Length and area generating functions for height-restricted Motzkin meanders

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(Received 13 October 2021; accepted 18 January 2022; published 2 February 2022)

We derive the length and area generating function of planar height-restricted forward-moving discrete paths of increments ± 1 or 0 with arbitrary starting and ending points, the so-called Motzkin meanders, and the more general length-area generating functions for Motzkin paths with markers monitoring the number of passages from the two height boundaries ("floor" and "ceiling") and the time spent there. The results are obtained by embedding Motzkin paths in a two-step anisotropic Dyck path process and using propagator, exclusion statistics, and bosonization techniques. We also present a cluster expansion of the logarithm of the generating functions that makes their polynomial structure explicit. These results are relevant to the derivation of statistical mechanical properties of physical systems such as polymers, vesicles, and solid-on-solid interfaces.

DOI: 10.1103/PhysRevE.105.024102

I. INTRODUCTION

Random walks of given length and area on planar lattices are of inherent mathematical and physical interest. In mathematics, their combinatorial properties, statistics, and generating functions are the subject of intense study. In physics, they arise either in actual diffusion processes near boundaries or, indirectly, in the quantum mechanics of particles moving in a periodic two-dimensional potential. The Hofstadter problem is the canonical example of the latter, leading to the famous "butterfly" energy spectrum [1].

In physical contexts, random walks are generated through the action of a Hamiltonian on the Hilbert space of the system. This connection was used to study the enumeration of closed walks of given length and (algebraic) area on the square lattice. Such walks are generated by the Hofstadter Hamiltonian, with the magnetic field playing the role of the variable dual to the area, and their properties can be derived from the study of the secular determinant of the Hamiltonian. The area enumeration generating function for walks of given length was derived in Ref. [2] in terms of a set of factors extracted from the secular determinant (the so-called Kreft coefficients [3]), leading to explicit albeit complicated expressions.

An interesting connection was made in Ref. [4] between a general class of two-dimensional walks and quantum mechanical particles obeying generalized *exclusion statistics* with exclusion parameter g depending on the type of walks ($g = 0$ for bosons, $g = 1$ for fermions, and higher g means a stronger exclusion beyond Fermi). The relevance of generalized quantum statistics to Calogero particles with inverse-square potential interactions was first pointed out in Ref. [5]. Exclusion statistics was proposed by Haldane [6] as a distillation of the statistical mechanical properties of Calogero-like spin systems. Exclusion statistics also emerges in the context of anyons projected on the lowest Landau level of a strong magnetic field [7] and has been extended to more general systems [8]. (For a review of exclusion statistics see Ref. [9].) Remarkably, the algebraic area considerations of a class of lattice walks directly map to the statistical mechanics of particular many-body systems with exclusion statistics [4].

In recent work [10] the Hamiltonian description of random walks and the exclusion statistics connection were used to study the generating function of a family of walks referred to as Dyck paths and their height-restricted generalizations [11–19].[1] These are walks on a two-dimensional lattice that propagate one step in the horizontal direction ("time") and one step either up or down in the vertical direction ("height") but without dipping below a "floor" at height zero nor exceeding a "ceiling" of maximal height. Paths that start and end at the floor are usually termed "excursions," while more general paths are "meanders." The Hamiltonian method for forward-moving paths is equivalent to the transition matrix formulation, which has been used in previous work to calculate the length generating function for such walks. In Ref. [10] these results were extended to length *and* area generating functions for meanders with arbitrary starting and ending points. Further, using the connection to exclusion statistics, the generating functions were expressed in terms of statistical mechanical properties of relatively simple particle systems with an equidistant energy spectrum that are amenable to a full solution by the technique of bosonization. Using a cluster expansion, an alternative form for the logarithm of the generating functions was derived in terms of sums over compositions (i.e., ordered partitions) of the

[1]The literature on Dyck and related Motzkin and Lukasiewicz paths is quite extensive. We refer the reader to T. Prellberg's site [20] and to the references in Ref. [21] for a comprehensive list of relevant papers.

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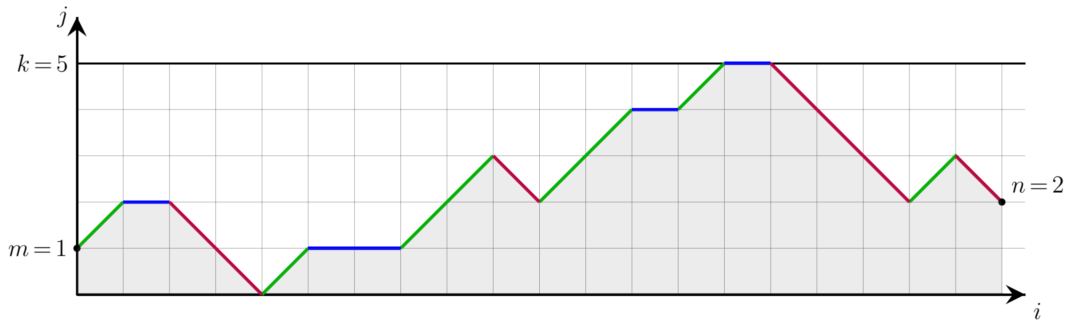

FIG. 1. A typical Motzkin path (meander) for $k = 5$, starting at $m = 1$ and ending at $n = 2$, with $l_u = 8$ (green) up-steps, $l_h = 5$ (blue) horizontal steps, and $l_d = 7$ (red) down-steps for a total length of $l = 20$ steps. The area under it is 49.5 plaquettes (shaded gray).

integer length of the path that made their polynomial structure explicit.

Motzkin paths are walks that can also propagate by a single step horizontally, in addition to up or down, with all remaining properties and definitions as in Dyck paths. Their combinatorial enumeration is given by the so-called Motzkin number [22] and they have been extensively studied from several points of view (see, e.g., Refs. [21,23–28]). Various physical systems can be mapped to Motzkin-like paths, such as solid-on-solid interfaces, vesicles, and, most straightforwardly, polymers. The floor represents a physical boundary that the polymer cannot cross, while the ceiling confines the polymer on a strip. The statistical mechanics of these polymers (or other systems) is determined by the combinatorics of Motzkin paths and, in general, exhibits phase transitions between diffuse and localized states.

Several combinatorial properties and generating functions of Motzking paths have been considered and studied. The full length *and* area generating function for Motzkin meanders, however, has apparently not been calculated (Ref. [26] comes closest to that goal, evaluating generating functions for excursions, that is, Motzkin paths starting and ending on the floor). In this paper we apply the techniques of Ref. [10] to calculate the length and area generating function of Motzkin paths, with different weights assigned to each kind of step (up, horizontal or down) and a set of additional variables probing their boundary properties. Although the basic methodology is the same as for Dyck paths, the application of exclusion statistics techniques in the Motzkin case presents additional challenges that require some new tricks. Nevertheless, the full generating function is derived in terms of determinants, related to exclusion-2 statistical systems and calculated via bosonization.

In the next section we set up the Hamiltonian description of Motzkin paths and express their generating functions in terms of matrix elements of the propagator, while in Sec. III we derive the basic determinant formula for the generating functions, including additional variables (markers), monitoring their passage and time spent on the floor or ceiling, and examine several special cases. In Sec. IV we introduce the anisotropic two-step Dyck process that generates Motzkin paths, review the exclusion statistics connection, and use it to express the basic building block of the generating functions, i.e., the secular determinant of the two-step process, in terms of grand partition functions and Chebyshev polynomials. In Sec. V we use cluster decomposition techniques to derive expressions for the logarithm of the generating functions of Motzkin paths. We conclude in Sec. VI with some remarks on previous work and directions for future research.

This is an opportune moment to log an apology to any mathematician readers. The introduction of concepts such as quantum exclusion statistics and bosonization may present for them an additional burden of familiarization and supplant a purely mathematical treatment that would eschew such schemes and jargon. We feel, nevertheless, that this approach, apart from reflecting the parochial point of view of the author, may add some physical context to the calculations and could be a source of inspiration and insight to those approaching the problem from other vantage points. (This paper is also written in a narrative style rather than the proposition-theorem format canonical to mathematics publications.)

II. MOTZKIN PATH HAMILTONIAN

Motzkin paths are forward-moving random walks on a square lattice on the first quadrangle of the plane consisting of points (i, j), $i = 0, 1, 2, \dots,$ $j = 0, 1, 2, \dots k$. The walk in each step moves one horizontal unit, $i \to i + 1$, and either 0 or 1 vertical units in either direction, $j \to j, j \pm 1$. (Walks where the horizontal step $j \to j$ is forbidden are called Dyck paths.) Paths can never dip below a lowest level ("floor") $j = 0$ nor exceed a maximum height ("ceiling") at $j = k$. Relevant quantitative features of the path are its starting and finishing heights m and n, respectively, the number of upward, horizontal, and downward steps l_u, l_h, l_d, as well as the total area vertically under the walk A, the total length of the walk being $l = l_u + l_h + l_d$ (see Fig. 1).

An object of special interest is the generating function of walks "packaging" the above quantitative features, defined as

$$G_{k,mn}(z_u, z_h, z_d, q) = \sum_{l_u,l_h,l_d,A}^{\infty} z_u^{l_u} z_h^{l_h} z_d^{l_d} q^A N_{k,mn;l_u,l_h,l_d,A}, \quad (2.1)$$

with $N_{k,mn;l_u,l_h,l_d,A}$ the number of walks with the given parameters. We can eliminate one of the dual variables z_u, z_d, right away: Clearly, $l_u - l_d = n - m$, and

$$G_{k,mn}(z_u, z_h, z_d, q) = z_d^{m-n} \sum_{l_u, l_h, A}^{\infty} (z_u z_d)^{l_u} z_h^{l_h} q^A N_{k,mn;l_u,l_h,l_u+m-n,A}. \tag{2.2}$$

So the dependence on z_d is trivial, the relevant variable being $z_u z_d$. The choice $z_d = 1$ could have been made, but we prefer the choice $z_u = z_d = z$, making the Hamiltonian symmetric and the generating function depend only on z, z_h, q. The calculation of $G_{k,mn}(z, z_h, q)$ will be the main focus of this paper.

Discrete forward-moving paths can be described in terms of a Hamiltonian (transition matrix) acting on a Hilbert space of dimensionality equal to the number of states (vertical positions) that the path can visit. Its structure encodes the allowed steps and keeps an account of the quantitative properties of the paths. The Hamiltonian for Motzkin paths of maximum height k can be expressed as the $(k+1)$-dimensional matrix

$$H_k = \begin{pmatrix} z_h & zq^{1/2} & 0 & 0 \ \cdots & 0 & 0 \\ zq^{1/2} & z_h q & zq^{3/2} & 0 \ \cdots & 0 & 0 \\ 0 & zq^{3/2} & z_h q^2 & zq^{5/2} \ \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \ \cdots & z_h q^{k-1} & zq^{k-1/2} \\ 0 & 0 & 0 & 0 \ \cdots & zq^{k-1/2} & z_h q^k \end{pmatrix}. \tag{2.3}$$

H_k includes the parameters z, z_h, and q of the generating function and is symmetric (and Hermitian, for real parameters, although this will be of no import for our considerations). In previous work on Dyck paths [10] the variable z dual to the length was an external multiplicative parameter, but here we prefer to include all dual variables in the Hamiltonian.

