Phase transition in the majority-vote model on time-varying networks

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Social interactions may affect the update of individuals' opinions. The existing models such as the majority-vote (MV) model have been extensively studied in different static networks. However, in reality, social networks change over time and individuals interact dynamically. In this work, we study the behavior of the MV model on temporal networks to analyze the effects of temporality on opinion dynamics. In social networks, people are able to both actively send connections and passively receive connections, which leads to different effects on individuals' opinions. In order to compare the impact of different patterns of interactions on opinion dynamics, we simplify them into two processes, that is, the single directed (SD) process and the undirected (UD) process. The former only allows each individual to adopt an opinion by following the majority of actively interactive neighbors, while the latter allows each individual to flip opinion by following the majority of both actively interactive and passively interactive neighbors. By borrowing the activity-driven time-varying network with attractiveness (ADA model), the two opinion update processes, i.e., the SD and the UD processes, are related with the network evolution. With the mean-field approach, we derive the critical noise threshold for each process, which is also verified by numerical simulations. Compared with the SD process, the UD process reaches a larger consensus level below the same critical noise. Finally, we also verify the main results in real networks.

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I. INTRODUCTION

In social networks, different types of interactions may have distinct and significant influence on opinion dynamics and decision-making [1], such as political affiliation [2,3] and rumor fermentation [4]. Commonly, people express and update their opinions through communication and discussion together. In opinion dynamics, a group of interacting individuals constantly update their opinions on the same issue based on various models to reach a consensus, polarization, or fragmentation in the final stage.

Many models have been proposed to study opinion dynamics, among which the majority vote (MV) model [5] is a famous nonequilibrium model, which has attracted much attention recently. It is a simple two-state stochastic model and displays a continuous order-disorder phase transition near the critical noise. In the MV model, the probability of the opinion flip that is controlled by the level of internal noise only depends on the sign of the sum of neighbors' opinions rather than on its exact value. In Ref. [5], the MV model with noise was first proposed on the square lattice, which belongs to the same universality class as the equilibrium Ising model [6]. By taking individuals' attributes into account, the MV model has been widely studied, such as visibility of the choice of neighbors' opinions [7], heterogeneous interactions of individuals [8], individuals with aging effects [9], individuals with more than two opinions [10,11], and the effect of inertia, which leads to the occurrence of a discontinuous transition [12,13].

The MV model on static network topologies has been explored, and the continuous order-disorder phase transition is observed in different network structures but with different critical noises [14]. The MV model was first proposed in square lattices [5]. Oliveira found that the critical exponents are the same as those of the Ising model on two-dimensional square lattices and the two models belong to the same universal class. Considering the randomness of the network, Pereira *et al.* [15] studied the MV model on ER networks and found that the increase of average connectivity improves the order of the system. After that, in Refs. [16,17], the "small-world" phenomenon is further studied on social networks. The critical transitions are dependent on the number of long-range interactions. Furthermore, due to the heterogeneity of real networks, critical noises on distinct network structures are determined by heterogeneous mean field [14,18] and quenched mean-field theory [19], respectively. From the above analysis, we see that network structure can definitely affect the opinion dynamics. However, most of previous work focused on static networks, ignoring the dynamic interactions between individuals, leading to the incomplete research of the opinion dynamics.

Temporal network is a natural framework for describing time-varying interactions [20,21], such as contact orders [22], burstiness [23], and lifetimes [24,25], which lead to completely different results from static networks. Therefore time-varying network structures have a significant impact on dynamics. However, little attention has been paid on opinion dynamics. Due to the complexity of the dynamic interactions between individuals, only the voter model (with binary opinions) [26] and the deffaunt model (with continuous opinions) [27] on temporal networks have been studied in depth. Hence, it is necessary to study the MV model on temporal networks.

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In social networks, various social interactions have different effects on opinion dynamics. People who are online may actively obtain information and passively receive information simultaneously. According to the information source, people generally have different attitudes on it. For example, while users of Facebook take the information they actively obtain and passively receive into consideration, those of the Internet Movie Database care more about the information they actively obtain and ignore passively received information, especially advertisements. Thus, individuals show diverse behavior in updating their opinions by considering different kinds of information. In Refs. [28–33], the impact of interaction types on the U.S. presidential campaign is deeply analyzed. It is found that some people only select traditional media content actively without considering the news from others, while others, especially the young, accept dissimilar perspectives passively. Their opinions are affected by actively interacted and passively interacted neighbors.

To characterize the features of the above mentioned interactions, we suppose that the update of an individual's opinion depends on his or her interaction patterns, which can be simplified into two processes, that is, the single directed (SD) process and the undirected (UD) process. In the SD process, individuals only care about their actively connected interactions, while in the UD process, individuals takes all kinds of interactions into consideration simultaneously.

With these ideas in mind, here we analyze the MV model with the SD and the UD process on activity-driven networks with attractiveness (ADA models) [34,35], where the interactions and the update of opinions are caused by the activity and attractiveness of individuals. With the heterogeneous mean-field theory, we derive the rate equation which rules the dynamics of the MV model on the ADA models. We calculate the critical points of noise $f_c^{\rm sd}$ and $f_c^{\rm ud}$ for the MV model with the SD process and the UD process, respectively. The analytical results show that in the MV model with the SD process, $f_c^{\rm sd}$ only depends on the number of actively interactive individuals at each time, while with the UD process, $f_c^{\rm ud}$ is related to the joint distribution of activity and attractiveness. The average activity and the attractiveness distribution have great influence on $f_c^{\rm ud}$. Finally, we conduct simulations on real networks to confirm the influence of activity and attractiveness on opinion dynamics.

