Diffusive majority-vote model

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We define a stochastic reaction-diffusion process that describes a consensus formation in a nonsedentary population. The process is a diffusive version of the majority-vote model, where the state update follows two stages: In the first stage, spins are allowed to jump to a random neighbor node with probabilities D_+ and D_- for the respective spin orientations, and in the second stage, the spins in the same node can change its values according to the majority-vote update rule. The model presents a consensus formation phase when the concentration is greater than a threshold value and a paramagnetic phase on the converse for equal diffusion probabilities, i.e., maintaining the inversion symmetry. Setting unequal diffusion probabilities for the respective spin orientations is the same as applying an external magnetic field. The system undergoes a discontinuous phase transition for concentrations higher than the critical threshold on the external field. The individuals that diffuse more dominate the stationary collective opinion.

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I. INTRODUCTION

We consider a modified definition of a widely studied consensus formation model, namely the majority-vote model [1–12]. Here, we want to investigate the effects of itinerant spins combined with a local majority rule [13] in the model dynamics and the presence of a random noise that describes Galam contrarians [14] and independence [15]. We call this definition the diffusive majority-vote (DMV) model, and it is a reaction-diffusion process [16–18]. In particular, we are interested in the effect of different diffusive taxes for itinerant spins on the consensus formation as a way to break the inversion ($\mathbb{Z}_2$) symmetry.

Our primary motivation is to introduce a consensus metapopulation model [19], where the lattice sites are not associated with one sedentary individual. Instead, the lattice sites are associated with locations where some diffusing individuals can reunite. Metapopulation models with diffusion can lead to spatial heterogeneity [19], allowing one individual to be influenced by different groups of friends or acquaintances [20,21], increasing the model realism. In addition, the model still preserves the main feature of lattice opinion models: Individuals with close opinions and ideas tend to stay together, and this is realistic in a way that the model can describe the spatial segregation of the individuals in an election, for example, where we can see local majorities.

In the DMV model, we can attach a spin variable to every individual with values $\sigma = \pm 1$, assuming between neighboring nodes with different probabilities D_+ and D_-, and a lattice site does not have any limit on the number of hosted individuals. The control variable is the concentration ρ, defined as the mean number of individuals per node. In addition, in the symmetric situation where $D_+ = D_-$, we have the inversion symmetry, and we can expect a ferromagnetic-paramagnetic transition for a local update rule.

A question is what is the effect on the $\mathbb{Z}_2$ symmetry breaking by turning the probabilities $D_+ \neq D_-$. Symmetry breaking, in this case, means favoring an opinion value, which can describe individuals more avid to spread its opinion. Symmetry breaking can also be interpreted as an external field, analogous to the mass media's influence over society. Our results can shed some light on the problem of election manipulation and the spreading of controversial opinions or conspiracy theories by organized groups, which has obvious implications for social systems.

The DMV model can determine if the Ising universality class is robust when combining the local update rule with diffusion. It is known that the epidemic models with local updates given by a contact process, when Brownian diffusion dominates transport, can have different critical exponents in lower dimensions owing to the additional conservation laws. The Janssen-Grassberger conjecture of directed percolation (DP) [22–24] predicts any short-range model with a fluctuating continuous phase transition from an active to a unique absorbing state, with a positive-definite order parameter and no additional symmetries, conserving laws or quenched randomness, falls into the DP universality class.

According to Janssen-Grassberger's conjecture, the universality class of an epidemic process can change if we include diffusion of a conserved number of particles. One example is the diffusive epidemic process (DEP), which models an epidemic spreading in a diffusive population [25–34]. The DEP presents a new universality class [33], where exponents in lower dimensions deviate from the exponents of the contact process (CP). The CP describes the spreading of epidemics with no permanent immunity and obeys the DP universality class.

In summary, our main objective is to study the critical behavior of the DMV model when changing the diffusive probabilities and noise parameter value, which enters the local update rule. In particular, the system presents a consensus phase for particle concentrations greater than a threshold value, which depends directly on the noise parameter. The difference of the diffusion probabilities acts as an external field. In Sec. II, we present the model definition and the relevant observables. In Sec. III, we discuss the main results. Finally, in Sec. IV, we present our conclusions.