We will assume that the Hamiltonian acts on a Hilbert space with basis elements $|j\rangle$ and produces a single step. Repeated action of H_k produces a superposition of all possible walks with weights as they appear in the generating function. Specifically, the left-action of H_k on the dual state $\langle j|$ produces the superposition

$$\langle j| H_k = zq^{j+1/2} \langle j+1| + z_h q^j \langle j| + zq^{j-1/2} \langle j-1| \tag{2.4}$$

with $|k+1\rangle \equiv 0 \equiv |-1\rangle$. (We chose H_k to act on the left to match time evolution from left to right to the ordering of operators.) Mapping the vertical position j to the Hilbert space element $|j\rangle$, we can interpret the action of H_k as producing a unit vertical step either up to $|j+1\rangle$, horizontally to $|j\rangle$, or down to $|j-1\rangle$. A single application of H_k corresponds to a unit step $i \to i+1$. The vertical area under such a step $(i, j) \to (i+1, j+\Delta j)$ measured in units of lattice plaquettes is $a = [j + (j+\Delta j)]/2 = j + \frac{1}{2}, j, j - \frac{1}{2}$ for an up, horizontal, or down step, respectively, and therefore the weighting factors in (2.4) are $z_i q^a$ with $z_i = z, z_h$ depending on the type of step. The repeated application H_k^l then produces a superposition of all possible Motzkin paths of l steps starting at height j, each path weighted by a factor arising from the products of the above coefficients in each step; that is, by a factor $z^{l_u} z_h^{l_h} z^{l_d} q^A$ with A the total area under the path. States $|0\rangle$ and $|k\rangle$ [corresponding to lattice points $(i, 0)$ and (i, k)] constitute a "floor" and a "ceiling."

The above correspondence of Motzkin paths with the action of H_k^l makes it clear that the m, n matrix element of H_k^l reproduces the sum of walks with l steps starting at height m and ending at height n weighted by their area and number of each type of steps,

$$\langle m| H_k^l |n\rangle = \sum_{l_u, l_h, l_d, A}^{\infty} z^{l_u + l_d} z_h^{l_h} q^A N_{k,mn;l_u,l_h,l_d,A}\, \delta(l_u + l_h + l_d - l), \tag{2.5}$$

and the full generating function becomes a matrix element of the "propagator" $(1 - H_k)^{-1}$,

$$G_{k,mn}(z, z_h, q) = \sum_{l=0}^{\infty} \langle m| H_k^l |n\rangle = \langle m| (1 - H_k)^{-1} |n\rangle \tag{2.6}$$

(we assumed small enough $|z_i|$ and $|q|$ for convergence of the sums). It is clear from this form that the generating function satisfies the convolution property

$$G_{k,mn}(z, z_h, q) = \sum_{j=0}^{k} G_{k,mj}(z, z_h, q) G_{k,jn}(z, z_h, q). \tag{2.7}$$

For later convenience, we will adopt the simplifying (and hopefully intuitive) convention that indices $k = \infty$ (no ceiling) and $mn = 00$ (excursions) are omitted, while indices $mn = kk$ (paths "hanging" from the ceiling) are replaced by overbar; that is,

$$G_{\infty,mn} = G_{mn}, \quad G_{k,00} = G_k, \quad G_{k,kk} = \overline{G}_k, \quad G_{\infty,00} = G. \tag{2.8}$$

In the following sections we will show that the above generating function can be expressed as a rational expression of determinants and will evaluate these determinants by connecting them to generalized quantum exclusion statistics of order 2 and using bosonization.

III. DETERMINANT FORMULA FOR THE GENERATING FUNCTION

Our goal is the evaluation of the matrix elements of the propagator matrix $(1 - H_k)^{-1}$ that appear in the generating function.

A. Basic result

The derivation proceeds much along the lines of the corresponding calculation for Dyck paths [10]. We define the secular matrix,

$$D_k(z_i,q) = 1 - H_k = \begin{pmatrix} 1-z_h & -zq^{1/2} & 0 & 0\cdots & 0 & 0 \\ -zq^{1/2} & 1-z_hq & -zq^{3/2} & 0\cdots & 0 & 0 \\ 0 & -zq^{3/2} & 1-z_hq^2 & -zq^{5/2}\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots & 1-z_hq^{k-1} & -zq^{k-1/2} \\ 0 & 0 & 0 & 0\cdots & -zq^{k-1/2} & 1-z_hq^k \end{pmatrix}, \tag{3.1}$$

with z_i denoting collectively z, z_h, as well as its determinant and matrix elements of its inverse (generating function),

$$F_k(z_i,q) = \det D_k(z_i,q), \qquad G_{k,mn}(z_i,q) = \langle m|\, D_k(z_i,q)^{-1}|n\rangle. \tag{3.2}$$

Clearly, $F_0(z_i,q) = 1 - z_h$, and we also define $F_{-1}(z_i,q) = 1$ and $F_k(z_i,q) = 0$ for $k \leqslant -2$.

$G_{k,mn}$ is calculated by the standard formula for the elements of the inverse of a matrix in terms of its cofactors. Applied to matrix $D_k(z_i,q)$ it yields

$$\langle m|\, D_k(z_i,q)^{-1}\,|n\rangle = (-1)^{m-n}\,\frac{\det D_k(z_i,q)_{(nm)}}{\det D_k(z_i,q)}, \tag{3.3}$$

where the complement $D_k(z_i,q)_{(nm)}$ is the matrix $D_k(z_i,q)$ with the nth row and mth column removed.

The denominator in the right-hand side is $F_k(z_i,q)$. The remaining determinant of $D_k(z_i,q)_{(nm)}$ can be related to simple secular determinants. First, observe that the secular matrix with its first n rows and columns truncated, denoted $D_k(z_i,q)_{[n]}$, is related to the secular matrix for a reduced k. Specifically,

$$D_k(z_i,q)_{[n]} = \begin{pmatrix} 1-z_hq^n & -zq^{n+1/2} & 0\ \cdots & 0 & 0 \\ -zq^{n+1/2} & 1-z_hq^{n+1} & -zq^{n+3/2}\cdots & 0 & 0 \\ 0 & -zq^{n+3/2} & 1-q^{n+2}\ \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0\ \cdots & 1-z_hq^{k-1} & -zq^{k-1/2} \\ 0 & 0 & 0\ \cdots & -zq^{k-1/2} & 1-z_hq^k \end{pmatrix} = D_{k-n}(z_iq^n,q). \tag{3.4}$$

Assuming, for now, $m \leqslant n$, it is easy to see that the complement $D_k(z_i,q)_{(nm)}$ becomes block-triangular of the form

$$D_k(z_i,q)_{(nm)} = \begin{pmatrix} D_{m-1}(z_i,q) & 0 & 0 \\ A & Q & 0 \\ B & C & D_k(z_i,q)_{[n+1]} \end{pmatrix} \tag{3.5}$$

with Q a lower-diagonal matrix. The crucial property of H_k and $D_k(z_i,q)$ that leads to this form of $D_k(z_i,q)_{(nm)}$ and Q is the fact that they are paradiagonal with two off-diagonals flanking the diagonal, a feature shared with Dyck paths. This block diagonal form of $D_k(z_i,q)_{(nm)}$ implies

$$\det D_k(z_i,q)_{(nm)} = \det D_{m-1}(z_i,q)\,\det Q\,\det D_k(z_i,q)_{[n+1]}, \tag{3.6}$$

where Q is lower-triangular with diagonal elements $-zq^{m+1/2}, -zq^{m+3/2}, \dots, -zq^{n-1/2}$ for $m < n$ and is completely absent if $m = n$. Therefore,

$$\det Q = (-z)^{n-m}\, q^{\frac{n^2-m^2}{2}}. \tag{3.7}$$

Putting everything together, and using (3.4), we finally obtain

$$\begin{aligned} G_{k,mn}(z_i,q) &= z^{n-m}\, q^{\frac{n^2-m^2}{2}}\,\frac{F_{m-1}(z_i,q)\,F_{k-n-1}(z_iq^{n+1},q)}{F_k(z_i,q)}, \quad n \geqslant m \\ &= z^{m-n}\, q^{\frac{m^2-n^2}{2}}\,\frac{F_{n-1}(z_i,q)\,F_{k-m-1}(z_iq^{m+1},q)}{F_k(z_i,q)}, \quad n \leqslant m, \end{aligned} \tag{3.8}$$

the second formula following from the symmetry $G_{k,mn}(z_i,q) = G_{k,nm}(z_i,q)$. The formula holds for all values of k, m, n under the conventions for F_k for negative values of k.

Formula (3.8) is our first main result. It is practically identical in form to the corresponding result for Dyck paths, the difference between the two kinds of paths being in the properties of the secular determinant $F_k(z_i,q)$. In the next section we will calculate this determinant by relating it to a two-step alternating Dyck process, which will allow us again to express it as the grand partition function of a quantum exclusion statistics system but for a more involved spectrum than the one in the Dyck case.

B. Special cases

A few interesting special cases are worth recording. For diagonal height-restricted paths starting and ending at the same height $m = n$ (higher excursions) we have

$$G_{k,nn}(z_i, q) = \frac{F_{n-1}(z_i, q)F_{k-n-1}(z_i q^{n+1}, q)}{F_k(z_i, q)}. \tag{3.9}$$

For excursions, in particular, $G_{k,00} = G_k$ becomes the ratio of determinants,

$$G_k(z_i, q) = \frac{F_{k-1}(z_i q, q)}{F_k(z_i, q)}, \tag{3.10}$$

and for "dual" paths hanging from the ceiling, $G_{k,kk}(z_i, q) = \overline{G}_k(z_i, q)$ becomes the simpler expression,

$$\overline{G}_k(z_i, q) = \frac{F_{k-1}(z_i, q)}{F_k(z_i, q)}. \tag{3.11}$$

Finally, for unrestricted Motzkin excursions (higher and floor ones) we have

$$G_{nn}(z_i, q) = F_{n-1}(z_i, q)G(z_i, q), \;\; G(z_i, q) = \frac{F(z_i q, q)}{F(z_i, q)}. \tag{3.12}$$

C. Duality and recursion relations

The secular matrix and determinant satisfy the duality relation

$$D_k(z_i q^k, q^{-1}) = \sigma D_k(z_i, q)\sigma \;\;\Rightarrow\;\; F_k(z_i q^k, q^{-1}) = F_k(z_i, q), \tag{3.13}$$

where $\sigma_{mn} = \delta_{m+n,k}$ is the reflection matrix. This expresses the symmetry of walks under vertical reflection around the median line at $k/2$, mapping $|n\rangle \to |k-n\rangle$. Equation (3.13) implies the corresponding duality relation for generating functions,

$$G_{k,mn}(z_i, q) = G_{k;k-n,k-m}(z_i q^k, q^{-1}). \tag{3.14}$$

Several generating function recursion relations can be deduced directly from the form itself of (3.8), irrespective of the form of $F_k(z_i, q)$. For instance,

$$\begin{aligned} G_{k,mn}(z_i, q) &= zq^{n-1/2}\, G_{k;m,n-1}(z_i, q)\, G_{k-n}(z_i q^n, q) \quad (m < n) \\ &= zq^{l+1/2}\, G_{k;l+1,n}(z_i, q)\, G_{l;m,l}(z_i, q) \quad (m \leqslant l < n). \end{aligned} \tag{3.15}$$

Further recursion relations, more specific to Motzkin paths, can be derived by expanding $\det D_k(z_i, q)$ in terms of its top row, as in the Dyck path case. We obtain

$$F_k(z_i, q) = (1 - z_h)F_{k-1}(z_i q, q) - z^2 q F_{k-2}(z_i q^2, q) \tag{3.16}$$

which leads to corresponding relations for $G_{k,nm}(z_i, q)$. Several such relations can be written, and we choose to present two: for generic paths, applying (3.16) to the term $F_{k-n-1}(z_i q^{n+1}, q)$ in (3.8) yields

$$\begin{aligned} (1 - z_h q^n)G_{k,mn}(z_i, q) &= zq^{n-1/2}\, G_{k;m,n-1}(z_i, q) \\ &\quad + zq^{n+1/2}\, G_{k;m,n+1}(z_i, q) \quad (m < n < k) \end{aligned} \tag{3.17}$$

and for excursions, dividing (3.16) by $F_k(z_i, q)$ yields

$$(1 - z_h)G_k(z_i, q) = 1 + z^2 q\, G_{k-1}(z_i q, q)\, G_k(z_i, q). \tag{3.18}$$

All the above recursion relations admit geometric interpretations in terms of decomposing paths into their parts. Figure 2 demonstrates the geometric significance of (3.18), which generalizes a similar construction for Dyck paths.

