Joint statistics of space and time exploration of one-dimensional random walks

J. Klinger,[1,2] A. Barbier-Chebbah,[1] R. Voituriez,[1,2] and O. Bénichou[1]
[1]Laboratoire de Physique Théorique de la Matière Condensée, CNRS/Sorbonne Université, 4 Place Jussieu, 75005 Paris, France
[2]Laboratoire Jean Perrin, CNRS/Sorbonne Université, 4 Place Jussieu, 75005 Paris, France

(Received 15 June 2021; revised 24 September 2021; accepted 9 February 2022; published 14 March 2022)

The statistics of first-passage times of random walks to target sites has proved to play a key role in determining the *kinetics* of space exploration in various contexts. In parallel, the number of distinct sites visited by a random walker and related observables has been introduced to characterize the *geometry* of space exploration. Here, we address the question of the *joint* distribution of the first-passage time to a target and the number of distinct sites visited when the target is reached, which fully quantifies the coupling between the kinetics and geometry of search trajectories. Focusing on one-dimensional systems, we present a general method and derive explicit expressions of this joint distribution for several representative examples of Markovian search processes. In addition, we obtain a general scaling form, which holds also for non-Markovian processes and captures the general dependence of the joint distribution on its space and time variables. We argue that the joint distribution has important applications to various problems, such as a conditional form of the Rosenstock trapping model, and the persistence properties of self-interacting random walks.

DOI: 10.1103/PhysRevE.105.034116

I. INTRODUCTION

Quantifying the efficiency of space exploration by random walkers is a key issue involved in a variety of situations. Applications range from reactive particles diffusing in the presence of catalytic sites, to living organisms looking for resources, to robots cleaning or demining a given area [1–3]. In this context, two important classes of observables have been considered.

First, the statistics of first-passage times (FPTs) to target sites of interest has proved to play a key role in determining the *kinetics* of space exploration [4–6]. The case of first-passage times in confined domains was found to be particularly relevant to assess the efficiency of target search processes, and it has led to an important activity [7–10]. Related observables, such as the cover time of a domain [11–13] or the occupation time of a subdomain, have also been considered in this context [14–17].

A second class of observables has been introduced to characterize the *geometry* of the territory explored by random walkers. In particular, the number of distinct sites visited (or the so called Wiener sausage in a continuous setting) by a random walker after n steps, which quantifies the overall territory swept by the random walker, has been the focus of many studies with a broad range of applications [1,18–20]. Notable extensions include the number of distinct sites visited by p independent walkers [19], the case of fractal geometries [21,22], or the case of random stopping times [23–26].

Even if it is clear that both classes of observables are coupled, the relation between the kinetic and geometric properties of exploration remains largely unexplored. (For the specific case of Brownian motion and biased Brownian motion, see [27].) Qualitatively, the first-passage time to a target of a generic stochastic process carries information about the territory visited before hitting the target: large values of the first-passage time imply large values of the visited territory. However, the quantitative determination of this coupling for general one-dimensional stochastic processes is still lacking.

Here, we address the question of the *joint* distribution of the first-passage time to a target and the number of distinct sites visited when the target is reached, which fully quantifies this coupling and gives access to a refined characterization of search trajectories. The joint law provides two conditional distributions, which allow us to answer quantitatively the following questions: (Q_1) What is the territory visited by a random walker knowing that it reached a target (and stopped or exited the domain) after a given time? (Q_2) How long does it take a random walker to reach a target knowing that it has visited a given number of distinct sites before? We anticipate that these quantities could have applications in various situations where only partial information—either kinetic or geometric—on trajectories is accessible.

II. SUMMARY OF THE RESULTS

We tackle this general question in the case of one-dimensional (1D) processes, and we determine the joint distribution $\sigma(s, n|s_0)$ of the FPT n at the target site 0 and the number s of distinct sites visited by a random walker starting from s_0 before reaching 0 for the first time [see Fig. 1(a), where x, t are the continuous counterparts of s, n]. (This should not be confused with the joint distribution of the maximum and the time for reaching the maximum of a Brownian motion derived in [28].) This 1D setting allows for a significant simplification since the number of visited sites is simply the farthest visited site s; in higher space dimensions, the

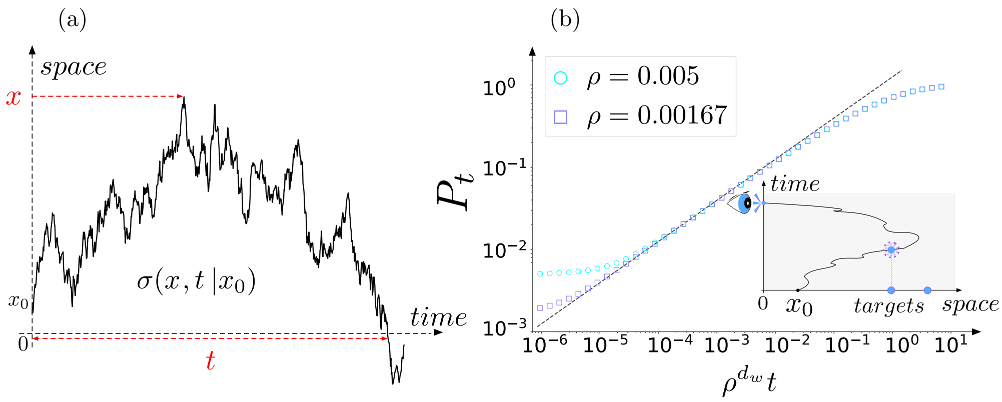

FIG. 1. (a) Starting from x_0, the random walker crosses 0 for the first time at time t, having explored up to a distance x from the origin. The joint law $\sigma(x, t|x_0)$ is the density probability function of such joint events. (b) Consider a random searcher evolving in a (grayed out) domain, filled with Poisson distributed targets. Having as the only information the time t of exit from the domain, we display the probability P_t of an encounter with at least one target in terms of the rescaled variable $\rho^{d_w}t$ in the case of a Brownian searcher of diffusion coefficient $D = 1/2$. Numerical integration of the exact result (13) (symbols) and the asymptotic scaling form (15) (dashed line) is shown.

geometry of the visited territory is complex, and the analysis requires dedicated methods that are left for further work. Our approach applies to general (space and time) discrete or continuous random walkers, evolving in a semi-infinite or finite domain, and it yields fully explicit expressions of $\sigma(s, n|s_0)$ for several representative examples of Markovian processes, such as simple symmetric and biased random walks, persistent random walks [1,18], or resetting random walks [29,30], whose definitions are recalled below. In addition, we derive a general scaling form of $\sigma(s, n|s_0)$ in the large s, n regime, which holds also for non-Markovian processes and captures the general dependence on s_0, s, n. Several applications of these central results are then discussed. First, we determine the efficiency of a schematic catalytic reaction [31] by deriving the probability that a catalytic particle diffusing in an open domain has activated at least one among Poisson distributed reactive sites before exiting the domain, knowing the exit time [see Fig. 1(b)]. Second, we show that knowledge of the joint distribution $\sigma(s, n|s_0)$ for simple random walks is actually required for determining first-passage properties of a class of strongly non-Markovian processes, namely self-interacting random walks [32–36]. More precisely, with the help of the joint distribution, we derive exactly the full large-time behavior of the FPT density of the so-called self-attracting walk (SATW) [37], which has been studied in the context of random search processes as a prototypical example of processes with long-range memory [38–41] and has important applications in the theoretical description of the trajectories of living organisms such as cells [42].

III. DISCRETE PROCESSES

We first consider the case of a general Markovian discrete (in space and time) process, which leaves no holes in its trajectory; in other words, the set of visited sites is assumed to be at all times the finite range $[\![s_{\min}, s_{\max}]\!]$ defined by the min ($s_{\min}$) and max ($s_{\max}$) values of the random walker's positions. This last hypothesis will hold for all processes presented in what follows. Denoting by $s_0 > 0$ the starting site, n the step at which the walker reaches the target 0 for the first time, and s the number of distinct sites visited up to this random stopping time, we derive a systematic procedure to obtain the joint law $\sigma(s, n|s_0)$. In turn, this joint law gives immediate access to the conditional probabilities mentioned above, i.e., (i) the distribution of the number s of distinct sites visited before reaching 0 knowing that the random walker has reached 0 at step n:

$$G_{sp}(s|n, s_0) = \frac{\sigma(s, n|s_0)}{\sum_{s'=s_0}^{\infty} \sigma(s', n|s_0)} \equiv \frac{\sigma(s, n|s_0)}{F_{\underline{0}}(n|s_0)}, \quad (1)$$

where $F_{\underline{0}}(n|s_0)$ is the usual FPT distribution to 0, and (ii) the distribution of the FPT to 0 knowing that s distinct sites have been visited before reaching 0:

$$G_{tm}(n|s, s_0) = \frac{\sigma(s, n|s_0)}{\sum_{n'=0}^{\infty} \sigma(s, n'|s_0)} \equiv \frac{\sigma(s, n|s_0)}{\mu_0(s|s_0)}, \quad (2)$$

where $\mu_0(s|s_0)$ is the distribution of the maximum s before reaching 0.

Let us denote $F_{\underline{0},s}(n|s_0)$ the probability that the walker reaches zero for the first time at step n, without ever reaching s, and make a partition over the rightmost site s' visited before reaching zero. Because the walker reaches 0 before s, one necessarily has $s' \in [\![s_0, s-1]\!]$, which yields $F_{\underline{0},s}(n|s_0) = \sum_{s'=s_0}^{s-1} \sigma(s', n|s_0)$. Note that this relation still holds for non-Markovian processes. Equivalently, we obtain the key relation

$$\sigma(s, n|s_0) = F_{\underline{0},s+1}(n|s_0) - F_{\underline{0},s}(n|s_0) \equiv D_s F_{\underline{0},s}(n|s_0), \quad (3)$$

which allows one to write the joint law σ explicitly in terms of the quantity $F_{\underline{0},s}(n|s_0)$.

We next provide a procedure based on backward equations to derive the probability $F_{\underline{0},s}(n|s_0)$ in the presence of two absorbing sites 0 and s for a given Markovian stochastic process. In this case, the propagator $P(s, n|s_0)$, i.e., the probability for the walker to be at site s after n steps, obeys the backward equation $P(s, n+1|s_0) = \mathcal{L}_{s_0}[P(s, n|s_0)]$ [1,43], obtained by partitioning over the first step of the walk, where $\mathcal{L}_{s_0}$ is a linear operator acting on s_0. For instance, in the case of a simple random walk, $\mathcal{L}_{s_0}[P(s,n|s_0)] = \frac{1}{2}P(s,n|s_0+1) + \frac{1}{2}P(s,n|s_0-1)$. It is easily seen that $F_{\underline{0},s}(n|s_0)$ obeys the same backward equation for $0 < s_0 < s$, and, introducing the generating function $\tilde{F}_{\underline{0},s}(\xi|s_0) = \sum_{n=0}^{\infty} \xi^n F_{\underline{0},s}(n|s_0)$, we obtain

$$\tilde{F}_{\underline{0},s}(\xi|s_0) = \xi \mathcal{L}_{s_0}[\tilde{F}_{\underline{0},s}(\xi|s_0)]. \tag{4}$$

Recalling that both 0 and s are absorbing boundaries, we have that, for any $n > 0$, $F_{\underline{0},s}(n|0 \text{ or } s) = 0$, whereas $F_{\underline{0},s}(0|0) = 1$ and $F_{\underline{0},s}(0|s) = 0$. In terms of generating functions, we obtain the following boundary conditions:

$$\tilde{F}_{\underline{0},s}(\xi|0) = 1, \;\; \tilde{F}_{\underline{0},s}(\xi|s) = 0. \tag{5}$$

Equation (4), completed by (5), fully determines $\tilde{F}_{\underline{0},s}(\xi|s_0)$. Making use of (3), we then derive the generating function of the joint law σ.