II. MODEL AND SCALING

A. DMV model

In the following, we present our definition of the DMV model. We consider a population W of walkers, given in terms of the concentration ρ as

$$W = \rho N, \tag{1}$$

where N is the number of the lattice nodes we assign a spin variable to each walker, assuming two values $\sigma = \pm 1$. The following rules define the Markovian chain:

(i) Initialization: We randomly distribute the population in the lattice. Every spin can randomly assume two values $\sigma = \pm 1$, and diffuse with respective probabilities $D_\pm$. We store the number of spins $+1$, and -1 in each node by using two arrays $S_\pm$ of size N. Every dynamical step is done in a unitary time interval and is composed of two stages:

(ii) Diffusion: All nodes are visited, and for each individual with spin $\sigma = \pm 1$ in the node i, one generates a random uniform number x, and if $x \leqslant D_\pm$, the individual jumps to a randomly chosen neighbor j, in such a way that the arrays are updated as follows,

$$\begin{aligned} S_\pm(i, t+dt) &= S_\pm(i, t) - 1, \\ S_\pm(j, t+dt) &= S_\pm(j, t) + 1. \end{aligned} \tag{2}$$

(iii) Reaction: Inside a node, the spins follow the two-state MV model update rule [1–3]. All nodes are visited, and for each spin $\sigma = \pm 1$ in the node i, we try a spin flip with a probability $\omega_\pm$, written as

$$\omega_\pm = \tfrac{1}{2}[1 \pm (1 - 2q)\Theta(S_+(i) - S_-(i))], \tag{3}$$

where $\Theta(x)$ is the *signal* function, associated with the node majority opinion

$$\Theta(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{4}$$

When computing the local majority, we sum only the other spins in the same node (excluding the spin we are trying the spin flip). However, this does not affect the critical behavior of the model. In the case of no local majority [$x = 0$ or $S_+(i) = S_-(i)$], the spin can change its opinion state with $\omega_\pm = 1/2$.

The noise parameter induces a no-consensus phase, analogous to the paramagnetic phase of magnetic materials. We repeat rules 2 and 3 by several predefined Monte Carlo (MC) steps, and for every repetition, we increase a time counter by one unit. Diffusion and reaction stages are done with the help of temporary arrays $S_\pm^{(t)}$ of size N, where we store the states of the lattice sites after the spin flips and jumps, respectively. The use of temporary arrays avoids the possibility of a spin jumping twice or more and eliminates the case of a spin undergoing twice or more spin flips in a time unit.

B. Observables and critical behavior

After describing the DMV dynamics, we present the needed observables to identify the critical behavior. The main observable is the magnetization

$$m = \frac{1}{W}\sum_{i}^{N}[S_+(i) - S_-(i)]. \tag{5}$$

From the moments of the time series of m, we can obtain the order parameter M, its respective susceptibility χ, and Binder's fourth-order cumulant U, which are given by [1]

$$\begin{aligned} M(q) &= \langle |m| \rangle, \\ \chi(q) &= N(\langle m^2 \rangle - \langle |m| \rangle^2), \\ U(q) &= 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}, \end{aligned} \tag{6}$$

respectively, where $|m|$ is the absolute value of m. All observables are functions of noise parameter q and concentration ρ.

The observables written in Eq. (6) obey the following finite-size scaling (FSS) relations in the case of symmetric $D_+ = D_-$,

$$\begin{aligned} M &= L^{-\beta/\nu} f_M[L^{1/\nu}(\rho - \rho_c)], \\ \chi &= L^{\gamma/\nu} f_\chi[L^{1/\nu}(\rho - \rho_c)], \\ U &= f_U[L^{1/\nu}(\rho - \rho_c)], \end{aligned} \tag{7}$$

where L is the linear size of the lattice. The number of nodes is $N = L^d$, where d is the dimension of the lattice. In the scaling relations given by Eq. (7), $1/\nu$, β/ν, and γ/ν are the critical exponent ratios, ρ_c is the critical concentration, and $f_{M,\chi,U}$ are the finite-size scaling functions. Identifying the external field as $h = |D_+ - D_-|$, we can conjecture the following scaling relation for the order parameter,

