Anomalous band center localization in the one-dimensional Anderson model with a disordered distribution of infinite variance

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We perform a detailed numerical study of the influence of distributions without a finite second moment on the Lyapunov exponent through the one-dimensional tight-binding Anderson model with diagonal disorder. Using the transfer matrix parametrization method and considering a specific distribution function, we calculate the Lyapunov exponent numerically and demonstrate its relation with the fractional lower order moments of the disorder probability density function. For the lower order of moments of disorder distribution with an infinite variance, we obtain the anomalous behavior near the band center.

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I. INTRODUCTION

Since Anderson's pioneer work [1] on the localization of electrons caused by randomly distributed impurities in 1958, the properties of disordered systems have fascinated scientists for more than 60 years. Compared with the great progress of strongly correlated systems in recent decades, peoples' understanding of disordered systems is still unsatisfactory. Only a few cases of low-dimensional disordered systems have analytic solutions [2].

In the one-dimensional finite chain Anderson model, the transport properties of electron wave function are characterized by the localization length ξ or the Lyapunov exponent γ, which is the reciprocal of the localization length. Many properties of disordered systems are related to this parameter. The Lyapunov exponent has no extensive mathematical form, and the analytic result of the Lyapunov exponent can be obtained only in the strong or weak disorder limits. It is very important to study the Lyapunov exponent for different disorder distributions and electron energies, to improve the understanding of the properties of disordered systems.

The one-dimensional Anderson model has been studied from different aspects. Researchers have explored the influence of different disorder strengths [3] and asymmetric disorder distribution [4] on the localization length. Most of these works are carried out by the perturbation expansion method on the variance of disorder distribution. In most statistical problems, the mean and variance are routinely used to describe a distribution. However, if the higher order moments of disorder strength do not exist, the perturbation expansion method is no longer applicable. A random variable X has an infinite variance and does not prevent the same variable from assuming finite values with probability 1. We need to find a new way to deal with this infinite variance situation.

Some researchers focus on stable distributions with heavy tails $1/x^{\alpha}$, which is possible to preserve scaling [5]. Such distributions have been found in various nature and social phenomena [6–8]. Only until recently is it possible to study these heavy-tails type disordered systems experimentally [9–11], which has aroused interest to study these phenomena again [12–16]. For instance, related researches on one-dimensional disordered systems have shown that the conductance distribution is completely determined by only two parameters [11–13]: the average of logarithm of conductance $\langle \ln G\rangle$ and the scaling exponent α of the power-law tail; all other details of the disorder configuration are irrelevant.

In the present work, we consider a distribution $(x^2 + \sigma^2)^{-(\alpha+1)/2}$ with $1 < \alpha \leqslant 2$. This disorder distribution does not have the finite second moment or variance, but the absolute mean $E[|X|]$ is defined. The noninteger moments of distributions are sometimes called fractional lower order

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moments. The distribution $(x^2+\sigma^2)^{-(\alpha+1)/2}$ covers the fractional lower order moments $E[X^\alpha]$ with the order of moments $\alpha \in (1, 2]$. Through this distribution we will study the behavior of the Lyapunov exponent for disorder distributions with a finite first order moment but with an infinite second order moment. We show detailed numerical results about the influence of fractional lower order α on the Lyapunov exponent γ.

II. TRANSFER MATRIX PARAMETRIZATION METHOD

The stationary Schrödinger equation of the one-dimensional Anderson model with diagonal disorder is

$$\epsilon_i\psi_i + t(\psi_{i+1}+\psi_{i-1}) = E\psi_i, \tag{1}$$

where ψ_i is the wave function on the ith lattice site, ϵ_i is the on-site energy, and E is the eigenenergy. In the following, the hopping energy t between the nearest neighbor lattice points is set to be the unit energy. The in-band energy is $E \in [-2, 2]$, and the energy out of band is $|E| > 2$. By introducing a two-component vector Ψ_i, we can rewrite the above equation with the transfer matrix $\mathbf{T}_i$,

$$\Psi_{i+1} = \begin{pmatrix}\psi_{i+1}\\ \psi_i\end{pmatrix} = \begin{pmatrix}E-\epsilon_i & -1\\ 1 & 0\end{pmatrix}\Psi_i = \mathbf{T}_i\Psi_i. \tag{2}$$