D. Top and bottom event markers

Before proceeding to the calculation of the secular determinant in the next section, we derive the expression of a generalization of the generating function that also keeps track of the times a path "hits" the floor $n = 0$ (a "touch-down"), the total time (in step units) that it spends on the floor ("creep-down"), the number of times it hits the ceiling $n = k$ ("touch-up"), and the total time it spends on the ceiling ("creep-up"). For example, the path of Fig. 2 has three touch-downs, one creep-down, one touch-up, and one creep-up. These are examples of various "markers" that we can add to monitor local properties of the paths.

Weighing each touch-down with a factor of t, each creep-down with a factor of s, each touch-up with a factor of $\bar{t}$, and each creep-up with a factor of $\bar{s}$, the generating function for paths from m to n with length l, area A, l_h horizontal steps, a touch-downs, b creep-downs, c touch-ups, and d creep-ups becomes

$$\hat{G}_{k,mn}(t, s; \bar{t}, \bar{s}\,|z_i, q) = \sum_{A,l,a,b,c,d=0}^{\infty} t^a s^b \bar{t}^c \bar{s}^d z^{l-l_h} z_h^{l_h} q^A N_{k,mn;l,l_h,A,a,b,c,d} \tag{3.19}$$

(we continue setting $z_u = z_d = z$). Clearly, $\hat{G}_{k,mn}(1, 1; 1, 1\,|z_i, q) = G_{k,mn}(z_i, q)$.

This generalization can be implemented in our Hamiltonian framework with minor modifications. Consider the Hamiltonian $\hat{H}_k$ ($k \geqslant 1$),

$$\hat{H}_k = \begin{pmatrix} s z_h & zq^{1/2} & 0 & 0 & \cdots & 0 & 0 \\ t z q^{1/2} & z_h q & zq^{3/2} & 0 & \cdots & 0 & 0 \\ 0 & zq^{3/2} & z_h q^2 & zq^{5/2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z_h q^{k-1} & \bar{t} z q^{k-1/2} \\ 0 & 0 & 0 & 0 & \cdots & zq^{k-1/2} & \bar{s} z_h q^k \end{pmatrix}, \tag{3.20}$$

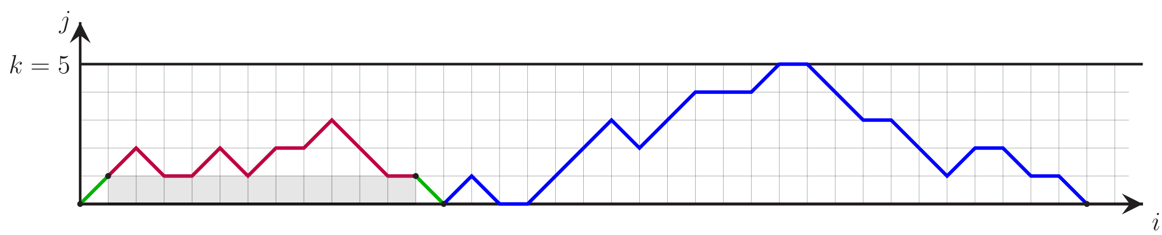

FIG. 2. An interpretation of (3.18), written as $G_k(z_i) = z_h G_k(z_i) + z^2 q\, G_{k-1}(z_i q)\, G_k(z_i) + 1$, as a "first passage" equation. The first step of excursions starting at (0,0) can be either horizontal or up. If it is horizontal (not shown in the figure), it contributes a factor of z_h and the remaining path is also a general excursion, accounting for the term $z_h G_k(z_i)$. If the first step is up (such a path of length 26 is depicted above), it can be decomposed into a path returning to the floor $j=0$ *for the first time* [first part (green and red) of path] and the remaining (blue) arbitrary excursion. For paths of length at least two, the first-passage path has one first step and one last step (green), each contributing a factor of $zq^{1/2}$. The remaining upper (red) part never dips below $j=1$ and can be interpreted as an excursion, but with a ceiling reduced by 1 and an area increased by its length (shaded plaquettes), contributing the factor $G_{k-1}(z_i q)$, the shift $z_i \to z_i q$ accounting for the extra area. The remaining (blue) path contributes $G_k(z_i)$. Finally, the trivial path of length zero cannot be decomposed and contributes the term 1. Relations (3.15) also admit a similar first-passage interpretation.

where $\hat{H}_k$ is the same as H_k but with the $|0\rangle\langle 0|$ element multiplied by s, the $|1\rangle\langle 0|$ element multiplied by t, the $|k-1\rangle\langle k|$ element multiplied by $\bar{t}$, and the $|k\rangle\langle k|$ element multiplied by $\bar{s}$. It should be obvious that $\hat{H}_k^l$ counts area-weighted paths of length l, as before, but also multiplies by an appropriate factor each up- or down-event. Therefore, as before,

$$\hat{G}_{k,mn}(t, s, \bar{t}, \bar{s}\,|z_i, q) = \sum_{l=0}^{\infty} \langle m|\, \hat{H}_k^l\, |n\rangle = \langle m|\, (1-\hat{H}_k)^{-1} |n\rangle. \tag{3.21}$$

Note that $\hat{H}_k$ is not symmetric for $t, \bar{t} \neq 1$. The asymmetry is due to the fact that paths entering or exiting the floor or the ceiling are weighted differently and implies $\hat{G}_{k,mn} \neq \hat{G}_{k,nm}$ if $m, n = 0, k$. Nevertheless, we can render $\hat{G}_{mn}$ fully symmetric by assigning an extra weight t to paths starting from the floor and an extra weight $\bar{t}$ to paths starting from the ceiling, and we shall adopt this convention. (Another alternative, often used in the literature, would be to not count the final touch-down or touch-up of paths ending at 0 or k. However, as we shall see, our symmetrization convention leads to more compact expressions.)

Denoting

$$\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q) = \det(1-\hat{H}_k), \tag{3.22}$$

a calculation entirely analogous to the $t=s=\bar{t}=\bar{s}=1$ case yields for $2 \leqslant m \leqslant n \leqslant k-2$,

$$\hat{G}_{k,mn}(t, s; \bar{t}, \bar{s}\,|z_i, q) = z^{n-m}\, q^{\frac{n^2-m^2}{2}}\, \frac{\hat{F}_{m-1}(t, s; 1, 1|z_i, q)\, \hat{F}_{k-n-1}(1, 1; \bar{t}, \bar{s}\,|z_i q^{n+1}, q)}{\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q)}. \tag{3.23}$$

$\hat{H}_k$ and $\hat{F}_k$ are not defined for $k<1$. Nevertheless, if we define

$$\hat{F}_0 = t + \bar{t} - t\bar{s}z_h - \bar{t}sz_h - t\bar{t} + t\bar{t}z_h, \quad \hat{F}_{-1} = t\bar{t},$$
$$\hat{F}_k = 0 (k \leqslant -2), \tag{3.24}$$

then (3.23) becomes valid for all values of n, m: For $m=1$ or $n=k-1$, $\hat{F}_0$ as defined in (3.24) reproduces the correct factors $1-sz_h$ or $1-\bar{s}z_h q^k$ arising from the evaluation of the corresponding matrix elements of $(1-\hat{H}_k)^{-1}$. For $n=k$, the last factor in the numerator is absent, but the matrix element includes an extra factor of $\bar{t}$ arising from the structure of the Q matrix appearing in the analogs of (3.5) and (3.6), which involves the $\langle k-1|\, \hat{H}_k\, |k\rangle$ element of H_k, and this factor is reproduced by $\hat{F}_{-1}(1, 1; \bar{t}, \bar{s})$ in (3.23). Finally, for $m=0$ the first factor in the numerator is absent, but by our symmetrization convention we must include an extra factor of t, which is reproduced by $\hat{F}_{-1}(s, t; 1, 1)$. Therefore, conventions (3.24) make formula (3.23) valid for the full range of values of m, n.

To relate $\hat{F}_k$ to F_k, we expand the determinant in terms of its top row. We obtain, for $k \geqslant 1$,

$$\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q) = (1-sz_h)\hat{F}_{k-1}(1, 1; \bar{t}, \bar{s}\,|qz_i, q) - t\, z^2 q \hat{F}_{k-2}(1, 1; \bar{t}, \bar{s}\,|q^2 z_i, q) \tag{3.25}$$

and combining with the same formula for $s=t=1$ yields

$$\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q) = t\,\hat{F}_k(1, 1; \bar{t}, \bar{s}\,|z_i, q) + (1-t-sz_h+tz_h)\hat{F}_{k-1}(1, 1; \bar{t}, \bar{s}\,|qz_i, q). \tag{3.26}$$

Similarly, expanding $\hat{F}_k$ in terms of its bottom row gives

$$\hat{F}_k(t, s; \bar{t}, \bar{s};\,|z_i, q) = (1-\bar{s}z_h q^k)\hat{F}_{k-1}(t, s; 1, 1|z_i, q) - \bar{t}\, z^2 q^{2k-1}\hat{F}_{k-2}(t, s; 1, 1|z_i, q) \tag{3.27}$$

and combining with the same formula for $\bar{t}=\bar{s}=1$ yields

$$\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q) = \bar{t}\,\hat{F}_k(t, s; 1, 1|z_i, q) + (1-\bar{t}-\bar{s}z_h q^k + \bar{t}z_h q^k)\hat{F}_{k-1}(t, s; 1, 1|z_i, q). \tag{3.28}$$

Finally, applying formula (3.26) for $\bar{t} = \bar{s} = 1$ and inserting in (3.28) (or vice versa) gives

$$\begin{aligned}\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q) &= t\,\bar{t}\,F_k(z_i, q) + t(1 - \bar{t} - \bar{s}z_h q^k + \bar{t}z_h q^k)F_{k-1}(z_i, q)\\ &\quad + \bar{t}(1 - t - sz_h + tz_h)F_{k-1}(z_i q, q)\\ &\quad + (1 - t - sz_h + tz_h)(1 - \bar{t} - \bar{s}z_h q^k + \bar{t}z_h q^k)F_{k-2}(z_i q, q).\end{aligned} \tag{3.29}$$

Note that, with the definitions (3.24), the above formula holds for all k, including $k < 1$. (This motivates the somewhat unintuitive form of $\hat{F}_{-1}$.)

We have thus related the secular determinant of the process with markers to that of the unmarked process, which will be calculated in the next section. Applying the above formula for the terms in (3.23), and also using the expression (3.8) for $G_{k,mn}$, leads to the relation for $\hat{G}_{k,mn} = \hat{G}_{k,nm}$ for $m \leqslant n$

$$\hat{G}_{k,mn}(t, s; \bar{t}, \bar{s}) = \frac{[t + A_0(t, s)\, G_{m-1}][\bar{t}\, G_{k,mn} + A_k(\bar{t}, \bar{s})\, \overline{G}_k\, G_{k-1,mn}]}{A_k(\bar{t}, \bar{s})[t + A_0(t, s)\, G_{k-1}]\overline{G}_k + \bar{t}[t + A_0(t, s)\, G_k]}, \tag{3.30}$$

where, for brevity, we suppressed the (common) arguments z_i, q and defined

$$A_r(t, s) = 1 - t + (t - s)\, z_h\, q^r. \tag{3.31}$$

Formula (3.30) is our second main result and expresses the generating function $\hat{G}_{k,mn}$ in terms of $G_{k,mn}$. It is conceivable that a relation between $\hat{G}_{k,mn}$ and $G_{k,mn}$ could be obtained combinatorially, with arguments similar to the ones of Fig. 2. However, the rather complicated form of (3.30) suggests that such an argument would be quite convoluted. Our Hamiltonian approach allowed for a relatively straightforward derivation of this relation without combinatorial ingenuity.