$$M = h^{1/\delta} g_M[h^{-1/(\beta\delta)}(\rho - \rho_c)]. \tag{8}$$

To obtain the relevant observables, we performed the dynamics on square lattices with sizes $L = 50$, $L = 60$, $L = 70$, $L = 80$, $L = 90$, and $L = 100$, and cubic lattices with sizes $L = 14$, $L = 16$, $L = 18$, $L = 20$, $L = 22$, and $L = 24$, all with periodic boundary conditions. In all simulations shown here, we used a noise value of $q = 0.1$. In general, for a finite value of the noise parameter, we obtain a threshold for $D_+ = D_-$ that increases with the noise parameter. We considered 10^6 MC steps to evolve the system in a stationary state and another 10^7 MC steps to collect 10^7 values of the opinion balance to measure the observables. We calculated error bars by using the "jackknife" resampling technique [35,36].

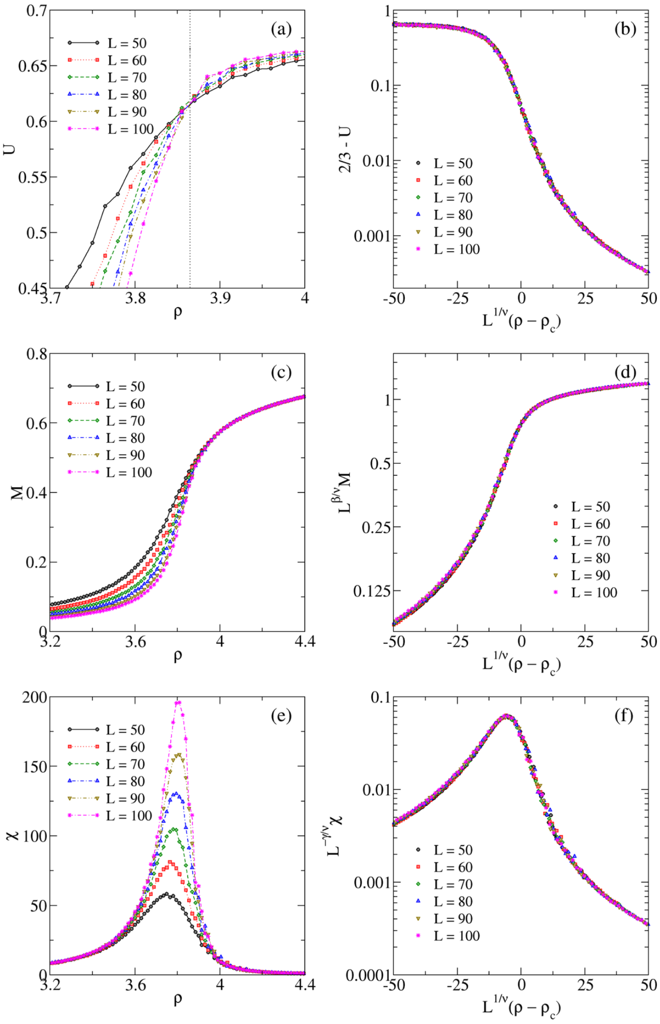

FIG. 1. Results of the averages given in Eq. (6) for the DMV model on a square lattice, with periodic boundary conditions, diffusion probabilities $D_+ = D_- = 0.5$, and noise parameter $q = 0.1$. In (a), (c), and (e), we show our numerical data for the Binder cumulant U, the order parameter M, and susceptibility χ for different lattice sizes. In (b), (d), and (f) we show the respective data collapses following Eq. (7), and the critical exponents given in Table I for two dimensions (2D). The estimated value of the critical threshold for the square lattice is $\rho_c = 3.862(5)$. Error bars are smaller than symbols and are not shown.

III. RESULTS AND DISCUSSION

We start by presenting the results for the relevant observables of the DMV model on the square and cubic lattices with periodic boundary conditions. In Figs. 1 and 2, we show the results for the cumulant, the order parameter, and its susceptibility for square and cubic lattices, respectively, with noise $q = 0.1$ and diffusion probabilities $D_+ = D_- = 0.5$. For a finite value of the noise parameter, and equal diffusion probabilities $D_+ = D_- = D$, the system undergoes a continuous phase transition by increasing the concentration.