This paper is organized as follows. In Sec. II, we introduce the evolution process of time-varying networks by the ADA models and the process of the MV model. In Sec. III, we provide a full description of the dynamics in the SD process and the UD process and exhibit the analytical results for the critical noise for each process. In Sec. IV, we present the numerical simulations, which show the consistency between the theoretical analysis and the simulation results. Then we apply the two processes on real network data to further verify our results. Finally, we summarize all the results and explore new perspectives in Sec. V.

II. THE MV MODEL ON TEMPORAL NETWORKS

In this section, we present the evolutionary process of the ADA model and the MV model with the SD process and the UD process.

A. ADA model

In the ADA models, we consider a fixed population of N individuals. Each individual is assigned with activity a, representing the probability of activating and interacting with others in each instantaneous network. In the original activity-driven network (AD model) [34], the activated individuals uniformly and randomly select other individuals for interaction. In order to make the network more consistent with real social networks, in Ref. [35], the authors proposed a time-varying network model by considering each individual's attractiveness following a given distribution. Hence, for individuals, the probability of being selected by other active individuals is positively correlated with attractiveness b. The activity a and attractiveness b are randomly assigned by the joint distribution $F(a,b)$. With these settings, the generation process of the time-varying network is demonstrated as follows [35]:

(i) At each time step t, start from a network G_t with N disconnected individuals;

(ii) With probability $a_i\Delta t$, each individual becomes active and selects m individuals to establish connections. Each individual is chosen according to his or her attractiveness, that is, the probability of individual j receiving connection is $\frac{b_j}{\sum_l b_l}$;

(iii) At the next time step $t+\Delta t$, delete all edges in the network G_t and restart the generation of a new instantaneous network $G_{t+\Delta t}$.

In the above model, only one edge is allowed between two individuals and self-loops are forbidden in each time step t. The duration of all interactions between individuals is Δt. Without loss generality, in the following, we set $\Delta t = 1$.

B. Majority-vote model

We study the opinion dynamics for the MV model by integrating the ADA models. For an active individual i, whose opinion is σ_i, we first determine the majority opinion of i's neighborhood. With probability $1-f$, the selected individual i adopts the majority opinion of his or her neighbors, and with probability f, i adopts the opposite opinion of the majority opinion. The probability f is called the noise parameter. The probability for σ_i flipping can be written as

$$\omega(\sigma_i) = \tfrac{1}{2}[1-(1-2f)\sigma_i S(\Theta_i)], \tag{1}$$

where $\Theta_i = \sum_{j\in\Omega(i)}\sigma_j$ represents the sum of i's neighbors' opinions. $\Omega(i)$ is the set of i's neighbors with whom i interacts. $S(x) = \mathrm{sgn}(x)$ if $x \neq 0$, and $S(0) = 0$. In the latter case, the probability of the opinion σ_i flipping to ± 1 is $1/2$. Then we consider two different opinion update processes based on the selection of interactive targets, as shown in Fig. 1: (i) SD process: At time t, individual i updates his or her opinion σ_i by computing the dominant opinion of the neighbors with whom he or she actively interacts. So $|\Omega_i(t)| = m$. (ii) UD process: At time t, individual i updates his or her opinion σ_i by computing the dominant opinion of both actively interacted and passively interacted neighbors. So $|\Omega_i(t)| = k_i(t)$, $k_i(t)$ is the instantaneous degree of i at time t. Compared with the SD process, in the UD process, the number of neighbors in $\Omega_i(t)$ is affected by the number of both actively connected and passively connected links, where the actively connected

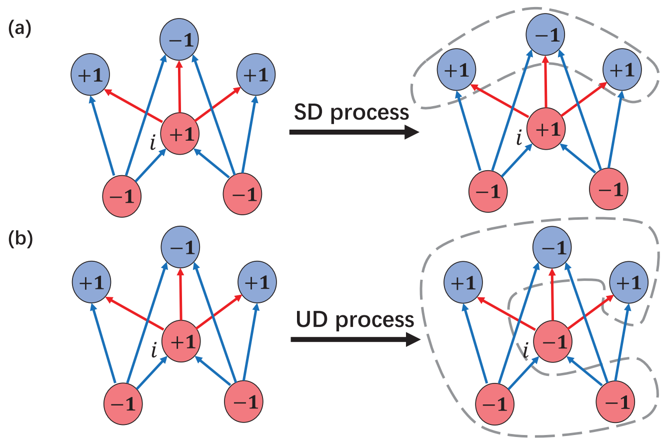

FIG. 1. The opinion dynamics of the MV model with the SD process and the UD process. Red and blue nodes represent active and inactive individuals, respectively. The red edges are generated by active individual i and the blue edges are generated by other active individuals. The individuals within the dashed circle are individual i's interactive neighbors. (a) The opinion flips in the MV model with the SD process. (b) The opinion flips in the MV model with the UD process.

links is fixed m and the passively received links are determined by attractiveness b_i. The increase of passively received links leads to heterogeneity of the number of neighbors.

