the network must be equal to or greater than $\hat{m}$. For instance, if we want to impose a power-law degree distribution $p(k) = ck^{-\gamma}$ this allows us to impose a minimum degree $\hat{m} \geqslant 1$ and to exclude isolated nodes for which $p(k)$ is not defined. However, also $\hat{m} = 0$ is allowed as long as $p(k)$ is well defined for $0 \leqslant k \leqslant K$. In order to build an exchangeable network ensemble we need to define a probability $\mathbb{P}(G)$ for any possible network $G = (V, E)$ in the ensemble described by the adjacency matrix $\mathbf{a}$. In order to ensure sparsity we impose that the total number of links, L, is fixed and given by $L = \langle k\rangle N/2$ and we impose that the probability of getting a degree sequence $\mathbf{k}$ is $P(\mathbf{k})$. To this end we assume that the probability of a network given its degree distribution, is uniform. Since the number of networks $\mathcal{N}$, with given degree sequence $\mathbf{k}$ can be expressed in terms of the entropy $\Sigma(\mathbf{k})$ of networks with degree sequence $\mathbf{k}$ as [7] $\mathcal{N} = \exp[\Sigma(\mathbf{k})]$, the probability of each single network G displaying a degree sequence $\mathbf{k}$ is therefore taken to be

$$\mathbb{P}(G) = P(\mathbf{k})e^{-\Sigma(\mathbf{k})}\delta\left(L, \sum_{i<j} a_{ij}\right), \tag{8}$$

where $\delta(x, y)$ indicates the Kronecker delta. For sparse networks the entropy $\Sigma(\mathbf{k})$ of networks with given degree sequence with $k_i \ll K_S$ obeys the Bender-Canfield formula [7,26,47,48]

$$\Sigma(\mathbf{k}) = \ln\left(\frac{(2L)!!}{\prod_{i=1}^{N} k_i!}\right) + o(N), \tag{9}$$

where in Eqs. (8) and (9) we indicate with $\mathbf{k} = \{k_1, k_2, \dots, k_N\}$ the degree sequence with k_i, the degree of node i, given by $k_i = \sum_{j=1}^{N} a_{ij}$. Note that the sparse exchangeable network ensemble greatly differs from the network ensemble with given expected degree sequence because in the exchangeable ensemble the constraints are global and not local. Indeed the expression for $\mathbb{P}(G)$ can be also given by

$$\mathbb{P}(G) = e^{-H(G)}\delta\left(L, \sum_{i<j} a_{ij}\right)\theta\left(K - \max_{i=1,\dots,N} k_i\right) \times \theta\left(\min_{i=1,\dots,N} k_i - \hat{m}\right), \tag{10}$$

with *Hamiltonian* $H(G)$ given by

$$H(G) = -\sum_{i=1}^{N} \ln p(k_i) + \Sigma(\mathbf{k}). \tag{11}$$

Using Eq. (9) for $\Sigma(\mathbf{k})$ we can derive the explicit expression for $H(G)$:

$$H(G) = -\sum_{i=1}^{N} \ln[p(k_i)k_i!] + \ln((2L)!!). \tag{12}$$

The Hamiltonian $H(G)$ is clearly a global variable that depends on all the nodes of the network where each node is treated on equal footing. Therefore, the expression of the Hamiltonian is clearly exchangeable as it is invariant upon a permutation of the node labels. In order to show that the marginal distribution is given by Eq. (5) we solve the model using the saddle-point method applied to a free energy expressed in terms of a functional order parameter. In order to perform this calculation, let us write the probability $\mathbb{P}(G)$ as

$$\mathbb{P}(G) = \frac{1}{(2L)!!}\sum_{\mathbf{k}}{}' \prod_{i=1}^{N}\left[k_i!p(k_i)\delta\left(k_i, \sum_{j=1}^{N} a_{ij}\right)\right] \times \delta\left(\sum_{i,j} a_{ij}, L\right), \tag{13}$$

where $\sum_{\mathbf{k}}'$ indicates the sum over all possible degree sequences with a maximum degree equal to or smaller than K and a minimum degree greater than or equal to $\hat{m}$. Expressing the Kronecker deltas in Eq. (13) with their integral form

$$\delta(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{\mathrm{i}\omega(x-y)}, \tag{14}$$

the partition function $Z = Z(h)$ can be expressed as

$$Z = \sum_{\mathbf{a}} \mathbb{P}(G)e^{-h\sum_{i<j} a_{ij}} = \frac{1}{(2L)!!}\sum_{\mathbf{a}}\sum_{\mathbf{k}}{}' \int \mathcal{D}\omega \int \frac{d\lambda}{2\pi} e^{G(\lambda,\boldsymbol{\omega},\mathbf{k},h)}, \tag{15}$$

with

$$G(\lambda, \boldsymbol{\omega}, \mathbf{k}, h) = \sum_{i=1}^{N}[\mathrm{i}\omega_i k_i + \ln(k_i!p(k_i))] + \mathrm{i}\lambda L + \frac{1}{2}\sum_{i,j}\ln(1 + e^{-\mathrm{i}\lambda-\mathrm{i}\omega_i-\mathrm{i}\omega_j-h}), \tag{16}$$

and $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$. Let us now introduce the functional order parameter indicating the density of nodes with degree $k_i = k$ and with $\omega_i = \omega$ [43,48,49],

$$c(\omega, k) = \frac{1}{N}\sum_{i=1}^{N}\delta(\omega - \omega_i)\delta(k, k_i). \tag{17}$$

By calculating the partition function Z in the sparse regime (i.e., $K \ll K_S$) with the saddle-point method (see Appendix A) we can derive for the functional order parameter $c(\omega, k)$ when $h \to 0$ the expression

$$c(\omega, k) = \frac{1}{2\pi}p(k)k!e^{i\omega k + e^{-i\omega}}. \tag{18}$$

This implies that the density of nodes of degree k is given by

$$\int d\omega c(\omega, k) = p(k). \tag{19}$$

Therefore, the degree of each node is fluctuating, but in the large network limit the degree distribution is given by $p(k)$ as desired. We are now in the position to evaluate the marginal probability p_{ij} of a link between node i and node j. A straightforward calculation (see Appendix A) leads to the expression of the marginal probability $p_{ij} = \langle a_{ij}\rangle$ in terms of the functional order parameter $c(\omega, k)$ leading to

$$p_{ij} = \frac{1}{\langle k\rangle N}\sum_{\hat{m}\leqslant k\leqslant K, \hat{m}\leqslant k'\leqslant K}\int_{-\pi}^{\pi}\frac{d\omega}{2\pi}\int_{-\pi}^{\pi}\frac{d\omega'}{2\pi}c(\omega, k) \times c(\omega', k')e^{-i\omega - i\omega'}.$$

From this expression, using Eq. (18) it follows immediately the expression for the marginal given by Eq. (5) leading to $p_{ij} = \langle k\rangle/N$ also if the marginal probability conditioned on the node degree (see Appendix A for a detailed derivation) is given by Eq. (6). Therefore, the marginal probability $p_{ij} = \langle k\rangle/N$ is the same for every node of the network and it is equal to the marginal probability in a Poisson-Erdős-Rényi network, but the degree distribution is $p(k)$; i.e., it can significantly differ from a Poisson distribution.

IV. METROPOLIS-HASTINGS ALGORITHM

The exchangeable ensemble of sparse networks can be obtained by implementing a simple Metropolis-Hastings algorithm using the network Hamiltonian given by Eq. (12). The Metropolis-Hastings algorithm for the exchangeable sparse networks is outlined below.

(1) Start with a network of N nodes having exactly $L = \langle k\rangle N/2$ links and in which the minimum degree is greater than or equal to $\hat{m}$ and the maximum degree is smaller than or equal to K.

(2) Iterate the following steps until equilibration:

(i) Choose randomly a random link $\ell = (i, j)$ between node i and j and choose a pair of random nodes (i', j') not connected by a link. By indicating with $\mathbf{a}$ the (symmetric) adjacency matrix of the network we have $a_{ij} = 1$ and $a_{i'j'} = 0$.

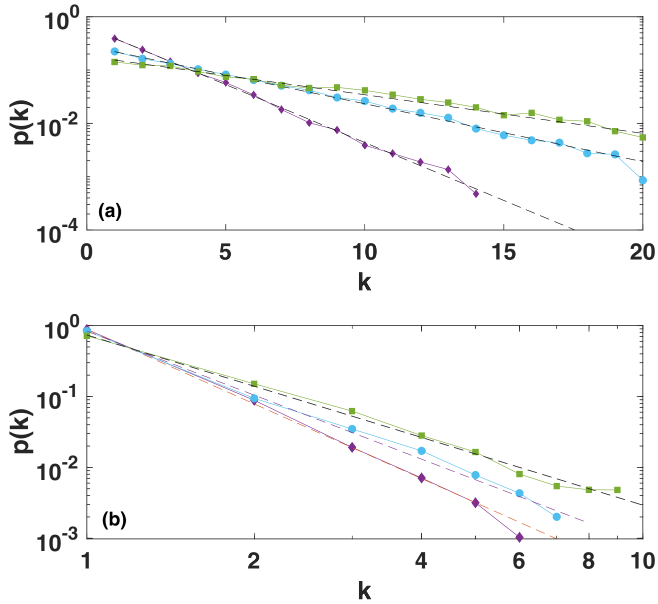

FIG. 1. Degree distributions of the exchangeable uncorrelated network ensembles generated by the Metropolis-Hastings algorithm. (a) The case of exponential degree distributions $p(k) = ce^{-k/k_0}$ with $k_0 = 2$ (green squares), $k_0 = 4$ (cyan circles), and $k_0 = 6$ (purple diamonds) (b) The case of power-law degree distributions with $p(k) = ck^{-\gamma}$ and $\gamma = 2.4$ (green squares), $\gamma = 3.0$ (cyan circles), and $\gamma = 3.5$ (purple diamonds). The dashed lines indicate the theoretical expectation. In all simulations the networks have $N = 2000$ nodes.

(ii) Draw a random number r from a uniform distribution in [0,1], i.e., $r \sim U(0, 1)$. Calculate the Hamiltonian $H_{\rm int} = H(\mathbf{a})$ for the network with adjacency matrix $\mathbf{a}$ of the network and calculate the Hamiltonian $H_{\rm fin} = H(\mathbf{a}')$ for the adjacency matrix $\mathbf{a}'$ in which the link between nodes (i, j) is removed and the link between the nodes (i', j') is inserted instead. If $r < \max(1, e^{-\Delta H})$ where $\Delta H = H_{\rm fin} - H_{\rm int}$ and if the move does not violate the conditions on the minimum and maximum degree of the network, remove the link (i, j) and insert the link (i', j'); i.e., update the following four elements of the adjacency matrix according to the rules: $a_{ij} \to 1 - a_{ij}$ and $a_{ji} \to 1 - a_{ji}$ and $a_{i'j'} \to 1 - a_{i'j'}$ and $a_{j'i'} \to 1 - a_{j'i'}$.

This algorithm can be used to generate exchangeable network ensembles with different degree distributions such as exponential distribution or power-law degree distribution (see Fig. 1).

This approach can be directly extended to treat sparse networks with given degree correlations, directed networks, bipartite networks, and also generalized network structures such as multiplex networks and simplicial complexes as we will describe in the following sections.

V. EXCHANGEABLE NETWORK ENSEMBLES WITH DEGREE CORRELATIONS

Degree correlations are an important characteristic of networks and have attracted large interest [50–53] because they encode additional network information not captured by the degree distribution. In particular in the literature different works have been proposed to model *labeled* networks with given degree correlations [12,48,54,55].

Here our aim is to construct a sparse exchangeable network ensemble of N nodes in which each node has degree k with probability $p(k)$ and every link between a node of degree k and a node of degree k' contributes to the probability of the network G by a factor $Q(k, k')$ where $Q(k, k') = Q(k, k')$ with $Q(k, k')$ independent of N. As we will see in the following, by changing the kernel function $Q(k, k')$ it is possible to select the nature of the degree correlations modulating the probability $p(k, k')$ of observing a link between a node of degree k and a node of degree k'. In order to define the exchangeable ensemble of correlated networks, we follow a derivation similar to the one considered in the previous section and we assign to each network G a probability $\mathbb{P}(G)$ given by

$$\mathbb{P}(G) = \prod_{i<j}[Q(k_i, k_j)]^{a_{ij}} \prod_{i=1}^{N} p(k_i) e^{-\Sigma(\mathbf{k})} \delta\left(L, \sum_{i<j} a_{ij}\right) \times \theta\left(K - \max_{i=1,\dots,N} k_i\right) \theta\left(\hat{m} - \min_{i=1,\dots,N} k_i\right), \quad (20)$$

where $\Sigma(\mathbf{k})$ is the entropy of the ensemble of correlated networks with degree sequence $\mathbf{k}$. The entropy $\Sigma(\mathbf{k})$ has been calculated in Refs. [48,56] and can be expressed as

$$\Sigma(\mathbf{k}) = \ln\left((2L)!! \prod_{i=1}^{N} \frac{[\gamma(k_i)]^{k_i}}{k_i!}\right) + o(N), \quad (21)$$

where $\gamma(k)$ is a function that satisfies

$$\gamma(k) = \frac{1}{\langle k \rangle} \sum_{k'} Q(k, k') p(k') \frac{k'}{\gamma(k')}. \quad (22)$$

The probability $\mathbb{P}(G)$ can be also written in Hamiltonian form as

$$\mathbb{P}(G) = e^{-H(G)} \delta\left(L, \sum_{i<j} a_{ij}\right) \theta\left(K - \max_{i=1,\dots,N} k_i\right) \times \theta\left(\hat{m} - \min_{i=1,\dots,N} k_i\right),$$

where the (exchangeable) Hamiltonian $H(G)$ is given by

$$H(G) = -\sum_{i<j} a_{ij} \ln Q(k_i, k_j) - \sum_{i=1}^{N} \ln p(k_i) + \Sigma(\mathbf{k}), \quad (23)$$

and $\Sigma(\mathbf{k})$ is given by Eq. (21). By studying this correlated network ensemble with statistical mechanics methods similar to the ones we have used in the case of the exchangeable uncorrelated network ensemble investigated earlier, we can show (see Appendix B for details) that the degree distribution is $p(k)$ and that the marginal probability of each link can be written as

$$p_{ij} = \sum_{\hat{m} \leqslant k \leqslant K, \hat{m} \leqslant k' \leqslant K} p(k) p(k') p(k, k') = \frac{\langle k \rangle}{N}, \quad (24)$$

with $p(k, k')$ indicating the probability of a link between a node of degree k and a node of degree k', with

$$p_{ij|k_i=k,k_j=k'} = p(k, k') = \frac{1}{\langle k \rangle N} Q(k, k') \frac{k k'}{\gamma(k)\gamma(k')}. \quad (25)$$

Note that for $Q(k, k') = 1$ it follows immediately from Eq. (22) that $\gamma(k) = 1$ and we recover the uncorrelated network ensemble. Equations (24) and (25) clearly reveal the exchangeable nature of this ensemble as the marginal probability of a link is independent of the node labels. However, the ensemble retains the ability to model sparse networks with arbitrary degree distribution and degree correlations.