Through the transfer matrix, we can express the wave function Ψ_L on the Lth site as

$$\Psi_L = \mathbf{M}_L\Psi_1 = \mathbf{T}_L\mathbf{T}_{L-1}\cdots\mathbf{T}_1\Psi_1. \tag{3}$$

The transfer matrix $\mathbf{T}_i$ is a real symplectic matrix, so is $\mathbf{M}_L$. We then can diagonalize $\mathbf{M}_L^T\mathbf{M}_L$ by an orthogonal matrix $\mathbf{U}(\theta_L)$,

$$\mathbf{U}(\theta_L)\mathbf{M}_L^T\mathbf{M}_L\mathbf{U}(-\theta_L) = \begin{pmatrix}e^{\lambda_L} & 0\\ 0 & e^{-\lambda_L}\end{pmatrix}, \tag{4}$$

with

$$\mathbf{U}(\theta_L) = \begin{pmatrix}\cos\theta_L & -\sin\theta_L\\ \sin\theta_L & \cos\theta_L\end{pmatrix}.$$

A recursion relation can be obtained from the above formula for sufficiently long chains in the localization regime,

$$\tan\theta_{L+1} = \frac{1}{v_L - \tan\theta_L}, \tag{5}$$

where we have defined $v_L = E - \epsilon_L$.

With the distribution of ϵ or v, we can obtain an integral equation for $p(\theta)$ [17],

$$p(\theta) = \frac{1}{\sin^2\theta}\int_{-\pi/2}^{\pi/2} p(\theta')p_v\left(\frac{1}{\tan\theta}+\tan\theta'\right)d\theta'. \tag{6}$$

Then the Lyapunov exponent is given by

$$\gamma = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty} p(\theta)p_v(v) \times \ln(1+v^2\cos^2\theta - v\sin 2\theta)d\theta\, dv. \tag{7}$$

For uncorrelated disorder, $p(\theta)$ and $p_v(v)$ are independent of each other, and the above equations are exact.

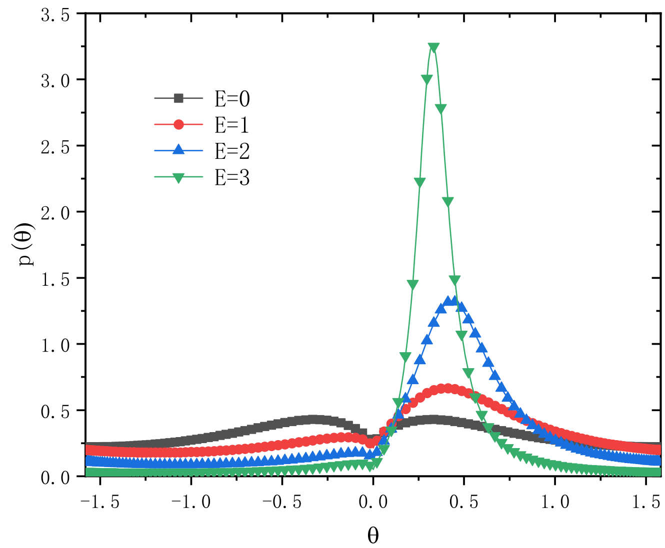

FIG. 1. $p(\theta)$ for typical in- and out-of-band energies E with $\alpha = 1.5$ and $\sigma = 1$.