As an illustration, we obtain in the case of a simple random walk (see Appendix A 1),

$$\tilde{\sigma}(s,\xi|s_0) = \frac{r_+ - r_-}{r_+^s - r_-^s}\frac{r_+^{s_0} - r_-^{s_0}}{r_+^{s+1} - r_-^{s+1}}, \tag{6}$$

where $r_\pm = \frac{1}{\xi}(1 \pm \sqrt{1-\xi^2})$. Further illustration is provided in Appendixes B 1 and B 2, where explicit expressions of $\tilde{\sigma}(s,\xi|s_0)$ are determined for the important examples of biased random walks (for which a step is taken to the right with probability p, and to the left with probability $1-p$), persistent random walks (for which each step is taken identical to the previous one with probability p) [44,45], and resetting random walks (for which at each step the walker has a probability λ to jump back to its initial position) [29,30,46,47]. Finally, in each case, a series expansion with respect to ξ gives access to an exact determination of $\sigma(s,n|s_0)$ (see Appendix A 5 for validation by numerical simulations), which constitutes the main result of this section; its physical implications are commented on below (see Secs. VI and VII).

IV. CONTINUOUS SPACE AND TIME

This method is easily adapted to continuous space and time (x,t) Markovian processes. Defining $F_{\underline{0},x}(t|x_0)$ as the probability density to reach 0 before x at time t, the continuous counterpart of Eq. (3) reads

$$\sigma(x,t|x_0) = D_x F_{\underline{0},x}(t|x_0), \tag{7}$$

where here D_x is the differential operator with respect to x, and the Laplace transform $\tilde{F}_{\underline{0},x}(p|x_0) = \int_0^\infty e^{-pt} F_{\underline{0},x}(t|x_0)dt$ satisfies the continuous counterpart of Eqs. (4) and (5) (see Appendix B 1). As an explicit example, for Brownian diffusion with diffusion coefficient D, it is found that the joint law is given by

$$\sigma(x,t|x_0) = \frac{2D\pi}{x^3}\sum_{k=1}^{\infty} e^{-(k\pi)^2 D\tau} k \sin(k\pi\tilde{x}_0) \times \left[2(k\pi)^2 D\tau - 2 - \frac{k\pi\tilde{x}_0}{\tan(k\pi\tilde{x}_0)}\right], \tag{8}$$

where $\tilde{x}_0 = \frac{x_0}{x}$ and $\tau = \frac{t}{x^2}$. Note that an alternative expression for this joint distribution can be found in [27]. Explicit expressions of $\tilde{\sigma}$ for other continuous Markov processes (biased diffusion and continuous resetting) are presented in Appendixes B 2 and B 3. Importantly, it is also shown in Appendixes B 3-C that our approach can be further extended to the case of continuous space but discrete time processes, also known as jump processes, as well as Markovian processes in confined domains.

V. GENERAL SCALING FORM

Beyond the case of Markovian processes, we now show that the joint law σ assumes a general scaling form for symmetric processes, which holds even in the non-Markovian case and elucidates its dependence on the parameters s, s_0, n. Because we are interested only in the large time and space limit, we adopt a continuous formalism and make use of the variables x, x_0, t. Extending an approach given in [48,49], we derive below a general scaling form for $F_{\underline{0},x}(t|x_0)$, which leads to the asymptotic behavior of $\sigma(x,t,x_0)$.

First, note that walkers reaching x before 0 do not contribute to the probability $F_{\underline{0},x}(t|x_0)$. Hence, for times shorter than the typical time $T_{\text{typ}} \propto x^{d_w}$ needed to reach x (which defines the walk dimension d_w of the process), $F_{\underline{0},x}(t|x_0)$ behaves as the first-passage time density $F_{\underline{0}}(t|x_0)$ in a semi-infinite domain, with a *single* target in 0. We now assume that this quantity has an algebraic decay with time for $t \to \infty$, quantified by the persistence exponent θ of the process: $F_{\underline{0}}(t|x_0) \sim k(x_0)t^{-(\theta+1)}$, where $k(x_0) \propto x_0^{d_w\theta}$ for $x_0 \gg 1$ [6]. Because almost all random walkers have either reached 0 or x at times $t \gg x^{d_w}$, we write

$$F_{\underline{0},x}(t|x_0) \sim F_{\underline{0}}(t|x_0) g\left(\frac{t}{x^{d_w}}\right) \sim k(x_0)t^{-(\theta+1)} g\left(\frac{t}{x^{d_w}}\right), \tag{9}$$

where g is a smooth cutoff function with $g(0) = 1$ and $g(y)$ vanishes for large y. Finally, with the help of (7), we obtain the general scaling form for the joint law in the scaling limit defined by $x \to \infty, t \to \infty$ with $\tau = t/x^{d_w}$ fixed:

$$\sigma(x,t|x_0) \sim \frac{h(x_0)}{x^{d_w(\theta+1)+1}} f(\tau), \tag{10}$$

where, defining $f_1(\tau) = -d_w g'(\tau)\tau^{-\theta}$ and $\mathcal{N} = \int_0^\infty f_1(\tau)d\tau$, we have $h(x_0) = k(x_0)\mathcal{N}$ and $f = f_1/\mathcal{N}$. In addition, $h(x_0) \propto x_0^{d_w\theta}$ for $x_0 \gg 1$, and $f(\tau)$ is a normalized process-dependent function.

Of note, integrating Eq. (10) over t recovers the distribution of the maximum $\mu_0(x|x_0) = h(x_0)x^{-(d_w\theta+1)}$ before reaching 0, in agreement with known results [48]. In turn, this provides a simple physical interpretation of $f(\tau)$. Making use of (2), we obtain the conditional density $G_{tm}(t|x,x_0) \sim \frac{1}{x^{d_w}} f(\tau)$. Thus, $f(\tau)$ is the density of the rescaled variable τ conditioned by the value of the maximum x. In particular, we stress that f is independent of x_0.

The general relation (10) is confirmed in Fig. 2 by numerical simulations for representative examples of both Markovian processes (simple random walks and Riemann walks, i.e., discrete space and time Levy flights [1]), and non-Markovian processes (fractional Brownian motion [50] and the random acceleration process [51]; see Appendixes D 1 and D 2 for definitions). Indeed, we find that the conditional density of the FPT knowing the territory covered, which *a priori* depends on the variables t, x, x_0, can in fact be rewritten as the distribution $f(\tau)$ of the single reduced variable τ, as shown by the data

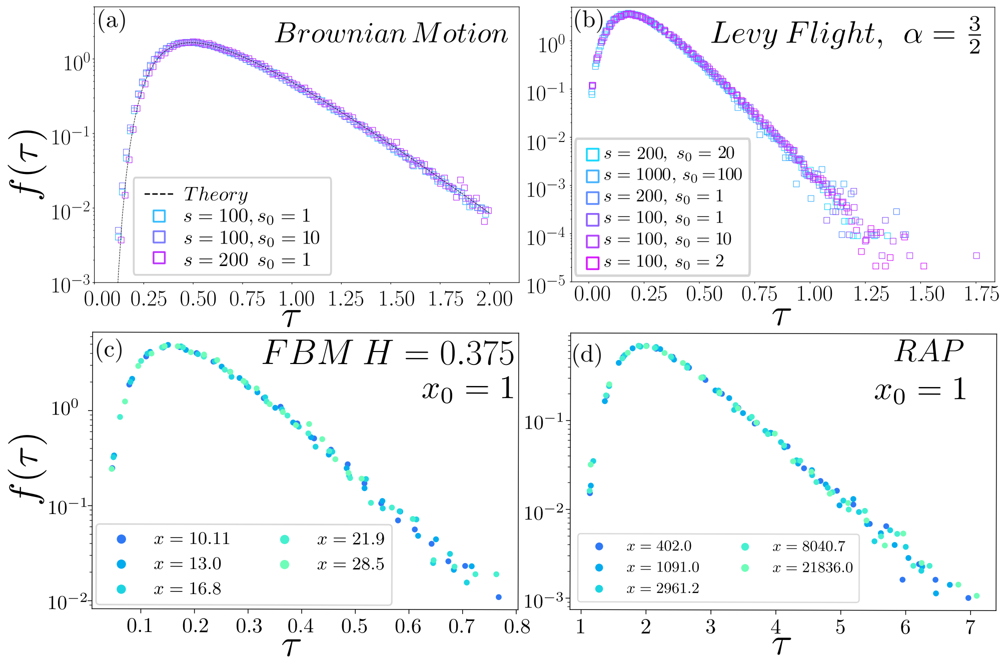

FIG. 2. Conditional distribution $f(\tau)$ of the rescaled variable τ (see the text). Distributions are drawn for fixed s (discrete space) or x (continuous space) and collapse. (a),(b) Markovian processes; (c),(d) non-Markovian processes. See Appendixes D 1 and D 2 for details on simulations.

collapse in the figure. Next, thanks to the exact Eq. (2), and the exact scaling of the distribution μ_0 of the maximum mentioned above [48], this observed scaling of f directly confirms (10).

In the case of diffusive random walks, $f(\tau)$ can be determined explicitly by taking $x \to \infty$ and $t \to \infty$ with τ fixed in Eq. (8):

$$f_{\rm BM}(\tau) = 2D\pi^2 \sum_{k=1}^{\infty} e^{-(k\pi)^2 D\tau} k^2[2(k\pi)^2 D\tau - 3]. \qquad (11)$$

Of note, this asymptotic conditional distribution holds for any symmetric Markovian random walk satisfying the central limit theorem.

Similarly (see Appendix D 3), the other conditional distribution defined in (1) can be written from (10) as $G_{sp}(x|t, x_0) \sim \frac{1}{t^{1/d_w}}\phi(\chi)$, where the density of the rescaled variable $\chi = x/t^{1/d_w}$ is given in terms of f by

$$\phi(\chi) = \frac{\chi^{-d_w(\theta+1)-1} f(\chi^{-d_w})}{\int_0^\infty u^{-d_w(\theta+1)-1} f(u^{-d_w}) du}. \qquad (12)$$

The agreement of this result with numerical simulations is shown in Appendix D.

VI. DISCUSSION

The above results yield both exact expressions of the joint law for Markovian processes and scaling forms for general non Markovian processes, and they have important implications. (i) The joint law, because it gives access to all correlation functions $\langle x^n t^m \rangle$, fully quantifies the coupling between the kinetics of space exploration and the territory explored by a random walker. This coupling manifests itself in the dependence of σ on the rescaled variable $\tau = t/x^{d_w}$. (ii) The joint law yields the conditional distributions G_{sp} [see (1)] and G_{tm} [see (2)], which provide new insights into the quantification of space exploration, and in particular explicit answers to the questions Q_1, Q_2 raised in the Introduction. Below, we further illustrate the importance of the joint law and turn to examples of applications of our results.