VI. EXCHANGEABLE SPARSE DIRECTED NETWORK ENSEMBLES

It is well known that a number of real networks including prey-predator interactions in ecology, financial contracts between banks, the World Wide Web, or directed online social networks like Twitter are actually directed. In directed networks a link (i, j) indicating a directed interaction from node i to node j is distinguished from the link (j, i). For instance, the existence of a prey-predator interaction between a species i (prey) and a species j (predator) is not typically reciprocated. The main difference between simple and directed networks is that in directed networks the adjacency matrix is not symmetric and for each node we can distinguish between the in-degree and the out-degree. Assuming that $a_{ij} = 1$ indicates the presence of a directed link from node i to node j, the in-degree and the out-degree of node i can be expressed as

$$k_i^{in} = \sum_{j=1}^{N} a_{ji}, \quad k_i^{\text{out}} = \sum_{j=1}^{N} a_{ij}. \quad (26)$$

Several previous work have modeled *labeled* directed network ensembles using statistical mechanics approaches [3,57–60]. Here we define the exchangeable uncorrelated ensemble of directed networks with joint degree distribution $p_d(k^{in}, k^{\text{out}})$ indicating the probability that a generic node i has degree $k_i^{in} = k^{in}$ and $k_i^{\text{out}} = k^{\text{out}}$. This distribution is arbitrary, but needs to satisfy $\langle k^{in} \rangle = \langle k^{\text{out}} \rangle$. In order to guarantee sparsity, we assume that the in-degree and the out-degree have a maximum value equal to or smaller than K with $K \ll K_s = \sqrt{\langle k^{in} \rangle N}$ and that they have a minimum value equal to or larger than $\hat{m}$. The exchangeable uncorrelated ensemble of directed networks assigns to each directed network G the probability $\mathbb{P}(G)$ given by

$$\mathbb{P}(G) = \prod_{i=1}^{N} p_d\left(k_i^{in}, k_i^{\text{out}}\right) e^{-\Sigma(\mathbf{k}^{in}, \mathbf{k}^{\text{out}})} \delta\left(L, \sum_{i,j} a_{ij}\right) \theta\left(K - \max_{i=1,\dots,N} k_i^{in}\right) \theta\left(K - \max_{i=1,\dots,N} k_i^{\text{out}}\right) \theta\left(\min_{i=1,\dots,N} k_i^{in} - \hat{m}\right) \theta\left(\min_{i=1,\dots,N} k_i^{\text{out}} - \hat{m}\right),$$

where the entropy $\Sigma(\mathbf{k}^{in}, \mathbf{k}^{\text{out}})$ is given by

$$\Sigma(\mathbf{k}) = \ln\left[\frac{L!}{\prod_{i=1}^{N}\left[k_i^{in}! k_i^{\text{out}}!\right]}\right] + o(N). \quad (27)$$

The probability $\mathbb{P}(G)$ admits a Hamiltonian expression as

$$\mathbb{P}(G) = e^{-H(G)}\delta\left(L, \sum_{i,j} a_{ij}\right)\theta\left(K - \max_{i=1,\dots,N} k_i^{in}\right)\theta\left(K - \max_{i=1,\dots,N} k_i^{\rm out}\right)\theta\left(\min_{i=1,\dots,N} k_i^{in} - \hat{m}\right)\theta\left(\min_{i=1,\dots,N} k_i^{\rm out} - \hat{m}\right), \quad (28)$$

where the Hamiltonian $H(G)$ of this ensemble is given by

$$H(G) = -\sum_{i=1}^{N} \ln\left(p_d\left(k_i^{in}, k_i^{\rm out}\right)k_i^{in}!k_i^{\rm out}!\right) + \ln(L!). \quad (29)$$

The statistical mechanics treatment of this model (see Appendix C) shows that the density of nodes with in-degree k^{in} and out-degree $k^{\rm out}$ is given by the desired joint distribution $p_d(k^{in}, k^{\rm out})$ although the marginal probability of each node is equal for each node and given by

$$p_{ij} = \sum_{k^{in},k^{\rm out}} p_{in}(k^{in})p_{\rm out}(k^{\rm out})\frac{k^{in}k^{\rm out}}{\langle k^{in}\rangle N}, \quad (30)$$

where

$$p_{in}(k^{in}) = \sum_{k^{\rm out}} p_d(k^{in}, k^{\rm out}),$$
$$p_{\rm out}(k^{\rm out}) = \sum_{k^{in}} p_d(k^{in}, k^{\rm out}). \quad (31)$$

Note that although the marginal probability is the same for each node the marginal probability of a directed link conditioned on the in-degree and the out-degree of its two end nodes is not; i.e.,

$$p_{ij|k_i^{\rm out}=k^{\rm out},k_j^{in}=k^{in}} = p(k^{in}, k^{\rm out}) = \frac{k^{in}k^{\rm out}}{\langle k^{in}\rangle N}. \quad (32)$$

VII. EXCHANGEABLE BIPARTITE NETWORK ENSEMBLES

Bipartite networks are another notable class of networks formed by two sets of nodes and interactions only existing between nodes of one class and nodes of the other class. Examples of bipartite networks are mutualistic networks in ecology [61], social networks between individuals, and taste or opinions, and in general can be used to partition a given set of nodes in different groups [62]. Interestingly bipartite networks are also called factor graphs and are widely used as the architecture supporting graphical models [49,63].

In this section we consider exchangeable ensembles of bipartite networks formed by two sets of nodes V and U with $|V| = N$ and $|U| = M$ with the condition

$$M = \alpha N, \quad (33)$$

with $\alpha > 0$ being a constant independent of N. We indicate with i the nodes belonging to the set V and with μ the nodes belonging to the set U. The structure of the bipartite network is determined by the $N \times M$ incidence matrix $b_{i\mu} = 1$ if there is a link between node i and node μ; otherwise, $b_{i\mu} = 0$. The degree of the nodes in V and in U is determined from the incidence matrix $\mathbf{b}$ according to the following equations:

$$k_i = \sum_{\mu=1}^{M} b_{i\mu}, \quad q_\mu = \sum_{i=1}^{N} b_{i\mu}. \quad (34)$$

Here we formulate the exchangeable sparse bipartite network ensemble designed in order to obtain bipartite networks in which the nodes in V have degree distribution $p(k)$ and the nodes in U have degree distribution $\hat{p}(q)$. These distributions can be arbitrary but must obey $N\langle k\rangle = M\langle q\rangle$ which implies $\langle k\rangle = \alpha\langle q\rangle$. Moreover, these ensembles have fixed number of links $L = \langle k\rangle N$ and the degree k (q) of the nodes in V (U) has a maximum smaller than or equal to $K \ll K_s = \sqrt{\langle k\rangle N}$ ($\hat{K} \ll K_S = \sqrt{\langle k\rangle N}$) and minimum degree greater or smaller than $\tilde{m}$ ($\hat{m}$). The probability $\mathbb{P}(G)$ for each bipartite network G is taken to be

$$\mathbb{P}(G) = \prod_{i=1}^{N} p(k_i)\prod_{\mu=1}^{M} \hat{p}(q_\mu)e^{-\Sigma(\mathbf{k},\mathbf{q})}\delta\left(L, \sum_{i,\mu} b_{i\mu}\right)$$
$$\times\,\theta\left(K - \max_{i=1,\dots,N} k_i\right)$$
$$\times\,\theta\left(\hat{K} - \max_{\mu=1,\dots,M} q_\mu\right)\theta\left(\min_{i=1,\dots,N} k_i - \tilde{m}\right)$$
$$\times\,\theta\left(\min_{\mu=1,\dots,M} q_\mu - \hat{m}\right).$$

The entropy of this ensemble is given by

$$\Sigma(\mathbf{k}, \mathbf{q}) = \ln\left[\frac{L!}{\prod_{i=1}^{N} k_i!\prod_{\mu=1}^{M} q_\mu!}\right] + o(N). \quad (35)$$

This ensemble is Hamiltonian as $\mathbb{P}(G)$ can be expressed as

$$\mathbb{P}(G) = e^{-H(G)}\delta\left(L, \sum_{i,\mu} b_{i\mu}\right)\theta\left(K - \max_{i=1,\dots,N} k_i\right)$$
$$\times\,\theta\left(\hat{K} - \max_{\mu=1,\dots,M} q_\mu\right)\theta\left(\min_{i=1,\dots,N} k_i - \tilde{m}\right)$$
$$\times\,\theta\left(\min_{\mu=1,\dots,M} q_\mu - \hat{m}\right)$$

with the Hamiltonian $H(G)$ given by

$$H(G) = -\sum_{i=1}^{N} \ln\left(p(k_i)k_i!\right) - \sum_{\mu=1}^{M} \ln\left(\hat{p}(q_\mu)q_\mu!\right) + \ln(L!). \quad (36)$$

This ensemble produces a network model that can be treated using statistical mechanics methods (see Appendix D) which clearly show that the nodes in V have degree distribution $p(k)$ and the nodes in U have degree distribution $\hat{p}(q)$. Both $p(k)$ and $\hat{p}(q)$ can be heterogeneous even if the marginal of every link is the same for every link and given by

$$p_{i\mu} = \sum_{k,q} p(k)\hat{p}(q)\frac{kq}{\langle k\rangle N}. \tag{37}$$

Note that the marginal probability $p_{i\mu|k_i=k,q_\mu=q} = p(k,q)$ of a link (i,μ) conditioned on the degrees $k_i = k$ and $q_\mu = q$ in this ensemble is given by

$$p_{i\mu|k_i=k,q_\mu=q} = p(k,q) = \frac{kq}{\langle k\rangle N}. \tag{38}$$

VIII. EXCHANGEABLE SPARSE MULTIPLEX NETWORKS

A large variety of complex systems including biological, social networks, and infrastructures are better described by multiplex networks [39,40,64] which are formed by a set of nodes connected by two or more networks indicating interactions of different nature and connotation. The different networks forming the multiplex networks are also called the *layers* of the multiplex network. Different works have investigated ensembles of *labeled* multiplex networks with different types of correlations between the layers, with weights of the links and with nontrivial spatial embedding [8,65,66].

Here our goal is to propose and investigate the properties of exchangeable sparse multiplex networks. To this end we can consider a multiplex network $\vec{G} = (G_1, G_2, \dots, G_M)$ formed by M layers $\alpha \in \{1, 2, \dots, M\}$ each determined by an adjacency matrix $\mathbf{a}^{[\alpha]}$ [39]. To keep the discussion simple we will assume that each adjacency matrix is undirected and unweighted. The degree $k_i^{[\alpha]}$ of each node i in layer $\alpha \in \{1, 2, \dots, M\}$ is determined by the equation

$$k_i^{[\alpha]} = \sum_{j=1}^{N} a_{ij}^{[\alpha]}. \tag{39}$$

An important feature of multiplex networks are multilinks $\vec{m} = (m^{[1]}, m^{[2]}, \dots, m^{[M]})$ (with $m^{[\alpha]} \in \{0, 1\}$) [8] indicating the pattern of connection between any two nodes. For instance, in a duplex network ($M = 2$) with two layers indicating mobile phone and email interaction the two nodes are connected by a multilink (1,0) if they only communicate via mobile phone, they are connected by a multilink (0,1) if they only communicate via email, and they are connected by a multilink (1,1) if they communicate both via mobile phone and email. In order to indicate if two nodes i and j are connected by a multilink $\vec{m}$ we can use the *multiadjacency matrices* $A^{\vec{m}}$ [8,39] whose element $A_{ij}^{\vec{m}}$ indicates whether node i and node j are connected by a multilink of type $\vec{m}$ ($A_{ij}^{\vec{m}} = 1$) or not ($A_{ij}^{\vec{m}} = 0$). The matrix elements of the multiadjacency matrices are given by

$$A_{ij}^{\vec{m}} = \prod_{\alpha=1}^{M} \left[a_{ij}^{[\alpha]} m^{[\alpha]} + \left(1 - a_{ij}^{[\alpha]}\right)(1 - m^{[\alpha]})\right]. \tag{40}$$

Since any two nodes can be connected only by a single multilink we have

$$\sum_{\vec{m}} A_{ij}^{\vec{m}} = 1. \tag{41}$$

Having defined the multiadjacency matrices, it is possible to introduce the definition of the *multidegree* $k_i^{\vec{m}}$ as the sum of multilinks $\vec{m}$ incident to the node i [8], i.e.,

$$k_i^{\vec{m}} = \sum_{j=1}^{N} A_{ij}^{\vec{m}}. \tag{42}$$

Using the approach described in this work we can either define exchangeable sparse multiplex networks in which each layer is independent of the other and has a given degree distribution (eventually dependent on the choice of the layer) or we can define exchangeable sparse multiplex networks in which the multidegree distribution is kept fixed.

The first case can be modeled by drawing each layer of the multiplex network independently from an exchangeable ensemble of uncorrelated simple networks. Given the simplicity of the approach here we neglect to treat this case in detail. The latter case can be modeled by an exchangeable multiplex network ensemble in which each node has a series of nontrivial multidegrees $\mathbf{k}_i^{\vec{m}}$ with $\vec{m} \neq \vec{0}$ [e.g., $\mathbf{k}_i^{\vec{m}} = (k_i^{(1,0)}, k_i^{(0,1)}, k_i^{(1,1)})$ in the case of $M = 2$ layers] with multidegree distribution $\tilde{\pi}(\mathbf{k}_i^{\vec{m}})$. Moreover, we impose that in the network there are exactly $L^{\vec{m}} = \langle k^{\vec{m}}\rangle N/2$ multilinks of type $\vec{m} \neq \vec{0}$ and that the multiplex is sparse; i.e., the multidegree $k_i^{\vec{m}}$ has a minimum value greater than or equal to $\hat{m}$ and a maximum value smaller than or equal to $K^{\vec{m}}$ with $K^{\vec{m}} \ll K_S^{\vec{m}} = \sqrt{\langle k^{\vec{m}}\rangle N}$. Therefore, the ensemble is defined by associating to each multiplex network of M layers $\vec{G} = (G_1, G_2, \dots, G_M)$ the probability

$$\mathbb{P}(\vec{G}) = P(\{k^{\vec{m}}\})e^{-\Sigma(\{k^{\vec{m}}\})} \prod_{\vec{m}\neq\vec{0}} \left[\delta\left(L^{\vec{m}}, \sum_{i<j} A_{ij}^{\vec{m}}\right) \times \theta\left(K^{\vec{m}} - \max_{i=1,\dots,N} k_i^{\vec{m}}\right)\theta\left(\min_{i=1,\dots,N} k_i^{\vec{m}} - \hat{m}\right)\right], \tag{43}$$

where $\{k^{\vec{m}}\}$ indicates the sequence of all the nontrivial multidegrees $\vec{m} \neq \vec{0}$ of every node i of the multiplex network, and where the entropy $\Sigma(\{k^{\vec{m}}\})$ is given by [8]

$$\Sigma(\{k^{\vec{m}}\}) = \ln\left(\prod_{\vec{m}\neq\vec{0}} \frac{(2L^{\vec{m}})!!}{\prod_{i=1}^{N} k_i^{\vec{m}}!}\right) + o(N). \tag{44}$$

Here $P(\{k^{\vec{m}}\})$ is given by the product of the probability that each node has multidegrees $\mathbf{k}_i^{\vec{m}}$:

$$P(\{k^{\vec{m}}\}) = \prod_{i=1}^{N} \tilde{\pi}\left(\mathbf{k}_i^{\vec{m}}\right). \tag{45}$$

This exchangeable multiplex network ensemble is Hamiltonian as the probability $\mathbb{P}(\vec{G})$ can be written as

$$\mathbb{P}(\vec{G}) = e^{-H(G)} \prod_{\vec{m}\neq\vec{0}} \left[\delta\left(L^{\vec{m}}, \sum_{i<j} A_{ij}^{\vec{m}}\right)\theta\left(K^{\vec{m}} - \max_{i=1,\dots,N} k_i^{\vec{m}}\right) \times \theta\left(\min_{i=1,\dots,N} k_i^{\vec{m}} - \hat{m}\right)\right], \tag{46}$$

with Hamiltonian $H(G)$ given by

$$H(G)=-\sum_{i=1}^{N}\ln\left(\tilde{\pi}\left(\mathbf{k}_i^{\vec{m}}\right)\right)-\sum_{i=1}^{N}\sum_{\vec{m}\neq\vec{0}}k_i^{\vec{m}}!+\sum_{\vec{m}\neq\vec{0}}\ln((2L^{\vec{m}})!).$$

Using a statistical mechanics treatment of this ensemble (see Appendix E) it can be shown that the marginal probability $p_{ij}^{\vec{m}}$ of observing a multilink $\vec{m}\neq\vec{0}$ between node i and node j is given by

$$p_{ij}^{\vec{m}}=\langle A_{ij}^{\vec{m}}\rangle=\sum_{k^{\vec{m}},k'^{,\vec{m}}}\tilde{\pi}(\mathbf{k}^{\vec{m}})\tilde{\pi}_{\vec{m}}(\mathbf{k}'^{,\vec{m}})p(k^{\vec{m}},k'^{,\vec{m}}), \quad (47)$$

where the marginal probability $p(k^{\vec{m}},k'^{,\vec{m}})$ of observing a multilink $\vec{m}\neq\vec{0}$ between node i of multidegree $k_i^{\vec{m}}=k^{\vec{m}}$ and node j of multidegree $k_j^{\vec{m}}=k'^{,\vec{m}}$ is given by

$$p_{ij|k_i^{\vec{m}}=k^{\vec{m}},k_j^{\vec{m}}=k'^{,\vec{m}}}=p(k^{\vec{m}},k'^{,\vec{m}})=\frac{k^{\vec{m}}(k'^{,\vec{m}})}{\langle k^{\vec{m}}\rangle N}. \quad (48)$$

IX. EXCHANGEABLE ENSEMBLE OF SPARSE SIMPLICIAL COMPLEXES

In recent years there has been a surge of interest in higher-order networks [41,42,45], including simplicial complexes and hypergraphs. Higher-order networks are able to capture the higher-order interactions present in a variety of complex systems including brain networks, social networks, and protein-interaction networks. Few works have proposed network ensembles for *labeled* simplicial complexes [43,67]. In this section we will propose and study exchangeable ensembles of sparse uncorrelated and correlated simplicial complexes.