We conclude with a few remarks and special cases. $\hat{G}_{k,mn}$ satisfies the floor-ceiling duality relation

$$\hat{G}_{k,mn}(t, s; \bar{t}, \bar{s}\,|z_i, q) = \hat{G}_{k;k-m,k-n}(\bar{t}, \bar{s}; t, s\,|z_i q^k, q^{-1}). \tag{3.32}$$

This is a consequence of the duality relation for $\hat{F}_k$,

$$\hat{F}_k(t, s; \bar{t}, \bar{s}\,|z_i, q) = \hat{F}_k(\bar{t}, \bar{s}; t, s\,|z_i q^k, q^{-1}), \tag{3.33}$$

which is obvious from the form of $\hat{H}_k$ (3.20) but also follows from (3.29) and the corresponding duality (3.13) for F_k. Expression (3.30) does not appear to respect this duality, as it does not look symmetric in t, s and $\bar{t}, \bar{s}$. However, this is an artifact of the specific form of the expression; using identities as derived in Sec. III C, and the fact that (3.30) is valid for $m \leqslant n$, duality is restored.

For $z_h = 0$, $\hat{G}_{k,mn}$ has no dependence on s and $\bar{s}$. Indeed, for $z_h = 0$ the process degenerates to Dyck paths, which cannot creep over the floor nor over the ceiling. For $\bar{t} = \bar{s} = 1$, only touch-downs and creep-downs are monitored. $\hat{G}_{k,mn}$ becomes

$$\hat{G}_{k,mn}(t, s; 1, 1) = G_{k,mn}\frac{t + [1 - t + (t - s)z_h]\, G_{m-1}}{t + [1 - t + (t - s)z_h]\, G_k} \tag{3.34}$$

and for $z_h = 0$ the above formula reproduces the result for Dyck paths with touch-downs obtained in Ref. [10].

For $s = t$, $\bar{s} = \bar{t}$, only the total number of lattice sites on the floor and on the ceiling are monitored. Remarkably, (3.30) does not involve z_h in that case, other than the implicit dependence through $G_{k,mn}$, so the same formula remains valid for the case of Dyck paths.

For $k = \infty$ (no ceiling), $\overline{G}_k = 0$. The dependence on $\bar{t}, \bar{s}$ drops, as expected, and $\hat{G}(t, s)$ is given by (3.34) with G (the unrestricted excursions generating function) instead of G_k in the denominator.

Finally, for

$$s = t + \frac{1-t}{z_h}, \quad \bar{s} = \bar{t} + \frac{1-\bar{t}}{z_h}q^k \;\Rightarrow\; \hat{G}_{k,mn} = G_{k,mn} \tag{3.35}$$

Remarkably, there is a two-parameter family of Hamiltonians that produce the same generating functions.

IV. TWO-STEP WALK AND EXCLUSION STATISTICS

The calculation of the secular determinant $F_k(z_i, q)$ and of $G_{k,mn}(z_i, q)$ can be most intuitively and conveniently performed through the connection of the random walk process with exclusion statistics, as was pointed out in Ref. [4].

Specifically, the secular determinant $\det(1 - \mathbf{M})$ of a paradiagonal matrix $\mathbf{M}$ with zero diagonal and two nonzero off-diagonals, one (with elements f_n) just above the diagonal and the other (with elements g_n) $g - 1$ steps below the diagonal, is given by the *grand partition function* of noninteracting particles of exclusion statistics g with single-particle statistical factors $s(n) = e^{-\beta(\varepsilon_n - \mu)}$, $n = 0, 1, 2, \ldots$ $(1/\beta = k_B T)$ given by

$$s(n) = -g_n\, f_n\, f_{n+1} \cdots f_{n+g-2}. \tag{4.1}$$

The spectral function $s(n)$ encodes the single-particle energy spectrum Boltzmann factor $e^{-\beta\varepsilon_n}$ together with the fugacity parameter $x = e^{\beta\mu}$. Exclusion g means that no more than one particle can occupy any set of g adjacent single-particle states.

The Dyck path Hamiltonian is of the above form. However, the Motzkin path matrix H_k in (2.3) is actually *not* of this form, since it has a nonvanishing diagonal. Nevertheless, it can be expressed as the grand partition function of a $g = 2$ exclusion statistics system. This is achieved by realizing Motzkin paths as two-step Dyck paths and calculating the generating function of the two-step process.

A. Two-step Dyck process

Consider a path realized by the alternation of two Dyck processes (see Fig. 3): one starting at even sites $(i, 2j)$ and jumping to odd sites, up to $(i + 1, 2j + 1)$ with weight z_1 or down to $(i + 1, 2j - 1)$ with weight z_2 and another starting at odd sites $(i, 2j + 1)$ and jumping to even sites up to $(i + 1, 2j + 2)$ with weight z_2 or down to $(i + 1, 2j)$ with weight z_1. The full process is height restricted with floor $j = 0$ and ceiling $j = 2k + 2$. The $(2k + 3)$-dimensional Hamilto-

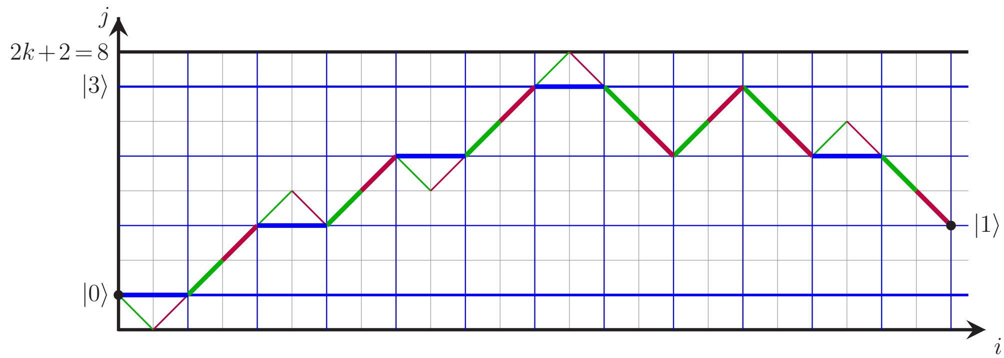

FIG. 3. The two-step Dyck process: (green) steps $(2i, 2j+1) \to (2i+1, 2j+1 \pm 1)$ with amplitudes z_1, z_2 alternating with (red) steps $(2i+1, 2j) \to (2i+2, 2j \pm 1)$ with reversed amplitudes z_2, z_1 on a lattice with ceiling 8. The points on the (blue) sublattice $(2i, 2j+1)$ generate a Motzkin path (thick meander) of length 12 from $m = 0$ to $n = 1$ with ceiling $k = 3$, such that $2k + 2 = 8$.

nian transition matrix of the full process is

$$H_{2D} = \begin{pmatrix} 0 & z_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ z_1 & 0 & z_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & z_2 & 0 & z_1 q_o & 0 & \cdots & 0 & 0 \\ 0 & 0 & z_1 q_o & 0 & z_2 q_o & \cdots & 0 & 0 \\ 0 & 0 & 0 & z_2 q_o & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & z_2 q_o^k \\ 0 & 0 & 0 & 0 & 0 & \cdots & z_2 q_o^k & 0 \end{pmatrix} \tag{4.2}$$

with q_o a new area-counting parameter. Denoting the basis states of this $(2k+3)$-dimensional Hilbert space $|j\rangle\rangle$, $j = 0, 1, \ldots, 2k+2$, H_{2D} acts on them as

$$\begin{aligned} \langle\langle 2j\,| H_{2D} &= z_1 q_o^j \langle\langle 2j+1\,| + z_2 q_o^{j-1} \langle\langle 2j-1\,| \\ \langle\langle 2j+1\,| H_{2D} &= z_2\, q_o^j \langle\langle 2j+2\,| + z_1 q_o^j \langle\langle 2j\,| \end{aligned} \tag{4.3}$$

with $|-1\rangle\rangle \equiv 0 \equiv |2k+3\rangle\rangle$.

It is clear that the path "distilled" from the above process by considering the height of the walk only at the odd sites $|2j+1\rangle\rangle$, $j = 0, 1, \ldots, k$ every second time step is a Motzkin path of restricted height k (see Fig. 3). Specifically, acting with H_{2D} twice on an odd state,

$$\begin{aligned} \langle\langle 2j+1\,| H_{2D}^2 &= z_1 z_2\, q_o^{2j+1} \langle\langle 2j+3\,| + \left(z_1^2 + z_2^2\right) q_o^{2j} \langle\langle 2j+1\,| \\ &\quad + z_1 z_2\, q_o^{2j-1} \langle\langle 2j-1\,|. \end{aligned} \tag{4.4}$$

Defining $|j\rangle = |2j+1\rangle\rangle$, $j = 0, 1, \ldots, k$, the above relation implies that H_{2D}^2 acts on the $(k+1)$-dimensional subspace $|j\rangle$ as the Motzkin Hamiltonian,

$$\langle j|\, H_{2D}^2 = z q^{j+1/2} \,\langle j+1| + z_h q^j \,\langle j| + z q^{j-1/2} \,\langle j-1|\,, \tag{4.5}$$

provided we identify

$$\begin{aligned} &q = q_o^2, \quad z = z_1 z_2, \\ &z_h = z_1^2 + z_2^2 \Rightarrow z_{1,2}^2 = \frac{z_h}{2} \pm \sqrt{\frac{z_h^2}{4} - z^2}. \end{aligned} \tag{4.6}$$

(Note that the choice of roots for z_1 and z_2 in (4.6), or equivalently the order of the two Dyck processes, is irrelevant; the Motzkin process does not depend on that choice.) We will also adopt the (z, ω) parametrization

$$\begin{aligned} &z_1^2 = z\omega, \quad \mathrm{Z}_2^2 = z\omega^{-1}, \\ &\omega = \frac{z_h}{2z} + \sqrt{\frac{z_h^2}{4z^2} - 1} \;\Rightarrow\; z_h = z(\omega + \omega^{-1}) \end{aligned} \tag{4.7}$$

so that powers of z count the total number of steps: $z^{l_u} z_h^{l_h} z^{l_d} = z^{l_u + l_h + l_d} (\omega + \omega^{-1})^{l_h}$.

We note that the complementary process restricted to even states also generates a Motzkin process of height $k+1$. The amplitudes of the steps $|0\rangle\rangle \to |0\rangle\rangle$ and $|2k+2\rangle\rangle \to |2k+2\rangle\rangle$, however, are truncated, since the intermediate steps $|0\rangle\rangle \to |-1\rangle\rangle$ and $|2k+2\rangle\rangle \to |2k+3\rangle\rangle$ are missing and do not contribute to the amplitude. The odd states, on the other hand, provide a faithful realization of Motzkin paths with the proper weights on identifying states and parameters as in (4.5) and (4.6).