The continuous transition is in the Ising universality class regarding the scaling behavior, which is unexpected at first glance because, as already mentioned in the Introduction, the reaction-diffusion version of the contact process, known as the DEP, has a different set of exponents from the CP defined on a sedentary population. According to the Janssen-Grassberger conjecture [22,23], the particle conservation laws of the DEP are responsible for changing the universality class. However, the introduction of diffusion while maintaining $\mathbb{Z}_2$ symmetry

TABLE I. Ising critical exponents. In our data collapses, we used the exact Ising exponents in 2D, and the best estimations in 3D to date, given in Ref. [37].

Critical exponents	Values in 2D	Values in 3D
ν	1	0.629971(4)
β	1/8	0.326419(3)
γ	7/4	1.237075(10)
δ	15	4.78984(1)

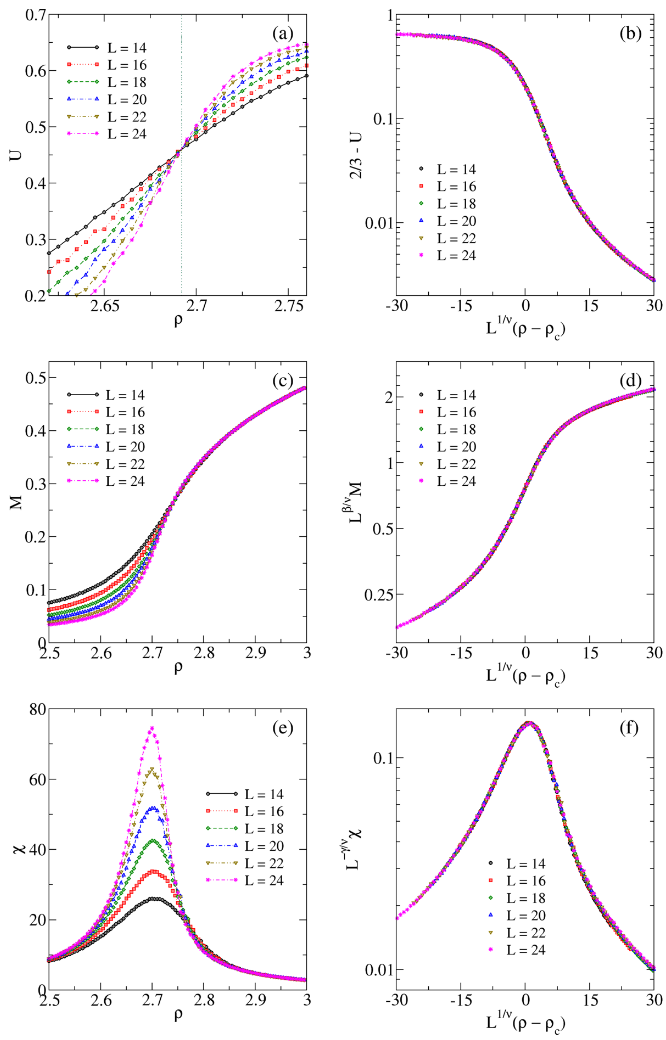

FIG. 2. The same as Fig. 1 for the simple cubic lattice with periodic boundary conditions. The estimated value of the critical threshold for the simple cubic lattice is $\rho_c = 2.692(5)$. The data collapses follow Eq. (7), and the critical exponents given in Table I for three dimensions (3D). Error bars are smaller than symbols and are not shown.