III. THEORETICAL ANALYSIS

To proceed a mean-field treatment, by assuming that individuals with the same activity a and attractiveness b are statistically equivalent, we define $q^{\pm}_{a,b}(t)$ as the probability that an individual with activity a and attractiveness b is in state ± 1 at time t. The probability that an individual with activity a and attractiveness b is chosen to be connected is proportional to his or her attractiveness, i.e., $bF(a,b)/\langle b\rangle$, where $\langle b\rangle$ is the average attractiveness. We denote $\theta^{\pm}(t)$ as the probability of randomly choosing an individual in state ± 1 at time t. The probabilities $q^{\pm}_{a,b}(t)$ and $\theta^{\pm}(t)$ satisfy the relation

$$\theta^{\pm}(t) = \sum_{a,b} \frac{b}{\langle b\rangle} F(a,b) q^{\pm}_{a,b}(t). \tag{2}$$

Then, for an individual with activity a and attractiveness b, the probability that the dominant opinion of his or her neighbors is ± 1 is defined as $\varphi^{\pm}_{a,b}(t)$, which is given by the cumulative binomial distribution

$$\varphi^{\pm}_{a,b}(t) = \sum_{n=\lceil |\Omega_{a,b}(t)|/2\rceil}^{|\Omega_{a,b}(t)|} \left(1 - \frac{1}{2}\delta_{n,|\Omega_{a,b}(t)|}\right) C^{n}_{|\Omega_{a,b}(t)|} [\theta^{\pm}(t)]^{n}[1-\theta^{\mp}(t)]^{|\Omega_{a,b}(t)|-n}, \tag{3}$$

where $\Omega_{a,b}(t)$ denotes the set of neighbors at time t, $\lceil \cdot \rceil$ is the ceiling function, and $\delta_{i,j}$ is the Kronecker symbol. If $i=j$, $\delta_{i,j}=1$; otherwise, $\delta_{i,j}=0$. $C^n_k = \frac{k!}{n!(k-n)!}$ is a binomial coefficient. Here each term $C^n_{|\Omega_{a,b}(t)|}$ in Eq. (3) calculates the probability of n neighbors in state $+1(-1)$ and $|\Omega_{a,b}(t)|-n$ neighbors in state $-1(+1)$ with associated probabilities $\theta^{+}(\theta^{-})$ and $\theta^{-}(\theta^{+})$, respectively. Thus, for an active individual with activity a and attractiveness b, the probability that his or her opinion takes value ± 1 can be expressed as $\psi^{\pm}_{a,b}(t)$, which is given by

$$\psi^{\pm}_{a,b}(t) = (1-f)\varphi^{\pm}_{a,b}(t) + f[1-\varphi^{\mp}_{a,b}(t)], \tag{4}$$

where the first term on the right denotes the probability that an active individual follows the majority rule with probability $1-f$ and the second one accounts for the probability that the active individual follows the minority rule at time t. The dynamical equations that determine the time evolution of the probability $q^{\pm}_{a,b}(t)$ are a function of the switching probability $\psi^{\pm}_{a,b}(t)$. These equations can be deduced by observing one of the following events occurs at each time step. (i) An individual with activity a and attractiveness b in state $+1$ is active. If he or she flips his or her opinion, then the rate at which $q^{+}_{a,b}(t)$ decreases and $q^{-}_{a,b}(t)$ increases will be determined with the probability $aq^{+}_{a,b}(t)[1-\psi^{+}_{a,b}(t)]$. If his or her opinion is not changed, then the rate of $q^{\pm}_{a,b}(t)$ does not change with the probability $aq^{+}_{a,b}(t)\psi^{+}_{a,b}(t)$. (ii) An individual with activity a and attractiveness b in state -1 is active. If he or she flips his or her opinion, then the rate at which $q^{-}_{a,b}(t)$ decreases and $q^{+}_{a,b}(t)$ increases will be determined with the probability $aq^{-}_{a,b}(t)[1-\psi^{-}_{a,b}(t)]$. If his or her opinion is not changed, then the rate of $q^{\pm}_{a,b}(t)$ does not change with the probability $aq^{-}_{a,b}(t)\psi^{-}_{a,b}(t)$.

Therefore, the rate equations for $q^{\pm}_{a,b}(t)$ can be written as

$$\frac{dq^{+}_{a,b}(t)}{dt} = -aq^{+}_{a,b}(t)[1-\psi^{+}_{a,b}(t)] + aq^{-}_{a,b}(t)[1-\psi^{-}_{a,b}(t)], \tag{5}$$

$$\frac{dq^{-}_{a,b}(t)}{dt} = -aq^{-}_{a,b}(t)[1-\psi^{-}_{a,b}(t)] + aq^{+}_{a,b}(t)[1-\psi^{+}_{a,b}(t)]. \tag{6}$$

In stationarity, by setting the change rate of $q^{\pm}_{a,b}(t)$ to zero, we obtain the following relationship:

$$q^{+}_{a,b}(1-\psi^{+}_{a,b}) = q^{-}_{a,b}(1-\psi^{-}_{a,b}). \tag{7}$$

Combining Eq. (7) with the conditions $q^{+}_{a,b}+q^{-}_{a,b}=1$ and $\psi^{+}_{a,b}+\psi^{-}_{a,b}=1$, we get

$$q^{+}_{a,b} = \psi^{+}_{a,b}. \tag{8}$$

This condition is necessary for stationarity. In fact, in stationarity, the probability of an individual in a given state does not change with time. The expected proportion of individuals in state $+1(-1)$ is equivalent to the probability that individuals change their state to $+1(-1)$.