It remains to connect the secular determinant of the Motzkin process with that of the two-step Dyck process. To this end, define the projection operator on odd states P and the projector on even states $\bar{P} = 1 - P$. Clearly,

$$\begin{aligned} &H_{2D} P = \bar{P} H_{2D}, \quad H_{2D} \bar{P} = P H_{2D}, \\ &P \bar{P} = \bar{P} P = 0, \quad P + \bar{P} = 1. \end{aligned} \tag{4.8}$$

The Motzkin Hamiltonian H_k and the "complementary" quasi-Motzkin Hamiltonian $\bar{H}_{k+1}$ are, up to zero modes when acting on the "wrong" subspace,

$$H_k = H_{2D}^2 P, \quad \bar{H}_{k+1} = H_{2D}^2 \bar{P} \tag{4.9}$$

and therefore

$$\begin{aligned} \det(1 - H_k) &= \det\left(1 - H_{2D}^2 P\right) = \det(1 - H_{2D} \bar{P} H_{2D} P) \\ &= \det(1 - H_{2D} P H_{2D} \bar{P}) = \det(1 - \bar{H}_{k+1}). \end{aligned} \tag{4.10}$$

The zero modes are irrelevant, due to the presence of the unit matrix, so the secular determinants of the Motzkin process and its complement in their respective subspaces are equal. Thus

$$\begin{aligned} \det\left(1 - H_{2D}^2\right) &= \det\left[\left(1 - H_{2D}^2\right) P + \left(1 - H_{2D}^2\right) \bar{P}\right] \\ &= \det(1 - H_k) \det(1 - \bar{H}_{k+1}) = \det(1 - H_k)^2. \end{aligned} \tag{4.11}$$

Finally, (4.8) implies that H_{2D} anticommutes with the parity matrix $\Sigma = \bar{P} - P,\ \Sigma^2 = 1$. Therefore,

$$\det(1 - H_{2D}) = \det(1 + \Sigma H_{2D} \Sigma) = \det[\Sigma(1 + H_{2D})\Sigma] = \det(1 + H_{2D}) \quad (4.12)$$

and

$$\det\left(1 - H_{2D}^2\right) = \det(1 - H_{2D}) \det(1 + H_{2D}) = \det(1 - H_{2D})^2. \quad (4.13)$$

Comparing with (4.11) we eventually obtain

$$\det(1 - H_{2D}) = \det(1 - H_k) \quad (4.14)$$

(the sign is fixed by continuity from $z_i = 0$).

The end result is that we can calculate the secular determinant $F_k(z_i, q)$ by evaluating instead the secular determinant of the two-step Dyck path Hamiltonian.

B. Two-step secular determinant and bosonization

The two-step Hamiltonian is of the $g = 2$ exclusion statistics form. The spectral parameters can be read off from the product of conjugate off-diagonal elements:

$$s(2n) = -z_1^2 q_o^{2n}, \quad s(2n+1) = -z_2^2 q_o^{2n}, \quad n = 0, 1, \ldots, k. \quad (4.15)$$

Calling, for ease of distinction, $s(2n) = \alpha(n)$, $s(2n+1) = \beta(n)$, and remembering that $z_{1,2}^2 = z\omega^{\pm 1}$, $q_o^2 = q$, the spectral parameters (4.15) are

$$\alpha(n) = -z\omega\, q^n, \quad \beta(n) = -z\omega^{-1} q^n, \quad n = 0, 1, \ldots, k. \quad (4.16)$$

The secular determinant $F_k(z_i, q)$ is the grand partition function of exclusion-2 particles in levels with spectral parameters $s(0), s(1), \ldots, s(2k+1)$ in that order. Particles placed on these levels must have at least one empty level between them. Calling $Z_{k,N}$ the N-body partition function in the above spectrum, the grand partition function $\mathcal{Z}_k$ becomes

$$\mathcal{Z}_k = \sum_{N=0}^{k+1} Z_{k,N} = \sum_{N=0}^{k+1} \sum_{\{0 \leqslant n_i \leqslant n_{i+1} - 2 \leqslant 2k-1\}} s(n_1) s(n_2) \cdots s(n_N), \quad (4.17)$$

where $n_i = 0, 1, \ldots, 2k+1$ mark the levels on which particles are placed, in increasing order, and the condition $n_i \leqslant n_{i+1} - 2$ in the partition function Z_N enforces exclusion-2 statistics. It is clear that at most $k+1$ particles can be accommodated in the available $2k+2$ levels.

In the case of Dyck paths, the spectral factors $s(n)$ corresponded to the equidistant spectrum of a truncated harmonic oscillator and the partition function could be found using bosonization. For general exclusion statistics g, bosonization is achieved by redefining the occupied level numbers $n_i \leqslant n_{i+1} - g$ as

$$n_i = \ell_i + g(i-1) \Rightarrow \ell_i \leqslant \ell_{i+1}. \quad (4.18)$$

This reduces the "gap" between successive occupied levels ℓ_i and ℓ_{i+1} by g, making the new occupation numbers ℓ_i obey

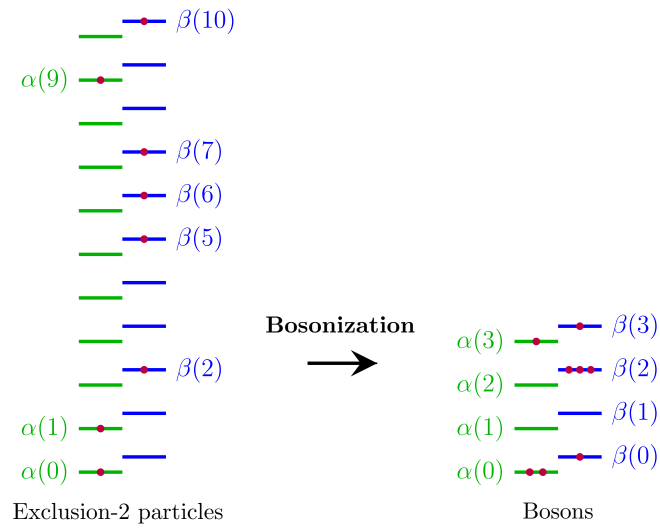

FIG. 4. An example of bosonization for a state of $g = 2$, $k = 10$, $N = 8$. In the exclusion-2 picture, on the left, there are 11 (green) α levels and 11 (blue) β levels. The distance between successive α or β levels represents a factor of q, while the distance between an α level and the next β level represents a factor of ω^{-2}. Particles, represented by (red) dots, cannot occupy the same or neighboring levels, irrespective of level type. Bosonization shrinks k to $k - (N-1) = 3$ and lowers the ith lowest particle by $i - 1$ steps in its own level type: $\alpha(0) \to \alpha(0)$, $\alpha(1) \to \alpha(1-1) = \alpha(0)$, $\beta(2) \to \beta(2-2) = \beta(0)$, $\beta(5) \to \beta(5-3) = \beta(2)$, and so on, leading to the bosonic state on the right. The number of particles in α levels ($N_\alpha = 3$) and β levels ($N_\beta = 5$) is preserved. The total "height" of particles has been reduced by $N(N-1)/2 = 28$, leading to a factor of $q^{N(N-1)/2} = q^{28}$ relating the $g = 2$ state to the bosonic state.

bosonic statistics. The N-body spectral factor becomes

$$s(n_1) s(n_2) \cdots s(n_N) = s(\ell_1) s(\ell_2 + g) \cdots s(\ell_N + (N-1)g). \quad (4.19)$$

In general, this is no simpler than the expression in terms of n_i. Bosonization becomes useful for an equidistant spectrum, for which $s(n) = aq^n$ for some a, q. In that case (4.19) gives

$$s(\ell_1) s(\ell_2 + g) \cdots s(\ell_N + (N-1)g) = q^{g\frac{N(N-1)}{2}} s(\ell_1) s(\ell_2) \cdots s(\ell_N) \quad (4.20)$$

and the N-body exclusion-g partition function $Z_N^{(g)}$ becomes the bosonic one $Z_N^{(B)}$ up to an overall coefficient

$$Z_N^{(g)} = q^{g\frac{N(N-1)}{2}} Z_N^{(B)}. \quad (4.21)$$

Remarkably, bosonization also works for the two-step spectrum of the Motzkin process, although the spectrum corresponding to $s(n)$ is not equidistant. The redefinition (4.18) for the two-step Dyck (i.e., Motzkin) case, with $g = 2$, involves a shift by an *even* integer, so it maps α levels to α levels and β levels to β levels (see Fig. 4):

$$\begin{aligned} n_i &= \ell_i + 2(i-1) \text{ so: } \ell_i = 2l_i : \\ s(n_i) &= \alpha(l_i + i - 1) = q^{i-1}\alpha(l_i) = q^{i-1} s(\ell_i) \\ \ell_i &= 2l_i + 1 : \ s(n_i) = \beta(l_i + i - 1) = q^{i-1}\beta(l_i) \\ &= q^{i-1} s(\ell_i). \end{aligned} \quad (4.22)$$

Consequently, $s(n_i) = q^{i-1}s(\ell_i)$ and the relation (4.21) remains valid. The range of ℓ_i, though, is reduced to $0, 1, \ldots, 2k+1-2(N-1)$. Therefore,

$$\mathcal{Z}_k = \sum_{N=0}^{k+1} Z_{k,N} = \sum_{N=0}^{k+1} q^{\frac{N(N-1)}{2}} Z^{(\alpha\beta)}_{k-N+1,N}, \tag{4.23}$$

where $Z^{(\alpha\beta)}_{k-N+1,N}$ is the partition function of bosons distributed to two towers of equidistant levels, $\alpha(n)$ and $\beta(n)$, with level numbers in each $l_i = 0, 1, \ldots, k-N+1$. The partition function for fixed numbers of particles N_α and N_β in each tower factorizes. The full grand partition function, however, does not, due to the factor $q^{N(N-1)/2}$ that involves $N_\alpha + N_\beta = N$, and the fact that each tower contains $k - N_\alpha - N_\beta + 1$ levels, coupling the two towers.

C. Calculation of the secular determinant

We now have all the components for calculating $\mathcal{Z}_k = \det(1 - H_{2D}) = F_k(z_i, q)$. The bosonic partition function of N particles in a truncated equidistant spectrum $s(n) = aq^n$, $n = 0, 1, \ldots, k$ is (see Ref. [10] for other alternative expressions)

$$Z^{(B)}_{k,N} = a^N \prod_{j=1}^{N} \frac{1-q^{j+k}}{1-q^j} = a^N \prod_{j=1}^{k} \frac{1-q^{j+N}}{1-q^j}. \tag{4.24}$$

Applying the above to towers of type α or β with spectra as in (4.16), for N_α and N_β particles and $k \to k-N+1$, $N = N_\alpha + N_\beta$, we have

$$\begin{aligned}
Z^{(\alpha)}_{k-N_\alpha-N_\beta+1,N_\alpha} &= (-z\omega)^{N_\alpha} \prod_{j=1}^{N_\alpha} \frac{1-q^{j+k-N_\alpha-N_\beta+1}}{1-q^j} \\
&= (-z\omega)^{N_\alpha} \prod_{j=1}^{k-N_\alpha-N_\beta+1} \frac{1-q^{j+N_\alpha}}{1-q^j} \\
Z^{(\beta)}_{k-N_\alpha-N_\beta+1,N_\beta} &= (-z\omega^{-1})^{N_\beta} \prod_{j=1}^{N_\beta} \frac{1-q^{j+k-N_\alpha-N_\beta+1}}{1-q^j} \\
&= (-z\omega^{-1})^{N_\beta} \prod_{j=1}^{k-N_\alpha-N_\beta+1} \frac{1-q^{j+N_\beta}}{1-q^j}
\end{aligned} \tag{4.25}$$

and

$$\mathcal{Z}_k = \sum_{N_\alpha,N_\beta=0}^{N_\alpha+N_\beta\leqslant k+1} q^{\frac{(N_\alpha+N_\beta)(N_\alpha+N_\beta-1)}{2}} Z^{(\alpha)}_{k-N_\alpha-N_\beta+1,N_\alpha} Z^{(\beta)}_{k-N_\alpha-N_\beta+1,N_\beta}. \tag{4.26}$$

Using either the first or the second expressions in (4.25) and changing summation variables we obtain for $\mathcal{Z}_k = F_k(z_i, q)$ the two alternative forms

$$\begin{aligned}
F_k(z_i, q) &= \sum_{N=0}^{k+1} (-z)^N \sum_{n=0}^{N} \omega^{N-2n} \prod_{j=1}^{n} \frac{q^{\frac{N-1}{2}} - q^{j+k-\frac{N-1}{2}}}{1-q^j} \\
&\quad \times \prod_{l=1}^{N-n} \frac{q^{\frac{N-1}{2}} - q^{l+k-\frac{N-1}{2}}}{1-q^l} \\
&= \sum_{N=0}^{k+1} (-z)^N q^{\frac{N(N-1)}{2}} \sum_{n=0}^{N} \omega^{N-2n} \\
&\quad \times \prod_{j=1}^{k-N+1} \frac{(1-q^{j+n})(1-q^{j+N-n})}{(1-q^j)^2}.
\end{aligned} \tag{4.27}$$