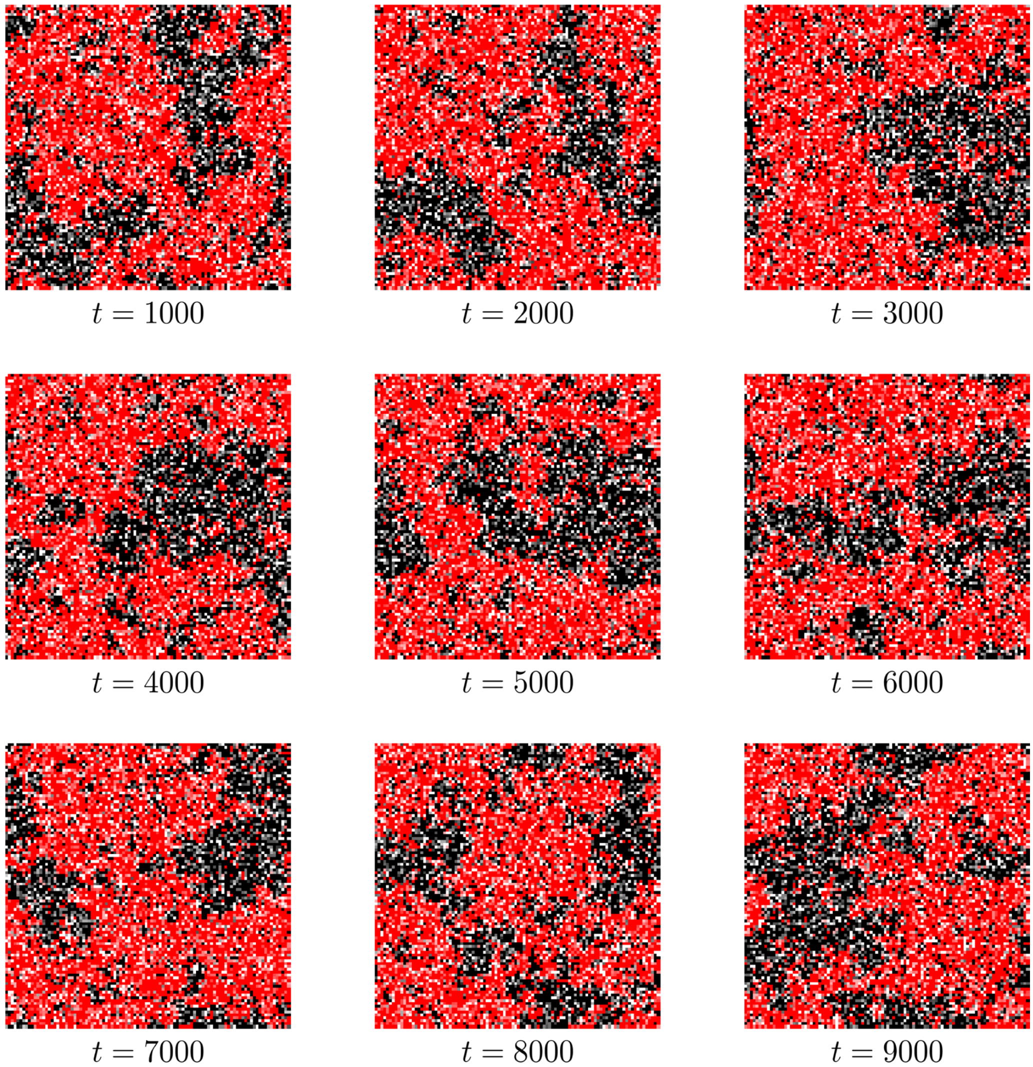

FIG. 3. Snapshots of DMV dynamics on a square lattice with periodic boundary conditions, with diffusion probabilities $D_+ = D_- = 0.5$, and noise parameter $q = 0.1$. A node is black if most of its spins have the value 1, red for the opposite, and white for no local majority. We simulated the dynamics for the value of the susceptibility maxima $\rho = 3.81(5)$. Note the presence of clusters, where walkers with the same opinion state tend to stay closer.

still leaves the system in the Ising universality class, as shown by the data collapses in Figs. 1 and 2. The exponents of the Ising universality class used in our data collapses are shown in Table I.

We show snapshots of the DMV dynamics in Fig. 3 for equal diffusive probabilities $D_+ = D_- = 0.5$ on the maxima of fluctuations for a square lattice, with $L = 100$, estimated at $\rho = 3.81(5)$. The dynamics generate clusters of populations with a defined polarization state, where spins with the same polarization tend to stay together, leading to spatial segregation. When close to the critical threshold, the system has clusters of various sizes.

We simulated the model for some diffusion probabilities $D_+ = D_- = D$. We estimate the critical thresholds by using the critical exponents in Table I, and the inspection of the data collapses when changing the ρ_c value yields an estimate for the critical threshold. The $\Delta\rho$ interval containing the cumulant crossings offers an estimate for the error bar for the critical threshold. In general, the threshold increases with the value of the noise parameter, indicating that an increasing noise strength induces the paramagnetic phase by the disorder. On the other hand, the growing concentration causes a consensus (ferromagnetic order). In Fig. 4, we show phase diagrams of the DMV model with a constant noise $q = 0.1$ in square and cubic lattices. The curve gives the critical thresholds for the model, where the system presents continuous phase transitions from a paramagnetic phase (shown in yellow) to a ferromagnetic phase (shown in blue). The regime at slow diffusion is almost linear on $1/D$.