A. Uncorrelated exchangeable ensembles of simplicial complexes

A d-dimensional simplex α is a set of $d+1$ nodes $\alpha=[i_0,i_1,\ldots,i_d]$ and indicates the higher-order interaction existing between these nodes. A pure d-dimensional simplicial complex $\mathcal{K}$ is formed by a set of d-dimensional simplices and by all the lower-dimensional simplices formed by any proper subsets of the nodes of these d-dimensional simplices.

A pure d-dimensional simplicial complex $\mathcal{K}$ has a structure that is fully determined by the adjacency tensor $\mathbf{a}$ of elements $a_\alpha=1$ if the d-dimensional simplex $\alpha=[i_0,i_1,\ldots,i_d]$ belongs to the simplicial complex, and with $a_\alpha=0$ otherwise. The generalized degree k_i of the generic node i [43,68] indicates the number of d-dimensional simplices incident to the node i and it can be expressed in terms of the adjacency tensor as

$$k_i=\sum_{\alpha\supset i}a_\alpha=\sum_{i_1<i_2<\cdots<i_d}a_{i_0i_1i_2\cdots\, i_d}. \quad (49)$$

The ensemble of *labeled* pure d-dimensional simplicial complexes with given generalized degree sequence $\mathbf{k}=(k_1,k_2,\ldots,k_N\}$ was studied in Ref. [43]. Here we consider the exchangeable ensemble of uncorrelated d-dimensional simplicial complexes. We indicate with $P(\mathbf{k})$ the probability assigned to observing a generalized degree sequence $\mathbf{k}$, with

$$P(\mathbf{k})=\prod_{i=1}^{N}[p(k_i)\theta(K_s-k_i)\theta(\hat{m}-k_i)]. \quad (50)$$

Therefore, the probability of the generalized degree sequence $\mathbf{k}$ factorizes in the product of the probability $p(k_i)$ that each node i has a generalized degree $k_i=k$. Moreover, we consider that the simplicial complexes are sparse, i.e., they have a *structural cutoff* [43]

$$K_S=\left(\frac{(\langle k\rangle N)^d}{d!}\right)^{1/(d+1)}. \quad (51)$$

This implies that the generalized degree of the nodes k_i has a maximum value $K\ll K_S$. Finally, we assume that each node has a generalized degree equal to or greater than $\hat{m}$. This ensemble is generated by associating to each simplicial complex $\mathcal{K}$ the probability $\mathbb{P}(\mathcal{K})$ given by

$$\mathbb{P}(\mathcal{K})=P(\mathbf{k})e^{-\Sigma(\mathbf{k})}\delta\left(S,\sum_{\alpha\in\mathcal{K}}a_\alpha\right), \quad (52)$$

where $S=\langle k\rangle N/(d+1)$ indicates the number of simplices in the simplicial complex and where $\Sigma(\mathbf{k})$ is the entropy of the ensemble with generalized degree sequence $\mathbf{k}$. In the presence of the structural cutoff, the entropy $\Sigma(\mathbf{k})$ of d-dimensional simplicial complexes with generalized degree sequence $\mathbf{k}$ is given by [43]

$$\Sigma(\mathbf{k})=\ln\left([(\langle k\rangle N)!]^{d/(d+1)}\frac{1}{\prod_{i=1}^N k_i!}(d!)^{-\langle k\rangle N/(d+1)}\right)+o(N).$$

It follows that the exchangeable ensemble of d-dimensional simplicial complexes can be obtained by considering the Hamiltonian simplicial complex ensemble

$$\mathbb{P}(\mathcal{K})=e^{-H(G)}e^{-\Sigma(\mathbf{k})}\delta\left(S,\sum_{\alpha\in\mathcal{K}}a_\alpha\right)\theta\left(K_S-\max_{i=1,2,\ldots,N}k_i\right)\times\theta\left(\min_{i=1,2,\ldots,N}k_i-\hat{m}\right), \quad (53)$$

with Hamiltonian $H(G)$ given by

$$H(G)=-\sum_{i=1}^{N}\ln p(k_i)+\Sigma(\mathbf{k}). \quad (54)$$

This ensemble is exchangeable and the marginal probability for each simplex α is given by (see Appendix F for the derivation)

$$p_\alpha=\sum_{\{k_0,k_1,\ldots,k_d}\left[\prod_{r=0}^{d}p(k_r)\right]p(k_0,k_1,\ldots,k_d)=d!\frac{\langle k\rangle}{N^d}, \quad (55)$$

where the marginal probability $p(k_0,k_1,\ldots,k_d)=p(\alpha=[i_0,i_2,\ldots,i_d]|k_{i_r}=k_r)$ of a simplex $\alpha=[i_0,i_2,\ldots,i_d]$ with the generic node i_r having degree $k_{i_r}=k_r$ is given by the uncorrelated expression [43]

$$p(\alpha=[i_0,i_2,\ldots,i_d]|k_{i_r}=k_r)=p(k_0,k_1,\ldots,k_d)=d!\frac{\prod_{r=0}^{d}k_r}{(\langle k\rangle N)^d}. \quad (56)$$

B. Correlated exchangeable simplicial complex ensemble

The final example of exchangeable ensemble is the ensemble of sparse correlated d-dimensional simplicial complexes in which each node has generalized degree k with probability $p(k)$ and each d simplex between $d+1$ nodes of generalized degrees $\mathbf{k}_\alpha = (k_0, k_1, \ldots, k_r)$ contributes to the probability of the simplicial complex by a term $Q(k_0, k_1, \ldots, k_d) = Q(\mathbf{k}_\alpha)$, where $Q(\mathbf{k}_\alpha)$ is invariant under any permutation of its arguments. Here we impose that the total number of d simplices is $S = \langle k\rangle N/(d+1)$ and that the maximum generalized degree of the simplicial complex is below or equal to K ensuring sparsity and the minimum generalized degree of the simplicial complex is greater than or equal to $\hat{m}$. To this end, we assign to each simplicial complex $\mathcal{K}$ a probability $\mathbb{P}(\mathcal{K})$ given by

$$\mathbb{P}(\mathcal{K}) = \prod_{\alpha\in\mathcal{K}}[Q(\mathbf{k}_\alpha)]^{a_\alpha}\prod_{i=1}^{N}p(k_i)e^{-\Sigma(\mathbf{k})}\delta\left(S, \sum_{\alpha\in\mathcal{K}}a_\alpha\right) \times\theta\left(K - \max_{i=1,\ldots,N}k_i\right)\theta\left(\hat{m} - \min_{i=1,\ldots,N}k_i\right), \qquad (57)$$

where $\Sigma(\mathbf{k})$ is the entropy of the ensemble of correlated networks with degree sequence $\mathbf{k}$ that can be expressed as

$$\Sigma(\mathbf{k}) = \ln\left([(\langle k\rangle N)!]^{d/(d+1)}(d!)^{-\langle k\rangle N/(d+1)}\prod_{i=1}^{N}\frac{[\gamma(k_i)]^{k_i}}{k_i!}\right) + o(N),$$

where $\gamma(k)$ is defined self-consistently by the equation

$$\gamma(k) = \frac{1}{\langle k\rangle^d}\sum_{k_1,k_2,\ldots,k_d}Q(k, k_1, k_2, \ldots, k_d) \times\prod_{r=1}^{d}\left[p(k_r)\frac{k_r}{\gamma(k_r)}\right]. \qquad (58)$$

The marginal probability of this ensemble is given by the exchangeable expression (see Appendix F for the statistical mechanics derivation)

$$p_\alpha = \sum_{k_0,k_1,\ldots,k_d}\left[\prod_{r=0}^{d}p(k_r)\right]p(k_0, k_1, \ldots, k_d) \qquad (59)$$

with $p(k_0, k_1, \ldots, k_d)$ expressing the marginal probability of a simplex connecting $d+1$ nodes with degrees $(k_0, k_1, \ldots, k_d)$:

$$p(k_0, k_1, \ldots, k_d) = \frac{d!}{(\langle k\rangle N)^d}\sum_{k_0,k_1,k_2,\ldots,k_d}Q(k_0, k_1, k_2, \ldots, k_d) \times\prod_{r=0}^{d}\left[\frac{k_r}{\gamma(k_r)}\right].$$

X. CONCLUSIONS

In this work we propose a statistical mechanics framework able to define sparse exchangeable network ensembles of a given number of nodes, N. Here by sparse we mean that the networks have a structural cutoff. This hypothesis is necessary for fully treating the model analytically but it can be removed as long as the entropy $\Sigma(\mathbf{k})$ is known and numerically estimated for every possible degree sequence $\mathbf{k}$ of the network. The network ensemble can be generated by a simple Metropolis-Hastings algorithm. This statistical mechanics approach is based on enforcing two global constraints, such as the total number of links and the value of the exchangeable Hamiltonian of the ensemble. Although every link has the same marginal probability, the ensemble can generate networks with very heterogeneous degree distribution. This implies that in order to impose a heterogeneous degree distribution we do not need as for the exponential random graphs to impose an extensive number of local constraints but two global constraints are actually sufficient. This approach is here shown to be generalizable to networks with degree correlations, to directed and bipartite networks, and to generalized network structures such as multilayer networks and simplicial complexes. This work provides a physical point of view for addressing the challenging problem of modeling exchangeable (but not projective) network ensembles. The model has wide applications as a null model of unlabeled networks. Indeed in applications it is true that exchangeability can be achieved by randomization of the network labels that can be performed by implementing a label reshuffling procedure; however, our theoretical contribution introduces sparse exchangeable network ensembles that are analytically tractable. In physics, one could think of obtaining the Maxwell distribution of velocity of the particles of a gas by drawing for each particle a velocity from a Gaussian distribution and then reshuffling the particle labels, yet being able to treat the gas without having to perform the node reshuffling procedure numerically has many advantages. Similarly, here we provide a statistical mechanics and analytically treatable formulation of exchangeable networks that can be potentially combined to other network analysis tools coming from statistics, network science, or machine learning.

In conclusion, we hope that this work will stimulate further theoretical and applied research at the frontier between physics, mathematics, and applications of network science as the formulation of a sparse exchangeable network model that is also projective would have applications in a number of fields ranging from data analysis and machine learning to sampling of networks, with profound ramifications in mathematics.

APPENDIX A: EXCHANGEABLE ENSEMBLE OF SPARSE SIMPLE NETWORKS

1. Treatment of the exchangeable ensemble of uncorrelated networks

In this section our goal is to solve the partition function $Z(h)$ [which for construction it is expected for $h = 0$ to take the value $Z(0) = 1$] for the exchangeable ensemble of simple networks given by Eq. (15). using the saddle-point equation deriving the expression of the functional order parameter $c(\omega, k)$.

Let us start by recalling the expression given in the main text for the partition function $Z(h)$ of this network ensemble,

$$Z(h) = \sum_{G}\mathbb{P}(G)e^{-h\sum_{i<j}a_{ij}} = \frac{1}{(2L)!!}\sum_{\mathbf{a}}{\sum_{\mathbf{k}}}'\int\mathcal{D}\omega\int\frac{d\lambda}{2\pi}e^{G(\lambda,\boldsymbol{\omega},\mathbf{k},h)}, \qquad (A1)$$

with

$$G(\lambda,\boldsymbol{\omega},\mathbf{k},h)=\sum_{i=1}^{N}[\mathrm{i}\omega_i k_i+\ln(k_i!p(k_i))]+\mathrm{i}\lambda L+\frac{1}{2}\sum_{i,j}\ln(1+e^{-\mathrm{i}\lambda-\mathrm{i}\omega_i-\mathrm{i}\omega_j-h}),\quad \text{(A2)}$$

and with $\mathcal{D}\omega=\prod_{i=1}^{N}[d\omega_i/(2\pi)]$. In Eq. (A1) and in the following we use the notation $\sum_{\mathbf{k}}'$ to indicate the sum over all the possible values of the degree of each node i satisfying $\hat{m}\leqslant k_i\leqslant K\ll K_s=\sqrt{\langle k\rangle N}$. Note that by construction we have $Z(h=0)=1$.

Let us now introduce the functional order parameter [43,48,49]

$$c(\omega,k)=\frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i),\quad \text{(A3)}$$

by enforcing its definition with a series of delta functions. Therefore, by assuming a discretization in ω in intervals of size $\Delta\omega$ we introduce for any value of (ω,k) the term

$$1=\int dc(\omega,k)\delta\left(c(\omega,k)-\frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)\right)=\int\frac{d\hat{c}(\omega,k)dc(\omega,k)}{2\pi/(N\Delta\omega)}\exp\left[\mathrm{i}\Delta\omega\hat{c}(\omega,k)[Nc(\omega,k)-\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)]\right].$$

After performing these operations, by imposing $2L=\langle k\rangle N$ where $\langle k\rangle=\sum_k kp(k)$, the partition function reads, in the limit $\Delta\omega\to 0$,

$$Z(h)=\frac{1}{(2L)!!}\sum_{\mathbf{k}}'\int\mathcal{D}c(\omega,k)\int\mathcal{D}\hat{c}(\omega,k)\int\frac{d\lambda}{2\pi}e^{Nf(\lambda,c(\omega,k),\hat{c}(\omega,k),h)}$$

with

$$f(\lambda,c(\omega,k),\hat{c}(\omega,k),h)=\mathrm{i}\int d\omega\sum_{\hat{m}\leqslant k\leqslant K}\hat{c}(\omega,k)c(\omega,k)+\mathrm{i}\lambda\langle k\rangle/2+\Psi+\ln\int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}p(k)k!e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)}\quad \text{(A4)}$$

where Ψ is given by

$$\Psi=\frac{N}{2}\int d\omega\int d\omega'\sum_{\hat{m}\leqslant k\leqslant K,\hat{m}\leqslant k'\leqslant K}c(\omega,k)c(\omega',k')\ln(1+e^{-\mathrm{i}\lambda-\mathrm{i}\omega-\mathrm{i}\omega'-h})$$

and where $\mathcal{D}c(\omega,k)$ is the functional measure $\mathcal{D}c(\omega,k)=\lim_{\Delta\omega\to 0}\prod_{\omega}\prod_{k}^{N}[dc(\omega,k)\sqrt{N\Delta\omega/(2\pi)}]$ and similarly $\mathcal{D}\hat{c}(\omega,k)=\lim_{\Delta\omega\to 0}\prod_{\omega}\prod_{k}^{N}[d\hat{c}(\omega,k)\sqrt{N\Delta\omega/(2\pi)}]$. Performing a Wick rotation in λ and assuming $z/N=e^{-\mathrm{i}\lambda}$ real and much smaller than one, i.e., $z/N\ll 1$ which is allowed in the sparse regime $K\ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi=\tfrac{1}{2}z\nu^2e^{-h},\quad \text{(A5)}$$

with

$$\nu=\int d\omega\sum_{\hat{m}\leqslant k\leqslant K}c(\omega,k)e^{-\mathrm{i}\omega}.\quad \text{(A6)}$$

The saddle-point equations determining the value of the partition function can be obtained by performing the (functional) derivative of $f(\lambda,c(\omega,k),\hat{c}(\omega,k),h)$ with respect to $c(\omega,k)$, $\hat{c}(\omega,k)$, and λ, obtaining for $h\to 0$

$$\begin{aligned}-\mathrm{i}\hat{c}(\omega,k)&=z\nu e^{-\mathrm{i}\omega},\\ c(\omega,k)&=\frac{\frac{1}{2\pi}p(k)k!e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)}}{\int\frac{d\omega'}{2\pi}\sum_{\hat{m}\leqslant k'\leqslant K}p(k')k'!e^{\mathrm{i}\omega'k'-\mathrm{i}\hat{c}(\omega',k')}},\\ z\nu^2&=\langle k\rangle.\end{aligned}\quad \text{(A7)}$$

Let us first calculate the integral

$$\int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}p(k)k!e^{-\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)}=\int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}k!p(k)e^{\mathrm{i}\omega k+z\nu e^{-\mathrm{i}\omega}},$$

where we have substituted the saddle-point expression for $\hat{c}(\omega,k)$. This integral can be also written as

$$\int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}k!p(k)e^{\mathrm{i}\omega k}\sum_{h=0}^{\infty}\frac{(z\nu)^h}{h!}e^{-\mathrm{i}\omega h}=\sum_{\hat{m}\leqslant k\leqslant K}p(k)(z\nu)^k=\langle(z\nu)^k\rangle.$$

Therefore, $c(\omega,k)$ at the saddle-point solution can be expressed as

$$c(\omega,k)=\frac{1}{2\pi}\frac{k!p(k)e^{\mathrm{i}\omega k+(z\nu)e^{-\mathrm{i}\omega}}}{\langle(z\nu)^k\rangle}.$$

With this expression, using a similar procedure we can express ν as

$$\nu=\int d\omega\sum_{\hat{m}\leqslant k\leqslant K}c(\omega,k)e^{-\mathrm{i}\omega}.\quad \text{(A8)}$$

Combining this equation with the third saddle-point equation

$$z\nu^2 = \langle k\rangle, \tag{A9}$$

it is immediate to show that $z\nu = 1$ is the solution with

$$z = \frac{1}{\langle k\rangle}, \quad \nu = \langle k\rangle. \tag{A10}$$

By inserting this expression in Eq. (A8) we get Eq. (18), i.e.,

$$c(\omega, k) = \frac{1}{2\pi} k! p(k) e^{\mathrm{i}\omega k + e^{-\mathrm{i}\omega}}. \tag{A11}$$

Calculating the partition function at the saddle point, we get $Z(h \to 0) = 1$.