Although ω can be complex, the above expressions are invariant under $\omega \to \omega^{-1} = \omega^*$ (on $n \to N-n$, for real z) and thus are real. In fact, we can use this property to express $F_k(z_i, q)$ in terms of Chebyshev polynomials $T_n(z_h/2z)$. Adding the expressions for ω and ω^{-1} in (4.27) and (4.28) we obtain

$$\begin{aligned}
F_k(z_i, q) &= \sum_{N=0}^{k+1} (-z)^N \sum_{n=0}^{N} T_{|2n-N|}\left(\frac{z_h}{2z}\right) \prod_{j=1}^{n} \frac{q^{\frac{N-1}{2}} - q^{j+k-\frac{N-1}{2}}}{1-q^j} \\
&\quad \times \prod_{l=1}^{N-n} \frac{q^{\frac{N-1}{2}} - q^{l+k-\frac{N-1}{2}}}{1-q^l} \\
&= \sum_{N=0}^{k+1} (-z)^N q^{\frac{N(N-1)}{2}} \sum_{n=0}^{N} T_{|2n-N|}\left(\frac{z_h}{2z}\right) \\
&\quad \times \prod_{j=1}^{k-N+1} \frac{(1-q^{j+n})(1-q^{j+N-n})}{(1-q^j)^2}.
\end{aligned} \tag{4.28}$$

Equations (4.27) and (4.28) are our third main result. Using the above expressions for $F_k(z_i, q)$ in (3.8), we obtain the generating function of Motzkin paths $G_{k,mn}(z_i, q)$.

Expressions (4.28) are explicit polynomials in z, z_h. For even N, only even Chebyshev polynomials appear with degrees 0 up to N, leading to polynomials in z, z_h of total degree N and terms from z_h^N to z^N. For odd N, only odd Chebyshev polynomials appear, leading again to total degree N polynomials in z, z_h but now with terms from z_h^N to $z^{N-1}z_h$. This is related to the fact that excursions with an odd number of steps cannot consist entirely of up and down steps (factors z) and must contain at least one horizontal step (z_h).

Finally, we point out that the products in the above formulas, as well as in other formulae in this paper, can be expressed in terms of the q-Pochhammer symbol $(a; q)_n$, but we will not do this transcription.

D. Checks and special cases

A few checks can be performed on formulas (4.27) and (4.28), reducing them to known cases.

(case a) Identical underlying Dyck processes: $z_1 = z_{\scriptscriptstyle D} q_{\scriptscriptstyle D}^{1/2}$, $z_2 = z_{\scriptscriptstyle D} q_{\scriptscriptstyle D}^{3/2}$, $q_o = q_D^2$. This reduces the two-step process (4.2) into a standard Dyck path process with parameters $z_{\scriptscriptstyle D}, q_{\scriptscriptstyle D}$. From (4.6) and (4.15) the above choices imply

$$\begin{aligned}
&q = q_{\scriptscriptstyle D}^4, \quad z = z_{\scriptscriptstyle D}^2 q_{\scriptscriptstyle D}^2, \quad Z_h = z_{\scriptscriptstyle D}^2 (q_{\scriptscriptstyle D} + q_{\scriptscriptstyle D}^3), \\
&\omega = q_{\scriptscriptstyle D}; \quad s(n) = -z_{\scriptscriptstyle D}^2 q_D^{2n+1},
\end{aligned} \tag{4.29}$$

where $s(n)$ is the standard equidistant spectrum of the Dyck process. Equation (4.27) reproduces the Dyck path determinant given in Ref. [10] on use of the identity

$$\sum_{n=0}^{N} q^n \prod_{j=1}^{k} \frac{(1-q^{2(j+n)})(1-q^{2(j+N-n)})}{(1-q^{2j})^2} = \prod_{j=1}^{2k+1} \frac{1-q^{j+N}}{1-q^j}. \tag{4.30}$$

(case b) $q = 1$: The generating function accounts only for length and type of steps. $F_k(z_i, 1)$ degenerates to

$$F_k(z_i, 1) = \sum_{N=0}^{k+1} (-z\omega)^N \sum_{n=0}^{N} \omega^{-2n} \binom{k+1-N+n}{n} \binom{k+1-n}{N-n}. \tag{4.31}$$

(case c) $z_h = 0$: The horizontal step is suppressed and the process degenerates into Dyck paths. Equation (4.16) implies $\omega = i$ and $\alpha(n) = -\beta(n) = -izq^n$. This actually eliminates all odd N from the sum in (4.27) and (4.28) and reproduces the Dyck path determinant, as demonstrated in the next subsection.

(case d) $z_h = z$, weighting all steps equally. Equation (4.16) implies

$$\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\pi/3}. \tag{4.32}$$

In this case, too, the determinant assumes a special form.

Cases c and d are instances of a subset of values of z, z_h for which the grand partition function admits a special interpretation and the determinant has special properties. We treat these special cases in the next subsection.

(case e) $k = \infty$ (no ceiling). In this case the formula simplifies to

$$F_k(z_i, q) = \sum_{N=0}^{k+1} (-z)^N q^{\frac{N(N-1)}{2}} \sum_{n=0}^{N} \prod_{j=1}^{n} \frac{\omega}{1-q^j} \prod_{l=1}^{N-n} \frac{\omega^{-1}}{1-q^l}. \tag{4.33}$$

E. A "dual" form of the determinant and cyclic cases

An alternative form for the secular determinant $F_k(z_i, q)$ can be obtained by considering the bosonized system, instead of two "vertical" towers of levels, $\alpha(n)$ and $\beta(n)$ with $k-N+2$ levels each, as $k-N+2$ "horizontal" sets of two levels each. Calling N_j the number of particles in set j with levels $\alpha(j), \beta(j)$ $(j = 0, 1, \dots, k-N+1)$, the N_j-particle bosonic partition function for set j is

$$\begin{aligned} Z^{(B)}_{j;N_j} &= \sum_{n=0}^{N_j} \alpha(j)^n \beta(j)^{N_j-n} \\ &= (-z\omega^{-1})^{N_j} q^{jN_j} \sum_{n=0}^{N_j} \omega^{2n} \\ &= (-z\omega^{-1})^{N_j} q^{jN_j} \frac{1-\omega^{2(N_j+1)}}{1-\omega^2}. \end{aligned} \tag{4.34}$$

Accounting for the factor $q^{N(N-1)/2}$ relating the bosonic to the exclusion-2 partition function, the grand partition function (secular determinant) becomes

$$F_k(z_i, q) = \sum_{N=0}^{k+1} (-z\omega^{-1})^N q^{\frac{N(N-1)}{2}} \sum_{\{\sum N_j = N\}} \prod_{j=0}^{k-N+1} q^{jN_j} \frac{1-\omega^{2(N_j+1)}}{1-\omega^2}. \tag{4.35}$$

This form may look less useful than (4.27) or (4.28), as it involves multiple sums, but is better in revealing the structure of the special systems we will study in the sequel. Enforcing the constraint $\delta(\sum_j N_j - N)$ in terms of an exponential integral, and harmlessly extending the summation range of the N_j to infinity, the above can also be rewritten as

$$F_k(z_i, q) = \int_0^{2\pi} d\theta \sum_{N=0}^{k+1} (-ze^{-i\theta})^N q^{\frac{N(N-1)}{2}} \times \prod_{j=0}^{k-N+1} \frac{1}{(1-\omega^{-1}e^{i\theta}q^j)(1-\omega\, e^{i\theta}q^j)}. \tag{4.36}$$

We now focus our attention to "cyclic" walks with parameters such that ω^2 is a root of unity; that is,

$$\omega = e^{i\pi p/r} \quad \text{or} \quad z_h = 2z\cos\frac{\pi p}{r}, \qquad r = 1, 2, \dots, \quad p, r \text{ coprime}, \tag{4.37}$$

where $p = 1, r = 2$ corresponds to case c of the previous subsection, while $p = 1, r = 3$ corresponds to case d. For such values of ω^2 the ω-dependent ratio inside the product in (4.35) is periodic in N_j with period r and vanishes for $N_j = -1 (\text{mod } r)$.

To capitalize on this property, we put $N_j = rn_j + \ell_j$, $n_j = 0, 1, \dots, \ell_j = 0, 1, \dots, r-2$; (4.35) becomes

$$F_k(z_i, q) = \sum_{N=0}^{k+1} (-z\omega^{-1})^N q^{\frac{N(N-1)}{2}} \sum_{\{ r\sum n_j + \sum \ell_j = N, \ \ell_j = 0,1,\dots,r-1 \}}^{N} \prod_{j=0}^{k-N+1} q^{rjn_j} \times \prod_{j=0}^{k-N+1} q^{j\ell_j} \frac{1-\omega^{2(\ell_j+1)}}{1-\omega^2}. \tag{4.38}$$

We can interpret n_j as counting bosons and ℓ_j as counting "parafermions" of order $r-2$ with the total number of bosons n and parafermions ℓ satisfying $N = rn + \ell$. By parafermions we mean particles with the property that up to $r-2$ of them can be placed in a single-particle level. Then the first product in (4.38) is the partition function of n bosons in levels q^{rj} while the second product is the corresponding parafermionic partition function. Reverting to (4.24) for the bosonic partition function, and putting $N = rn + \ell$,

$$F_k(z_i, q) = \sum_{n,\ell} (-z\omega^{-1})^{rn+\ell} q^{\frac{(rn+\ell)(rn+\ell-1)}{2}} Z^{\{r-2\}}_{k-rn-\ell+1}(\ell) \times \prod_{j=1}^{n} \frac{1-q^{r(j+k-rn-\ell+1)}}{1-q^{rj}} \tag{4.39}$$

with $Z^{\{r-2\}}_{k-rn-\ell+1}(\ell)$ the parafermionic partition function for ℓ particles

$$Z^{(r-2)}_{k-rn-\ell+1}(\ell)=\sum_{\substack{\ell_j=0\\ \{\sum\ell_j=\ell\}}}^{r-2}\prod_{j=0}^{k-rn-\ell+1}q^{j\ell_j}\frac{1-\omega^{2(\ell_j+1)}}{1-\omega^2}. \tag{4.40}$$

Wecan now examine some special cases.

(case c) $r=2, p=1\Rightarrow\omega=i$: Only $\ell_j=0$ survives, so there are no parafermions. We get

$$\begin{aligned}F_k(z,z_h=0,q)&=\sum_n(zi)^{2n}q^{n(2n-1)}\prod_{j=1}^n\frac{1-q^{2(j+k-2n+1)}}{1-q^{2j}}\\&=\sum_n(-z^2)^n(q^2)^{n(n-1)}\\&\quad\times\prod_{j=1}^n(q^2)^{1/2}\frac{1-(q^2)^{j+k-2n+1}}{1-(q^2)^j}.\end{aligned} \tag{4.41}$$

This is precisely the secular determinant of the Dyck path process calculated in Ref. [10] with length parameter z^2 and area parameter q^2, as expected.