A. The MV model with the SD process

In the MV model with the SD process, only neighbors that the individual actively interacts with are possible to alter his or her opinion. Since an active individual only actively interacts with m neighbors, we have $|\Omega^{\rm sd}_{a,b}(t)| = m$. Observing Eq. (3), we find that the expression of $\psi^{+}_{a,b}(t)$ only depends on the number of actively connected links m, while the joint distribution of activity and attractiveness has no effect on it. Thus,

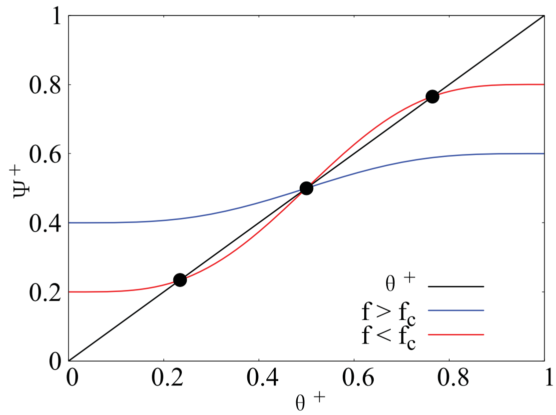

FIG. 2. The steady solutions of $\theta^+ = \psi^+$. When $f < f_c$, $\theta^+ = \psi^+$ has three solutions. One fixed solution is $\theta^+ = \frac{1}{2}$ corresponding to completely disordered state, and the other two represent symmetric ordered state solutions. When $f > f_c$, there is only one solution at $\theta^+ = \frac{1}{2}$.

for $\forall a, b \neq 0$, we set $\varphi^{\pm}_{a,b}(t) = \varphi^{\pm}(t)$ and $\psi^{\pm}_{a,b}(t) = \psi^{\pm}(t)$. By inserting Eq. (8) into Eq. (2), we get

$$\theta^+(t) = \sum_{a,b} \frac{b}{\langle b\rangle} F(a,b)\psi^+(t) = \psi^+(t). \tag{9}$$

In stationary state, when $t \to \infty$, $\frac{d\theta^+}{dt} = 0$ and we set $\theta^+(t) = \theta^+_s$. Since θ^+_s is a stable point, we have $\theta^+_s = \psi^+(\theta_s)$. Let us analyze the relationship between θ^+ and ψ^+. According to Eq. (4), $\psi^+(t)$ is also a function of f. When f is less than the critical value f_c, Eq. (9) has three solutions, as shown in Fig. 2. One is $\theta^+ = \frac{1}{2}$ corresponding to a disorder phase, and the other two correspond to the symmetric ordered phases. If $f > f_c$, then there is only one solution at $\theta^+ = \frac{1}{2}$. Therefore, for the MV model with the SD process, the critical noise, defined as $f^{\rm sd}_c$, is determined by the condition that the derivation of ψ^+ with θ^+ equals to one at $f = f^{\rm sd}_c$, i.e.,

$$\left.\frac{d\psi^+}{d\theta^+}\right|_{\theta^+=\frac{1}{2}} = 1. \tag{10}$$

According to Eq. (3) and Eq. (4), after some simple algebra, the expression of critical noise $f^{\rm sd}_c$ reads as

$$f^{\rm sd}_c = \frac{1}{2} - \frac{1}{2^{2-m} m C^{\lceil \frac{m-1}{2}\rceil}_{m-1}}. \tag{11}$$

Using Stirling's approximation for $m \to \infty$, $C^{\lceil \frac{m-1}{2}\rceil}_{m-1} \to 2^{m-1}/\sqrt{m\pi/2}$, Eq. (11) can be simplified as

$$f^{\rm sd}_c \approx \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{m}}. \tag{12}$$

From Eq. (12), we see that $f^{\rm sd}_c$ only depends on the number of actively connected links m and increases with it. If $m = N-1 \to \infty$, then we have $f^{\rm sd}_c = \frac{1}{2}$. By observing Eq. (12), we find that the expression of $f^{\rm sd}_c$ is similar to that on the random regular (RR) networks [15], where all individuals have the same degree k and the critical noise only depends on it.

B. The MV model with the UD process

In the MV model with the UD process, active individuals interact with both actively interacted and passively interacted neighbors simultaneously, and we have $|\Omega^{\rm ud}_{a,b}(t)| \approx k_{a,b}$, where $k_{a,b}$ is the instantaneous degree [36], read as

$$\begin{aligned} k_{a,b} &= m + \frac{N\langle a\rangle mb}{N\langle b\rangle} \\ &= m + \frac{m\langle a\rangle b}{\langle b\rangle}, \end{aligned} \tag{13}$$

which is independent of time and consists of two parts, i.e., the number of actively connected links m and the number of passively connected links $\frac{m\langle a\rangle b}{\langle b\rangle}$ [36]. Since $k_{a,b}$ does not depend on a, we simplify the expression $k_{a,b}$ as $k_{\cdot,b}$. Observing Eq. (3) and Eq. (4) that the variable $|\Omega^{\rm ud}_{a,b}(t)| \approx k_{\cdot,b}$ only depends on attractiveness b and the number of actively connected links m, we set $\varphi^{\pm}_{a,b}(t) = \varphi^{\pm}_{\cdot,b}(t)$ and $\psi^{\pm}_{a,b}(t) = \psi^{\pm}_{\cdot,b}(t)$.