Application—Conditional Rosenstock problem. The above results provide as a by-product an explicit solution to a conditional version of the celebrated Rosenstock problem [1,31]. We consider a catalytic diffusing particle that enters a one-dimensional chemical reactor at x_0 and leaves it at 0; it is assumed that the time t spent in the reactor is observed. The reactor contains Poisson-distributed pointlike reactive sites of density ρ, which trigger a reaction upon encounter with the catalytic particle, whose diffusive dynamics is assumed to remain unchanged upon reaction [see Fig. 1(b)]. The efficiency of such a schematic catalytic reaction can be quantified by the probability P_t that the catalytic particle has activated at least one reactive site before exiting the domain, knowing the exit time t. This is readily obtained as

$$P_t = \int_0^\infty (1 - e^{-\rho x}) G_{sp}(x|t, x_0) dx. \qquad (13)$$

The determination of P_t thus requires G_{sp}, and therefore the joint law. Making use of the general scaling (10), we obtain the large-time scaling behavior:

$$P_t \underset{t\to\infty}{\sim} \int_0^\infty (1 - e^{-\rho t^{1/d_w} u}) \phi(u) du; \qquad (14)$$

this shows that P_t is asymptotically a function of the reduced variable $\rho t^{1/d_w}$ only, with $P_t \propto \rho t^{1/d_w}$ for $\rho t^{1/d_w} \to 0$. Equation (14) provides, thanks to (12), an explicit determination of

P_t for all processes for which σ (and thus f) is known, and in particular elucidates its dependence on the exit time t from the domain [see Fig. 1(b)]. On the example of Brownian motion, one obtains [for $x_0^2/D \ll t \ll 1/(D\rho^2)$]

$$P_t \sim \sqrt{\pi}\rho(Dt)^{\frac{1}{2}}. \tag{15}$$

VII. APPLICATION—SELF-INTERACTING WALKERS

Next, we show that the joint law can be needed to obtain the first-passage time distribution. This is the case of self-interacting random walks, which are defined generically as random walks whose jump probabilities at time n depend on the full set of visited sites at earlier times $n' < n$. We focus on the example of the one-dimensional self-attracting walk (SATW) [37], which has been studied in the context of random search processes as a prototypical example of process with long-range memory, and has recently proved to be relevant to describe the dynamics of motile cells [42]. At each time step, if both its neighboring sites have already been visited, the random walker hops on either of them with probability $1/2$. However, if one of them has never been visited, it is chosen with probability β. Note that this can either be an attractive effect ($\beta < 1/2$) or a repulsive one ($\beta > 1/2$). Since the dynamics of the walk is completely determined by the location of unvisited sites, the determination of the first-passage time distribution requires the knowledge of all times at which unvisited sites have been discovered. Denoting here $F_{0,\underline{s}}(n|s_0)$ the probability to reach s before 0 for the first time at step n, knowing that the sites $[\![1, s-1]\!]$ have already been visited, the generating function of $\sigma(s, n|1)$ can be written as

$$\tilde{\sigma}(s, \xi|1) = \frac{\xi}{2}\left(\prod_{s'=3}^{s}\tilde{F}_{0,\underline{s'}}(\xi|s'-1)\right)\tilde{F}_{\underline{0},s+1}(\xi|s). \tag{16}$$

Solving for $\tilde{F}_{0,\underline{s}}(\xi|s_0)$ yields an explicit expression of $\tilde{\sigma}$ (see Appendixes E 1 and E 2). For large s and n, with $\tau = \frac{n}{s^2}$ fixed, this yields $\sigma(s, n|s_0) = h(s_0)s^{-\frac{1-\beta}{\beta}-3}f_{\mathrm{SATW}}(\tau)$, where $h(1) = \frac{\Gamma(-2+2/\beta)}{\Gamma(-1+1/\beta)}\frac{(1-\beta)}{\beta}$ and $h(s_0) \propto s_0^{\frac{1-\beta}{\beta}}$ for large s_0. [Since for the SATW $d_w = 2$ and $\theta = \frac{1-\beta}{2\beta}$ [52], the joint law obeys the general scaling form (10).] Finally, the conditional distribution f_{SATW} is defined by its strikingly simple Laplace transform:

$$\tilde{f}_{\mathrm{SATW}}(p) = \int_0^\infty e^{-pu}f_{\mathrm{SATW}}(u)du = \left(\frac{\sqrt{2p}}{\sinh(\sqrt{2p})}\right)^{\frac{1}{\beta}}. \tag{17}$$

The FPT distribution is finally deduced from $\sigma(s, n|1)$ and yields the following exact asymptotics (see Appendix E 6):

$$F_{\underline{0}}(n|s_0 = 1) \underset{n\to\infty}{\sim} \frac{\Gamma\left(\frac{2}{\beta}-1\right)}{\Gamma\left(\frac{1}{2\beta}-\frac{1}{2}\right)}2^{-\frac{1+\beta}{2\beta}}n^{-\frac{1-\beta}{2\beta}-1}. \tag{18}$$

While the n decay is in agreement with the recent determination of the persistent exponent of the SATW relying on a different approach [52], this formalism based on the joint law gives access to the explicit expression of the prefactor for this strongly non-Markovian process.

VIII. CONCLUSION

We have proposed a general method to derive explicit expressions of the joint distribution of the first-passage time to a target and the number of distinct sites visited when the target is reached for one-dimensional random walks. This method yields explicit expressions for several representative examples of Markovian search processes. Furthermore, we showed that the dependence of the joint distribution on its space and time variables is captured by a general scaling form, which holds even for non-Markovian processes. We argue that the joint distribution could have applications in various situations where only partial information—either kinetic or geometric—on trajectories is accessible; in addition, it appears to be a useful technical tool that for instance can give access to persistence properties of self interacting random walks.

ACKNOWLEDGMENT

We thank Eric Brunet for drawing our attention to Ref. [27] concerning the specific case of Brownian motion.

APPENDIX A: DISCRETE SPACE AND TIME PROCESSES—JOINT LAWS

1. Normal diffusion

In this Appendix, we present the derivation of the joint law for the discrete symmetric diffusion process. Following the main notations, we obtain the backward equation:

$$F_{0,\underline{s}}(n+1, s_0) = \tfrac{1}{2}[F_{0,\underline{s}}(n, s_0+1) + F_{0,\underline{s}}(n, s_0-1)] \tag{A1}$$

and going into the generating function formalism with $\tilde{F}_{0,\underline{s}}(\xi, s_0) = \sum_{n=0}^{\infty}\xi^n F_{0,\underline{s}}(n, s_0)$,

$$\frac{1}{\xi}\tilde{F}_{0,\underline{s}}(\xi, s_0) = \frac{1}{2}[\tilde{F}_{0,\underline{s}}(\xi, s_0+1) + \tilde{F}_{0,\underline{s}}(\xi, s_0-1)]. \tag{A2}$$

Enforcing the boundary conditions $\tilde{F}_{0,\underline{s}}(\xi, s) = 1$, $\tilde{F}_{0,\underline{s}}(\xi, 0) = 0$, and denoting $r_\pm = \frac{1}{\xi}(1 \pm \sqrt{1-\xi^2})$, we obtain

$$\tilde{F}_{0,\underline{s}}(\xi, s_0) = \frac{r_+^{s_0} - r_-^{s_0}}{r_+^{s} - r_-^{s}}. \tag{A3}$$

Note that $\tilde{F}_{\underline{0},s}(\xi|s_0) = \tilde{F}_{0,\underline{s}}(\xi|s-s_0)$. Applying the method, we obtain

$$\begin{aligned}\tilde{\sigma}(s, \xi|s_0) &= \tilde{F}_{\underline{0},s+1}(\xi|s_0) - \tilde{F}_{\underline{0},s}(\xi|s_0)\\ &= \frac{r_+ - r_-}{r_+^s - r_-^s}\frac{r_+^{s_0} - r_-^{s_0}}{r_+^{s+1} - r_-^{s+1}}(r_+r_-)^{s-s_0}\\ &= \frac{r_+ - r_-}{r_+^s - r_-^s}\frac{r_+^{s_0} - r_-^{s_0}}{r_+^{s+1} - r_-^{s+1}}.\end{aligned} \tag{A4}$$

2. Biased diffusion

In this section we only give the steps required to derive $\tilde{F}_{0,\underline{s}}(\xi|s_0)$, as the joint law is directly obtained from differentiation. Denoting p the probability to take a step to the right, the backward equation on F reads

$$p\tilde{F}_{0,\underline{s}}(\xi|s_0+1)+\frac{1}{\xi}\tilde{F}_{0,\underline{s}}(\xi|s_0)+(1-p)\tilde{F}_{0,\underline{s}}(\xi|s_0-1)=0. \tag{A5}$$

Denoting the roots $r_\pm=\frac{1}{2p\xi}(1\pm\sqrt{1-4pq\xi^2})$, we obtain

$$\tilde{F}_{0,\underline{s}}(\xi,s_0)=\frac{r_+^{s_0}-r_-^{s_0}}{r_+^{s}-r_-^{s}}. \tag{A6}$$

3. Persistent walk

Let us now turn to the case of the persistent walk, where each step is taken identical to the previous one with probability p [44,45]. As in the case of the biased random walk, we only present the derivation of $\tilde{F}_{0,\underline{s}}(\xi|s_0)$. Let us denote by u and v the function F conditioned on the direction of the previous step. Denoting a the probability of taking the first step toward the right, we obtain our first equation:

$$F_{0,\underline{s}}(n+1,s_0)=au_{0,\underline{s}}(n,s_0+1)+(1-a)v_{0,\underline{s}}(n,s_0-1). \tag{A7}$$

Going into the generating function formalism yields

$$\xi^{-1}\tilde{F}_{0,\underline{s}}(\xi,s_0)=a\tilde{u}_{0,\underline{s}}(\xi,s_0+1)+(1-a)\tilde{v}_{0,\underline{s}}(\xi,s_0-1). \tag{A8}$$

Dropping the ξ dependence for brevity, we also obtain a set of equations for $\tilde{u}$ and $\tilde{v}$:

$$\tilde{u}_{0,\underline{s}}(s_0+2)-\frac{\frac{1}{\xi}-\xi+2p\xi}{p}\tilde{u}_{0,\underline{s}}(s_0+1)+\tilde{u}_{0,\underline{s}}(s_0)=0,$$
$$\tilde{v}_{0,\underline{s}}(s_0)=\frac{1}{1-p}\left[\frac{1}{\xi}\tilde{u}_{0,\underline{s}}(s_0+1)-p\tilde{u}_{0,\underline{s}}(s_0+2)\right]. \tag{A9}$$

As expected, taking ξ to 0 yields back the governing equations for the splitting probability to reach s before 0. The quantity $\tilde{u}$ obeys a second-order difference equation whose roots read

$$r_\pm=\frac{\frac{1}{\xi}-\xi+2p\xi}{2p}\pm\frac{\sqrt{\left(\frac{1}{\xi}-\xi+2p\xi\right)^2-4p^2}}{2p} \tag{A10}$$

yielding

$$\tilde{u}_{0,\underline{s}}(s_0)=Ar_+^{s_0}+Br_-^{s_0}. \tag{A11}$$

Imposing the two boundary conditions $\tilde{u}_{0,\underline{s}}(s)=1$ and $\tilde{v}_{0,\underline{s}}(0)=0$, one obtains the following result:

$$A=\frac{(p\xi r_--1)r_-}{r_-^s r_+-p\xi r_-^s r_+^2-r_-r_+^s+p\xi r_-^2r_+^s},$$
$$B=-\frac{(p\xi r_+-1)r_+}{r_-^s r_+-p\xi r_-^s r_+^2-r_-r_+^s+p\xi r_-^2r_+^s}. \tag{A12}$$

We are now in possession of all the terms required to perform a series expansion of $\tilde{F}$ and thus obtain $\sigma(s,n|s_0)$.