We focus on the critical behavior as a function of the diffusion probabilities and analyze what happens at the $D_+ \neq D_-$ case. The system behavior follows the FSS relation given in Eq. (8) where the external magnetic field is $h = |D_+ - D_-|$.

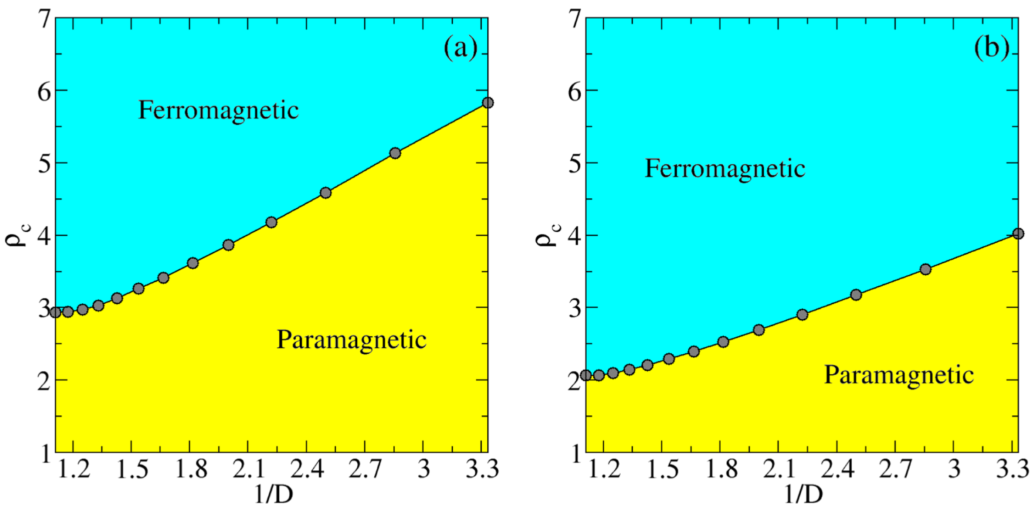

FIG. 4. We show the phase diagrams for the DMV model's constant noise parameter $q = 0.1$ on the square and cubic lattices in (a) and (b), respectively. The critical threshold increases when decreasing the diffusion probability $D_+ = D_- = D$ and presents an approximately linear dependence in the slow diffusion (small D) regime. The curve gives the critical thresholds ρ_c as functions of $1/D$ for the continuous phase transitions between the paramagnetic phase (shown in yellow) to the ferromagnetic phase (shown in blue). In the blue region (ferromagnetic phase), for a constant $\rho > \rho_c$, the system undergoes a discontinuous phase transition as a function of $D_+ - D_-$, analogous to the Ising model isotherms.

We show results as functions of the external field in Fig. 5 for a square lattice with $L = 100$ and a cubic lattice with $L = 24$ with periodic boundary conditions. Note that the critical behavior is still consistent with the Ising universality class, where the critical isotherm exponent δ values for 2D and 3D are given in Table I.

In this way, the system has a discontinuous phase transition for $\rho > \rho_c$ as a function of the difference between the diffusion probabilities. At the critical threshold $\rho = \rho_c$, we have the critical isotherm where M scales as $h^{1/\delta}$. We can state that an organized group of individuals that are more avid to spread their own opinion is similar to a mass media influence over society described by an external magnetic field. The external field dictates the direction of the net magnetization as shown by the histogram of the time series with 10^6 values of m for a DMV model with different diffusive probabilities, depicted in Fig. 6. The interchange of different values reflects the histogram, changing the sign of the most probable value of the net magnetization.

In addition, the maxima separations of the histograms from $m = 0$ increase with the concentration so that if we grow the population of walkers, we produce a greater net magnetization. For $\rho > \rho_c$, the histogram of a time series shows a bimodal pattern. Note that the histogram is not the actual stationary distribution because our sampling is just an importance sampling. The actual stationary distribution could be obtained by a "flat histogram" algorithm, such as the Wang-Landau algorithm [38] for equilibrium systems. Such an algorithm is nonexistent for nonequilibrium systems as far as we know. However, the histogram shows a metastable state, indicating a discontinuous transition on the external field h.