By inserting Eq. (8) into Eq. (2), we get

$$\theta^+(t) = \sum_{a,b} \frac{b}{\langle b\rangle} F(a,b)\psi^+_{\cdot,b}(t). \tag{14}$$

Like the MV model with the SD process, when the system reaches stationarity, the critical noise $f^{\rm ud}_c$ can be solved by setting $\frac{d\theta^+}{dt} = 0$ at $\theta^+ = \frac{1}{2}$, that is,

$$\sum_{a,b} \frac{b}{\langle b\rangle} F(a,b)\left(1 - 2f^{\rm ud}_c\right)2^{1-2k_{\cdot,b}} k_{\cdot,b} C^{\lceil k_{\cdot,b}/2\rceil}_{k_{\cdot,b}-1} = 1. \tag{15}$$

After some simple algebra, we can express the critical noise $f^{\rm ud}_c$ as

$$f^{\rm ud}_c = \frac{1}{2} - \frac{1}{2}\frac{1}{\sum_{a,b}\frac{b}{\langle b\rangle}F(a,b)2^{1-k_{\cdot,b}}k_{\cdot,b}C^{\lceil (k_{\cdot,b}-1)/2\rceil}_{k_{\cdot,b}-1}}. \tag{16}$$

Due to the fact that $k_{\cdot,b} \approx m + \frac{m\langle a\rangle b}{\langle b\rangle}$ and if $k \to \infty$, $C^{\lceil \frac{k-1}{2}\rceil}_{k-1} \to 2^{k-1}/\sqrt{k\pi/2}$, we can obtain the approximate expression of $f^{\rm ud}_c$,

$$f^{\rm ud}_c \approx \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{(\langle b\rangle)^{3/2}}{\sqrt{m}\sum_{a,b} bF(a,b)\sqrt{\langle b\rangle + \langle a\rangle b}}, \tag{17}$$

from which we see that the critical noise $f^{\rm ud}_c$ is related to the number of actively connected links m, the average activity $\langle a\rangle$, and the attractiveness distribution. When we take the attractiveness of individuals in the original AD models as a constant value, i.e., $F(b) = b_0$, the instantaneous degree becomes $k_{a,b} = m + m\langle a\rangle$ and the critical noise can be expressed as

$$f^{\rm ud}_c \approx \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{m + m\langle a\rangle}}, \tag{18}$$

from which we find that the critical noise is dependent on $k_{a,b} = m + m\langle a\rangle$, where m is the number of actively connected links and $m\langle a\rangle$ is the expectation of the number of passively received links. Like the MV model with the SD process, the effect of instantaneous degree $k_{a,b}$ on the AD model is similar to that of connectivity degree in RR networks [15], which represents the number of neighbors interacting with an active individual. Hence, the complicated expression of the critical noise is due to the heterogeneity of attractiveness on temporal networks.

IV. SIMULATION RESULTS

In this section, we present the exact solutions of critical noise in the MV model with the SD and the UD process, respectively. We perform Monte Carlo (MC) simulations as well as the equation iterations to verify the correctness of the theoretical analysis. To do so, we generate temporal networks with the ADA models, where activity and attractiveness are independent, i.e., $F(a,b) = F(a)F(b)$ and they both follow the pow-law distribution $F(a) \sim a^{-\gamma_a}$, $F(b) \sim b^{-\gamma_b}$ as empirically observed in Refs. [16,35]. Each simulation starts with a completely ordered state, that is, for $\forall i \in N$, $\sigma_i(t=0) = +1$. At each MC step, first, each active node is randomly chosen once and then flips opinion with the probability according to the process described in Eq. (1).

Our main observation measurement is the "magnetization" M, which represents the consensus level of the system. First, the instantaneous magnetization M_t at time t is calculated as

$$M_t = \left| \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t) \right|, \tag{19}$$

where N is the network size and $\sigma_i = \pm 1$. Then M can be expressed as the temporal average of M_t,

$$M = [\overline{M_t}],\ t \in (t_{\text{final}/2}, t_{\text{final}}), \tag{20}$$

where $\overline{\cdots}$ represents the temporal average value calculated for the interval $(t_{\text{final}/2}, t_{\text{final}})$ in stationarity. t_{final} is the end time of the simulation, and $[\cdots]$ is the aggregate average of different simulation results and network structures. If all individuals are in the same opinion state, then we have $M = 1$. If the individuals are divided into two opposite populations of equal size, where the majority opinion does not exist, then we have $M = 0$. Hence M can be regarded as the consensus level of the system. To obtain the critical noise f_c for the numerical exact solution, we need to calculate the Binders fourth-order cumulant U [16,37], which is defined as

$$U = 1 - \frac{[\overline{M_t^4}]}{3[\overline{M_t^2}]^2}. \tag{21}$$

The critical noise f_c is calculated by detecting the critical point $f = f_c$, where the curves $U = U(f)$ obtained for distinct networks sizes N intercept each other. In the simulations, f_c is obtained by three distinct network sizes with $N = 1000, 5000, 10\,000$. For example, as shown in Fig. 3, in the SD process, the curves intercept each other at critical noise $f_c^{\text{sd}} = 0.167$ for $m = 3$.

A. The MV model with the SD process

First, we explore the influence of the SD process on the MV model on the ADA models. Since the activity and attractiveness joint distribution does not affect the consensus level at stationarity, we use a normalized activity and attractiveness distribution with a power-law form, $F(a) \sim a^{-2.1}$ and $F(b) \sim b^{-2.1}$ with $\langle a \rangle = \langle b \rangle = 0.2$. Without specification, the network size is set as $N = 10^4$.

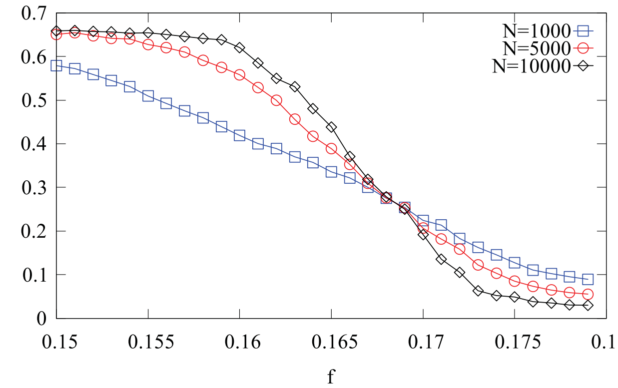

FIG. 3. In the SD process, Binder's fourth-order cumulant U on the ADA models with $m = 3$. The network sizes are 1000, 5000, and 10 000, and the results are averaged over 20 network configurations. The critical point is estimated as f_c^{sd}=0.167.