4. Resetting walk

We finally turn to the discrete resetting process. At each step, the walker either jumps back to its starting position with probability λ or chooses one of its neighbors with probability $\frac{1-\lambda}{2}$ [29,30,46,47]. Denoting s_p the starting point of the walker, the backward equation for F reads, in the generating function formalism,

$$\tilde{F}_{0,\underline{s}}(\xi|s_0+1)-\frac{2}{\xi(1-\lambda)}\tilde{F}_{0,\underline{s}}(\xi|s_0)+\tilde{F}_{0,\underline{s}}(\xi|s_0-1)=\frac{2\lambda}{(1-\lambda)}\tilde{F}_{0,\underline{s}}(\xi|s_p) \tag{A13}$$

with the associated boundary conditions $\tilde{F}_{0,\underline{s}}(\xi|0)=0$ and $\tilde{F}_{0,\underline{s}}(\xi|s)=1$. Denoting $G(s_1,s_2)$ the Green function such that

$$G(s_1+1,s_2)-\frac{2}{\xi(1-\lambda)}G(s_1,s_2)+G(s_1-1,s_2)=\delta_{s_1,s_2}, \tag{A14}$$

one can show that

$$G(s_1,s_2)=1_{s_1\leqslant s_2}G_-(s_1,s_2)+1_{s_1>s_2}G_+(s_1,s_2), \tag{A15}$$

where

$$G_-(s_1,s_2)=A(s_2)^{-1}(r_+^{s_1}-r_-^{s_1}),$$
$$G_+(s_1,s_2)=A(s_2)^{-1}\frac{r_+^{s_2}-r_-^{s_2}}{r_+^{s_2}-r_-^{s_2}\frac{r_+^s}{r_-^s}}\left(r_+^{s_1}-r_-^{s_1}\frac{r_+^s}{r_-^s}\right), \tag{A16}$$

with

$$r_\pm=\frac{1}{\xi(1-\lambda)}\pm\sqrt{\left[\frac{1}{\xi(1-\lambda)}\right]^2-1},$$
$$A(s_2)=(r_+^{s_2}-r_-^{s_2})\left(r_+^{s_2}-r_-^{s_2}\frac{r_+^s}{r_-^s}\right)^{-1}\left(r_+^{s_2+1}-r_-^{s_2+1}\frac{r_+^s}{r_-^s}\right)-\frac{2}{\xi(1-\lambda)}(r_+^{s_2}-r_-^{s_2})+(r_+^{s_2-1}-r_-^{s_2-1}). \tag{A17}$$

Introducing the homogeneous solution

$$h_{0,\underline{s}}(s_0)=\frac{r_+^{s_0}-r_-^{s_0}}{r_+^s-r_-^s}, \tag{A18}$$

we finally obtain, with s_p the starting point,

$$\tilde{F}_{0,\underline{s}}(\xi|s_0)=h_{0,\underline{s}}(s_0)-\frac{h(s_p)2\lambda\sum_{s_2}G(s_0,s_2)}{1-\lambda+2\lambda\sum_{s_2}G(s_p,s_2)}. \tag{A19}$$

5. Numerical validation

We display in Fig. 3 the agreement between the theoretical expressions of the joint law derived in Appendixes A 1–A 4 and numerical simulations.

For each random process, the generating function of the joint law is derived from the expressions of $\tilde{F}_{0,\underline{s}}(\xi|s_0)$ given above, according to $\tilde{\sigma}(s,\xi|s_0)=\tilde{F}_{\underline{0},s+1}(\xi|s_0)-\tilde{F}_{\underline{0},s}(\xi|s_0)$. Since $\tilde{\sigma}(s,\xi|s_0)=\sum_{k=0}^{\infty}\sigma(s,n|s_0)\xi^n$, for given s and n, we perform a series expansion in ξ up to order n and read off the desired coefficient. Although $\sigma(s,n|s_0)$ is only defined for discrete values of n, we choose to represent the theory with continuous lines to increase readability and help differentiate from the simulated data.

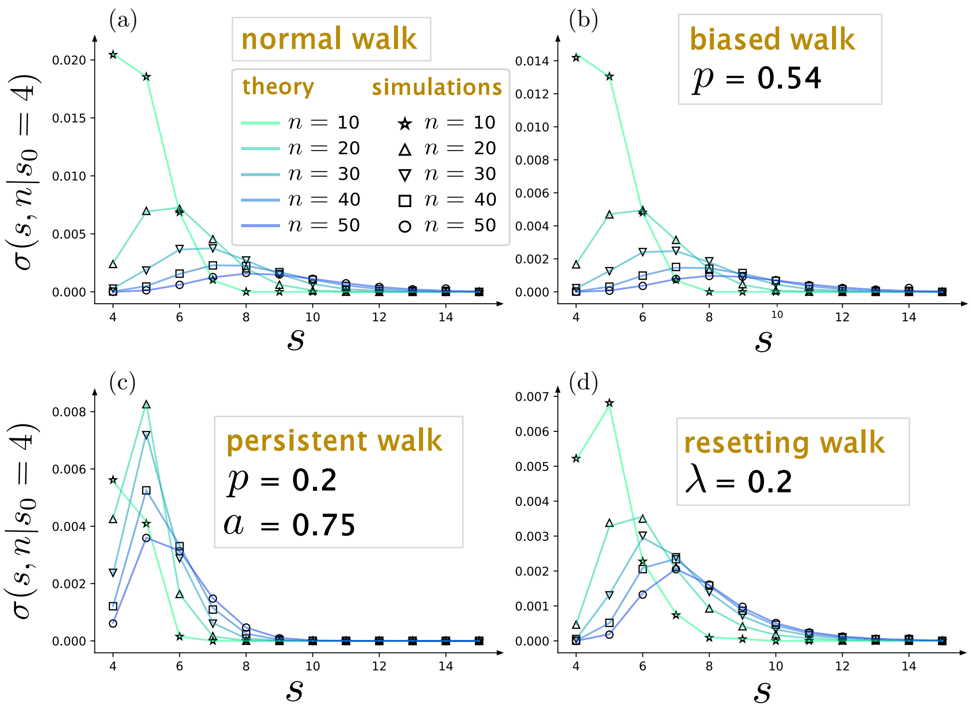

FIG. 3. Space and time joint law for various discrete stochastic one-dimensional processes—one can show that the biased case is exactly proportional to the isotropic case. For each s and n we estimate the quantity $\sigma(s, n|s_0)$ by averaging over 10^6 random walks. (a) Symmetric nearest-neighbor random walk. (b) Biased random walk: each step is taken rightward with probability p. (c) Persistent random walk: each step is taken identical to the previous one with probability p. The first step is taken rightward with probability a. (d) Resetting walk: at each time step, the walker resets to its initial position with probability λ or hops symmetrically on one of its nearest neighbors.

APPENDIX B: CONTINUOUS SPACE AND TIME PROCESSES—JOINT LAWS

1. Backward equation for $F_{0,\underline{x}}(t|x_0)$ in Brownian diffusion

For classical continuous diffusion, denoting the propagator $c(x, t|x_0)$ as the probability for the walker to be at position x at time t starting from x_0, one has the following backward equation:

$$\frac{\partial}{\partial t}c(x, t|x_0) = D\frac{\partial^2}{\partial^2 x_0}c(x, t|x_0), \tag{B1}$$

where D is the diffusive coefficient. Denoting the Laplace transform $\tilde{F}_{0,\underline{x}}(p|x_0) = \int_0^\infty e^{-pt}F_{0,\underline{x}}(t|x_0)dt$, we have

$$p\tilde{F}_{0,\underline{x}}(p|x_0) = D\frac{\partial^2}{\partial^2 x_0}\tilde{F}_{0,\underline{x}}(p|x_0). \tag{B2}$$

Solving for F with $\tilde{F}_{0,\underline{x}}(p|0) = 0$ and $\tilde{F}_{0,\underline{x}}(p|x) = 1$, we obtain

$$\tilde{F}_{0,\underline{x}}(p|x_0) = \frac{\sinh\left[\sqrt{\frac{p}{D}}x_0\right]}{\sinh\left[\sqrt{\frac{p}{D}}x\right]}. \tag{B3}$$

Once again, $\tilde{F}_{\underline{0},x}(p|x_0) = \tilde{F}_{0,\underline{x}}(p|x - x_0)$ and, partitioning over the maximum reached during a trajectory exiting at 0 before x, one obtains

$$\tilde{F}_{\underline{0},x}(t|x_0) = \int_{x_0}^{x}\sigma(x', t|x_0)dx' \tag{B4}$$

and thus the desired relation

$$\sigma(x, t|x_0) = D_x F_{\underline{0},x}(t|x_0). \tag{B5}$$

2. Biased diffusion

For biased diffusion with diffusive coefficient D and rightward drift coefficient v, we obtain, in the Laplace formalism,

$$\tilde{\sigma}^v(x, s|x_0) = e^{\frac{-vx_0}{2D}}\frac{|v|}{2D}\sqrt{1+\frac{4sD}{v^2}}\frac{\sinh\left(\frac{x_0|v|}{2D}\sqrt{1+\frac{4sD}{v^2}}\right)}{\sinh\left(\frac{x|v|}{2D}\sqrt{1+\frac{4sD}{v^2}}\right)^2}. \tag{B6}$$

Once inverted, this expression takes the following form:

$$\sigma^v(x, t|x_0) = e^{-\frac{vx_0}{2D}-\frac{v^2 t}{4D}}\frac{2D\pi}{x^3}\sum_{k=1}^{\infty}e^{-(k\pi)^2\tau}k \times\left[2(k\pi)^2\tau - 2 - \frac{k\pi\tilde{x}_0}{\tan(k\pi\tilde{x}_0)}\right]\sin(k\pi\tilde{x}_0) \tag{B7}$$

or, in a simpler manner,

$$\sigma^v(x, t|x_0) = e^{-\frac{vx_0}{2D}}e^{-\frac{v^2 t}{4D}}\sigma^0(x, t|x_0), \tag{B8}$$

where σ^0 denotes the joint-law for symmetric diffusion. Note that this expression can be found in [27] in the form of another infinite series.

3. Continuous resetting

We consider here the continuous resetting diffusion process, which diffuses with coefficient D and resets to its initial position x_p with resetting rate λ [29,30,46,47]. The backward equation for the propagator $c(x,t)$ reads

$$\frac{\partial}{\partial t}c(x,t|x_0) = D\frac{\partial^2}{\partial^2 x_0}c(x,t|x_0) + \lambda(c(x,t|x_p) - c(x,t|x_0)). \tag{B9}$$

The same equation holds for $F_{0,\underline{x}}(t|x_0)$, and, going into the Laplace domain, we obtain

$$\frac{\partial^2}{\partial^2 x_0}\tilde{F}_{0,\underline{x}}(p|x_0) - \omega^2\tilde{F}_{0,\underline{x}}(p|x_0) = -\frac{\lambda}{D}\tilde{F}_{0,\underline{x}}(p|x_p), \tag{B10}$$

where $\omega^2 = \frac{\lambda+p}{D}$. Using the Green's function method, we are able to solve for $\tilde{F}$, and derivation with respect to x finally yields the joint law in the Laplace domain,

$$\begin{aligned}\tilde{\sigma}(x,p|x_0) &= D\omega^3[p + \lambda\cosh(\omega(x-x_0))] \\ &\times \frac{\sinh(\omega x_0)}{\sinh^2(\omega x)[p+\lambda\cosh(\omega x)][\sinh(\omega(x-x_0)) + \sinh(\omega x_0)]^2}.\end{aligned} \tag{B11}$$

4. Bounded domains

Let us consider a continuous walker evolving in a bounded domain $[0, L]$ with an absorbing boundary at $x = 0$ and a reflective one at $x = L$. Let us denote $\sigma_{\text{bounded}}(x,t|x_0)$ as the associated joint distribution. For $x < L$, one has $\sigma_{\text{bounded}}(x,t|x_0) = \sigma_{\text{free}}(x,t|x_0)$, since the walker never sees the reflecting boundary. However, when one inquires specifically about $x = L$, the joint law has to be modified. Defining $F_{\underline{0},L}^{\text{bounded}}(t|x_0)$ as the probability for the walker in the interval to reach 0 for the first time at time t, one has

$$\sigma_{\text{bounded}}(L,t|x_0) = F_{\underline{0},L}^{\text{bounded}}(t|x_0) - F_{\underline{0},L}(t|x_0). \tag{B12}$$

In the particular case of symmetric diffusion in an interval, $F_{\underline{0},L}^{\text{bounded}}(t|x_0)$ can be computed and the joint law reads, in the Laplace domain,

$$\tilde{\sigma}_{\text{bounded}}(L,p|x_0) = \frac{2\sinh\left[\sqrt{\frac{p}{D}}x_0\right]}{\sinh\left[\sqrt{\frac{p}{D}}2L\right]}, \tag{B13}$$

which can be inverted in real space:

$$\sigma_{\text{bounded}}(L,t|x_0) = \frac{D\pi}{L^2}\sum_{k=1}^{\infty} e^{-(k\pi)^2\tau}(-1)^{k+1}k\sin(k\pi\tilde{x}), \tag{B14}$$

where $\tau = \frac{Dt}{4L^2}$ and $\tilde{x} = \frac{x_0}{2L}$.