(case d) $r=3, p=1\Rightarrow\omega=e^{i\pi/3}$: Here $\ell_j=0,1$, and parafermions become ordinary fermions. The fermionic partition function $Z^{(F)}_{k-3n-\ell+1}(\ell)$ can itself be bosonized. Omitting the intermediate steps, the final result is

$$\begin{aligned}F_k(z,z,q)&=\sum_{n,\ell}(-1)^nz^{3n+\ell}q^{\frac{3n(3n-1)}{2}+3n\ell+\ell(\ell-1)}\\&\quad\times\prod_{j=1}^n\frac{1-q^{3(j+k-3n-\ell+1)}}{1-q^{3j}}\prod_{s=1}^{\ell}\frac{1-q^{s+k-3n-2\ell+2}}{1-q^s}\end{aligned} \tag{4.42}$$

(the summation in n,ℓ is over the values for which the summand does not vanish). This expression is preferable to the general expression (4.27) or (4.28) only in that it is manifestly real and a polynomial in z,q with integer coefficients.

V. CLUSTER EXPRESSIONS OF GENERATING FUNCTIONS

In the previous section we obtained relatively explicit formulas for $F_k(z_i,q)$ and therefore for $G_{k,mn}(z_i,q)$. Their form, however, is rather complicated, and their dependence on z_h through ω is obscured. In this section we will take further advantage of the connection to exclusion statistics to express the logarithm of the generating function $\ln G_{k,mn}$ in terms of cluster coefficients.

A. Cluster coefficients

For a grand partition function $\mathcal{Z}$, cluster coefficients b_a, $a=1,2,\dots$, are defined in terms of the expansion of the grand potential $\ln\mathcal{Z}(x)$ in terms of the fugacity parameter $x=e^{\beta\mu}$,

$$\ln\mathcal{Z}(x)=\ln\left(\sum_{N=0}^\infty x^NZ_N\right)=\sum_{a=1}^\infty x^a\,b_a. \tag{5.1}$$

In our case, extracting the factor $x^N=(-z)^N$ out of the N-body partition function, $Z_{k,N}=(-z)^N\tilde Z_{k,N}$ [see Eqs. (4.27) and (4.28)] as a fugacity parameter, we have the cluster expansion

$$\begin{aligned}\ln F_k(z_i,q)&=\ln\left[\sum_{N=0}^{k+1}(-z)^N\tilde Z_{k,N}(\omega,q)\right]\\&=\sum_{a=1}^\infty(-z)^a\,b_{k,a}(\omega,q).\end{aligned} \tag{5.2}$$

The expression of the cluster coefficients for general exclusion statistics g was derived in [2,4]. For exclusion $g=2$, relevant to our case, with spectral parameter $s(r)$, they are expressed as a sum over all compositions of the integer a and read

$$\begin{aligned}(-z)^ab_{k,a}&=(-1)^{a-1}\sum_{\substack{l_1,l_2,\dots,l_j;\,j\leqslant 2k+2\\ \text{compositions of }a}}c_2(l_1,l_2,\dots,l_j)\\&\quad\times\sum_{r=0}^{2k+2-j}\prod_{i=1}^js^{l_i}(r+i-1).\end{aligned} \tag{5.3}$$

(Compositions are partitions where the order of terms also matters.) In our case, since the spectrum has $2k+2$ states, only compositions with at most $2k+2$ components are possible. The combinatorial coefficients $c_2(l_1,l_2,\dots,l_j)$ depend only on the composition and the statistics, and for $g=2$ they are

$$\begin{aligned}c_2(l_1,l_2,\dots,l_j)&=\frac{1}{l_1}\prod_{i=1}^{j-1}\binom{l_i+l_{i+1}-1}{l_{i+1}}\\&=\frac{\prod_{i=1}^{j-1}(l_i+l_{i+1}-1)!}{\prod_{i=2}^{j-1}(l_i-1)!\prod_{i=1}^jl_i!}.\end{aligned} \tag{5.4}$$

The dependence of $b_{k,a}$ on z_i and q is entirely through the dependence of $s(r)$ on these parameters.

B. Cluster expansion of the generating function

The logarithm of the generating function $\ln G_{k,mn}(z_i,q)$ follows from (3.8) as

$$\begin{aligned}\ln G_{k,mn}(z_i,q)&=(n-m)\ln z+\frac{n^2-m^2}{2}\ln q\\&\quad+\ln F_{m-1}(z_i,q)+\ln F_{k-n-1}(z_iq^{n+1},q)\\&\quad-\ln F_k(z_i,q)\end{aligned} \tag{5.5}$$

for $m\leqslant n$ and similarly for $m\geqslant n$. The terms in the second line are all given by (5.2) and (5.3), with a common $c_2(l_1,l_2,\dots,l_j)$, differing only in the last sums over r in (5.3). These sums for the three terms can be brought to a common form by noticing that the dependence of the spectral factors $s(r)$ on z_i and q implies

$$s(q^mz_i;r)=s(z_i;r+2m) \tag{5.6}$$

as is clear from (4.15) or (4.16). The three sums combine as

$$\ln G_{k,mn}(z_i,q) = (n-m)\ln z + \frac{n^2-m^2}{2}\ln q + \sum_{a=1}^{\infty}(-1)^{a-1}\sum_{\substack{l_1,l_2,\dots,l_j\\ \text{compositions of } a}} c_2(l_1,l_2,\dots,l_j) \times\left(\sum_{r=0}^{2m-j}+\sum_{r=2n+2}^{2k+2-j}-\sum_{r=0}^{2k+2-j}\right)\times\prod_{i=1}^{j}s^{l_i}(r+i-1) \tag{5.7}$$

and telescoping the sums we finally obtain

$$\ln G_{k,mn}(z_i,q) = (n-m)\ln z + \frac{n^2-m^2}{2}\ln q + \sum_{a=1}^{\infty}(-1)^{a}\sum_{\substack{l_1,l_2,\dots,l_j\\ \text{compositions of } a}} c_2(l_1,l_2,\dots,l_j) \times\sum_{r=\max(2m+1-j,0)}^{\min(2k+2-j,2n+1)}\prod_{i=1}^{j}s^{l_i}(r+i-1) \tag{5.8}$$

with the understanding that sums vanish when their lower limit exceeds their upper limit. It is clear that only compositions with length j up to $2k+2$ will contribute. Since $s(r)$ is proportional to z, it is clear that the above sum is an expansion in terms of z^a.

To derive an explicit expression, we separate the sums over r in (5.8) into even ($r=2s$) and odd ($r=2s+1$) terms. After some manipulations we obtain the form

$$\ln G_{k,mn}(z_i,q) = (n-m)\ln z + \frac{n^2-m^2}{2}\ln q + \sum_{a=1}^{\infty} z^a P_{k,mn;a}(q;\omega) \tag{5.9}$$

with

$$P_{k,mn;a}(q;\omega) = \sum_{\substack{l_1,l_2,\dots,l_j\\ \text{compositions of } a}} c_2(l_1,l_2,\dots,l_j)\, q^{\frac{1}{2}\sum_{i=1}^{j}(i-1)l_i-\frac{1}{2}S(l_2,l_4,\dots)} \times\left\{\omega^{a-2S(l_2,l_4,\dots)}\sum_{s=\max(\lfloor m+1-\frac{j}{2}\rfloor,0)}^{\min(\lfloor k+1-\frac{j}{2}\rfloor,n)} q^{sa} + \omega^{-a+2S(l_2,l_4,\dots)}\, q^{S(l_2,l_4,\dots)}\sum_{s=\max(\lfloor m+\frac{1-j}{2}\rfloor,0)}^{\min(\lfloor k+\frac{1-j}{2}\rfloor,n)} q^{sa}\right\}, \tag{5.10}$$

where $\lfloor\cdot\rfloor$ is the "floor" (integer part) function and $S(l_2,l_4,\dots)$ is the sum of even-order l_i

$$S(l_2,l_4,\dots) = \sum_{i=1}^{\lfloor \frac{j}{2}\rfloor} l_{2i}. \tag{5.11}$$

$P_{k,mn;a}(q;\omega)$ is necessarily real, although this is rather obscured in the expression (5.10): For ω complex, each term in (5.10) is complex, and terms from different compositions $l_1,\dots,l_j$ combine to a real. The actual form of $c_2(l_1,\dots,l_j)$ is needed to achieve reality.

$P_{k,mn;a}(q;\omega)$ is a polynomial in q since the fractional powers of q combine to an integer:

$$\tfrac{1}{2}\sum_{i=1}^{j}(i-1)l_i - \tfrac{1}{2}S(l_2,l_4,\dots) = l_3+l_4+2l_5+2l_6+\dots. \tag{5.12}$$

An examination of the terms in (5.10) determines its maximal power in q (degree) as

$$\text{max power in } q \text{ of} P_{k,mn;a}(q) = \begin{cases} an+\lfloor\frac{a^2}{4}\rfloor, & a\leqslant 2k-2n\\ ak-(k-n)^2, & a>2k-2n\end{cases}. \tag{5.13}$$

The minimal power in q (with nonzero coefficient) can also be extracted:

$$\text{min power in } q \text{ of} P_{k,mn;a}(q) = \begin{cases} am-\lfloor\frac{a^2}{4}\rfloor, & a\leqslant 2m\\ m^2, & a>2m\end{cases}. \tag{5.14}$$

The above results have clear geometric interpretations (see Fig. 5): The prefactors in the expression (3.8) for $G_{k,mn}(z,\omega,q)$, leading to the log terms in (5.9), correspond to the minimal length $n-m$ that a path connecting points at heights m and n can have, and the minimal area $n^2/2-m^2/2$ that such a straight path will have. Terms z^a correspond to an additional length a over the minimal one, and the exponential $\exp P_{k,mn;a}(q;\omega)$ accounts for the additional area of such nonminimal paths.

Since the degree of $P_{k,mn;a}(q;\omega)$ (5.13) is a convex function of a, the degree of the corresponding terms of order z^a in $\exp P_{k,mn;a}(q;\omega)$ is the same as that of $P_{k,mn;a}(q;\omega)$. Therefore, (5.13) gives the maximal excess area of a path of length $n-m+a$: The upper expression corresponds to a "roof" path that cannot touch the ceiling, for which the height restriction is irrelevant, while the lower expression corresponds to a "flattened roof" path that grazes the ceiling.

Similarly, since the lowest power of $P_{k,mn;a}(q;\omega)$ in (5.14) is a concave function of a, the minimal power in q of the term of order z^a in $\exp P_{k,mn;a}(q;\omega)$ is the same as that of $P_{k,mn;a}(q;\omega)$. Therefore, Eq. (5.14) gives the minimal excess area of a path of length $n-m+a$: The upper expression corresponds to a "gorge" path that cannot touch the floor, while the lower expression corresponds to a "valley" path that creeps on the floor (see Fig. 5).

The above expressions depend only on n (for maximal area) or m (for minimal area). However, the expressions of the total maximal or minimal area in terms of the total length $l=n-m+a$ become symmetric in n, m:

$$A_{\max} = \begin{cases} \lfloor\left(\frac{m+n+l}{2}\right)^2\rfloor - \frac{m^2+n^2}{2}, & l+m+n\leqslant 2k\\ k(m+n+l-k)-\frac{m^2+n^2}{2}, & l+m+n>2k\end{cases}, \tag{5.15}$$

$$A_{\min} = \begin{cases} \frac{m^2+n^2}{2} - \lfloor\left(\frac{m+n-l}{2}\right)^2\rfloor, & l\leqslant m+n\\ \frac{m^2+n^2}{2}, & l>m+n\end{cases}. \tag{5.16}$$

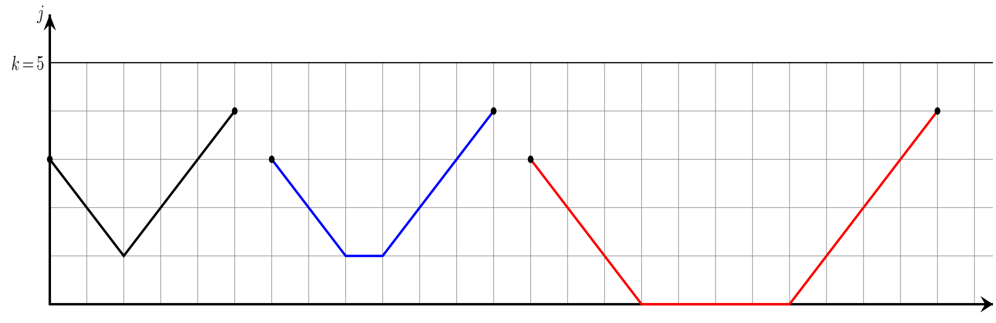

FIG. 5. Minimal area Motzkin meanders of various lengths from $m = 3$ to $n = 4$: The first one (black) of length $l = 5$ is an unrestricted "gorge," while the second one (blue) of length $l = 6$ necessitates a horizontal step at the bottom. Their area of 11.5 and 12.5 is given by the top expression in (5.16). The third one (red) of length $l = 11$ is a "valley" restricted by the floor. Its area of 12.5 is given by the bottom expression in (5.16). Paths of maximal area restricted by the ceiling have similar but inverted shapes.