Figure 4 shows the relationship between magnetization M and noise parameter f in stationarity for different m. The dots denote the MC simulation results, and the solid curves are the solutions by iterating equations Eqs. (5) and (6). Triangle symbols represent the critical noise obtained by Eq. (12). The high consistency between simulation results and the iterative results shows the correctness of the theoretical analysis. Furthermore, when the value of noise parameter f is relatively small, the system is in an ordered state, where M tends to 1. With the increase of noise parameter f, the magnetization M will continuously decrease to zero near the critical value f_c^{sd}. Then, in the region of $f > f_c^{\text{sd}}$, there is no dominant opinion in the system. State ± 1 can be obtained with equal probability 0.5.

In the inset of Fig. 4, we show the errors between the MC simulation results and the theoretical analysis from Eq. (12).

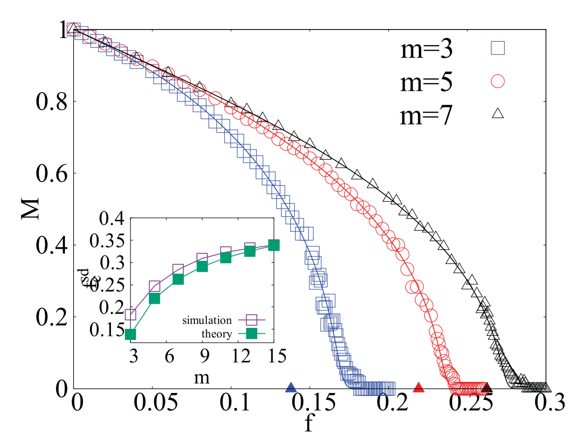

FIG. 4. The magnetization M of the MV model with the SD process on the ADA models with $m = 3, 5, 7$. M is obtained from 10 network configurations for the MC simulation. Dots are the MC simulation results and solid lines are the numerical solutions from Eqs. (5) and (6). Inset: The critical value of noise intensity f_c^{sd} increases monotonically with m. "Simulation" represents the MC simulation results and "theory" represents the theoretical results from Eq. (11). Other parameters are set as $F(a) \sim a^{-2.1}$ and $F(b) \sim a^{-2.1}$ with $\langle a \rangle = 0.2$ and $\langle b \rangle = 0.2$.

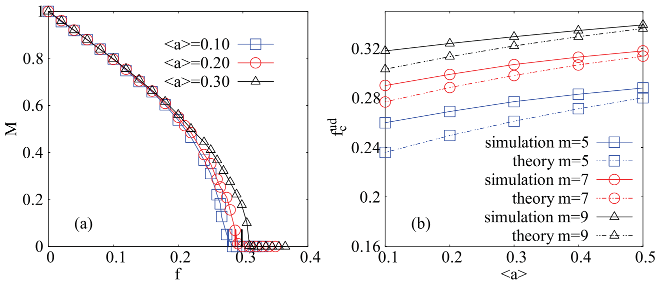

FIG. 5. The impacts of average activity $\langle a\rangle$ and the number of actively connected edges m on the critical noise f in the MV model with the UD process. (a) The magnetization M is plotted as a function of noise intensity f for distinct values of $\langle a\rangle$. Dotlines represent the MC simulations. (b) The critical noise f_c^{ud} versus the $\langle a\rangle$ for distinct values of m. "Simulation" represents the MC simulation results and "theory" represents the theoretical results from Eq. (17). Other parameters are set as $F(a) \sim a^{-2.1}$ and $F(b) \sim b^{-2.1}$ with $\langle b\rangle = 0.2$.

We see that with the increase of m, the critical noise threshold f_c^{sd} increases and the errors decreases. It indicates that actively interacting with more people promotes the consensus in the system. When m is relatively large, the instantaneous network becomes denser and further increase in m has little influence on the critical noise f_c^{sd}, i.e., $m = N - 1 \to \infty,\ f_c^{\mathrm{sd}} \to 0.5$.

B. The MV model with the UD process

In this subsection, we show the phase transitions in the MV model with the UD process. Active individuals interact with both actively connected and passively connected neighbors at the same time, which leads to the variety of the number of different interactive individuals.

First, we illustrate the effect of f under different activity averages $\langle a\rangle$ on M. Here we perform simulations on the ADA models with $F(a) \sim a^{-2.1}$, $F(b) \sim b^{-2.1}$, and $m = 7$. In Fig. 5(a), we show the magnetization M as a function of f for several distinct $\langle a\rangle$. By increasing the noise parameter f, the magnetization M continuously decreases to zero near the critical value f_c^{ud}. Comparing the results of different $\langle a\rangle$, we find that smaller $\langle a\rangle$ can decrease the critical noise required to reach the disordered state to a certain extent. As predicted by the theory analysis, the critical noise f_c^{ud} increases with $\langle a\rangle$, which indicates that more individuals who actively interact with each other will promote the consensus, thus leading to a small critical noise f_c^{ud}. More details are shown in Fig. 5(b), where we present the relationship between f_c^{ud} and $\langle a\rangle$ for different m. Like the SD process, f_c^{ud} also increases with m, and the errors between simulation results and theoretical results become less with the increase of $\langle a\rangle$ and m. When interacting with more individuals at each time, the results of both the SD process and the UD process tend to be a consensus state. It indicates that more active interactions between individuals will promote the consensus in the network.