APPENDIX C: DISCRETE TIME AND CONTINUOUS SPACE PROCESSES—JOINT LAWS

1. General derivation of the joint law

For any jump process defined as a process performing a random jump at every discrete time n, the relation between F and σ can always be written as

$$F_{\underline{0},x}(n|x_0) = \int_{x_0}^{x}\sigma(y,n|x_0)dy, \tag{C1}$$

which simply reflects the fact that the maximum of the trajectory necessarily lies in the interval $[x_0, x]$. However, $F_{\underline{0},x}(n|x)$ can have nonzero values for jump processes, since we require the walker to strictly cross 0 or x in order to terminate its trajectory. Writing

$$D_x F_{\underline{0},x}(n|x_0) = \sigma(x,n|x_0) \tag{C2}$$

would yield

$$F_{\underline{0},x}(n|x_0) - F_{\underline{0},x_0}(n|x_0) = \int_{x_0}^{x}\sigma(y,n|x_0)dy. \tag{C3}$$

Hence the need to correct the previous equation with the Heaviside and δ function in the following way to recover Eq. (C1):

$$H(x-x_0)D_x F_{\underline{0},x}(n|x_0) + \delta(x-x_0)F_{\underline{0},x_0}(n|x_0) = \sigma(x,n|x_0). \tag{C4}$$

2. Example of the Laplace distributed jump process

As an example, we provide the exact result in the generating function formalism for a jump process whose jumps are distributed according to $p(l) \propto e^{-\gamma|l|}$ [43]:

$$\begin{aligned}\tilde{F}_{\underline{x},0}(\xi|x_0) = {} & [\sinh(\gamma\sqrt{1-\xi}x_0) \\ & + \sqrt{1-\xi}\cosh(\gamma\sqrt{1-\xi}x_0)] \\ & \times x\xi((2-\xi)\sinh(\gamma\sqrt{1-\xi}x) \\ & + 2\sqrt{1-\xi}\cosh(\gamma\sqrt{1-\xi}x))^{-1}\end{aligned} \tag{C5}$$

from which the generating function of the joint law is obtained with the help of Eq. (C4).

APPENDIX D: NON-MARKOVIAN PROCESSES

1. Fractional Brownian motion

a. Definition

The fractional Brownian motion is a Gaussian process with a constant mean (here set at x_0) and a correlation function given by

$$\langle[X(t) - X_0][X(t') - X_0]\rangle = K[t^{2H} + t'^{2H} + |t-t'|^{2H}], \tag{D1}$$

where H is the Hurst exponent and K here is set to 1. The FBM is non-Markovian for $H \neq 1/2$ $(0 < H < 1)$ and by definition of the correlations is anomalous since $H \neq 1/2$. However, the process performs stationary increments and displays a persistent exponent shown to be equal to $\theta = 1 - H$ [6,53].

b. Numerical method

The algorithm used to sample the 1D FBM trajectories is based on the circulant matrix method and is detailed in [54–56]. This method allows generating exact trajectories with a constant time step $\Delta t \approx 1.2 \times 10^{-4}$, until a maximal time, here taken to be $t_{\max} = 4000$. By reason of the scale-invariant property, we fixed $x_0 = 1$ without loss of generality. Since this process is defined in both continuum space and time, the exact numerical measurement of the conditional law evaluated at a fixed maximum x or FPT t needed to be approximated. To evaluate $f(\tau)$ at fixed t, we kept all the trajectories for which the FPT fell on the

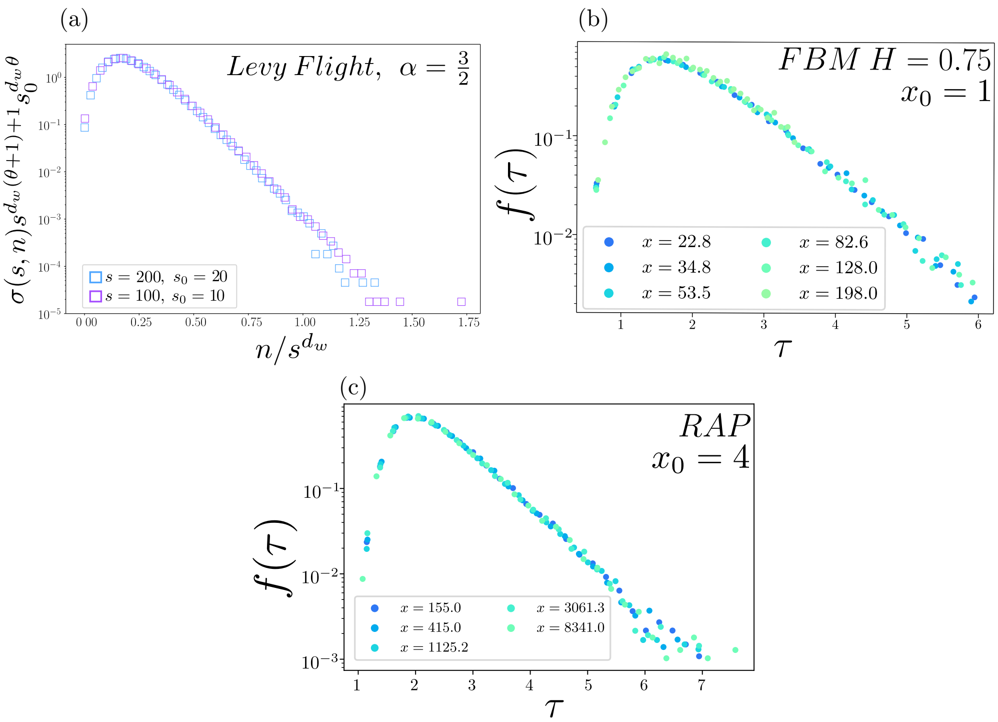

FIG. 4. (a) Data collapse of the joint law $\sigma(s, n|s_0)$ as predicted in the main text for a Riemann walk of Levy exponent $\frac{3}{2}$. (b) Conditional law of the rescaled variable for a FBM process ($H = 0.75$) starting from $x_0 = 1$. (c) Conditional law of the rescaled variable for a RAP process starting from $x_0 = 4$.

interval $[t - dt, t + dt]$, where dt is chosen as minimal as possible keeping a convenient realization number (>2000). Similarly for $\Phi(\chi)$, we only kept the trajectories for which $x \in [x - dx, x + dx]$. Here we recap the interval chosen for each curve drawn in both the main text and the Appendix. Figure 2 in the main text shows $H = 0.375$: [($x = 10.11$, $dx = 0.674$), ($x = 13.0$, $dx = 0.867$), ($x = 16.8$ $dx = 1.12$), ($x = 22.0$, $dx = 1.46$), ($x = 29.0$, $dx = 1.9$)]. Figure 4 in Appendix D shows $H = 0.75$: [($x = 22.8$, $dx = 1.52$), ($x = 34.8$, $dx = 2.38$), ($x = 53.5$, $dx = 3.57$), ($x = 82.6$, $dx = 5.51$), ($x = 128.0$, $dx = 8.53$), ($x = 198.0$, $dx = 13.2$)]. Figure 5 in Appendix D shows $H = 0.75$: [($t = 145$, $dt = 4$), ($t = 252$, $dt = 12$), ($t = 438$, $dt = 16$)].

2. Random acceleration process

a. Definition

We consider here the one-dimensional process $X(t)$ defined by

$$\frac{d^2}{dt^2}X(t) = \xi(t), \tag{D2}$$

where $\xi(t)$ is a Gaussian white noise with zero mean and $\langle\xi(t)\xi(t')\rangle = K\delta(t - t')$ with K an appropriate constant, here set to 1. Obviously this process is non-Markovian, and it has been shown that it performs a superdiffusive regime characterized by $\langle X^2(t)\rangle \underset{t\to\infty}{\sim} t^3$. Besides, the process is known for performing a compact exploration characterized by a persistent exponent $\theta = 1/4$ [6]. Despite its aging properties characterized by nonstationary increments defined by

$$\langle[X(t + T) - X(T)]^2\rangle \propto t^2 T, \tag{D3}$$

we argue that the scaling law of joint law, given in the main text, still applies.

b. Numerical method

The stochastic trajectories of the random acceleration processes are generated by means of the algorithm introduced in [51], which generates the exact probability function with a discrete time step. Since the velocity grows with t as $\sqrt{t}$, we reduce the time step at each iteration in order to keep the discrete space interval covered small. To do so, we used $\Delta t = \frac{0.05}{n^{1/3}}$, where n is the number of steps. By reason of the scale-invariant property, we fixed $x_0 = 1$ without loss of generality. Similarly to the FBM, we used the trajectories for which t (similarly x) fell in a convenient interval. Here we recap the interval chosen for each curve drawn in both the main text and the Appendix. Figure 2 in the main text shows *RAP*: [($x = 26.8$, $dx = 5$), ($x = 1091.0$, $dx = 72.7$), ($x = 2961.2$, $dx = 197$), ($x = 8040.7$, $dx = 536$), ($x = 21836.0$, $dx = 1456$)]. Figure 4 in Appendix D shows *RAP*: [($x = 155.0, dx = 10.3$), ($x = 415.0, dx = 27.7$), ($x = 1125.2$, $dx = 75.01$), ($x = 3061.3, dx = 204.8$), ($x = 8341.0, dx = 556.1$)]. Figure 5 in Appendix D shows *RAP*: [($t = 442$, $dt = 5$), ($t = 869.7$, $dt = 10$), ($t = 1711.26$, $dt = 50$), ($t = 3367.08$, $dt = 80$)].

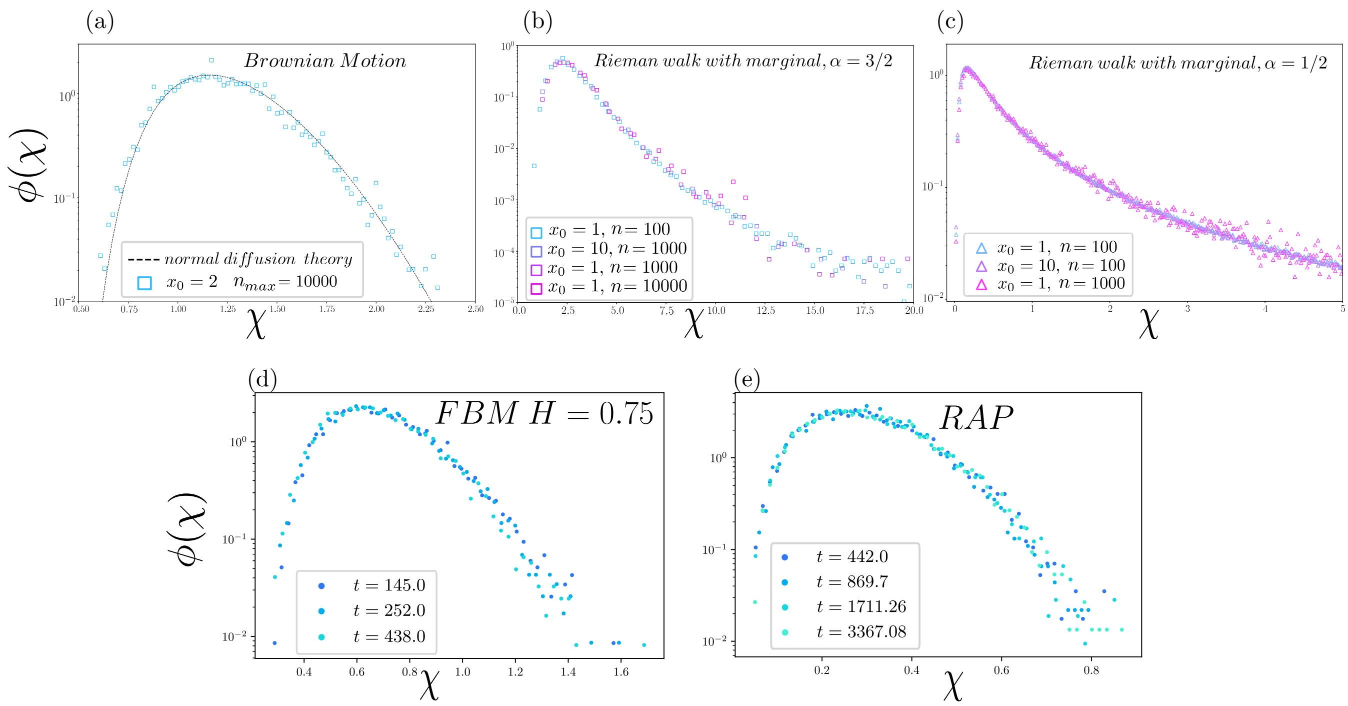

FIG. 5. Conditional law of the rescaled variable $\chi = s/n^{\frac{1}{d_w}}$ for (a) Brownian motion of diffusive coefficient $\frac{1}{2}$, (b) Riemann walk of Levy exponent $\frac{3}{2}$, and (c) Riemann walk of Levy exponent $\frac{1}{2}$. Each distribution is drawn for fixed n and collapses at large s to a process-dependent but x_0-independent function. Similarly the conditional law of the rescaled variable $\chi = x/t^{\frac{1}{d_w}}$ at fixed FPT t collapses at large x for (d) a FBM process ($H = 0.75$), and (e) a RAP.