There is no general solution for $p(\theta)$ except for some special cases, such as when $p_v(v)$ is the Cauchy distribution. Therefore we consider a probability density function

$$p_v(v,\alpha) = \frac{\sigma^\alpha}{B(\frac{\alpha}{2},\frac{1}{2})}\left(\frac{1}{(E-v)^2+\sigma^2}\right)^{(\alpha+1)/2}, \tag{8}$$

with

$$B\left(\frac{\alpha}{2},\frac{1}{2}\right) = \frac{\Gamma[(1+\alpha)/2]}{\sqrt{\pi}\Gamma(\alpha/2)},$$

where $B(x, y)$ and $\Gamma(x)$ are the beta function and the gamma function, respectively. In particular, the mean exists if and only if $\alpha > 1$, and the variance exists if and only if $\alpha > 2$. When $\alpha = 1$, we recover the Cauchy distribution,

$$p_v(v,1) = \frac{1}{\pi}\frac{\sigma}{(v-E)^2+\sigma^2}, \tag{9}$$

and in the limit $\alpha \to \infty$, the density function $p_v(v,\alpha)$ converges to the Gaussian distribution $\exp[\frac{-(\alpha+1)(E-v)^2}{2\sigma^2}]$. In addition, when $\sigma \to \infty$, the distribution decays with a power law. Therefore this distribution not only represents an interpolation between the Gaussian distribution and the Cauchy distribution, but also provides an approach to study the power-law-like distribution.

In order to get the density function $p(\theta)$, we solve the integral equation (6) numerically. The accuracy of numerical calculation is approximately 10^{-7}. In Fig. 1, four curves of $p(\theta)$ are plotted for different values of energy E, where we have chosen $\alpha = 1.5$ and $\sigma = 1$. Figure 1 shows that $p(\theta)$ is symmetrical about the origin $\theta = 0$ when $E = 0$. As the energy E increases, the distribution of θ becomes asymmetric and tends to be the δ function. This is consistent with previous studies [17].

To compare with previous results for weak disorder distributions with small variances, we plot $p(\theta)$ for in-band energies in Fig. 2, where we set $\alpha = 1.5$ and $\sigma = 0.1$. As the energy decreases, the peak of $p(\theta)$ becomes lower and lower.

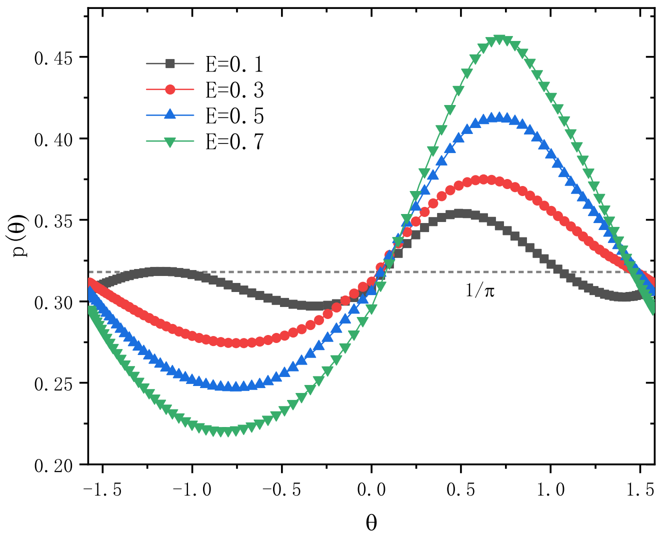

FIG. 2. Density function $p(\theta)$ for small energies E under the conditions $\alpha = 1.5$ and $\sigma = 0.1$.