2. Calculation of the marginal probability of a link

For calculating the marginal probability p_{ij} of a link between node i and node j in the exchangeable network ensemble we first note that given that the ensemble has an exchangeable Hamiltonian, the marginal probability of a link should be the same for every link of the network, i.e., $p_{ij} = \tilde{p}$. In order to obtain $\tilde{p}$ we can simply derive the free energy $F = Nf$ with f given by Eq. (A4) with respect to the auxiliary field h obtaining

$$\begin{aligned}\frac{N(N-1)}{2}\tilde{p} &= -\left.\frac{\partial(Nf)}{\partial h}\right|_{h=0} = -\left.\frac{\partial(N\Psi)}{\partial h}\right|_{h=0}\\ &= \frac{N}{2} z \int d\omega \int d\omega' \\ &\quad\times \sum_{\hat{m}\leqslant k\leqslant K; \hat{m}\leqslant k'\leqslant K} c(\omega, k) c(\omega', k') e^{-\mathrm{i}\omega - \mathrm{i}\omega'},\end{aligned}$$

from which, inserting the saddle-point value of $c(\omega, k)$ and z and performing the integrals, we get, for $N \gg 1$,

$$p_{ij} = \tilde{p} = \sum_{\hat{m}\leqslant k\leqslant K} \sum_{\hat{m}\leqslant k'\leqslant K} p(k)p(k') \frac{kk'}{\langle k\rangle N} = \frac{\langle k\rangle}{N}. \tag{A12}$$

3. Expression of the marginal probability of a link conditioned on the degrees of its two end nodes

In this paragraph our goal is to derive the expression of the probability $p_{ij|k_i=k,k_j=k'} = p(k, k')$ of a link between node i and node j in the exchangeable network ensemble conditioned on the degree of the two end nodes. The expression for $p_{ij|k_i=k,k_j=k'}$ can be obtained by showing that the probability $\hat{\pi}_{ij}$ that node i is connected to node j in any network ensemble enforcing a given degree sequence $\mathbf{k}$ (the configuration model) is given by

$$\begin{aligned}\hat{\pi}_{ij} &= \sum_{\mathbf{a}} a_{ij} \prod_{r=1}^{N} \delta\left(k_r - \sum_{s=1}^{N} a_{rs}\right) \delta\left(L - \sum_{r<s} a_{rs}\right) e^{-\Sigma(\mathbf{k})}\\ &= \frac{k_i k_r}{\langle k\rangle N}\end{aligned}$$

as long as the maximum degree of the network, K, is much smaller than the structural cutoff, i.e., $K \ll K_S$. Since $\hat{\pi}_{ij}$ only depends on the degrees k_i and k_j of its two end nodes, in this ensemble the probability $\hat{\pi}(k, k')$ of any link between any two nodes of degree k and degree k' takes the expression

$$\hat{\pi}_{ij|k_i=k,k_j=k'} = \hat{\pi}(k, k') = \frac{kk'}{\langle k\rangle N}, \tag{A13}$$

as long as $K \ll K_S$. The exchangeable network model is essentially an ensemble in which we can get very different degree distributions but each network G with a given distribution $\mathbf{k}$ is weighted by $P(\mathbf{k})\exp[-\Sigma(\mathbf{k})]$. Therefore, the we can express $p_{ij|k_i=k,k_j=k'}$ as

$$\begin{aligned}p_{ij|k_i=k,k_j=k'} = p(k, k') &= \frac{\sum_{\mathbf{k}|k_i=k,k_j=k'} \prod_{r=1}^{N} p(k_r)\hat{\pi}(k_i, k_j)}{p(k)p(k')}\\ &= \hat{\pi}(k, k') = \frac{kk'}{\langle k\rangle N}.\end{aligned} \tag{A14}$$

Let us now derive Eq. (A13) for the ensemble in which we fix the degree sequence of the network (for the other examples of ensemble the derivation is similar and we will omit for space constraints). To this end we consider the partition function

$$\begin{aligned}\tilde{Z}(\mathbf{h}) = \sum_{\mathbf{a}} \exp\left[-\sum_{i<j} h_{k_i,k_j} a_{ij}\right]\left[\prod_{r=1}^{N} \delta\left(k_r - \sum_{s=1}^{N} a_{rs}\right)\right]\\ \times \delta\left(L - \sum_{r<s} a_{rs}\right) e^{-\Sigma(\mathbf{k})},\end{aligned} \tag{A15}$$

where we have introduced some auxiliary fields $\mathbf{h} = \{h_{k,k'}\}$ where each different auxiliary field $h_{k,k'}$ is associated to the links between nodes of degree k and degree k'. Here the entropy $\Sigma(\mathbf{k})$ of the network with given degree sequence with $k_i \ll K_S$ obeys the Bender-Canfield formula [7,26,47,48]

$$\Sigma(\mathbf{k}) = \ln\left(\frac{(2L)!!}{\prod_{i=1}^{N} k_i!}\right) + o(N), \tag{A16}$$

where in Eqs. (A15) and (A16) we indicate with k_i the degree of node i given by $k_i = \sum_{j=1}^{N} a_{ij}$. Expressing the Kronecker delta in Eq. (A15) in integral form we get for the partition function $\tilde{Z}(\mathbf{h})$ of this network ensemble

$$\tilde{Z}(\mathbf{h}) = \frac{1}{(2L)!!} \sum_{\mathbf{a}} \int \mathcal{D}\omega \int \frac{d\lambda}{2\pi} e^{\tilde{G}(\lambda, \boldsymbol{\omega}, \mathbf{k}, \mathbf{h})}, \tag{A17}$$

with

$$\begin{aligned}\tilde{G}(\lambda, \boldsymbol{\omega}, \mathbf{k}, \mathbf{h}) &= \sum_{i=1}^{N} [\mathrm{i}\omega_i k_i + \ln(k_i!)] + \mathrm{i}\lambda L\\ &\quad + \frac{1}{2}\sum_{i,j} \ln(1 + e^{-\mathrm{i}\lambda - \mathrm{i}\omega_i - \mathrm{i}\omega_j - h_{k_i,k_j}}),\end{aligned} \tag{A18}$$

and with $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$ in Eq. (A17). By indicating with N_k the fraction of nodes with degree k, let us introduce the functional order parameters [43,48,49]

$$c_k(\omega) = \frac{1}{N_k} \sum_{i=1}^{N} \delta(\omega - \omega_i)\delta(k, k_i), \tag{A19}$$

determining the fraction of nodes of degree k that are associated to $\omega_i = \omega$. Therefore, by assuming a discretization in ω in intervals of size $\Delta\omega$ we introduce for any value of ω and k

the term

$$1 = \int dc_k(\omega)\delta\left(c_k(\omega) - \frac{1}{N_k}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)\right)$$
$$= \int \frac{d\hat{c}_k(\omega)dc_k(\omega)}{2\pi/(N_k\Delta\omega)}\exp\left[\mathrm{i}\Delta\omega\hat{c}_k(\omega)\left[N_k c_k(\omega) - \sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)\right]\right].$$

After performing these operations, by imposing $2L = \langle k\rangle N$ where $\langle k\rangle = \sum_k kp(k)$, the partition function reads, in the limit $\Delta\omega \to 0$,

$$\tilde{Z}(\mathbf{h}) = \frac{1}{(2L)!!}\int \mathcal{D}c_k(\omega)\int \mathcal{D}\hat{c}_k(\omega)\int \frac{d\lambda}{2\pi}e^{N\tilde{f}(\lambda,c_k(\omega),\hat{c}_k(\omega),\mathbf{k},\mathbf{h})}$$

with

$$\tilde{f}(\lambda, c_k(\omega), \hat{c}_k(\omega), \mathbf{k}, \mathbf{h}) = \mathrm{i}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K}\tilde{P}(k)\hat{c}_k(\omega)c_k(\omega) + \mathrm{i}\lambda\langle k\rangle/2 + \Psi + \sum_{\hat{m}\leqslant k\leqslant K}\tilde{P}(k)\ln\int\frac{d\omega}{2\pi}k!e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}_k(\omega)}, \tag{A20}$$

where we have indicated with $\tilde{P}(k) = N_k/N$ and where Ψ is given by

$$\Psi = \frac{N}{2}\sum_{\hat{m}\leqslant k\leqslant K,\hat{m}\leqslant k'\leqslant K}\tilde{P}(k)\tilde{P}(k')\int d\omega\int d\omega' c_k(\omega)c_{k'}(\omega')\ln(1+e^{-\mathrm{i}\lambda-\mathrm{i}\omega-\mathrm{i}\omega'-h_{k,k'}})$$

and where $\mathcal{D}c_k(\omega)$ is the functional measure $\mathcal{D}c_k(\omega) = \lim_{\Delta\omega\to 0}\prod_\omega\prod_k[dc_k(\omega)\sqrt{N_k\Delta\omega/(2\pi)}]$ and similarly $\mathcal{D}\hat{c}_k(\omega) = \lim_{\Delta\omega\to 0}\prod_\omega\prod_k[d\hat{c}_k(\omega)\sqrt{N_k\Delta\omega/(2\pi)}]$. Performing a Wick rotation in λ and assuming $z/N = e^{-i\lambda}$ real and much smaller than one, i.e., $z/N \ll 1$, which is allowed in the sparse regime $K \ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi = \frac{1}{2}z\sum_{\hat{m}\leqslant k\leqslant K}\sum_{\hat{m}\leqslant k'\leqslant K}\tilde{P}(k)\tilde{P}(k')\nu_k\nu_{k'}e^{-h_{k,k'}}, \tag{A21}$$

with

$$\nu_k = \int d\omega c_k(\omega)e^{-\mathrm{i}\omega}. \tag{A22}$$

For later convenience let us also define ν as

$$\nu = \sum_{\hat{m}\leqslant k\leqslant K}\tilde{P}(k)\int d\omega c_k(\omega)e^{-\mathrm{i}\omega}. \tag{A23}$$

The saddle-point equations determining the value of the partition function can be obtained by performing the (functional) derivative of $f(\lambda, c_k(\omega), \hat{c}_k(\omega), \mathbf{k}, \mathbf{h})$ with respect to $c_k(\omega)$, $\hat{c}_k(\omega)$, and λ, obtaining for $h_{k,k'} \to 0$

$$-\mathrm{i}\hat{c}_k(\omega) = z\nu e^{-\mathrm{i}\omega},$$
$$c_k(\omega) = \frac{\frac{1}{2\pi}k!e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}_k(\omega)}}{\int\frac{d\omega'}{2\pi}k!e^{\mathrm{i}\omega' k-\mathrm{i}\hat{c}_k(\omega')}},$$
$$z\nu^2 = \langle k\rangle. \tag{A24}$$

Let us first calculate the integral

$$\int\frac{d\omega}{2\pi}k!e^{-\mathrm{i}\omega k-\mathrm{i}\hat{c}_k(\omega)} = \int\frac{d\omega}{2\pi}k!e^{\mathrm{i}\omega k+z\nu e^{-\mathrm{i}\omega}}, \tag{A25}$$

where we have substituted the saddle-point expression for $\hat{c}_k(\omega)$. This integral can be also written as

$$\int\frac{d\omega}{2\pi}e^{\mathrm{i}\omega k}\sum_{h=0}^{\infty}\frac{(z\nu)^h}{h!}e^{-\mathrm{i}\omega h} = (z\nu)^k. \tag{A26}$$

Therefore, $c_k(\omega)$ at the saddle-point solution can be expressed as

$$c(\omega, k) = \frac{1}{2\pi}\frac{k!e^{\mathrm{i}\omega k+(z\nu)e^{-\mathrm{i}\omega}}}{(z\nu)^k}. \tag{A27}$$

With this expression, using a similar procedure we can express ν as

$$\nu = \int d\omega\sum_{m\leqslant k\leqslant K}\tilde{P}(k)c_k(\omega)e^{-\mathrm{i}\omega}$$
$$= \sum_{\hat{m}\leqslant k\leqslant K}\frac{k\tilde{P}(k)}{(z\nu)} = \frac{\langle k\rangle}{z\nu}. \tag{A28}$$

Therefore, this equation reduces to the third saddle-point equation,

$$z\nu^2 = \langle k\rangle. \tag{A29}$$

It is immediate to show that $z\nu = 1$ is a solution with

$$z = \frac{1}{\langle k\rangle}, \quad \nu = \langle k\rangle. \tag{A30}$$

By inserting this expression in Eq. (A27) we get Eq. (18), i.e.,

$$c_k(\omega) = \frac{1}{2\pi}k!e^{\mathrm{i}\omega k+e^{-\mathrm{i}\omega}}. \tag{A31}$$

The marginal probability $\hat{\pi}(k, k')$ of a link between a node of degree k and a node of degree k' can be expressed as

$$N_k N_{k'}\hat{\pi}(k, k') = \left.\frac{\partial N\tilde{f}}{\partial h_{k,k'}}\right|_{\mathbf{h}=\mathbf{0}}, \tag{A32}$$

leading to

$$\hat{\pi}(k,k') = \frac{z}{N}\int d\omega \int d\omega' c_k(\omega)c_{k'}(\omega')e^{-\mathrm{i}\omega-\mathrm{i}\omega'} = \frac{kk'}{\langle k\rangle N}.$$

It follows that

$$p(k,k') = \hat{\pi}(k,k') = \frac{kk'}{\langle k\rangle N}. \tag{A33}$$

APPENDIX B: EXCHANGEABLE ENSEMBLE OF SPARSE SIMPLE NETWORKS WITH DEGREE CORRELATIONS

Treatment of the exchangeable ensemble of sparse correlated simple networks

In this section our goal is to treat the exchangeable ensemble of sparse correlated networks in which each node has degree k with probability $p(k)$ and each link between a node of degree k and a node of degree k' contributes to the partition function by a term $Q(k,k') = Q(k',k)$. Here we impose that the total number of links $L = \langle k\rangle N/2$ and that the maximum degree of the network is below or equal to K and the minimum degree of the network is greater than or equal to $\hat{m}$. For simplicity of notation we take the auxiliary field $h = 0$ from the beginning and we express the partition function Z of the exchangeable ensemble of sparse correlated networks as

$$Z = \sum_{\mathbf{a}}\sum_{\mathbf{k}}{}' \prod_{i<j} Q(k_i,k_j)]^{a_{ij}} e^{-\Sigma(\mathbf{k})} \prod_{i=1}^{N}\delta\left(k_i, \sum_{j=1}^{N} a_{ij}\right) \times \delta\left(L, \sum_{i<j} a_{ij}\right),$$

with the entropy $\Sigma(\mathbf{k})$ given by

$$\Sigma(\mathbf{k}) = \ln\left((2L)!!\prod_{i=1}^{N}\frac{[\gamma(k_i)]^{k_i}}{k_i!}\right) + o(N), \tag{B1}$$

where $\gamma(k)$ is determined by the self-consistent equation

$$\gamma(k) = \frac{1}{\langle k\rangle}\sum_{\hat{m}\leqslant k'\leqslant K} Q(k,k')p(k')\frac{k'}{\gamma(k')}. \tag{B2}$$