3. From one conditional to another

In this section, we are interested in the other marginal, namely the distribution of the maximum of the trajectory knowing its return time, which we will denote as $G_{sp}(x|t, x_0)$. Let us focus on the scaling form of such a marginal. Recall that, in the large x and t limit,

$$\sigma(x, t|x_0) \sim \frac{h(x_0)}{x^{d_w(\theta+1)+1}} f\left(\frac{t}{x^{d_w}}\right), \tag{D4}$$

where f is normalized to 1 and can be identified as the conditional probability density function of the rescaled variable $\tau = t/x^{d_w}$, and "$\sim$" stands for mathematical equivalence in the large x and t limit. Let us now transform the above given expression in the following way:

$$\sigma(x, t|s_0) \sim \frac{h(x_0)}{t^{\theta+1+1/d_w}} \left(\frac{t}{x^{d_w}}\right)^{1+\theta+1/d_w} f\left(\frac{t}{x^{d_w}}\right). \tag{D5}$$

Defining now $g(u) = u^{-[d_w(\theta+1)+1]} f(u^{-d_w})$ and $\phi(u) = \frac{g(u)}{\int_0^\infty g(x)dx}$, we obtain

$$\sigma(x, t|s_0) \sim \frac{h(x_0)}{t^{\theta+1}} \left[\int_0^\infty g(u)du\right] \frac{1}{t^{\frac{1}{d_w}}} \phi\left(\frac{x}{t^{\frac{1}{d_w}}}\right). \tag{D6}$$

ϕ is normalized, so we immediately read off the desired marginal:

$$G_{sp}(x|t, s_0) \sim \frac{1}{t^{\frac{1}{d_w}}} \phi\left(\frac{x}{t^{\frac{1}{d_w}}}\right) \tag{D7}$$

with

$$\phi(\chi) = \frac{\chi^{-d_w(\theta+1)-1} f(\chi^{-d_w})}{\int_0^\infty u^{-d_w(\theta+1)-1} f(u^{-d_w}) du}, \tag{D8}$$

whose scaling form is verified in Fig. 5. Note that the first-passage time density reads

$$F_{\underline{0}}(t|x_0) \sim \frac{h(x_0)}{t^{\theta+1}} \int_0^\infty g(u)du. \tag{D9}$$

APPENDIX E: SATW WALK

1. Splitting probability of the SATW

Here we derive the splitting probability $\pi_{0,\underline{s}}$ of the SATW process starting at site 1 to hit site s before hitting site 0. In view of determining $\pi_{0,\underline{s}}$, it is convenient to parametrize its dynamics in terms of the number of distinct sites already visited. To do so, we define $\pi_{0,\underline{s}}(s')$ as the splitting probability of a walker currently at s' with a set of already visited sites $\mathcal{D}_s = [1, s-1]$ to hit site s before hitting site 0. It is then clear that $\pi_{0,\underline{s}}$ verifies the following recurrence relation:

$$\pi_{0,\underline{s+1}} = \pi_{0,\underline{s}} \pi_{0,\underline{s+1}}(s). \tag{E1}$$

Noticing that inside the visited territory D_s the walker performs a classical symmetric nearest-neighbor random walk, we have, for $0 < s' < s$,

$$\pi_{0,\underline{s}}(s') = \tfrac{1}{2}\pi_{0,\underline{s}}(s'-1) + \tfrac{1}{2}\pi_{0,\underline{s}}(s'+1). \tag{E2}$$

The solution of (E2) can be written

$$\pi_{0,\underline{s}}(s') = \lambda + \mu s, \tag{E3}$$

where λ and μ can be deduced from the boundary conditions at the extremities of the visited area,

$$\pi_{0,\underline{s}}(1) = (1-\beta)\,\pi_{0,\underline{s}}(2),$$
$$\pi_{0,\underline{s}}(s-1) = \beta + (1-\beta)\,\pi_{0,\underline{s}}(s-2). \tag{E4}$$

This yields

$$\pi_{0,\underline{s+1}}(s) = 1 - \frac{1-\beta}{2+\beta(s-3)}. \tag{E5}$$

Combining the solution (E3) with (E4) and (E1) yields finally the full splitting probability, :

$$\pi_{0,\underline{s}} = \prod_{s'=1}^{s-1}\left(1 - \frac{1-\beta}{2+\beta(s'-3)}\right) \underset{s\to\infty}{\sim} \frac{\Gamma(-2+2/\beta)}{\Gamma(-1+1/\beta)} s^{-\frac{1-\beta}{\beta}}. \tag{E6}$$

Let us notice that the distribution of the maximum can be deduced from the splitting probability:

$$\mu(s|1) = \pi_{0,\underline{s}}(1 - \pi_{0,\underline{s+1}}(s)), \tag{E7}$$

$$\mu(s|1) \underset{s\to\infty}{\sim} \frac{\Gamma(-2+2/\beta)}{\Gamma(-1+1/\beta)} \frac{1-\beta}{\beta} s^{-\frac{1-\beta}{\beta}-1}. \tag{E8}$$

2. Generative function of the joint law

Let us derive the generative function of the joint distribution $\sigma(s,n|s_0)$ of the maximum s reached during the exploration and the FPT n to 0, starting from s_0.

To do so, we make a partition on the discovery times of the first s sites. Denoting $F_{0,\underline{s}}(n|s_0)$ the probability to reach site s before 0 for the first time at step n starting from s_0 knowing that the sites $\{1,\dots,s-1\}$ have already been visited, we obtain in the generating function formalism:

$$\tilde{\sigma}(s,\xi|1) = \frac{\xi}{2}\left(\prod_{s'=3}^{s} \tilde{F}_{0,\underline{s'}}(\xi|s'-1)\right)\tilde{F}_{\underline{0},s+1}(\xi|s). \tag{E9}$$

For $1 < s_0 < n$, notice that inside the bulk of visited sites the walkers perform a simple random walk enforcing the following recurrence relation for $F_{0,\underline{s}}(n|s_0)$:

$$\tilde{F}_{0,\underline{s}}(\xi|s_0) = \frac{\xi}{2}\tilde{F}_{0,\underline{s}}(\xi|s_0+1) + \frac{\xi}{2}\tilde{F}_{0,\underline{s}}(\xi|s_0-1). \tag{E10}$$

Solving the recurrence equation leads to

$$\tilde{F}_{0,\underline{s}}(\xi|s_0) = \lambda r_1^{s_0} + \mu r_2^{s_0} \tag{E11}$$

with

$$r_1 = \frac{1}{\xi} - \frac{\sqrt{1-\xi^2}}{\xi},$$
$$r_2 = \frac{1}{\xi} + \frac{\sqrt{1-\xi^2}}{\xi}. \tag{E12}$$

We now deduce λ and μ using the boundary condition,

$$\tilde{F}_{0,\underline{s}}(\xi|1) = \xi(1-\beta)\tilde{F}_{0,\underline{s}}(\xi|2),$$
$$\tilde{F}_{0,\underline{s}}(\xi|s-1) = \beta\xi + (1-\beta)\xi\tilde{F}_{0,\underline{s}}(\xi|s-2). \tag{E13}$$

After a few lines of algebra, we finally get

$$\frac{\tilde{F}_{0,\underline{s}}(\xi|s-1)}{\beta\xi} = \frac{r_1^{s-3}[r_1-(1-\beta)\xi] - r_2^{s-3}[r_2-(1-\beta)\xi]}{r_1^{s-4}[r_1-(1-\beta)\xi]^2 - r_2^{s-4}[r_2-(1-\beta)\xi]^2}. \tag{E14}$$

For completeness, we derive the specific cases $s=2,3$,

$$\tilde{F}_{0,\underline{2}}(\xi|1) = \frac{\xi}{2},$$
$$\tilde{F}_{0,\underline{3}}(\xi|2) = \frac{\beta\xi}{1-(1-\beta)^2\xi^2}. \tag{E15}$$

We finally derive the last excursion,

$$\tilde{F}_{\underline{0},s+1}(\xi|s) = \beta\xi\frac{[1-(1-\beta)r_2\xi] - [1-(1-\beta)r_1\xi]}{r_1^{s-3}[r_1-(1-\beta)\xi]^2 - r_2^{s-3}[r_2-(1-\beta)\xi]^2}. \tag{E16}$$

Combining the different terms leads to

$$\tilde{\sigma}(\xi,s|1) = \frac{\beta^{s-1}\xi^s}{2[1-(1-\beta)^2\xi^2]} \times\left(\prod_{i=0}^{s-4}\frac{r_1^{i+1}[r_1-(1-\beta)\xi] - r_2^{i+1}[r_2-(1-\beta)\xi]}{r_1^{i}[r_1-(1-\beta)\xi]^2 - r_2^{i}[r_2-(1-\beta)\xi]^2}\right) \times \frac{[1-(1-\beta)r_2\xi] - [1-(1-\beta)r_1\xi]}{r_1^{s-3}[r_1-(1-\beta)\xi]^2 - r_2^{s-3}[r_2-(1-\beta)\xi]^2}. \tag{E17}$$

3. Laplace transformed distribution of the rescaled variable τ conditioned on the value of the maximum large time and large distance asymptotic behavior

In this section, we aim to derive the asymptotic scaling of the first-passage time distribution conditioned by the maximum value s. To do so, we need to determine the asymptotic scaling of the different terms in (E17), divided by their associated splitting probability obtained in (E6) and (E7).

Let us start by considering a given excursion of the joint law (E14), from which we extract the conditional law of a single excursion, dividing by the associated splitting probability,

$$\pi_{0,\underline{i+4}}(i+3)^{-1}\tilde{F}_{0,\underline{i+4}}(\xi|i+3) = \frac{1}{1-\frac{1-\beta}{\beta i+2}}\beta\xi\frac{r_1^{i+1}[r_1-(1-\beta)\xi] - r_2^{i+1}[r_2-(1-\beta)\xi]}{r_1^{i}[r_1-(1-\beta)\xi]^2 - r_2^{i}[r_2-(1-\beta)\xi]^2} = \frac{1}{1-\frac{1-\beta}{\beta i+2}}\left(\beta\xi\frac{r_1[r_1-(1-\beta)\xi] - \frac{r_2^i}{r_1^i}r_2[r_2-(1-\beta)\xi]}{[r_1-(1-\beta)\xi]^2 - \frac{r_2^i}{r_1^i}[r_2-(1-\beta)\xi]^2}\right). \tag{E18}$$

We now change ξ in e^{-u}, taking the discrete Laplace transform of the conditional law (E18) at fixed i. In view of determining the asymptotic behavior at large times and space of (E18), we first need to draw two observations:

(i) We are concerned with the joint limit $u \to 0$, $i \to \infty$. Thus, we aim to extract the lowest dependency (i.e., minimal exponent a) on the coupled variable iu^a which will emerge, because it will drive the asymptotic behavior of the conditional law.

(ii) By definition of the conditional law, the series expansion at large i of $\tilde{F}_{0,\underline{i+4}}(\xi|i+3)$ and $\pi_{0,\underline{i+4}}(i+3)$ cancels in the limit $u \to 0$ at fixed i.