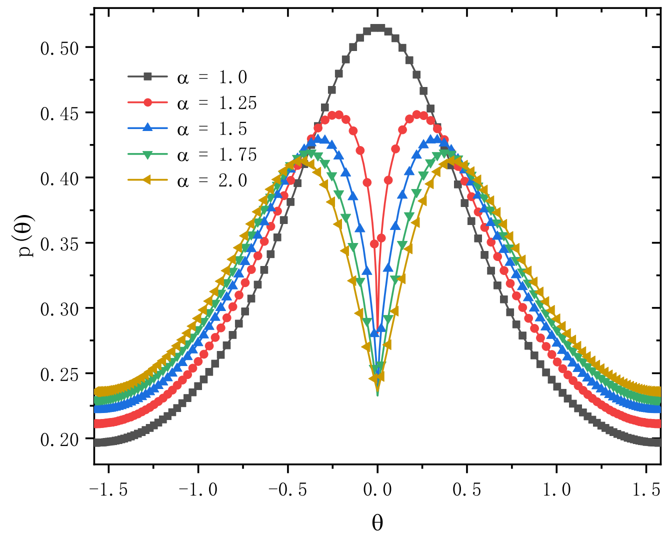

FIG. 3. $p(\theta)$ for typical α's with $E = 0$ and $\sigma = 1.0$.

These curves are close to the following relation:

$$p(\theta) = \frac{\sqrt{4 - E^2}}{2\pi(1 - E/2\sin 2\theta)}, \tag{10}$$

which corresponds to the $p(\theta)$ of uncorrelated disorder at a finite E in the weak disorder limit [3,18–21]. We find that within our numerical error $p(\theta)$ is given by Eq. (10) when $\sigma \to 0$ for $\alpha = 1.5$. Here the point is not that $p(\theta)$ is given by Eq. (10) in the weak disorder limit when we fix σ and let α go to infinity, which is natural. Here the disorder has no higher moments except the mean. We find that $p(\theta)$ is given by Eq. (10) when we fix α and let σ go to zero.

III. ANOMALY OF THE LYAPUNOV EXPONENT NEAR THE BAND CENTER

Now we study the band center region with energies around zero. We demonstrate the behavior of $p(\theta)$ for $E = 0$ first. In Fig. 3, we plot $p(\theta)$ for various α's under $E = 0$ and $\sigma = 1$. We see that the probability density function $p(\theta)$ is bimodal, which is different from the unimodal result of the Cauchy distribution. $p(\theta)$ for the Gaussian distribution is also bimodal. In Fig. 3 the $p(\theta)$ for $1 < \alpha \leqslant 2$ looks more like the result of the Gaussian distribution except the singularity at $\theta = 0$.

To obtain an analytical expression of $p(\theta)$ around $\theta = 0$, we change the integral variable $d\theta'$ into $d\tan\theta'$ in the integral equation (6), then

$$p(\theta) = \int \frac{p(\theta')}{1 + \tan^2\theta'} \frac{p_v\left(\frac{1}{\tan\theta} + \tan\theta'\right)}{\sin^2\theta} d\tan\theta'.$$

In the limit of $\theta \to 0$, only two parts of the integration need to be considered when $\alpha \geqslant 1$: the $\theta' = 0$ peak for $p(\theta')/(1 + \tan^2\theta')$, and the $E - \frac{1}{\tan\theta} - \tan\theta' = 0$ peak for p_v. We obtain the behavior of the probability density function $p(\theta)$ around $\theta = 0$ when $\alpha \geqslant 1$,

$$p(\theta) = p\left(\frac{\pi}{2}\right) + \frac{\sigma^\alpha}{B(\frac{\alpha}{2}, \frac{1}{2})}|\sin\theta|^{\alpha-1}, \tag{11}$$

which is demonstrated by numerical results in Fig. 3.

In the weak disorder limit, the probability density function $p(\theta)$ given by the traditional perturbation theory is a stable form [3,21]

$$p(\theta) = \frac{1}{K(1/\sqrt{2})\sqrt{3 + \cos 4\theta}}, \tag{12}$$

where $K(1/\sqrt{2}) \approx 1.85$ is the complete elliptic integral of the first kind. In Fig. 4 we plot $p(\theta)$ for small σ's at $E = 0$ and $\alpha = 1.1$, and we find that $p(\theta)$ is not close to Eq. (12) for small σ's at $E = 0$. In the case of $1 < \alpha \leqslant 2$, $p(\theta)$ around $\theta = 0$ is in a V shape, which can be seen in Fig. 4. We calculate further $p(\theta)$ for $\alpha > 2$, and find $p(\theta)$ tends to Eq. (12) smoothly when σ goes to zero. This indicates that the existence of second moment has a great influence on $p(\theta)$.