IV. CONCLUSIONS

We presented a consensus formation model in a system composed of nonsedentary individuals that can interact in the same node of a lattice or network according to the majority rule. The model assigns different diffusion probabilities for the other individual opinion states, and it includes the presence of a random noise that describes Galam contrarians [14] and independence [15]. The DMV is a reaction-diffusion process, where the transport is dominated by Brownian diffusion, and the reactions are done by following a local majority rule.

In the case of $\mathbb{Z}_2$ symmetry, i.e., equal diffusion probabilities, the system presents a continuous phase transition from a paramagnetic phase to a ferromagnetic state, with a global consensus when increasing the number of individuals. Unlike the DP universality class, introducing a conserved metapopulation of diffusive spins does not affect the Ising universality class. In this way, the model predicts that isolation can potentialize local disagreements and induce a global dissensus state. Increasing concentration causes a contrary effect of the Galam contrarians, i.e., preserving the global consensus. As expected, the critical threshold increases with the noise parameter and decreases with the equal diffusion probabilities.

When breaking the $\mathbb{Z}_2$ symmetry, the greater diffusion probability determines the dominant (consensus) opinion state, which is consistent with the fact that the dominant opinion state is the opinion state of the more eager individuals to convince others. The external magnetic field is the absolute value of the difference between the diffusion probabilities, analogous to the influence of mass media, which has obvious implications for election manipulation and the spreading of organized groups' controversial opinions or conspiracy theories.