By expressing the Kronecker deltas in integral form,

$$\delta(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{\mathrm{i}\omega(x-y)}, \tag{B3}$$

we get

$$Z = \frac{1}{(2L)!!}\sum_{\mathbf{k}}{}' \int \mathcal{D}\boldsymbol{\omega} \int \frac{d\lambda}{2\pi} e^{G(\boldsymbol{\omega},\lambda,\mathbf{k})}, \tag{B4}$$

where $G(\boldsymbol{\omega},\lambda,\mathbf{k})$ is given by

$$G(\boldsymbol{\omega},\lambda,\mathbf{k}) = \sum_{i=1}^{N}[\mathrm{i}\omega_i k_i + \ln(k_i!p(k_i)) - k_i\ln\gamma(k_i)] + \mathrm{i}\lambda L + \frac{1}{2}\sum_{i,j}\ln(1 + Q(k_i,k_j)e^{-\mathrm{i}\lambda-\mathrm{i}\omega_i-\mathrm{i}\omega_j}), \tag{B5}$$

and where $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$. Let us now introduce the functional order parameter [43,48,49]

$$c(\omega,k) = \frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i), \tag{B6}$$

by enforcing its definition with a series of delta functions. Therefore, by assuming a discretization in ω in intervals of size $\Delta\omega$ we introduce for every (ω,k) the term

$$1 = \int dc(\omega,k)\delta\left(c(\omega,k) - \frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)\right) = \int \frac{d\hat{c}(\omega,k)dc(\omega,k)}{2\pi/(N\Delta\omega)} e^{\mathrm{i}\Delta\omega\hat{c}(\omega,k)[Nc(\omega,k)-\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)]}. \tag{B7}$$

After performing these operations, by imposing $2L = \langle k\rangle N$ where $\langle k\rangle = \sum_k kp(k)$, the partition function reads in the limit $\Delta\omega \to 0$

$$Z = \frac{1}{(2L)!!}\sum_{\mathbf{k}}{}' \int \mathcal{D}c(\omega,k)\int \mathcal{D}\hat{c}(\omega,k)\int \frac{d\lambda}{2\pi} e^{Nf(\lambda,c(\omega,k),\hat{c}(\omega,k))}$$

with

$$f(\lambda,c(\omega,k),\hat{c}(\omega,k)) = \mathrm{i}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K}\hat{c}(\omega,k)c(\omega,k) + \mathrm{i}\lambda\langle k\rangle/2 + \Psi + \ln\int d\omega \sum_{\hat{m}\leqslant k\leqslant K} p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)}, \tag{B8}$$

where Ψ is given by

$$\Psi = \frac{N}{2}\int d\omega \int d\omega' \sum_{\hat{m}\leqslant k\leqslant K,\hat{m}\leqslant k'\leqslant K} c(\omega,k)c(\omega',k')Q(k,k')\ln(1 + e^{-\mathrm{i}\lambda-\mathrm{i}\omega-\mathrm{i}\omega'}), \tag{B9}$$

where $\mathcal{D}c(\omega,k)$ and $\mathcal{D}\hat{c}(\omega,k)$ have the same definition then in the simple uncorrelated case. Performing a Wick rotation in λ and assuming $z/N = e^{-\mathrm{i}\lambda}$ real and much smaller than one, i.e., $z/N \ll 1$, which is allowed in the sparse regime $K \ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi = \frac{z}{2}\int d\omega \int d\omega' \sum_{\hat{m}\leqslant k\leqslant K, \hat{m}\leqslant k'\leqslant K} c(\omega,k)c(\omega',k')Q(k,k')e^{-\mathrm{i}\omega-\mathrm{i}\omega'}.$$

The saddle-point equations determining the value of the partition function read

$$-\mathrm{i}\hat{c}(\omega,k) = ze^{-\mathrm{i}\omega}\int d\omega' \sum_{\hat{m}\leqslant k'\leqslant K} Q(k,k')c(\omega',k')e^{-\mathrm{i}\omega'},$$

$$c(\omega,k) = \frac{\frac{1}{2\pi}p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)}}{\frac{1}{2\pi}\int d\omega' \sum_{\hat{m}\leqslant k'\leqslant K} p(k')\frac{k'!}{[\gamma(k')]^{k'}}e^{\mathrm{i}\omega' k'-\mathrm{i}\hat{c}(\omega',k')}},$$

$$z\int d\omega\int d\omega' \sum_{\hat{m}\leqslant k\leqslant K, \hat{m}\leqslant k'\leqslant K} c(\omega,k)c(\omega',k')Q(k,k')e^{-\mathrm{i}\omega-\mathrm{i}\omega'} = \langle k\rangle. \tag{B10}$$

Let us define $\tilde{\gamma}(k)$ as

$$\tilde{\gamma}(k) = z\int d\omega' \sum_{\hat{m}\leqslant k'\leqslant K} Q(k,k')c(\omega',k')e^{-\mathrm{i}\omega'}. \tag{B11}$$

With this definition we have

$$-i\hat{c}(\omega,k) = \tilde{\gamma}(k)e^{-\mathrm{i}\omega}. \tag{B12}$$

Let us first calculate the integral

$$\frac{1}{2\pi}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K} p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)} = \frac{1}{2\pi}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K}\frac{k!}{[\gamma(k)]^k}p(k)e^{\mathrm{i}\omega k+\tilde{\gamma}(k)e^{-\mathrm{i}\omega}}, \tag{B13}$$

where we have substituted the saddle-point expression for $\hat{c}(\omega,k)$. This integral can be also written as

$$\int d\omega \sum_{\hat{m}\leqslant k\leqslant K}\frac{k!}{\gamma(k)^k}p(k)e^{\mathrm{i}\omega k}\sum_{h=0}^{\infty}\frac{(\tilde{\gamma}(k))^h}{h!}e^{-\mathrm{i}\omega h} = \sum_{\hat{m}\leqslant k\leqslant K} p(k)\left(\frac{\tilde{\gamma}(k)}{\gamma(k)}\right)^k.$$

Let w indicate the value of this integral, i.e.,

$$w = \sum_{\hat{m}\leqslant k\leqslant K} p(k)\left(\frac{\tilde{\gamma}(k)}{\gamma(k)}\right)^k. \tag{B14}$$

The functional order parameter $c(\omega,k)$ at the saddle-point solution can be expressed as

$$c(\omega,k) = \frac{1}{2\pi w}\frac{k!p(k)}{[\gamma(k)]^k}e^{\mathrm{i}\omega k+\tilde{\gamma}(k)e^{-\mathrm{i}\omega}}. \tag{B15}$$

With this expression, using a similar procedure we can express $\tilde{\gamma}(k)$ as

$$\begin{aligned}\tilde{\gamma}(k) &= z\int d\omega' \sum_{\hat{m}\leqslant k'\leqslant K} Q(k,k')c(\omega',k')e^{-\mathrm{i}\omega'}\\ &= \frac{z}{w}\sum_{m\leqslant k'\leqslant K} Q(k,k')p(k')\frac{k'}{\tilde{\gamma}(k')}\left(\frac{\tilde{\gamma}(k')}{\gamma(k')}\right)^{k'}.\end{aligned} \tag{B16}$$

Combining this equation with the third saddle-point equation we get

$$\begin{aligned}&z\int d\omega\int d\omega' \sum_{\hat{m}\leqslant k\leqslant K}\sum_{\hat{m}\leqslant k'\leqslant K} Q(k,k')c(\omega,k)c(\omega',k')e^{-\mathrm{i}\omega-\mathrm{i}\omega'}\\ &= \frac{1}{w}\sum_{m\leqslant k\leqslant K} p(k)k\left(\frac{\tilde{\gamma}(k)}{\gamma(k)}\right)^k = \langle k\rangle.\end{aligned} \tag{B17}$$

Given that $\gamma(k)$ is defined through Eq. (B2), it follows that

$$\tilde{\gamma}(k) = \gamma(k), \quad w = 1, \quad z = \frac{1}{\langle k\rangle}. \tag{B18}$$

Finally using Eqs. (F32) we can derive the final expression for $c(\omega,k)$ given by

$$c(\omega,k) = \frac{1}{2\pi}\frac{k!p(k)}{[\gamma(k)]^k}e^{\mathrm{i}\omega k+\gamma(k)e^{-\mathrm{i}\omega}}. \tag{B19}$$

From this equation of the functional order parameter we can derive the marginal for each link of the network which is given by

$$p_{ij} = \frac{1}{N}\int d\omega\int d\omega' \sum_{\hat{m}\leqslant k\leqslant K, \hat{m}\leqslant k'\leqslant K} c(\omega,k)c(\omega',k')Q(k,k') \times e^{-\mathrm{i}\omega-\mathrm{i}\omega'},$$

yielding

$$p_{ij} = \sum_{\hat{m}\leqslant k\leqslant K, \hat{m}\leqslant k'\leqslant K} p(k)p(k')p(k,k'). \tag{B20}$$

Here $p(k,k')$ indicates the probability of a link between node i and node j conditioned to the degree of the two nodes $k_i = k$

and $k_j = k'$, i.e.,

$$p_{ij|k_i=k,k_j=k'} = p(k,k') = \frac{1}{\langle k\rangle N} Q(k,k') \frac{kk'}{\gamma(k)\gamma(k')}. \tag{B21}$$

Note that for $Q(k,k')=1$ it follows that $\gamma(k)=1$, and for $Q(k,k')=kk'$ it follows that $\gamma(k)=k$ and hence in both cases we recover the exchangeable network ensemble of simple uncorrelated networks.

APPENDIX C: EXCHANGEABLE ENSEMBLE OF SPARSE DIRECTED NETWORKS

Derivation of the marginal probability

In this section our goal is the solve the partition function Z for the exchangeable ensemble of directed networks using the saddle-point equation expression for the marginal probability of a link. For simplicity for this ensemble we put the auxiliary fields $h=0$ from the beginning and we express the partition function Z as

$$Z = \sum_{\mathbf{a}} \sum_{\mathbf{k}^{in}}{}' \sum_{\mathbf{k}^{\text{out}}}{}' e^{-H(G)} \prod_{i=1}^{N} \left[\delta\left(k_i^{in} - \sum_{j=1}^{N} a_{ji}\right) \delta\left(k_i^{\text{out}} - \sum_{j=1}^{N} a_{ij}\right)\right] \delta\left(L, \sum_{i,j} a_{ij}\right). \tag{C1}$$

By expressing the Kronecker delta's in integral form,

$$\delta(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{\mathrm{i}\omega(x-y)}, \tag{C2}$$

we get

$$Z = \frac{1}{L!} \sum_{\mathbf{a}} \sum_{\mathbf{k}^{in}}{}' \sum_{\mathbf{k}^{\text{out}}}{}' \int \mathcal{D}\omega \int \mathcal{D}\hat{\omega} \int \frac{d\lambda}{2\pi} e^{G(\lambda,\boldsymbol{\omega},\hat{\boldsymbol{\omega}},\mathbf{k}^{in},\mathbf{k}^{\text{out}})},$$

with

$$G(\lambda,\boldsymbol{\omega},\hat{\boldsymbol{\omega}},\mathbf{k}^{in},\mathbf{k}^{\text{out}}) = \sum_{i=1}^{N} \left[\mathrm{i}\omega_i k_i^{in} + \mathrm{i}\hat{\omega}_i k_i^{\text{out}} + \ln\left(k_i^{in}! k_i^{\text{out}}! p_d\left(k_i^{in}, k_i^{\text{out}}\right)\right)\right] + \mathrm{i}\lambda L + \sum_{i,j} \ln(1 + e^{-\mathrm{i}\lambda - \mathrm{i}\omega_i - \mathrm{i}\hat{\omega}_j}), \tag{C3}$$

and with $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$, and $\mathcal{D}\hat{\omega} = \prod_{i=1}^{N}[d\hat{\omega}_i/(2\pi)]$. Let us now introduce the functional order parameter [43,48,49]

$$c(\omega,\hat{\omega},k^{in},k^{\text{out}}) = \frac{1}{N}\sum_{i=1}^{N} \delta(\omega-\omega_i)\delta(\hat{\omega}-\hat{\omega}_i)\delta\left(k^{in},k_i^{in}\right)\delta\left(k^{\text{out}},k_i^{\text{out}}\right),$$

by enforcing its definition with a series of delta functions by introducing the conjugated order parameter $\hat{c}(\omega,\hat{\omega},k^{in},k^{\text{out}})$ and by imposing $L = \langle k^{in}\rangle N = \langle k^{\text{out}}\rangle N$ where $\langle k^{in}\rangle = \sum_{k^{in},k^{\text{out}}} k^{in} p_d(k^{in},k^{\text{out}})$, $\langle k^{\text{out}}\rangle = \sum_{k^{in},k^{\text{out}}} k^{\text{out}} p_d(k^{in},k^{\text{out}})$. The partition function reads

$$Z = \frac{1}{L!} \sum_{\mathbf{k}^{in}}{}' \sum_{\mathbf{k}^{\text{out}}}{}' \int \mathcal{D}c(\omega,\hat{\omega},k^{in},k^{\text{out}}) \int \mathcal{D}\hat{c}(\omega,\hat{\omega},k^{in},k^{\text{out}}) \int \frac{d\lambda}{2\pi} e^{Nf} \tag{C4}$$

with

$$\begin{aligned} f = {} & \mathrm{i}\int d\omega \int d\hat{\omega} \sum_{\hat{m}\leqslant k^{in}\leqslant K, \hat{m}\leqslant k^{\text{out}}\leqslant K} \hat{c}(\omega,\hat{\omega},k^{in},k^{\text{out}}) c(\omega,\hat{\omega},k^{in},k^{\text{out}}) + \mathrm{i}\lambda\langle k^{in}\rangle \\ & + \Psi + \ln \int \frac{d\omega}{2\pi} \sum_{\hat{m}\leqslant k^{in}\leqslant K; \hat{m}\leqslant k^{\text{out}}\leqslant K} p_d(k^{in},k^{\text{out}}) k^{in}! k^{\text{out}}! \exp[\mathrm{i}\omega k^{in} + \mathrm{i}\hat{\omega} k^{\text{out}} - \mathrm{i}\hat{c}(\omega,\hat{\omega},k^{in},k^{\text{out}})], \end{aligned} \tag{C5}$$

where Ψ is given by

$$\Psi = \frac{N}{2}\int d\omega \int d\omega' \sum_{\hat{m}\leqslant k^{in}\leqslant K, \hat{m}\leqslant k^{\text{out}}\leqslant K} \sum_{\hat{m}\leqslant k'^{,in}\leqslant K, \hat{m}\leqslant k'^{,out}\leqslant K} c(\omega,\hat{\omega},k^{in},k^{\text{out}}) c(\omega',\hat{\omega}',k'^{,in},k'^{,out}) \ln(1 + e^{-\mathrm{i}\lambda - \mathrm{i}\omega - \mathrm{i}\hat{\omega}'}),$$

and where $\mathcal{D}c(\omega,\hat{\omega},k^{in},k^{\text{out}})$ and $\mathcal{D}\hat{c}(\omega,\hat{\omega},k^{in},k^{\text{out}})$ are functional measures. Performing a Wick rotation in λ and assuming $z/N = e^{-\mathrm{i}\lambda}$ real and much smaller than one, i.e., $z/N \ll 1$, which is allowed in the sparse regime $K \ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi = z\nu\hat{\nu}, \tag{C6}$$

with

$$\nu = \int d\omega \int d\hat{\omega} \sum_{\hat{m}\leqslant k^{in}\leqslant K;\hat{m}\leqslant k^{\rm out}\leqslant K} c(\omega,\hat{\omega},k^{in},k^{\rm out})e^{-{\rm i}\omega},$$
$$\hat{\nu} = \int d\omega \int d\hat{\omega} \sum_{\hat{m}\leqslant k^{in}\leqslant K;\hat{m}\leqslant k^{\rm out}\leqslant K} c(\omega,\hat{\omega},k^{in},k^{\rm out})e^{-{\rm i}\hat{\omega}}. \tag{C7}$$