Since in the original AD model it does not take into account the heterogeneity of individuals' attractiveness [27], we can take the attractiveness of individuals in the AD models as a constant value, i.e., $F(b) = b_0$. As shown in Fig. 6(a), compared with the AD model with constant b_0, nonuniform attractiveness leads to a larger critical noise f_c^{ud}. This is because the heterogeneity of attractiveness strengthens the influence of individuals who have large attractiveness. Thus, when updating opinions, neighboring individuals have to consider the opinions of these influential individuals. Furthermore, compared with the results with $\gamma_b = 2.5$, the network with higher attractiveness heterogeneity promotes a higher consensus level ($\gamma_b = 2.1$). More detailed results are presented in Fig. 6(b), which shows that the heterogeneity of attractiveness promotes the transformation of magnetization near phase transitions under different $\langle a\rangle$ and the errors between simulation results and theoretical results become smaller with the increase of $\langle a\rangle$ and γ_b. γ_b determines the heterogeneity of attractiveness distribution and a larger γ_b implies that less heterogeneous distribution of attractiveness achieves a lower critical noise f_c^{ud}. It indicates that in reality, improving the credibility of the media and increasing the official influence can effectively guide the trend of public opinion and avoid chaos.

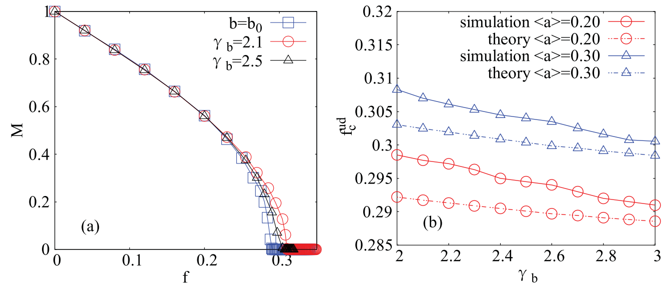

FIG. 6. The impact of attractiveness heterogeneity γ_b on the critical noise f in the MV model with the UD process. Larger γ_b represents that the network has lower attractiveness heterogeneity. (a) The magnetization M as a function of the noise f for several values of γ_b with $\langle a\rangle = 0.2$. Dotlines represent the MC simulations. (b) The critical noise f_c^{ud} versus γ_b for distinct values of $\langle a\rangle$. "Simulation" represents the MC simulation results and "theory" represents the theoretical results from Eq. (17). Other parameters are set as $F(a) \sim a^{-2.1}$ with $\langle b\rangle = 0.2$ and $m = 7$.

In addition, the comparison results of the SD process and the UD process are shown in Fig. 7. It highlights the important notable feature that the UD process (solid points) tends not only to higher consensus level and larger M than the SD

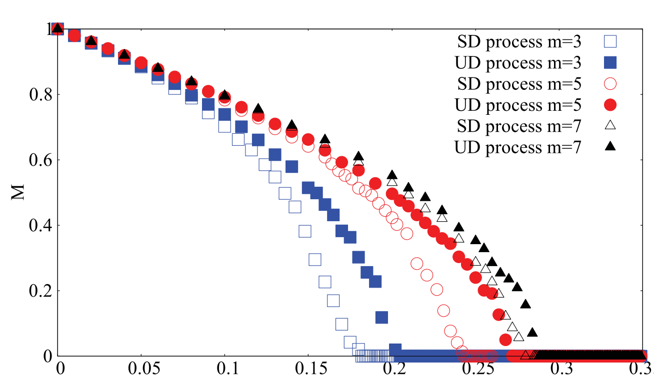

FIG. 7. Comparison of the SD process and the UD process of the MV model. Dots are the MC simulation results. The parameters are set as $F(a) \sim a^{-2.1}$, $F(b) \sim b^{-3.1}$ with $\langle a\rangle = 0.2$ and $\langle b\rangle = 0.2$,

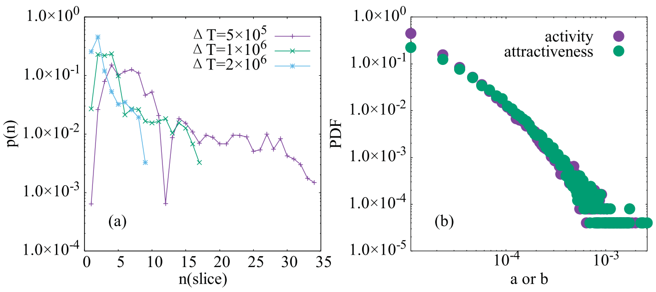

FIG. 8. Statistical properties of the CollegeMsg Data Set. (a) Fraction of active interaction for different time slice ΔT. The proportion of interaction is more than 0.01 at most cumulative time steps. (b) Cumulative distributions of activity and attractiveness for the CollegeMsg Data Set.

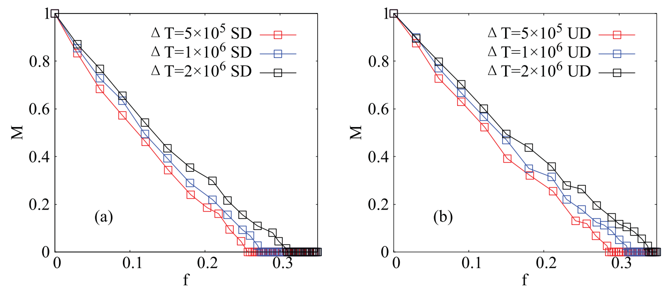

FIG. 9. The magnetization M versus the noise parameter f for different time slice ΔT. (a) The SD process; (b) The UD process.

process (hollow points) but also to hold the consensus level up against larger noise.