In Fig. 5, we plot the Lyapunov exponent $\gamma(E)$ within the band for different α's under $\sigma = 0.1$. We can see that as the parameter α increases, the Lyapunov exponent γ becomes smaller and closer to the result of the Gaussian distribution.

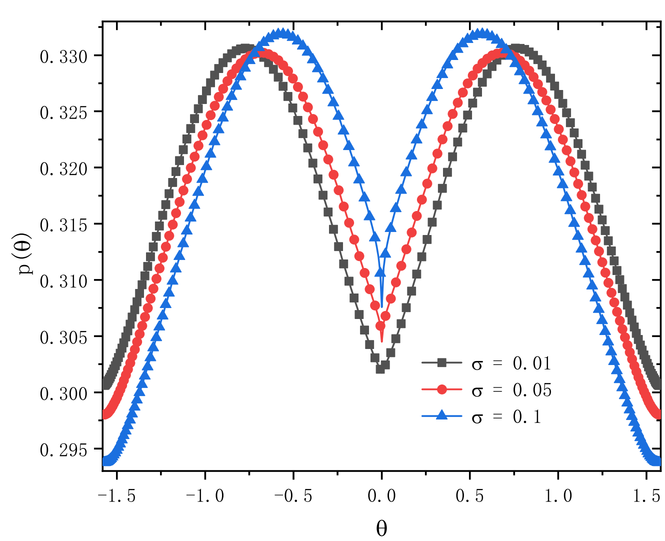

FIG. 4. $p(\theta)$ for decreasing σ's with $E = 0$ and $\alpha = 1.1$.

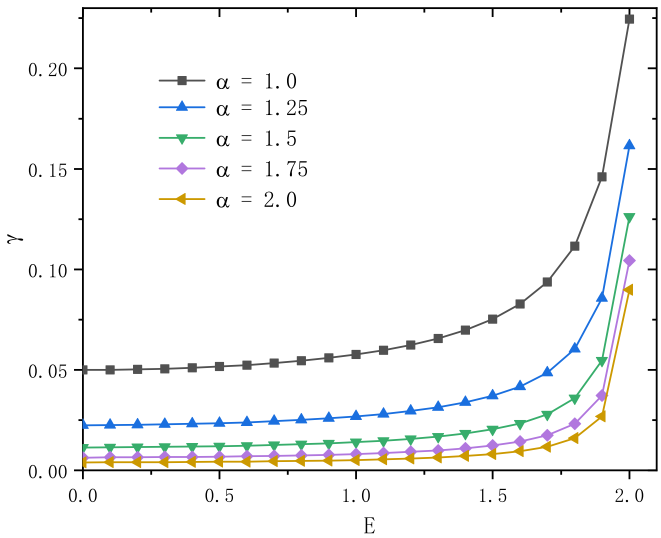

FIG. 5. Lyapunov exponent γ for typical α's under $\sigma = 0.1$.

In the case of the Cauchy distribution, the analytical result of the Lyapunov exponent has been obtained [2]:

$$\gamma(E, \sigma) = \text{arccosh}\frac{\sqrt{(2+E)^2+\sigma^2}+\sqrt{(2-E)^2+\sigma^2}}{4}; \tag{13}$$

our numerical results for the Cauchy distribution are in good agreement with this, and the accuracy is about 10^{-6}.

To demonstrate the anomalous behavior near the band center, we plot $\gamma(E, \alpha)$ for different E's and α's but at the same $\sigma = 0.05$. For a more intuitive comparison, in Fig. 6 we plot the results using $y = \gamma(E, \alpha)/\gamma(0.1, \alpha)$. We can see from the picture that the anomalous behavior of the band center exists when $\alpha > 1$, and the anomaly strongly depends on the higher moments of the distribution function. For $\alpha \in (1, 2]$, the disorder distribution p_v possesses an infinite variance. Figure 6 demonstrates the anomalous behavior near the band center.