The saddle-point equations determining the value of the partition function can be obtained by performing the (functional) derivative of $f(\lambda, c(\omega,k), \hat{c}(\omega,k))$ with respect to $c(\omega,\hat{\omega},k^{in},k^{\rm out})$, $\hat{c}(\omega,\hat{\omega},k^{in},k^{ouy})$, and λ, obtaining

$$-{\rm i}\hat{c}(\omega,\hat{\omega},k^{in},k^{\rm out}) = z\hat{\nu}e^{-{\rm i}\omega}+z\nu e^{-{\rm i}\hat{\omega}},$$
$$c(\omega,\hat{\omega},k^{in},k^{\rm out}) = \frac{\frac{1}{(2\pi)^2}p_d(k^{in},k^{\rm out})k^{in}!k^{out}!\exp[{\rm i}\omega k^{in}+{\rm i}\hat{\omega}k^{\rm out}-{\rm i}\hat{c}(\omega,\hat{\omega},k^{in},k^{\rm out})]}{\int\frac{d\omega'}{2\pi}\int\frac{d\hat{\omega}'}{2\pi}\sum_{\hat{m}\leqslant k^{\prime,in}\leqslant K;\hat{m}\leqslant k^{\prime,out}\leqslant K}p_d(k^{\prime,in},k^{\prime,out})k^{\prime,in}!k^{\prime,out}!\exp[{\rm i}\omega' k^{\prime,in}+{\rm i}\hat{\omega}'k^{\prime,out}-{\rm i}\hat{c}(\omega',\hat{\omega}',k^{\prime,in},k^{\prime,out})]},$$
$$z\nu\hat{\nu} = \langle k^{in}\rangle. \tag{C8}$$

Let us first calculate the integral

$$\int\frac{d\omega}{2\pi}\int\frac{d\hat{\omega}}{2\pi}\sum_{\hat{m}\leqslant k^{in}\leqslant K;\hat{m}\leqslant k^{\rm out}\leqslant K}p_d(k^{in},k^{\rm out})k^{in}!k^{\rm out}!\exp[{\rm i}\omega k^{in}+{\rm i}\hat{\omega}k^{\rm out}-{\rm i}\hat{c}(\omega,\hat{\omega},k^{in},k^{\rm out})], \tag{C9}$$

where we have substituted the saddle-point expression for $\hat{c}(\omega,k)$. Expanding the exponential and proceeding as in the simple uncorrelated case we get

$$\int\frac{d\omega}{2\pi}\int\frac{d\hat{\omega}}{2\pi}\sum_{\hat{m}\leqslant k^{in}\leqslant K;\hat{m}\leqslant k^{\rm out}\leqslant K}p_d(k^{in},k^{\rm out})k^{in}!k^{\rm out}!\exp[{\rm i}\omega k^{in}+{\rm i}\hat{\omega}k^{\rm out}-{\rm i}\hat{c}(\omega,\hat{\omega},k^{in},k^{\rm out})]$$
$$=\sum_{\hat{m}\leqslant k^{in}\leqslant K}\sum_{\hat{m}\leqslant k^{\rm out}\leqslant K}p_d(k^{in},k^{\rm out})(z\hat{\nu})^{k^{in}}(z\nu)^{k^{\rm out}} = \langle(z\hat{\nu})^{k^{in}}(z\nu)^{k^{\rm out}}\rangle. \tag{C10}$$

Therefore, $c(\omega,k)$ at the saddle-point solution can be expressed as

$$c(\omega,k) = \frac{1}{(2\pi)^2}\frac{k^{in}!k^{\rm out}!p_d(k^{in},k^{\rm out})e^{{\rm i}\omega k+(z\hat{\nu})e^{-{\rm i}\omega}+(z\nu)e^{-{\rm i}\hat{\omega}}}}{\langle(z\hat{\nu})^{k^{in}}(z\nu)^{k^{\rm out}}\rangle}. \tag{C11}$$

With this expression, using a similar procedure we can express ν as

$$\nu = \frac{1}{\langle(z\hat{\nu})^{k^{in}}(z\nu)^{k^{\rm out}}\rangle}\sum_{\hat{m}\leqslant k^{in}\leqslant K}\sum_{\hat{m}\leqslant k^{\rm out}\leqslant K}k^{in}p_d(k^{in},k^{\rm out})(z\hat{\nu})^{k^{in}-1}(z\nu)^{k^{\rm out}},$$
$$\hat{\nu} = \frac{1}{\langle(z\hat{\nu})^{k^{in}}(z\nu)^{k^{\rm out}}\rangle}\sum_{\hat{m}\leqslant k^{in}\leqslant K}\sum_{\hat{m}\leqslant k^{\rm out}\leqslant K}k^{\rm out}p_d(k^{in},k^{\rm out})(z\hat{\nu})^{k^{in}}(z\nu)^{k^{\rm out}-1}. \tag{C12}$$

Combining this equation with the third saddle-point equation

$$z\nu\hat{\nu} = \langle k^{in}\rangle = \langle k^{\rm out}\rangle, \tag{C13}$$

it is immediate to show that $z\nu = z\hat{\nu} = 1$ is a solution with

$$z = \frac{1}{\langle k^{in}\rangle}, \quad \nu = \hat{\nu} = \langle k^{in}\rangle = \langle k^{\rm out}\rangle. \tag{C14}$$

By inserting this expression in Eq. (C11) we get

$$c(\omega,\hat{\omega},k^{in},k^{\rm out}) = \frac{1}{(2\pi)^2}k^{in}!k^{\rm out}!p_d(k^{in},k^{\rm out})\exp[{\rm i}\omega k^{in}+{\rm i}\hat{\omega}k^{\rm out}+e^{-i\omega}+e^{-{\rm i}\hat{\omega}}]. \tag{C15}$$

From this equation we can conclude that the networks of these ensembles have heterogeneous degree distribution, as the density of nodes of in-degree k^{in} and out-degree $k^{\rm out}$ is given the desired joint probability distribution, i.e.,

$$\int d\omega\int d\hat{\omega}c(\omega,\hat{\omega},k^{in},k^{\rm out}) = p_d(k^{in},k^{\rm out}). \tag{C16}$$

However, the marginal for each link is the same and given by Eq. (30) with the marginal probability of a link conditioned on the degrees of its two end nodes given by Eq. (32).

APPENDIX D: EXCHANGEABLE ENSEMBLE OF SPARSE BIPARTITE NETWORKS

Derivation of the marginal probability

In this section our goal is the solve the partition function Z for the exchangeable ensemble of bipartite networks using the saddle-point equation expression for the marginal probability of a link. The partition function Z of this network ensemble, where for simplicity we have put the auxiliary field $h = 0$ from the beginning, is given by

$$Z = \sum_{\mathbf{a}} \sum_{\mathbf{k}}{}' \sum_{\mathbf{q}}{}' e^{-H(G)} \delta\left(L, \sum_{i,\mu} a_{i\mu}\right) \prod_{i=1}^{N}\left[\delta\left(k_i - \sum_{\mu=1}^{M} b_{i\mu}\right)\right] \prod_{\mu=1}^{M}\left[\delta\left(q_\mu - \sum_{i=1}^{N} b_{i\mu}\right)\right]. \tag{D1}$$

By expressing the Kronecker delta's in integral form,

$$\delta(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{\mathrm{i}\omega(x-y)}, \tag{D2}$$

we get

$$Z = \sum_{\mathbf{a}} \mathbb{P}(G) = \frac{1}{L!}\sum_{\mathbf{a}} \sum_{\mathbf{k}}{}' \sum_{\mathbf{q}}{}' \int \mathcal{D}\omega \int \mathcal{D}\hat{\omega} \int \frac{d\lambda}{2\pi} e^{G(\lambda, \boldsymbol{\omega}, \hat{\boldsymbol{\omega}}, \mathbf{k}, \mathbf{q})},$$

with

$$G(\lambda, \boldsymbol{\omega}, \hat{\boldsymbol{\omega}}, \mathbf{k}, \mathbf{q}) = \sum_{i=1}^{N}[\mathrm{i}\omega_i k_i + \ln(k_i! p(k_i))] + \sum_{\mu=1}^{M}[\mathrm{i}\hat{\omega}_\mu q_\mu + \ln(q_\mu! \hat{p}(q_\mu))] + \mathrm{i}\lambda L + \sum_{i,\mu}\ln(1 + e^{-\mathrm{i}\lambda - \mathrm{i}\omega_i - \mathrm{i}\hat{\omega}_\mu}), \tag{D3}$$

and with $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$, and $\mathcal{D}\hat{\omega} = \prod_{\mu=1}^{M}[d\hat{\omega}_i/(2\pi)]$. Let us now introduce the two functional order parameters [43,48,49]

$$c(\omega, k) = \frac{1}{N}\sum_{i=1}^{N}\delta(\omega - \omega_i)\delta(k, k_i), \quad \sigma(\hat{\omega}, q) = \frac{1}{M}\sum_{\mu=1}^{M}\delta(\hat{\omega} - \hat{\omega}_\mu)\delta(q, q_\mu), \tag{D4}$$

by enforcing their definition with a series of delta functions involving $c(\omega, k)$ and $\sigma(\omega, k)$ and their conjugated order parameters $\hat{c}(\omega, k)$ and $\hat{\sigma}(\omega, k)$. Imposing also $L = \langle k\rangle N = \langle q\rangle M$ where $\langle k\rangle = \sum_k k p(k)$, $\langle q\rangle = \sum_q q\hat{p}(q)$, the partition function reads

$$Z = \frac{1}{L!}\sum_{\mathbf{k}}{}' \sum_{\mathbf{q}}{}' \int \mathcal{D}\hat{c}(\omega, k) \int \mathcal{D}c(\omega, k) \int \mathcal{D}\hat{\sigma}(\hat{\omega}, q) \int \mathcal{D}\sigma(\hat{\omega}, q) \int \frac{d\lambda}{2\pi} e^{Nf} \tag{D5}$$

with

$$\begin{aligned} f = {}& \mathrm{i}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K} \hat{c}(\omega, k)c(\omega, k) + i\alpha \int d\hat{\omega} \sum_{\hat{m}\leqslant q\leqslant \hat{K}} \hat{\sigma}(\hat{\omega}, q)\sigma(\hat{\omega}, q) + \mathrm{i}\lambda\langle k\rangle \\ & + \Psi + \ln\int \frac{d\omega}{2\pi} \sum_{\hat{m}\leqslant k\leqslant K} p(k)k! \exp[\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega, k)] + \alpha \ln\int \frac{d\hat{\omega}}{2\pi} \sum_{\hat{m}\leqslant q\leqslant \hat{K}} \hat{p}(q)q! \exp[\mathrm{i}\hat{\omega} q - \mathrm{i}\hat{\sigma}(\hat{\omega}, q)], \end{aligned} \tag{D6}$$

and Ψ is given by

$$\Psi = \frac{\alpha N}{2}\int d\omega \int d\hat{\omega} \sum_{\hat{m}\leqslant k\leqslant K, \hat{m}\leqslant q\leqslant \hat{K}} c(\omega, k)\sigma(\hat{\omega}, q)\ln(1 + e^{-\mathrm{i}\lambda - \mathrm{i}\omega - \mathrm{i}\hat{\omega}}),$$

where $\mathcal{D}\hat{c}(\omega, k)$, $\mathcal{D}c(\omega, k)$, $\mathcal{D}\hat{\sigma}(\hat{\omega}, q)$, and $\mathcal{D}\sigma(\hat{\omega}, q)$ are functional measures. Performing a Wick rotation in λ and assuming $z/N = e^{-\mathrm{i}\lambda}$ real and much smaller than one, i.e., $z/N \ll 1$, which is allowed in the sparse regime $K \ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi = z\alpha\nu\hat{\nu}, \tag{D7}$$

with

$$\nu = \int d\omega \sum_{\hat{m}\leqslant k\leqslant K} c(\omega, k)e^{-\mathrm{i}\omega}, \quad \hat{\nu} = \int d\hat{\omega} \sum_{\hat{m}\leqslant q\leqslant \hat{K}} \sigma(\hat{\omega}, q)e^{-\mathrm{i}\hat{\omega}}. \tag{D8}$$

The saddle-point equations determining the value of the partition function can be obtained by performing the (functional) derivative of $f(\lambda, c(\omega, k), \hat{c}(\omega, k))$ with respect to $c(\omega, \hat{\omega}, k^{in}, k^{\mathrm{out}})$, $\hat{c}(\omega, \hat{\omega}, k^{in}, k^{\mathrm{out}})$, and λ, obtaining

$$-\mathrm{i}\hat{c}(\omega,k)=\alpha z\hat{\nu}e^{-\mathrm{i}\omega},\quad c(\omega,k)=\frac{\frac{1}{(2\pi)}p(k)k!\exp\left[\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)\right]}{\int\frac{d\omega'}{2\pi}\sum_{m\leqslant k'\leqslant K}p(k')k'!\exp\left[\mathrm{i}\omega'k'-\mathrm{i}\hat{c}(\omega',k')\right]},$$
$$-\mathrm{i}\hat{\sigma}(\hat{\omega},q)=z\nu e^{-\mathrm{i}\hat{\omega}},\quad \sigma(\hat{\omega},q)=\frac{\frac{1}{2\pi}\hat{p}(q)q!\exp\left[\mathrm{i}\hat{\omega}q-\mathrm{i}\hat{\sigma}(\hat{\omega},q)\right]}{\int\frac{d\hat{\omega}'}{2\pi}\sum_{\hat{m}\leqslant q'\leqslant\hat{K}}\tilde{p}(q')q'!\exp\left[\mathrm{i}\hat{\omega}'q'-\mathrm{i}\hat{\sigma}(\hat{\omega}',q')\right]},\quad z\alpha\nu\hat{\nu}=\langle k\rangle. \tag{D9}$$

Let us first calculate the integrals

$$\int\frac{d\omega}{2\pi}\sum_{\bar{m}\leqslant k\leqslant K}p(k)k!\exp\left[\mathrm{i}\omega k-\mathrm{i}\hat{c}(\omega,k)\right]=\sum_{\bar{m}\leqslant k\leqslant K}p(k)(\alpha z\hat{\nu})^k=\langle(\alpha z\hat{\nu})^k\rangle,$$
$$\int\frac{d\hat{\omega}}{2\pi}\sum_{\hat{m}\leqslant q\leqslant\hat{K}}\hat{p}(q)q!\exp\left[\mathrm{i}\hat{\omega}q-\mathrm{i}\hat{\sigma}(\hat{\omega},q)\right]=\sum_{\hat{m}\leqslant q\leqslant\hat{K}}\hat{p}(q)(z\hat{\nu})^q=\langle(z\nu)^q\rangle, \tag{D10}$$

where we have substituted the saddle-point expression for $\hat{c}(\omega,k)$ and $\hat{\sigma}(\hat{\omega},q)$ and we have followed the same procedure as for calculating the corresponding integrals in the previous case. It follows that $c(\omega,k)$ and $\sigma(\hat{\omega},q)$ at the saddle-point solution can be expressed as

$$c(\omega,k)=\frac{1}{2\pi}\frac{k!p(k)e^{\mathrm{i}\omega k+(\alpha z\hat{\nu})e^{-\mathrm{i}\omega}}}{\langle(\alpha z\hat{\nu})^k\rangle},\quad \sigma(\hat{\omega},q)=\frac{\alpha}{2\pi}\frac{q!\hat{p}(q)e^{\mathrm{i}\hat{\omega}q+(z\nu)e^{-\mathrm{i}\hat{\omega}}}}{\langle(z\nu)^q\rangle}. \tag{D11}$$

With this expression, using a similar procedure as in the precedent integrals, we can express ν as

$$\nu=\frac{1}{\langle(\alpha z\hat{\nu})^k\rangle}\sum_{\tilde{m}\leqslant k\leqslant K}kp(k)(\alpha z\hat{\nu})^{k-1},\quad \hat{\nu}=\frac{1}{\langle(z\nu)^q\rangle}\sum_{\hat{m}\leqslant q\leqslant\hat{K}}q\hat{p}(q)(z\nu)^{q-1}. \tag{D12}$$

Combining this equation with the third saddle-point equation

$$\alpha z\nu\hat{\nu}=\langle k\rangle=\alpha\langle q\rangle, \tag{D13}$$

it is immediate to show that $z\nu=\alpha z\hat{\nu}=1$ is a solution with

$$z=\frac{1}{\langle k\rangle},\quad \nu=\langle k\rangle,\quad \hat{\nu}=\langle q\rangle. \tag{D14}$$

By inserting this expression in Eq. (D11) we get

$$c(\omega,k)=\frac{1}{2\pi}k!p(k)\exp[\mathrm{i}\omega k+e^{-\mathrm{i}\omega}],\quad \sigma(\hat{\omega},q)=\frac{1}{2\pi}q!\hat{p}(q)\exp[\mathrm{i}\hat{\omega}q+e^{-\mathrm{i}\hat{\omega}}]. \tag{D15}$$

From this equation we can conclude that the networks of these ensembles have heterogeneous degree distribution, as the density of nodes in V with degree k is given by $p(k)$ while the density of nodes in U having degree q is given by $\hat{p}(q)$, i.e.,

$$\int d\omega\int d\hat{\omega}c(\omega,\hat{\omega},k)=p(k),\quad \int d\omega\int d\hat{\omega}\sigma(\omega,\hat{\omega},1)=\hat{p}(q). \tag{D16}$$

However, the marginal for each link is the same and given by Eq. (37) with the marginal probability of a link conditioned on the degrees of its two end nodes given by Eq. (38).