It can be checked that the above reproduce the actual areas of the maximal and minimal area paths. When $m + n + l$ is odd, in which case the top or bottom of unrestricted paths acquires a single horizontal link, the floor function adds a correction of $\pm\frac{1}{4}$ corresponding to the truncated triangular part.

VI. CONCLUSIONS AND DISCUSSION

The use of a Hamiltonian framework in combination with the two-step construction, exclusion statistics, and bosonization allowed for the derivation of generating functions for Motzkin paths including dual variables for length, area, as well as various top and bottom "events." The relation to quantum statistics of exclusion $g = 2$ identified in Ref. [10] for Dyck paths persists in the case of Motzkin paths, allowing the calculation of these functions and providing cluster decomposition techniques used to derive alternative expressions for the logarithm of the generating functions in terms of sums of compositions.

It is interesting to compare our results with other work done in the literature and put them in the proper context. The generating functions studied in this paper are essentially embedded in the general class of Motzkin polynomials, defined as sums over paths with height- and step-specific weights [21,23]. Specifically, if each up, horizontal, or down step ending at height j is assigned a variable u_j, h_j, and $d_j = 1$, respectively, and walks are weighted by the product of such variables, then (using a notation consistent with our conventions) $P_{l|k}^{(m,n)}(\{u_j\},\{h_j\})$ is a polynomial in the variables h_0, h_j, u_j, $j = 1, 2, 3, \dots$, arising from summing the weights of all meanders of length l starting at m and ending at n with a ceiling k, while $P_l(\{u_j\},\{h_j\})$ is the corresponding sum over unrestricted excursions. It should be clear that $G_{\infty,00} = G$ is related to P_l as

$$G(z_i, q) = \sum_{l=0}^{\infty} P_l(\{u_j\},\{h_j\}),$$
$$\text{with } u_j = z^2 q^{2j-1}, \quad h_j = z_h q^{j+1/2}. \tag{6.1}$$

[An analogous relation between $G_{k,mn}$ and $P_{l|k}^{(m,n)}$ fails, since the convention $d_j = 1$ only considers the area of up-links, which is not related to that of down-links if $m \neq n$.] In fact, an explicit combinatorial expression for the polynomial P_l involving multiple sums was given in Refs. [21,23]. Adapting it to our case and notation it implies

$$G(z_i, q) = \sum_{\{m_j,n_j\geqslant 0\}} \frac{1}{(1-z_h)^{n_1+1}} \prod_{i=1}^{\infty} \binom{m_i + n_{i+1} + n_i - 1}{m_i,\ \ n_{i+1}} \times (z^2 q^{2i-1})^{n_i} (z_h q^i)^{m_i} \tag{6.2}$$

with the usual definition of the choose-symbol

$$\binom{M}{N, K} = \frac{M!}{N!\,K!\,(M-N-K)!}. \tag{6.3}$$

(The restriction $\sum_i 2n_i + \sum_j m_j = l$ in Ref. [21] is relaxed since we sum over all l. We also rearranged the factors and performed the sum over m_0.) This is a compact-looking expression, but in fact it involves infinitely many sums. (We also have no expression for the more general $G_{k,mn}$.) Our formulas (4.27) and (4.28), on the other hand, involve only double sums. We can, actually, evaluate the sums in (6.2) sequentially, in the order $m_1, n_1, m_2, n_2 \dots$, but at the end we obtain an infinitely nested expression, essentially the Rogers-Ramanujan-like continued fraction that will be given in the sequel.

Generating functions counting the number of horizontal steps l_h were considered in Ref. [24] by including a weight t^{l_j} but not the remaining steps nor the area. Interestingly, in the same paper the first and second moments of the area under the paths were considered and shown to satisfy specific recursion relations. These would correspond to the $O(\epsilon)$ and $O(\epsilon^2)$ terms in an ϵ expansion of our $G(z = 1, z_h = 1, q = e^{\epsilon})$. Quantities related to path area were also considered in Ref. [28].

The work whose results come closest to ours is Ref. [26], which studied Motzkin excursions (floor-to-floor paths), also counting touch-ups and touch-downs (although nor creep-ups and creep-downs), and calculated the length-area generating function. The results were obtained by combinatorially deriving recursion relations analogous to (3.18) and solving them using various ingenious mathematical techniques. The resulting generating function is then a ratio of quantities expressed in terms of generalized hypergeometric functions. These results should match our results (3.10), (3.30) (with $m = n = 0$ and thus $G_{m-1} = 0$, and $s = \bar{s} = 1$), and, in particular, (4.27) and (4.28). However, the expressions in (4.27) and (4.28) look quite different than the ones in Ref. [26], involving sums of Chebyshev polynomials rather than generalized hypergeometric functions. Such differences of form are expected, given the nontrivial nature of the results and the various identities that could be used in reshaping them, and already [26] noted that their results for the half-plane ($k = \infty$, no ceiling) and the slit (k finite) looked "dramatically different." The reconciliation between our results and those of Ref. [26] is an open interesting mathematical task.

The generating functions derived in the present work satisfy several recursion relations, as stated in Sec. III C. Such relations for Motzkin polynomials were derived in several papers and invariably lead to expressions related to the Rogers-Ramanujan continued fraction. Specifically, the recursion relation (3.18) for $G_k(z, q)$ can be iterated leading to a continued fraction. To put it in a clean form, we define

$$\lambda = \omega + \omega^{-1} = \frac{z_h}{z}, \quad w = z^{-1}, \quad u = q^{-1},$$
$$g_k(w) = zG_k(z, \omega, q). \tag{6.4}$$

In this parametrization, the recursion relation becomes

$$g_k(w) = \frac{1}{w - \lambda - g_{k-1}(uw)}. \tag{6.5}$$

Iterating this relation with the final condition $G_{-1}(z, q) = 0$ we obtain

$$g_k(w) = \frac{1}{w - \lambda - \frac{1}{wu - \lambda - \frac{1}{\cdots \frac{1}{wu^k - \lambda}}}}. \tag{6.6}$$

This is a truncated version of a continued fraction related to the Rogers-Ramanujan's identity. Other forms, involving directly G_k, are readily obtainable.

We conclude with some possible directions for future research. The most interesting and relevant next task would be to use the results in this work to derive physical properties of statistical systems described in terms of Motzkin paths. Several systems can be mapped to such paths, the canonical one being linear polymers on the plane in a slit (represented by the space between the floor and ceiling of the paths), with potential adsorbing interactions whenever the polymer bounces off or sticks to the two boundaries (our $t, s, \bar{t}, \bar{s}$ terms in Sec. III D). As a physical model, weighting by the area underneath the path accounts for polymers with a different solvent from the area above the path by assigning an energy depending on the area, q playing the role of the corresponding Boltzmann factor. The area-length generating function then reproduces the partition function of the polymer, and the free energy of the model can be determined from that generating function and would determine critical transition properties of the model (as in Ref. [13] for $q = 1$). Such critical properties for $q \neq 1$ have not been explored, to the best of our knowledge, and constitute an important open problem and obvious next step.

On the mathematical side, the results in this work most likely admit further refinement and elaboration. For instance, the explicit expressions for F_k, $G_{k,mn}$, and $\hat{G}_{k,mn}$ presented in this paper are not unique, as they satisfy several identities and recursion relations, and alternative forms are possible, as we already noted in the comparison with the results in [26]. In addition, it would be desirable to have expressions for the cluster coefficients involving directly z, z_h as opposed to z, ω, as in (4.28) for the partition function, making the dependence on z_h clearer. Such a rewriting remains to be achieved.

The enumeration of Motzkin paths according to their length can be expressed in terms of trinomial coefficients in the expansion of $(x + 1 + x^{-1})^l$, the term x^n identifying the (unrestricted from floor or ceiling) Motzkin paths of total climb n (or descent, if $n < 0$). This method was used in Ref. [13] to find the length generating function ($q = 1$) of unrestricted Motzkin paths ($k = \infty$) and derive critical properties of the corresponding statistical mechanical model. The use of this technique, however, becomes cumbersome when dealing with restricted paths ($k < \infty$) and fails to address the more interesting case of length *and* area generating functions ($q \neq 1$). In a related development, the expressions of length generating functions ($q = 1$) for restricted Dyck, Motzkin, and more general paths with a number of possible up and down steps, and arbitrary weights associated to each kind of step, have been related to skew-Schur functions [19]. Including the area counting variable q would generalize these generating functions to q-deformed versions of skew-Schur functions, both in the case of Dyck paths and for Motzkin paths. This points to a possible generalization of the trinomial method involving q-deformed polynomial expressions. Such a generalization and the related skew-Schur functions and their properties remain an interesting topic for further mathematical study.

The $q = 1$ limit of Dyck or Motzkin paths is intimately related to compositions of a large number of SU(2) spins: Spin-$\frac{1}{2}$ individual spins are related to Dyck paths, while Motzkin paths correspond to spin-1. In Ref. [29], the combinatorics and statistics of such compositions of general spin-s components were studied using generating function and partition function techniques, and a corresponding large-N phase transition was identified. Symmetric (bosonic) and antisymmetric (fermionic) spin compositions were also studied in Ref. [29] and led to novel statistical properties. The existence of a ceiling at $n = k$ for paths would correspond to deforming the spin group to the "quantum group" $SU(2)_Q$ with $Q = \exp[2\pi i/(k+1)]$, which has irreducible representations of dimension up to $k + 1$. It would be interesting to further explore the connection between the two systems (spins and paths). Investigating the physical meaning of weighting the spin compositions with an exponential factor proportional to the "area" of the specific composition channel, as for paths, and, conversely, the concept of symmetric or antisymmetric weighing of paths, as for fermionic or bosonic spins, and the possibility of a phase transition in the statistics of paths, are fascinating topics that deserve further exploration.

Finally, there are other generalizations of paths that have been studied in the literature. For instance, "colored" Motzkin paths in which each link can come in one of several "colors," and k-Motzkin paths in which horizontal steps are of length k have been considered. Further, paths with more general increments, such as Lukasiewicz paths, have been studied, and there are other possible generalizations that have not. All such paths can be treated in the Hamiltonian framework, and their generating functions can be related to the secular Hamiltonian of the process, as in Secs. II and III of this paper. However, explicit expressions for the generating function of such systems would require the evaluation of secular determinants as in Sec. IV of this paper, which may not be tractable. It appears that at least a class of such walks can be related to quantum exclusion statistics, but for exclusion higher than 2 and for more general one-body spectra. The statistical mechanics of general-g exclusion systems with an arbitrary discrete energy spectrum have recently been derived using techniques closely related to the ones in the present work [30]. Using these techniques, the generating functions and statistics of generalized paths could be derived. We defer a full treatment of these cases to a future publication.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. 2112729 and by PSC-CUNY Grant No. 6D136-0001. I also thank Thomas Prellberg and the two anonymous referees for constructive comments that helped improve the manuscript.

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