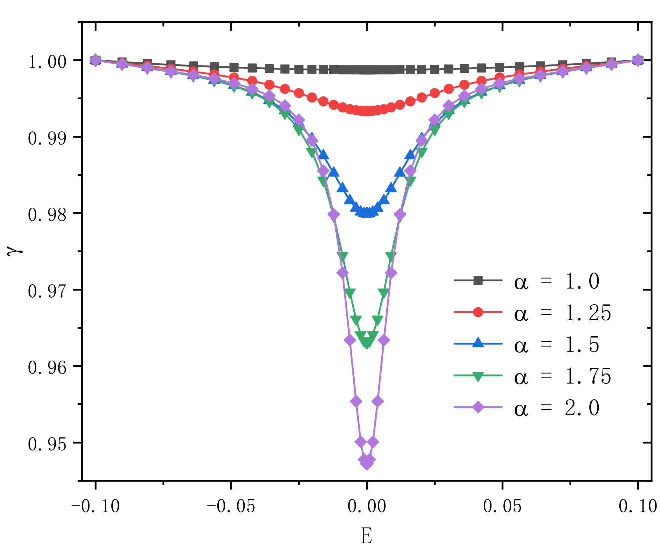

FIG. 6. Lyapunov exponent $\gamma(E)$ near the band center under $\sigma = 0.05$ for different α's. The vertical axis is in unit $\gamma(0.1)$ for each α.

It is not immediately clear whether the numerically obtained behavior of the Lyapunov exponent should be classified as "anomalous." In the model studied in this work, neither analytical expression or clear understanding, nor physical interpretation is available for the Lyapunov exponent for $1 < \alpha \leqslant 2$ at zero energy. For the weak disorder with $\alpha > 2$, the Thouless formula and the Lyapunov exponent in the neighborhood of the band center can be obtained by standard perturbation techniques. The band-center anomaly consists in a discrepancy between the Thouless formula for a finite in-band energy and the Lyapunov exponent in the neighborhood of the band center. This discrepancy is due to resonance effects. In Figs. 5 and 6 we see that the Lyapunov exponent for $1 < \alpha \leqslant 2$ as a function of energy is similar to perturbation results for the weak disorder with $\alpha > 2$. Considering the different behavior of the Lyapunov exponent for a finite energy and in the neighborhood of the band center, we expect that the different behavior for the $1 < \alpha \leqslant 2$ case also comes from resonance effects, which is similar to that of the $\alpha > 2$ case. Therefore, like the well-known band-center anomaly for $\alpha > 2$, we call this behavior for $1 < \alpha \leqslant 2$ the anomalous band center localization for a disorder distribution of infinite variance. The site-energy distribution in this study decays with a power law, and the exponent α determines how fast the tails of the distribution decay in the model. In Fig. 6 we see the band-center anomaly is in connection with disorder distributions ($1 < \alpha \leqslant 2$) possessing well-defined absolute means but infinite variances.

IV. CONCLUSION

In this work, we numerically calculated the probability density function $p(\theta)$ and the Lyapunov exponent γ in the one-dimensional Anderson model with diagonal disorder. We investigated the effect of fractional lower order moments of the random potential density function. We found the anomalous behavior near the band center for the order of moments $\alpha \in (1, 2]$, where the disorder distributions possess infinite variances.

Taking the existence of the mean value and the nonexistence of the second moment of disorder probability density function as bounds, where $1 < \alpha \leqslant 2$, we found $p(\theta)$ is bimodal and V shaped for small σ's at $E = 0$. This $p(\theta)$ is different from the result given by weak disorder perturbation for disorder distributions with small second moments. For $1 < \alpha \leqslant 2$, in the limit of $\sigma \to 0$, we found $p(\theta)$ at an in-band finite energy is the same as the result given by weak disorder perturbation for disorder distributions with small second moments.

Since when $\alpha < 1$, the numerical result of $p(\theta)$ is singular at the origin, the behavior in this situation needs further research in the future.

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