APPENDIX E: EXCHANGEABLE ENSEMBLE OF SPARSE MULTIPLEX NETWORKS

Treatment of the exchangeable ensemble of sparse multiplex networks

In this section our goal is the solve the partition function Z for the exchangeable ensemble of multiplex networks. The partition function Z of this multiplex network ensemble is given by

$$Z(\mathbf{h})=\sum_{\mathbf{A}}\sum_{\{\mathbf{k}^{\vec{m}}\}}{}^{'}e^{-H(G)}e^{-\sum_{ij}\sum_{\vec{m}\neq\vec{0}}h^{\vec{m}}A_{ij}^{\vec{m}}}\prod_{\vec{m}\neq\vec{0}}\left[\delta\left(L^{\vec{m}},\sum_{i,j}A_{ij}^{\vec{m}}\right)\prod_{i=1}^{N}\delta\left(k_i^{\vec{m}}-\sum_{j=1}^{N}A_{ji}^{\vec{m}}\right)\right]. \tag{E1}$$

Here and in the following we use the notation $\sum_{\mathbf{k}}'$ to indicate the sum over all the possible values of the degree of each node i satisfying $\hat{m}\leqslant k_i^{\vec{m}}\leqslant K^{\vec{m}}\ll K_s^{\vec{m}}=\sqrt{\langle k^{\vec{m}}\rangle N}$. By expressing the Kronecker delta's in integral form,

$$\delta(x,y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega e^{\mathrm{i}\omega(x-y)}, \tag{E2}$$

we get for the partition function Z of this network ensemble

$$Z(\mathbf{h}) = \frac{1}{\prod_{\vec{m}\neq\vec{0}}(2L^{\vec{m}})!!}\sum_{\mathbf{A}}\sum_{\{\mathbf{k}^{\vec{m}}\}}{}^{\prime}\int\mathcal{D}\boldsymbol{\omega}\int\frac{d\lambda}{2\pi}e^{G(\lambda,\boldsymbol{\omega},\mathbf{k}^{\vec{m}},\mathbf{h})}, \tag{E3}$$

with

$$G(\lambda,\boldsymbol{\omega},\{k^{\vec{m}}\},\mathbf{h}) = \sum_{i=1}^{N}\left\{\sum_{\vec{m}\neq\vec{0}}\left[\mathrm{i}\omega_i^{\vec{m}}k_i^{\vec{m}} + \ln\left(k_i^{\vec{m}}!\right)\right] + \ln\tilde{\pi}\left(\mathbf{k}_i^{\vec{m}}\right)\right\} + \mathrm{i}\sum_{\vec{m}\neq\vec{0}}\lambda^{\vec{m}}L^{\vec{m}} + \frac{1}{2}\sum_{i,j}\ln\left(1+\sum_{\vec{m}\neq\vec{0}}e^{-\mathrm{i}\lambda^{\vec{m}}-\mathrm{i}\omega_i^{\vec{m}}-\mathrm{i}\omega_j^{\vec{m}}-h^{\vec{m}}}\right),$$

and with $\mathcal{D}\boldsymbol{\omega} = \prod_{i=1}^{N}\prod_{\vec{m}\neq\vec{0}}[d\omega_i^{\vec{m}}/(2\pi)]$. Let us now introduce the functional order parameter [43,48,49]

$$c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}) = \frac{1}{N}\sum_{i=1}^{N}\prod_{\vec{m}\neq\vec{0}}\delta\left(\omega^{\vec{m}}-\omega_i^{\vec{m}}\right)\delta\left(k^{\vec{m}},k_i^{\vec{m}}\right), \tag{E4}$$

by enforcing its definition with a series of delta functions. By imposing $2L^{\vec{m}} = \langle k^{\vec{m}}\rangle N$, where $\langle k^{\vec{m}}\rangle = \sum_{k^{\vec{m}}} k^{\vec{m}}p(k^{\vec{m}})$, we get

$$Z(\mathbf{h}) = \frac{1}{\prod_{\vec{m}\neq\vec{0}}(2L^{\vec{m}})!!}\sum_{\{\mathbf{k}^{\vec{m}}\}}{}^{\prime}\int\mathcal{D}c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})\int\mathcal{D}\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})\int\frac{d\lambda}{2\pi}e^{Nf(\lambda,c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}),\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}),\mathbf{h})} \tag{E5}$$

with

$$\begin{aligned} f(\lambda,c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}),\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}),\mathbf{h})) &= \mathrm{i}\int d\boldsymbol{\omega}\sum_{\mathbf{k}^{\vec{m}}}{}^{\prime}\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}) + \mathrm{i}\sum_{\vec{m}\neq\vec{0}}\lambda^{\vec{m}}\langle k^{\vec{m}}\rangle/2 + \Psi \\ &\quad + \ln\int\frac{d\boldsymbol{\omega}}{(2\pi)^W}\sum_{\mathbf{k}^{\vec{m}}}{}^{\prime}\tilde{\pi}(\mathbf{k}^{\vec{m}})\left[\prod_{\vec{m}\neq\vec{0}}k^{\vec{m}}!e^{\mathrm{i}\omega^{\vec{m}}k^{\vec{m}}}\right]e^{-\mathrm{i}\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})}, \end{aligned} \tag{E6}$$

where $W = 2^M - 1$ indicates the number of nontrivial multilinks $\vec{m}\neq\vec{0}$ and Ψ is given by

$$\Psi = \frac{N}{2}\int d\boldsymbol{\omega}\int d\boldsymbol{\omega}^{\prime}\sum_{\mathbf{k}^{\vec{m}},\mathbf{k}^{\prime,\vec{m}}}{}^{\prime}c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})c(\boldsymbol{\omega}^{\prime},\mathbf{k}^{\prime,\vec{m}})\ln\left(1+\sum_{\vec{m}\neq\vec{0}}e^{-\mathrm{i}\lambda^{\vec{m}}-\mathrm{i}\omega^{\vec{m}}-\mathrm{i}\omega^{\prime\vec{m}}-h^{\vec{m}}}\right)$$

and where $\mathcal{D}c(\boldsymbol{\omega},\mathbf{k})$ and $\mathcal{D}\hat{c}(\boldsymbol{\omega},\mathbf{k})$ are functional measures. Performing a Wick rotation in λ and assuming $z^{\vec{m}}/N = e^{-i\lambda^{\vec{m}}}$ real and much smaller than one, i.e., $z^{\vec{m}}/N \ll 1$, which is allowed in the sparse regime $K^{\vec{m}} \ll K_S^{\vec{m}}$, we can linearize the logarithm and express Ψ as

$$\Psi = \frac{1}{2}\sum_{\vec{m}\neq\vec{0}}z^{\vec{m}}[\nu(\vec{m})]^2e^{-h^{\vec{m}}}, \tag{E7}$$

with

$$\nu(\vec{m}) = \int d\boldsymbol{\omega}\sum_{\mathbf{k}^{\vec{m}}}{}^{\prime}c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})e^{-\mathrm{i}\omega^{\vec{m}}}. \tag{E8}$$

The saddle-point equations determining the value of the partition function can be obtained by performing the (functional) derivative of $f(\lambda,c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}),\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}))$ with respect to $c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})$, $\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})$ and $\lambda^{\vec{m}}$, obtaining, for $h^{\vec{m}}\to 0$,

$$\begin{aligned} -\mathrm{i}\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}) &= \sum_{\vec{m}\neq\vec{0}}z^{\vec{m}}\nu(\vec{m})\exp[-\mathrm{i}\omega^{\vec{m}}-h^{\vec{m}}], \\ c(\boldsymbol{\omega},\mathbf{k}^{\vec{m}}) &= \frac{\frac{1}{(2\pi)^W}\tilde{\pi}(\mathbf{k}^{\vec{m}})\prod_{\vec{m}\neq\vec{0}}[k^{\vec{m}}!\exp[\mathrm{i}\omega^{\vec{m}}k^{\vec{m}}]]\exp[-\mathrm{i}\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})]}{\int\frac{d\boldsymbol{\omega}^{\prime}}{(2\pi)^W}\sum_{\mathbf{k}^{\prime}}^{\prime,\vec{m}}\tilde{\pi}(\mathbf{k}^{\prime\vec{m}})\prod_{\vec{m}\neq\vec{0}}[k^{\prime,\vec{m}}!\exp[\mathrm{i}\omega^{\prime,\vec{m}}k^{\prime\vec{m}}]]\exp[-\mathrm{i}\hat{c}(\boldsymbol{\omega}^{\prime},\mathbf{k}^{\prime,\vec{m}})]}, \\ z^{\vec{m}}[\nu(\vec{m})]^2 &= \langle k^{\vec{m}}\rangle. \end{aligned} \tag{E9}$$

By proceeding like in the previous examples, we can perform the integral

$$\int\frac{d\boldsymbol{\omega}}{(2\pi)^W}\sum_{\mathbf{k}}{}^{\prime}\tilde{\pi}(\mathbf{k}^{\vec{m}})\prod_{\vec{m}\neq\vec{0}}[k^{\vec{m}}!\exp[-\mathrm{i}\omega^{\vec{m}}k^{\vec{m}}]]\exp[-\mathrm{i}\hat{c}(\boldsymbol{\omega},\mathbf{k}^{\vec{m}})] = \sum_{\mathbf{k}^{\vec{m}}}{}^{\prime}\tilde{\pi}(\mathbf{k}^{\vec{m}})\prod_{\vec{m}\neq\vec{0}}(z^{\vec{m}}\nu(\vec{m}))^{k^{\vec{m}}} = w. \tag{E10}$$

Therefore, $c(\boldsymbol{\omega}, \mathbf{k}^{\vec{m}})$ at the saddle-point solution can be expressed as

$$c(\boldsymbol{\omega}, \mathbf{k}^{\vec{m}}) = \frac{1}{(2\pi)^W w}\tilde{\pi}(\mathbf{k}^{\vec{m}})\prod_{\vec{m}\neq\vec{0}}\exp[\mathrm{i}\omega^{\vec{m}}k^{\vec{m}} + z^{\vec{m}}\nu(\vec{m})e^{-\mathrm{i}\omega^{\vec{m}}}].$$

With this expression, using a similar procedure we can express $\nu(\vec{m})$ as

$$\nu(\vec{m}) = \int d\boldsymbol{\omega}\sum_{\mathbf{k}}{}' c(\boldsymbol{\omega}, \mathbf{k}^{\vec{m}})e^{-\mathrm{i}\omega^{\vec{m}}} = \frac{1}{a}\sum_{\mathbf{k}}{}'\tilde{\pi}(\mathbf{k}^{\vec{m}})k^{\vec{m}}[z^{\vec{m}}\nu(\vec{m})]^{k^{\vec{m}}-1}\prod_{\vec{m}'\neq\vec{m},\vec{0}}[z^{\vec{m}'}\nu(\vec{m}')]^{k^{\vec{m}'}}. \quad \text{(E11)}$$

Combining this equation with the third saddle-point equation, it is immediate to show that $z^{\vec{m}}\nu(\vec{m}) = 1$ is a solution with

$$z^{\vec{m}} = \frac{1}{\langle k^{\vec{m}}\rangle}, \quad \nu(\vec{m}) = \langle k^{\vec{m}}\rangle, \quad w = 1. \quad \text{(E12)}$$

By inserting this expression in Eq. (E11) we get

$$c(\boldsymbol{\omega}, \mathbf{k}^{\vec{m}}) = \frac{1}{2\pi}\tilde{\pi}(\mathbf{k}^{\vec{m}})\prod_{\vec{m}\neq\vec{0}}\{k^{\vec{m}}!\exp[\mathrm{i}\omega^{\vec{m}}k^{\vec{m}} + e^{-\mathrm{i}\omega^{\vec{m}}}]\}. \quad \text{(E13)}$$

From this expression, by proceeding like in the simple network case, we can derive that each node of the network has multidegrees $\mathbf{k}^{\vec{m}}$ with a probability $\tilde{\pi}(\mathbf{k}^{\vec{m}})$ and that the marginal probability of multilinks is given by Eqs. (47) and (48).

APPENDIX F: EXCHANGEABLE ENSEMBLE OF SPARSE SIMPLICIAL COMPLEXES

1. Derivation of the marginal probability of a simplex in the uncorrelated exchangeable simplicial complex ensembles

In this section our goal is the solve the partition function $Z(h)$ [which for construction is expected to take the value $Z(h=0)=1$] for the exchangeable ensemble of uncorrelated simplicial complexes. Let us start by defining the partition function $Z(h)$ of this simplicial complex ensemble as

$$Z(h) = \sum_{\mathbf{a}}\mathbb{P}(\mathcal{K})e^{-h\sum_{\alpha\in\mathcal{K}}a_\alpha} = \hat{C}\sum_{\mathbf{a}}\sum_{\mathbf{k}}{}'\int\mathcal{D}\omega\int\frac{d\lambda}{2\pi}e^{G(\lambda,\boldsymbol{\omega},\mathbf{k},h)}, \quad \text{(F1)}$$

with $\hat{C} = ([d!]^{\langle k\rangle N/(d+1)}/[(\langle k\rangle N)!]^{d/(d+1)})$ and

$$G(\lambda, \boldsymbol{\omega}, \mathbf{k}, h) = \sum_{i=1}^{N}[\mathrm{i}\omega_i k_i + \ln(k_i!p(k_i))] + \mathrm{i}\lambda\langle k\rangle/(d+1) + \sum_{\alpha\in\mathcal{K}}\ln(1+e^{-\mathrm{i}\sum_{r\subset\alpha}\omega_r - \mathrm{i}\lambda - h}), \quad \text{(F2)}$$

and with $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$. In Eq. (F1) and in the following we use the notation $\sum_{\mathbf{k}}'$ to indicate the sum over all the possible values of the generalized degree of each node i satisfying $m \leqslant k_i \leqslant K \ll K_S$. Let us now introduce the functional order parameter [43,48,49]

$$c(\omega, k) = \frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k, k_i), \quad \text{(F3)}$$

by enforcing its definition with a series of delta functions. By assuming a discretization in ω in intervals of size $\Delta\omega$ we then introduce for each choice of (ω, k) the term

$$\begin{aligned}1 &= \int dc(\omega,k)\delta\left(c(\omega,k) - \frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)\right)\\ &= \int\frac{d\hat{c}(\omega,k)dc(\omega,k)}{2\pi/(N\Delta\omega)}\exp\left[\mathrm{i}\Delta\omega\hat{c}(\omega,k)[Nc(\omega,k) - \sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)]\right].\end{aligned}$$

After performing these operations, by imposing $(d+1)S = \langle k\rangle N$ where $\langle k\rangle = \sum_k kp(k)$, the partition function reads, in the limit $\Delta\omega \to 0$,

$$Z(h) = \frac{[d!]^{\langle k\rangle N/(d+1)}}{[(\langle k\rangle N)!]^{d/(d+1)}}\sum_{\mathbf{k}}{}'\int\mathcal{D}c(\omega,k)\int\mathcal{D}\hat{c}(\omega,k)\int\frac{d\lambda}{2\pi}e^{Nf(\lambda,c(\omega,k),\hat{c}(\omega,k),h)} \quad \text{(F4)}$$

with

$$f(\lambda, c(\omega,k), \hat{c}(\omega,k), h) = \mathrm{i}\int d\omega \sum_k \hat{c}(\omega,k)c(\omega,k) + \mathrm{i}\lambda\langle k\rangle/(d+1) + \Psi + \ln \int \frac{d\omega}{2\pi} \sum_{\hat{m}\leqslant k\leqslant K} p(k)k!e^{\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega,k)},$$

where Ψ for $K \ll K_S$ can be approximated by

$$\Psi = \frac{N^d}{(d+1)!}e^{-h-\mathrm{i}\lambda}\prod_{r=0}^{d}\left[\sum_{m\leqslant k_r\leqslant K}\int d\omega_r c(\omega_r,k_r)e^{-\mathrm{i}\omega_r}\right]$$

and where $\mathcal{D}c(\omega,k)$ is the functional measure $\mathcal{D}c(\omega,k) = \lim_{\Delta\omega\to 0}\prod_\omega \prod_k^N [dc(\omega,k)\sqrt{N\Delta\omega/(2\pi)}]$ and similarly $\mathcal{D}\hat{c}(\omega,k) = \lim_{\Delta\omega\to 0}\prod_\omega \prod_k^N [d\hat{c}(\omega,k)\sqrt{N\Delta\omega/(2\pi)}]$. Performing a Wick rotation in λ and assuming $z/N^d = e^{-\mathrm{i}\lambda}$ real and much smaller than one, i.e., $z/N^d \ll 1$, which is allowed in the sparse regime $K \ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi = \frac{1}{(d+1)!}z\nu^{d+1}e^{-h}, \tag{F5}$$

with

$$\nu = \int d\omega \sum_{\hat{m}\leqslant k\leqslant K} c(\omega,k)e^{-\mathrm{i}\omega}. \tag{F6}$$