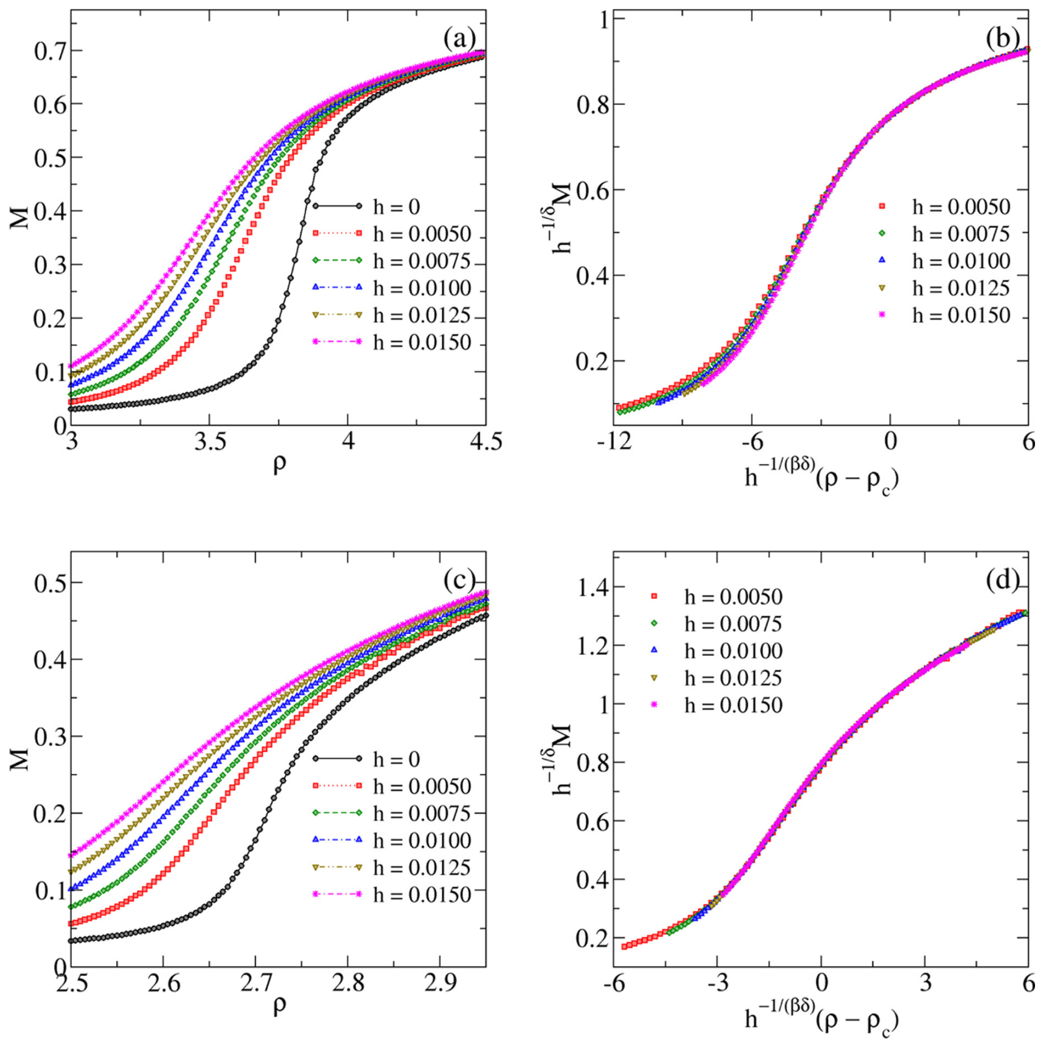

FIG. 5. In (a), we show magnetizations for the DMV model on a square lattice of size $L = 100$ with periodic boundary conditions. We consider the case $D_+ = 0.5$, $D_- = 0.5 + h$, and noise parameter $q = 0.1$, where the external field is identified as $h = |D_+ - D_-|$. In (b), we show the respective data collapse following the FSS relation given in Eq. (8), the critical threshold $\rho_c = 3.862(5)$, and the critical isotherm exponent δ for 2D, given in Table I. In (c) and (d), we show the same of (a) and (b), respectively, for a cubic lattice of size $L = 24$ with periodic boundary conditions where the critical threshold is $\rho_c = 2.692(5)$. Error bars are smaller than the symbols and are not shown.

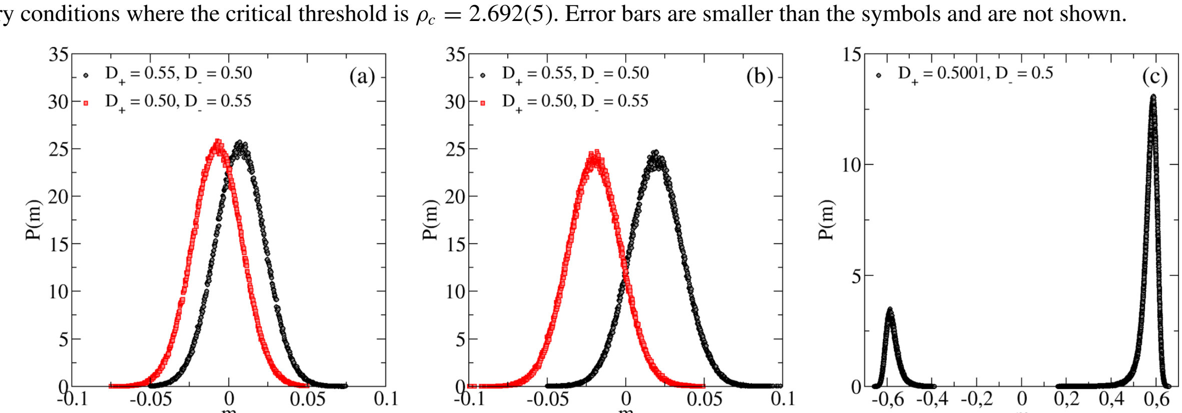

FIG. 6. In (a), we show two histograms of the time series of m for the stationary evolution of DMV on a square lattice with $D_+ = 0.55$, and $D_- = 0.5$ (black circles), and with $D_+ = 0.5$, and $D_- = 0.55$ (red circles). For both data sets, the noise parameter is $q = 0.1$, $L = 100$, and the concentration is $\rho = 1$. In (b), we have the same parameters as (a) except the concentration, which is $\rho = 1.5$. In (c), we show the histogram of a time series for a concentration $\rho = 4 > \rho_c$. Note that the sign of the most probable value of m follows one of the most diffusing particles. The most probable value of m increases with the concentration, which can be seen from the spacing between the maxima of the histograms in (a) and (b). Eventually, for $\rho > \rho_c$, the system undergoes a discontinuous phase transition on the external field h, where the histogram presents a metastable state and a bimodal shape.

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