Hence, we propose to rewrite (E18) as

$$\frac{\tilde{F}_{0,\underline{i+4}}(\xi|i+3)}{\pi_{0,\underline{i+4}}(i+3)} = \left(1 - \sum_{k=1}^{\infty} \frac{1-\beta}{\beta}\left(-\frac{2}{\beta}\right)^{k-1} i^{-k}\right)^{-1} \times \left(1 + \sum_{k=1}^{\infty} u^{ka} A_k(iu^a)\right), \tag{E19}$$

with $A_k(iu^a) \underset{u\to 0}{\sim} -\frac{1-\beta}{\beta}(-\frac{2}{\beta})^{k-1}(iu^a)^{-k}$. We propose to derive the exponent a by making an expansion at the lowest order in u of $(\frac{r_2}{r_1})^i$ which appears in (E18),

$$\frac{r_2^i}{r_1^i} = \exp[i\ln(1 - 2\sqrt{2u} + 4u - 3\sqrt{2}u^{3/2})] = \exp\left(-2\sqrt{2u}i - iO(u^{3/2})\right). \tag{E20}$$

Noticing that the dependency i only appears in the above ratio, we conclude that $a = 1/2$. In the following, we will neglect any term of the form $O(iu^{3/2})$ since it will always be negligible compared to $i\sqrt{u}$ when $u \to 0, i \to \infty$. Injecting (E20) in rightmost term of (E18), we obtain after a few lines of algebra,

$$\left(\beta e^{-u}\frac{r_1[r_1 - (1-\beta)e^{-u}] - \frac{r_2^i}{r_1^i} r_2[r_2 - (1-\beta)e^{-u}]}{[r_1 - (1-\beta)e^{-u}]^2 - \frac{r_2^i}{r_1^i}[r_2 - (1-\beta)e^{-u}]^2}\right) = 1 - \frac{1-\beta}{\beta}\frac{\sqrt{2u}}{\tanh(i\sqrt{2u})} + O(u)H(i^2u). \tag{E21}$$

Injecting the expansion (E21) obtained above in (E19) leads to

$$\frac{\tilde{F}_{0,\underline{i+4}}(\xi|i+3)}{\pi_{0,\underline{i+4}}(i+3)} = \left(1 - \sum_{k=1}^{\infty} \frac{1-\beta}{\beta}\left(-\frac{2}{\beta}\right)^{k-1} i^{-k}\right)^{-1} \times \left(1 - \frac{1-\beta}{\beta}\frac{\sqrt{2u}}{\tanh(i\sqrt{2u})} + O(u)H(i^2u)\right). \tag{E22}$$

In the limit $i\sqrt{u} \to 0$, the last term displays a series expansion along i which cancels each term of the splitting series expansion. Finally, for $i \to \infty$ with ui^2 fixed, we note that the last term is negligible compared to $\sqrt{u}$. We thus obtain

$$\frac{\tilde{F}_{0,\underline{i+4}}(\xi|i+3)}{\pi_{0,\underline{i+4}}(i+3)} = 1 + \frac{1-\beta}{\beta i}\left(1 - \frac{\sqrt{2u}i}{\tanh(i\sqrt{2u})}\right) + R(u,i), \tag{E23}$$

where we have included the additional terms of both series expansion $O(u)H_1(ui^2) + O(i^{-2})$ inside the variable noted $R(u,i)$ [$H_1(ui^2)$ taking into account the cross product of the first-order terms]. One should note that $R(u,i) \underset{ui^2\to 0}{\to} 0$ but also $R(u,i) \underset{ui^2\sim O(1)}{\to} O(i^{-2}) + o(\sqrt{u})$. In other words, the last term is negligible for $ui^2 \sim O(1)$ and $i \to \infty$. Considering the full product in (E17), we obtain in the large-time limit in Laplace,

$$P(S) = \frac{\left(\beta \prod_{i=0}^{s-4} \frac{r_1^{i+1}[r_1-(1-\beta)e^{-u}] - r_2^{i+1}[r_2-(1-\beta)e^{-u}]}{r_1^i[r_1-(1-\beta)e^{-u}]^2 - r_2^i[r_2-(1-\beta)e^{-u}]^2}\right)}{\prod_{i=0}^{s-4}\left(1 - \frac{1-\beta}{2+\beta i}\right)} \underset{us^2\sim O(1), u\to 0}{\sim} \exp\left\{\int_{i=0}^{s} \ln\left[1 + \frac{1-\beta}{i\beta}\left(1 - \frac{\sqrt{2u}i}{\tanh(i\sqrt{2u})}\right)\right]di\right\}, \tag{E24}$$

where we have legitimately expressed the sum as an integral since all the terms with $i^2 \ll u$ are negligible. We also have neglected $R(u,i)$ in the summation since the contribution of $O(i^{-2})$ is the rest of a converging series starting at a typical number given by $O(u^{-1/2})$.

Noticing the following simplification after expressing $i = i'/s$:

$$\int_0^1 \frac{1}{i'}\left(1 - \frac{i'\sqrt{2us^2}}{\tanh(i'\sqrt{2us^2})}\right)di' = \ln\left(\frac{\sqrt{2us^2}}{\sinh(2us^2)}\right), \tag{E25}$$

we finally find in the large-s limit, with s^2u fixed,

$$P(S) \underset{us^2\sim O(1), u\to 0}{\sim} \left(\frac{\sqrt{2us^2}}{\sinh(\sqrt{2us^2})}\right)^{\frac{1-\beta}{\beta}}. \tag{E26}$$

We finally consider the last excursion of (E17) when the walker finds the target without discovering any other new site. The asymptotic scaling of this last excursion normalized by its own splitting probability reads

$$\frac{2+\beta(s-3)}{1-\beta} \frac{\beta e^{-u}(1-\beta)(r_1e^{-u} - r_2e^{-u})}{r_1^{s-3}[r_1 - (1-\beta)e^{-u}]^2 - r_2^{s-3}[r_2 - (1-\beta)e^{-u}]^2} \underset{us^2\sim O(1), u\to 0}{\sim} \frac{\sqrt{2us^2}}{\sinh(\sqrt{2us^2})}. \tag{E27}$$

Lastly, we consider the first two excursions of (E17), which, normalized by their own splitting probability, do not express any dependency on the variable su^2:

$$2\frac{(1-1-\beta)^2}{\beta}\frac{e^{-u}}{2}\frac{\beta e^{-u}}{1-(1-\beta)^2e^{-2u}} = 1 + O(u). \tag{E28}$$

Combining (E28), (E27), and (E26) leads to the asymptotic expression of the Laplace transform of the conditional distribution $f_{\mathrm{SATW}}(\tau)$, which has been obtained in an another context [57]:

$$\tilde{f}_{\mathrm{SATW}}(p) = \left(\frac{\sqrt{2p}}{\sinh(\sqrt{2p})}\right)^{1/\beta}. \tag{E29}$$

The numerical inverses of Laplace transform for the SATW model are obtained using the Stehfest method [58]; see [59]. Several other numerical methods have also been tested and were in complete agreement with our prediction.

4. Continuous limit of the conditional distribution at large times and large distance

We wish here to give some precision on how to go from the initial discrete setup to a large time/large distance limit.

Recall that the discrete Laplace transform reads

$$\begin{aligned}\tilde{\sigma}(u|s,1) &= \sum_{n=0} e^{-un}\sigma(n|s,1)\\ &= \sum_{n=0} e^{-un}\frac{\phi(n/s^2)}{n}\\ &= \int_0^\infty e^{-un}\phi(n/s^2)\frac{dn}{n}\\ &= \int_0^\infty e^{-us^2\tau}\phi(\tau)\frac{d\tau}{\tau}\\ &= \mathcal{L}\left(\frac{\phi(\tau)}{\tau}\right)(us^2). \end{aligned}\tag{E30}$$

Thus defining $f_{\rm SATW}(\tau)$ as the inverse Laplace transform of

$$\tilde{f}_{\rm SATW}(u) = \left(\frac{\sqrt{2u}}{\sinh(\sqrt{2u})}\right)^{\frac{1}{\beta}}\tag{E31}$$

we finally have for the conditional law

$$G_{tm}(n|s,1) = f_{\rm SATW}(n/s^2)s^{-2}.\tag{E32}$$

Or equivalently expressing the conditional distribution along the rescaled variable $\tau = n/s^2$:

$$G_{tm}(\tau|s,1) = f_{\rm SATW}(\tau).\tag{E33}$$

5. Exact inversion of $f_{\rm SATW}(\tau)$ for some β values

When $\beta = \frac{1}{n}$ with $n \in \mathrm{N}^*$, the standard residue method can be used to invert $\tilde{f}_{\rm SATW}$. We provide here a few exact results: $\beta = \frac{1}{3}$,

$$\begin{aligned}f_{\rm SATW}(\tau) = \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2\tau}{2}}\bigg[&\frac{1}{2}(-1)^{k+1}k^6\pi^6\tau^2 - \frac{9}{2}(-1)^{k+1}k^4\pi^4\tau\\ &+\frac{1}{2}(-1)^{k+1}k^4\pi^4 + 6(-1)^{k+1}k^2\pi^2\bigg].\end{aligned}\tag{E34}$$

$\beta = \frac{1}{4}$,

$$\begin{aligned}f_{\rm SATW}(\tau) = \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2\tau}{2}}\bigg[&\frac{k^8\pi^8\tau^3}{6} - 3k^6\pi^6\tau^2 + \frac{25k^4\pi^4\tau}{2}\\ &+\frac{2k^6\pi^6\tau}{3} - \frac{10k^4\pi^4}{3} - 10k^2\pi^2\bigg].\end{aligned}\tag{E35}$$

6. Large-time asymptotics of the survival probability of the SATW

Here we derive the scaling with n of the survival probability $F_{\underline{0}}(n|s_0 = 1)$. We rewrite $F_{\underline{0}}(n|s_0 = 1)$ as a partition over the number of distinct sites discovered before reaching the target:

$$F_{\underline{0}}(n|s_0 = 1) = \sum_{s=1}^{\infty}\sigma(n,s|1) = \sum_{s=1}^{\infty}\mu(s|1)G_{tm}(n|s,1).\tag{E36}$$

In the large-s limit, considering (E8) for $\mu(s|1)$ leads to

$$\begin{aligned}F_{\underline{0}}(n|s_0 = 1) \sim &\frac{\Gamma(-2+2/\beta)}{\Gamma(-1+1/\beta)}\frac{(1-\beta)}{\beta}n^{-\frac{1-\beta}{2\beta}-1}\\ &\times\sum_{s=1}^{\infty}G_{tm}(n|s,1)\frac{n^{\frac{1-\beta}{2\beta}+1}}{s^{\frac{1-\beta}{\beta}+1}}.\end{aligned}\tag{E37}$$

Remembering that $G_{tm}(n|s,1)$ is scale-invariant in the large-n limit, negligible for $n \gg s^2$ and $n \ll s^2$, we rewrite $G_{tm}(n|s,1) = \frac{f(n/s^2)}{s^2}$ and, taking the continuum limit with $\tau = n/s^2$, we obtain

$$F_{\underline{0}}(n|s_0 = 1) \sim \frac{\Gamma(-2+2/\beta)}{\Gamma(-1+1/\beta)}\frac{(1-\beta)}{2\beta}A(\beta)n^{-\frac{1-\beta}{2\beta}-1},\tag{E38}$$

where we have defined $A(\beta)$ as

$$A(\beta) = \int_0^\infty f_{\rm SATW}(\tau)\tau^{\frac{1-\beta}{2\beta}}d\tau.\tag{E39}$$

We now propose to compute the prefactor of the first-passage time density in the case where $n = \frac{1}{\beta}$ is an odd integer. Let us observe that in this case, $\tau^{\frac{1-\beta}{2\beta}} = \tau^m$ with $m = \frac{n-1}{2}$ some integer. What is more, for any given function g,

$$\begin{aligned}\int_0^\infty g(\tau)\tau^m d\tau &= (-1)^m\frac{d^m}{d^m s}\int_0^\infty e^{-s\tau}g(\tau)d\tau\bigg|_{s=0}\\ &= (-1)^m\frac{d^m}{d^m s}\mathcal{L}(g(\tau))(s)\bigg|_{s=0}.\end{aligned}\tag{E40}$$