The saddle-point equations determining the value of the partition function can be obtained by performing the (functional) derivative of $f(\lambda, c(\omega,k), \hat{c}(\omega,k), h)$ with respect to $c(\omega,k)$, $\hat{c}(\omega,k)$, and λ, obtaining for $h \to 0$

$$-\mathrm{i}\hat{c}(\omega,k) = \frac{z}{d!}\nu^d e^{-\mathrm{i}\omega}, \quad c(\omega,k) = \frac{\frac{1}{2\pi}p(k)k!e^{\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega,k)}}{\int\frac{d\omega'}{2\pi}\sum_{m\leqslant k'\leqslant K}p(k')k'!e^{\mathrm{i}\omega' k' - \mathrm{i}\hat{c}(\omega',k')}}, \quad \frac{z}{d!}\nu^{d+1} = \langle k\rangle. \tag{F7}$$

Let us first calculate the integral

$$\int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}p(k)k!e^{-\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega,k)} = \int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}k!p(k)e^{\mathrm{i}\omega k + (z\nu^d/d!)e^{-\mathrm{i}\omega}}, \tag{F8}$$

where we have substituted the saddle-point expression for $\hat{c}(\omega,k)$. This integral can be also written as

$$\int\frac{d\omega}{2\pi}\sum_{\hat{m}\leqslant k\leqslant K}k!p(k)e^{\mathrm{i}\omega k}\sum_{h=0}^{\infty}\frac{(z\nu^d/d!)^h}{h!}e^{-\mathrm{i}\omega h} = \sum_{\hat{m}\leqslant k\leqslant K}p(k)\left(\frac{z}{d!}\nu^d\right)^k = \left\langle\left(\frac{z}{d!}\nu^d\right)^k\right\rangle. \tag{F9}$$

Therefore, $c(\omega,k)$ at the saddle-point solution can be expressed as

$$c(\omega,k) = \frac{1}{2\pi}\frac{k!p(k)\exp[\mathrm{i}\omega k + (z\nu^d/d!)e^{-\mathrm{i}\omega}]}{\langle(z\nu^d/d!)^k\rangle}. \tag{F10}$$

With this expression, using a similar procedure we can express ν as

$$\nu = \int d\omega\sum_{k\leqslant K}c(\omega,k)e^{-\mathrm{i}\omega} = \frac{1}{\langle(z\nu^d/d!)^k\rangle}\sum_{k\leqslant K}kp(k)(z\nu^d/d!)^{k-1}.$$

Combining this equation with the third saddle-point equation

$$\frac{z}{d!}\nu^{d+1} = \langle k\rangle, \tag{F11}$$

it is immediate to show that $z\nu^d/d! = 1$ is a solution with

$$z = \frac{d!}{\langle k\rangle^d}, \quad \nu = \langle k\rangle. \tag{F12}$$

By inserting this expression in Eq. (F10) we get

$$c(\omega,k) = \frac{1}{2\pi}k!p(k)e^{i\omega k + e^{-i\omega}}. \tag{F13}$$

Calculating the partition function at the saddle point, we get $Z(h\to 0) = 1$. For calculating the marginal distribution p_α of a simplex α in the exchangeable network ensemble we first note that given that the ensemble has an exchangeable Hamiltonian, the marginal probability of a simplex should be the same for every simplex of the simplicial complex, i.e., $p_\alpha = \tilde{p}$. In order to obtain $\tilde{p}$ we can simply derive the free energy $F = Nf$ with f given by Eq. (F5) with respect to the auxiliary field h, obtaining

$$\binom{N}{d+1}\tilde{p} = -\frac{\partial(Nf)}{\partial h}\bigg|_{h=0} = -\frac{\partial(N\Psi)}{\partial h}\bigg|_{h=0} = \frac{N}{(d+1)!}z\left[\int d\omega \sum_{\hat{m}\leqslant k\leqslant K} c(\omega,k)e^{-\mathrm{i}\omega}\right]^{d+1}, \tag{F14}$$

from which, by approximating the binomial

$$\binom{N}{d+1} \simeq \frac{N^{d+1}}{(d+1)!} \tag{F15}$$

for $N \gg 1$ and d finite, and inserting the saddle-point value of $c(\omega,k)$ and z, we get, for $N \gg 1$,

$$p_\alpha = \tilde{p} = \sum_{\{k_r\}|m\leqslant k_r\leqslant K}\left[\prod_{r=0}^{d} p(k_r)\right] p(k_0 k_1,\ldots,k_d), \tag{F16}$$

with

$$p_{\alpha=[i_0,i_1,\ldots,i_d]|k_{i_r}=k_r} = p(k_0k_1,\ldots,k_d) = d!\frac{\prod_{r=0}^{d} k_r}{(\langle k\rangle N)^d}. \tag{F17}$$

2. Derivation of the marginal probability of a simplex in the correlated sparse exchangeable ensemble of simplicial complexes

The partition function of the exchangeable ensemble of sparse correlated simplicial complexes can be written as

$$Z(h) = \sum_{\mathbf{a}} e^{-h\sum_\alpha a_\alpha}\sum_{\mathbf{k}}{}^{'}\prod_{\alpha\in\mathcal{K}} Q\big(k_{i_0},k_{i_1},\ldots,k_{i_d}\big)\big]^{a_\alpha} e^{-\Sigma(\mathbf{k})}\delta\left(k_i,\sum_{i_1<i_2<\ldots i_d}^{N} a_{i,i_1,\ldots,i_d}\right)\delta\left(S,\sum_{\alpha\in\mathcal{K}}a_\alpha\right), \tag{F18}$$

with the entropy $\Sigma(\mathbf{k})$ given by

$$\Sigma(\mathbf{k}) = \ln\left((\langle k\rangle N)!]^{d/(d+1)}\frac{1}{(d!)^{-\langle k\rangle N/(d+1)}}\right) + \ln\left(\prod_{i=1}^{N}\frac{[\gamma(k_i)]^{k_i}}{k_i!}\right) + o(N), \tag{F19}$$

where $\gamma(k)$ is determined by the self-consistent Eq. (58). By expressing the Kronecker delta's in integral form,

$$\delta(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{\mathrm{i}\omega(x-y)}, \tag{F20}$$

we get

$$Z(h) = \frac{(d!)^{-\langle k\rangle N/(d+1)}}{((\langle k\rangle N)!]^{d/(d+1)}}\sum_{\mathbf{k}}{}^{'}\int \mathcal{D}\boldsymbol{\omega}\int\frac{d\lambda}{2\pi}e^{G(\boldsymbol{\omega},\lambda,\mathbf{k},h)}, \tag{F21}$$

where $G(\boldsymbol{\omega},\lambda,\mathbf{k})$ is given by

$$G(\boldsymbol{\omega},\lambda,\mathbf{k},h) = \sum_{i=1}^{N}[\mathrm{i}\omega_i k_i + \ln(k_i!p(k_i)) - k_i\ln\gamma(k_i)] + \mathrm{i}\lambda\langle k\rangle/(d+1) + \sum_{\alpha\in\mathcal{K}}\ln(1+Q(k_0,k_1,\ldots,k_d)e^{-\mathrm{i}\lambda-\mathrm{i}\sum_{r\subset\alpha}\omega_r - h}),$$

and where $\mathcal{D}\omega = \prod_{i=1}^{N}[d\omega_i/(2\pi)]$. Let us now introduce the functional order parameter [43,48,49]

$$c(\omega,k) = \frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i), \tag{F22}$$

by enforcing its definition with a series of delta functions introducing for each choice of (ω,k) the term

$$1 = \int dc(\omega,k)\delta\left(c(\omega,k) - \frac{1}{N}\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)\right) = \int\frac{d\hat{c}(\omega,k)dc(\omega,k)}{2\pi/(N\Delta\omega)}e^{\mathrm{i}\Delta\omega\hat{c}(\omega,k)[Nc(\omega,k)-\sum_{i=1}^{N}\delta(\omega-\omega_i)\delta(k,k_i)]}. \tag{F23}$$

After performing these operations, by imposing $(d+1)S = \langle k\rangle N$ where $\langle k\rangle = \sum_k kp(k)$, the partition function reads

$$Z = \frac{1}{N}\sum_{\mathbf{k}}{}^{'}\int\mathcal{D}c(\omega,k)\int\mathcal{D}\hat{c}(\omega,k)\int\frac{d\lambda}{2\pi}e^{Nf(\lambda,c(\omega,k),\hat{c}(\omega,k),h)}$$

with

$$f(\lambda, c(\omega,k), \hat{c}(\omega,k), h) = \mathrm{i}\int d\omega \sum_k \hat{c}(\omega,k)c(\omega,k) + \mathrm{i}\lambda\langle k\rangle/(d+1) + \Psi + \ln\int d\omega \sum_{\hat{m}\leqslant k\leqslant K} p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega,k)},$$

where Ψ is given by

$$\Psi = \frac{N^d}{(d+1)!}\sum_{\mathbf{k}=(k_0,k_1,\ldots,k_d)|m\leqslant k_r\leqslant K}\int d\hat{\mathcal{D}}\omega \prod_r c(\omega_r,k_r)Q(k_0,k_1,\ldots,k_d)\ln(1+e^{-\mathrm{i}\lambda - h - \mathrm{i}\sum_{r\subset\alpha}\omega_r}), \tag{F24}$$

where $\mathcal{D}c(\omega,k)$ and $\mathcal{D}\hat{c}(\omega,k)$ are functional measures. Performing a Wick rotation in λ and assuming $z/N^d = e^{-\mathrm{i}\lambda}$ real and much smaller than one, i.e., $z/N^d \ll 1$, which is allowed in the sparse regime $K \ll K_S$, we can linearize the logarithm and express Ψ as

$$\Psi = \frac{z}{(d+1)!}\sum_{\mathbf{k}=(k_0,k_1,\ldots,k_d)|m\leqslant k_r\leqslant K} Q(k_0,k_1,\ldots,k_d)\prod_{r=0}^{d}\nu(k_r),$$

where

$$\nu(k) = \int d\omega c(\omega,k)e^{-\mathrm{i}\omega}. \tag{F25}$$

The saddle-point equations determining the value of the partition function read for $h \to 0$

$$-\mathrm{i}\hat{c}(\omega,k) = e^{-\mathrm{i}\omega}\tilde{\gamma}(k), \quad c(\omega,k) = \frac{\frac{1}{2\pi}p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega,k)}}{\frac{1}{2\pi}\int d\omega' \sum_{\hat{m}\leqslant k\leqslant K} p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega' k - \mathrm{i}\hat{c}(\omega',k)}}, \quad \Psi = \frac{\langle k\rangle}{d+1}, \tag{F26}$$

where $\tilde{\gamma}(k)$ is

$$\tilde{\gamma}(k) = \frac{z}{d!}\sum_{\mathbf{k}=(k_1,\ldots,k_d)|\hat{m}\leqslant k_r\leqslant K} Q(k,k_1,\ldots,k_d)\prod_{r=1}^{d}\left[\int d\omega_r c(\omega_r,k_r)e^{-\mathrm{i}\omega_r}\right]. \tag{F27}$$

Let us first calculate the integral

$$\frac{1}{2\pi}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K} p(k)\frac{k!}{[\gamma(k)]^k}e^{\mathrm{i}\omega k - \mathrm{i}\hat{c}(\omega,k)} = \frac{1}{2\pi}\int d\omega \sum_{\hat{m}\leqslant k\leqslant K}\frac{k!}{[\gamma(k)]^k}p(k)e^{\mathrm{i}\omega k + \tilde{\gamma}(k)e^{-\mathrm{i}\omega}},$$

where we have substituted the saddle-point expression for $\hat{c}(\omega,k)$. This integral can be also written as

$$\int d\omega \sum_{\hat{m}\leqslant k\leqslant K}\frac{k!}{\gamma(k)^k}p(k)e^{\mathrm{i}\omega k}\sum_{h=0}^{\infty}\frac{(\tilde{\gamma}(k))^h}{h!}e^{-\mathrm{i}\omega h} = \sum_{\hat{m}\leqslant k\leqslant K} p(k)\left(\frac{\tilde{\gamma}(k)}{\gamma(k)}\right)^k.$$

Let w indicate the value of this integral, i.e.,

$$w = \sum_{\hat{m}\leqslant k\leqslant K} p(k)\left(\frac{\tilde{\gamma}(k)}{\gamma(k)}\right)^k. \tag{F28}$$

The functional order parameter $c(\omega,k)$ at the saddle-point solution can be expressed as

$$c(\omega,k) = \frac{1}{2\pi w}\frac{k!p(k)}{[\gamma(k)]^k}e^{\mathrm{i}\omega k + \tilde{\gamma}(k)e^{-\mathrm{i}\omega}}. \tag{F29}$$

With this expression, using a similar procedure we can express $\tilde{\gamma}(k)$ as

$$\tilde{\gamma}(k) = \frac{z}{d!w^d}\sum_{\mathbf{k}=(k_1,k_2,\ldots,k_r)|\hat{m}\leqslant k'\leqslant K} Q(k_0,k_1,\ldots,k_d)\prod_{r=1}^{d}\left[p(k_r)\frac{k_r}{\tilde{\gamma}(k_r)}\left(\frac{\tilde{\gamma}(k_r)}{\gamma(k_r)}\right)^{k_r}\right]. \tag{F30}$$

Combining this equation with the third saddle-point equation, we get

$$\Psi = \frac{1}{w}\sum_{m\leqslant k\leqslant K}\left[p(k)k\left(\frac{\tilde{\gamma}(k)}{\gamma(k)}\right)^k\right] = \langle k\rangle. \tag{F31}$$

Given that $\gamma(k)$ is defined through Eq. (58), it follows that

$$\tilde{\gamma}(k) = \gamma(k), \quad w = 1, \quad z = \frac{d!}{\langle k\rangle^d}. \tag{F32}$$

Finally using Eqs. (F32) we can derive the final expression for $c(\omega, k)$ given by

$$c(\omega, k) = \frac{1}{2\pi}\frac{k!p(k)}{[\gamma(k)]^k}e^{i\omega k+\gamma(k)e^{-i\omega}}. \tag{F33}$$

From this equation of the functional order parameter we can derive the marginal for each link of the network, which is given by

$$p_\alpha = \frac{d!}{(\langle k\rangle N)^d}\sum_{\mathbf{k}=(k_0,k_1,\ldots,k_d)|\hat{m}\leqslant k_r\leqslant K} Q(k_0, k_1, \ldots, k_r)\prod_{r=0}^{d}\left[\int d\omega_r c(\omega_r, k_r)e^{-i\omega_r}\right], \tag{F34}$$

yielding

$$p_\alpha = \sum_{\mathbf{k}|\hat{m}\leqslant k\leqslant K}\left[\prod_{r=0}^{d} p(k_r)\right]p(k_0, k_1, \ldots, k_r). \tag{F35}$$

Here $P(k_0, k_1, \ldots, k_r)$ indicates the probability of a link between node i and node j conditioned to the degree of the two nodes $k_i = k$ and $k_j = k'$, i.e.,

$$p_{\alpha=[i_0,i_1,\ldots,i_d]|k(i_r)=k_r} = p(k_0, k_1, \ldots, k_r) = \frac{d!}{(\langle k\rangle N)^d}Q(k_0, k_1, \ldots, k_r)\prod_{r=0}^{d}\left[\frac{k_r}{\gamma(k_r)}\right]. \tag{F36}$$

Note that for $Q(k_0, k_1, \ldots, k_r) = 1$ it follows that $\gamma(k) = 1$, and for $Q(k_0, k_1, \ldots, k_r) = \prod_{r=0}^{d} k_r$ it follows that $\gamma(k) = k$ and hence in both cases we recover the exchangeable ensembles of uncorrelated simplicial complexes.

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