All the above simulation results show that the critical noise required to reach the disordered state is higher for the UD process. In the SD process, the consensus level only depends on m, since individuals only consider actively connected neighbors' opinions, while in the UD process, since active individuals interact with both actively connected and passively connected neighbors simultaneously, the number of passively received neighbors is affected by attractiveness distribution. Consequently, the consensus level M is controlled by the number of actively connected links and the hub individuals with high attractiveness.

C. Real networks

In reality, active individuals interact with different individuals on networks. As a next step, we study the process of opinion dynamics on the real networks with the CollegeMsg Data Set. The number of nodes is 1899, representing the website individuals. The number of time-varying edges is 59 836, representing the real-time replies between individuals. Multiple edges between two nodes are possible and denote multiple interactions. The data are available in Ref. [38].

The contact sequences can be represented by the triples $\chi[i, j, t]$, indicating that individual i actively interacts with individual j at time t. The activity and attractiveness of individiual i can be calculated by $a_i = c_{i,\mathrm{out}}/\sum_j c_{j,\mathrm{out}}$ and $b_i = c_{i,\mathrm{in}}/\sum_j c_{j,\mathrm{in}}$, where $c_{j,\mathrm{in}}$ and $c_{j,\mathrm{out}}$ represent the total number of actively and passively connected edges in the whole sequence, respectively. The distribution of activity and attractiveness is shown in Fig. 8(b).

Since the finite duration of the datasets is not sufficient to reach stationarity of opinion dissemination, we employ the method of sequence replication to solve this problem [35]. The contact sequences are repeated on a regular basis, defining a new extended characteristic function as $\chi_e^{\mathrm{SRep}}(i, j, t) = \chi(i, j, t \bmod T)$, where T is the total duration of the contact sequence.

According to the time stamp of the original interaction sequence, since the connectivity of the network is too low to update the whole opinions, we use the method of equal time interval division to slice the original sequence and integrate it with all the time series in the slice into a time sequence snapshot. The size of the time interval determines the number of active individuals and the number of edges during the time slice. In order to explore the influence of the time interval, we set the size of the time interval as follows: $\Delta T = 1 \times 10^5$, 1×10^6, and 2×10^6. As shown in Fig. 8(a), with the increase of ΔT, the number of active individuals in the time accumulation network increases accordingly. The proportion of average active individuals is more than 0.01. In addition, as shown in Fig. 8(b), we find that the activity distribution and the attractiveness distribution are heterogeneous, which indicates that individuals in the network have different tendencies of active interaction and passive interaction.

Results of numerical simulations of the MV model with the SD process and the UD process are summarized in Fig. 9. Both the SD process and the UD process display the continuous transitions. We note that the magnetization M is higher in the UD process than in the SD process. Significantly, in Fig. 9(a), different critical noise f_c^{sd} on the SD process are observed. In the previous analysis of the MV model on the ADA models, we conclude that interacting with more people simultaneously will promote the consensus level in the system. A larger ΔT implies more individuals and more interactions occur simultaneously, which leads to the increase of magnetization M and the critical noise f_c^{sd}. As shown in Fig. 9(b), we obtain similar result for the MV model with the UD process. It is worth mentioning that f_c^{ud} for the UD process is larger than f_c^{sd} for the SD process under the same configuration. The UD process is a two-way interaction pattern which promotes the interactions between individuals leading to various number of neighbors for different individuals.

Overall, the results of the MV model on real networks are similar to the results on the artificial networks. The SD process is an actively interacting mode where individuals only actively achieve information, such as review sites, Yelp, and resource sharing websites. We note that consensus can be prevented by employing the SD process to avoid the adverse effects of popular opinion, such as the spread of fake news. For the UD process, considering the influence of heterogeneity of attractiveness, we can improve the influence and appeal of official media to guide the trend of public opinion in a timely manner.

V. CONCLUSION

In this work, based on the observation that individuals update their opinions by considering the neighbors with whom they actively and passively interact, we formulate the MV model with the SD process and the UD process on the ADA models, respectively. In fact, in the SD process, the probability that an active individual alters his or her state only depends on actively connected neighbors, while it is dependent on both actively connected and passively connected neighbors in the UD process.

With the heterogeneous mean-field method, we obtain the critical noise for the SD process and the UD process, respectively. In the SD process, the consensus level only depends on the number of links that active individuals send. More links sent by active individuals will promote the consensus level. In the UD process, individuals update their opinion by considering both the neighbors that they actively connect with and those that they passively received, determined by the attractiveness. Thus, the critical noise parameter is determined by the number of active links, average activity, and the attractiveness distribution. A heterogeneous attractiveness distribution leads to the existence of influential individuals whose opinion will promote the consensus. Similarly, more active individuals also promote the consensus in the network.

Compared with previous work on static networks, our study aims at understanding the role of dynamical interaction between individuals in the mode of opinion dynamics. In this respect, we consider two different modes of interactions and obtain continuous phase transitions, which is different from those on static networks [14]. In the SD process, the critical noise is similar to that of RR network, while in the UD process, under some special conditions, such as homogeneous attractiveness, the critical noise can also be recovered to a similar result as in RR networks. However, several questions remain open from the perspective of the current opinion models and the temporal networks in general. Extensions of the present study to networks structures with more real social network characteristics are possible, for example, the number of neighbors that active individuals actively connect with is different. In addition, models for opinion dynamics adapted to different social situations deserves further study. We hope this work could provide new inspiration for all these diverse directions.

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