Injecting the expression of $f_{\rm SATW}$ yields

$$\begin{aligned}A(\beta) &= (-1)^m\frac{d^m}{d^m s}\mathcal{L}\left\{\mathcal{L}^{-1}\left[\left(\frac{\sqrt{2s}}{\sinh(\sqrt{2s})}\right)^n\right](\tau)\right\}(s)\bigg|_{s=0}\\ &= (-1)^m\frac{d^m}{d^m s}\left(\frac{\sqrt{2s}}{\sinh(\sqrt{2s})}\right)^n\bigg|_{s=0}.\end{aligned}\tag{E41}$$

Defining now the *Norlund polynomials* or *higher-order Bernoulli polynomials* by their generating function [60]:

$$\left[\frac{t}{e^t-1}\right]^l e^{xt} = \sum_{n=0}^{\infty}B_n^l(x)\frac{t^n}{n!},\tag{E42}$$

we obtain by identification

$$A\left(\frac{1}{2m+1}\right) = B_{2m}^{2m+1}\left(\frac{2m+1}{2}\right)\frac{2^{3n}m!}{2m!},\tag{E43}$$

and using known properties of the *Norlund polynomials* [60], we finally obtain

$$A\left(\frac{1}{2m+1}\right) = (2m-1)!!\tag{E44}$$

We now claim the result for arbitrary β,

$$A(\beta) = \frac{2^{\frac{1-\beta}{2\beta}}\Gamma\left(\frac{1}{2\beta}\right)}{\sqrt{\pi}},\tag{E45}$$

yielding the final result,

$$F_{\underline{0}}(n|s_0 = 1) \underset{n\to\infty}{\sim} \frac{\Gamma\left(\frac{2}{\beta}-1\right)}{\Gamma\left(\frac{1}{2\beta}-\frac{1}{2}\right)}2^{-\frac{1+\beta}{2\beta}}n^{-\frac{1-\beta}{2\beta}-1}.\tag{E46}$$

Consistency with simulations is displayed in Fig. 6.

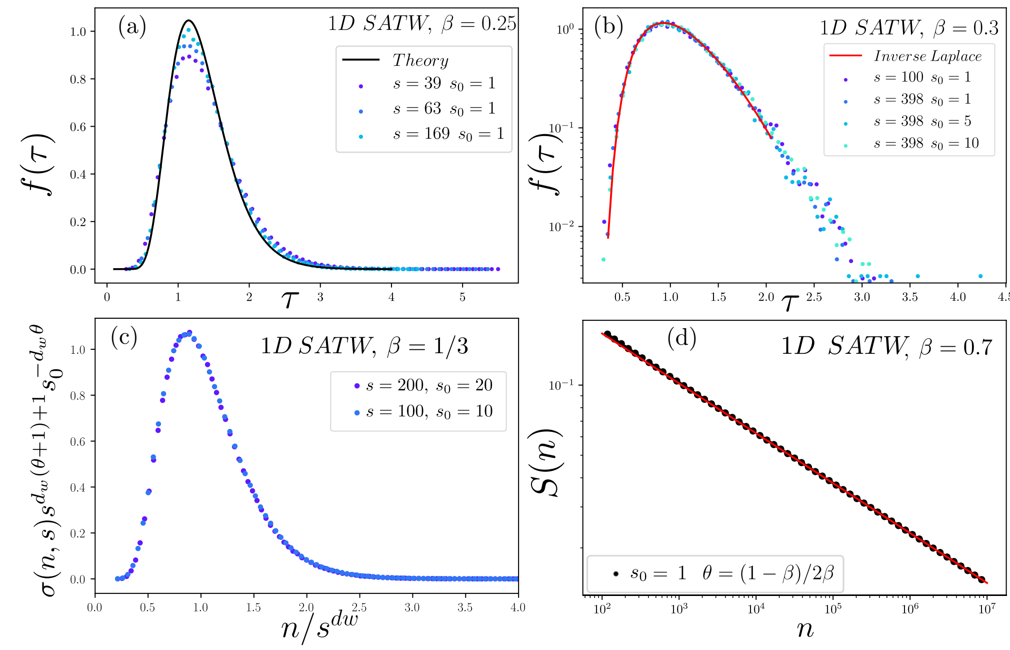

FIG. 6. (a) $\beta = \frac{1}{4}$—conditional law of the rescaled variable τ, plotted against the analytical prediction of Eq. (E35). The convergence for higher values of s is apparent. (b) $\beta = 0.3$—conditional law of the rescaled variable τ plotted against the numerical laplace inverse. Each distribution is drawn for a fixed distance s and collapses at large times to a β-dependent scaling function. Importantly, this function is independent of s_0. (c) $\beta = \frac{1}{3}$—joint law collapse along the scaling predicted by Eq. (10). Note that the collapse is independent both of s and s_0. (d) $\beta = 0.7$—large-time survival probability for a walker starting at $s_0 = 1$, plotted against the analytical prediction (E46).

[1] B. Hughes, *Random Walks and Random Environments* (Oxford University Press, New York, 1995).
[2] O. Bénichou, C. Loverdo, M. Moreau, and R. Voituriez, Rev. Mod. Phys. **83**, 81 (2011).
[3] G. M. Viswanathan, E. P. Raposo, and M. G. E. da Luz, Phys. Life Rev. **5**, 133 (2008).
[4] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, England, 2001).
[5] R. Metzler, G. Oshanin, and S. Redner, *First Passage Problems: Recent Advances* (World Scientific, Singapore, 2014).
[6] A. J. Bray, S. N. Majumdar, and G. Schehr, Adv. Phys. **62**, 225 (2013).
[7] S. Condamin, O. Bénichou, V. Tejedor, R. Voituriez, and J. Klafter, Nature (London) **450**, 77 (2007).
[8] O. Bénichou and R. Voituriez, Phys. Rep. **539**, 225 (2014).
[9] A. F. Cheviakov, M. J. Ward, and R. Straube, Multiscale Mod. Simul. **8**, 836 (2010).
[10] Z. Schuss, A. Singer, and D. Holcman, Proc. Natl. Acad. Sci. (USA) **104**, 16098 (2007).
[11] M. J. A. M. Brummelhuis and H. J. Hilhorst, Physica A **176**, 387 (1991).
[12] M. J. A. M. Brummelhuis and H. J. Hilhorst, Physica A **185**, 35 (1992).
[13] M. Chupeau, O. Benichou, and R. Voituriez, Nat. Phys. **11**, 844 (2015).
[14] O. Bénichou, M. Coppey, M. Moreau, P. H. Suet, and R. Voituriez, Europhys. Lett. **70**, 42 (2005).
[15] G. H. Weiss and P. P. Calabrese, Physica A **234**, 443 (1996).
[16] S. Burov and E. Barkai, Phys. Rev. Lett. **98**, 250601 (2007).
[17] S. Condamin, V. Tejedor, R. Voituriez, O. Benichou, and J. Klafter, Proc. Natl. Acad. Sci. (USA) **105**, 5675 (2008).
[18] G. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
[19] H. Larralde, P. Trunfio, S. Havlin, H. E. Stanley, and G. H. Weiss, Nature (London) **355**, 423 (1992).
[20] A. M. Berezhkovskii, Y. A. Makhnovskii, and R. A. Suris, J. Stat. Phys. **57**, 333 (1989).
[21] A. Blumen, J. Klafter, and G. Zumofen, in *Optical Spectroscopy of Glasses*, edited by I. Zschokke (Reidel, Dordrecht, 1986).
[22] D. Ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, Cambridge, 2000).
[23] I. Dayan and S. Havlin, J. Phys. A **25**, L549 (1992).
[24] M. J. Kearney and S. N. Majumdar, J. Phys. A **38**, 4097 (2005).
[25] P. L. Krapivsky, S. N. Majumdar, and A. Rosso, J. Phys. A **43**, 315001 (2010).
[26] J. Klinger, R. Voituriez, and O. Bénichou, Phys. Rev. E **103**, 032107 (2021).
[27] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion—Facts and Formulae* (Birkhäuser, Basel, 1996).
[28] J. Randon-Furling and S. N. Majumdar, J. Stat. Mech.: Theor. Exp. (2007) P10008.
[29] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011).
[30] M. R. Evans, S. N. Majumdar, and G. Schehr, J. Phys. A **53**, 193001 (2020).
[31] H. B. Rosenstock, SIAM J. Appl. Math. **9**, 169 (1961).
[32] D. J. Amit, G. Parisi, and L. Peliti, Phys. Rev. B **27**, 1635 (1983).

[33] R. Pemantle, Prob. Surv. **4**, 1 (2007).
[34] P. Grassberger, Phys. Rev. Lett. **119**, 140601 (2017).
[35] J. G. Foster, P. Grassberger, and M. Paczuski, New J. Phys. **11**, 023009 (2009).
[36] A. Stevens and H. G. Othmer, SIAM J. Appl. Math. **57**, 1044 (1997).
[37] V. B. Sapozhnikov, J. Phys. A **27**, L151 (1994).
[38] D. Boyer and P. D. Walsh, Philos. Trans. R. Soc. A **368**, 5645 (2010).
[39] D. Boyer, M. C. Crofoot, and P. D. Walsh, J. R. Soc., Interface **9**, 842 (2012).
[40] L. Börger, B. D. Dalziel, and J. M. Fryxell, Ecol. Lett. **11**, 637 (2008).
[41] A. Falcón-Cortés, D. Boyer, L. Giuggioli, and S. N. Majumdar, Phys. Rev. Lett. **119**, 140603 (2017).
[42] J. d'Alessandro, A. Barbier-Chebbah, V. Cellerin, O. Bénichou, R.-M. Mège, R. Voituriez, and B. Ladoux, Nat. Commun. **12**, 4118 (2021).
[43] N. V. Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[44] M. H. Ernst, J. Stat. Phys. **53**, 191 (1988).
[45] V. Tejedor, R. Voituriez, and O. Bénichou, Phys. Rev. Lett. **108**, 088103 (2012).
[46] S. N. Majumdar, S. Sabhapandit, and G. Schehr, Phys. Rev. E **92**, 052126 (2015).
[47] L. Kusmierz, S. N. Majumdar, S. Sabhapandit, and G. Schehr, Phys. Rev. Lett. **113**, 220602 (2014).
[48] S. N. Majumdar, A. Rosso, and A. Zoia, Phys. Rev. Lett. **104**, 020602 (2010).
[49] N. Levernier, O. Bénichou, T. Guérin, and R. Voituriez, Phys. Rev. E **98**, 022125 (2018).
[50] B. B. Mandelbrot and J. Vanness, SIAM Rev. **10**, 422 (1968).
[51] D. J. Bicout and T. W. Burkhardt, J. Phys. A **33**, 6835 (2000).
[52] A. Barbier-Chebbah, O. Benichou, and R. Voituriez, Phys. Rev. E **102**, 062115 (2020).
[53] G. M. Molchan, Commun. Math. Phys. **205**, 97 (1999).
[54] C. R. Dietrich and G. N. Newsam, SIAM J. Sci. Comput. **18**, 1088 (1997).
[55] R. B. Davies and D. S. Harte, Biometrika **74**, 95 (1987).
[56] A. T. A. Wood and G. Chan, J. Comput. Graph. Statist. **3**, 409 (1994).
[57] L. Serlet, Stoch. Proc. Appl. **123**, 110 (2013).
[58] https://library.wolfram.com/infocenter/MathSource/2691/.
[59] H. Stehfest, Commun. ACM **13**, 47 (1970).
[60] A. Adelberg, in *Applications of Fibonacci Numbers: Volume 7*, edited by G. E. Bergum, A. N. Philippou, and A. F. Horadam (Springer, Dordrecht, 1998), pp. 1–8.