Mean first-passage time for random walks on random growth tree networks

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As opposed to most previous works focusing on random walks on deterministic tree networks, in this paper, we pay more attention on random growth tree networks. Specifically, we propose two distinct types of random growth tree networks $\mathcal{T}(1;t,p)$ and $\mathcal{T}(2;t,p)$ where t represents time step and p is a probability parameter ($0 \leqslant p \leqslant 1$). Tree $\mathcal{T}(1;t,p)$ is iteratively built in a fractal manner; however, $\mathcal{T}(2;t,p)$ is generated using a nonfractal operation. Then we study random walks on tree network $\mathcal{T}(i;t,p)$ ($i = 1, 2$) and derive the analytical solution to mean first-passage time $\langle \mathcal{F}_{\mathcal{T}(i;t,p)} \rangle$. The results suggest that two growth ways have remarkably different influence on parameters $\langle \mathcal{F}_{\mathcal{T}(i;t,p)} \rangle$. More precisely, the fractal-growth manner makes topological structure of tree network more loose than the nonfractal one and thus increases drastically mean first-passage time in the large graph limit. Finally, we extensively conduct experimental simulations, and the results demonstrate that computer simulations are in strong agreement with the theoretical analysis.

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I. INTRODUCTION

Nowadays, complex network, as a powerful tool, has attracted increasing attention from various disciplines ranging from applied mathematics and theoretical computer science to statistical physics and even to social science [1]. One of the most important reasons for this is that a great variety of complex systems, such as scientific collaboration networks [2], metabolic networks [3], protein-protein interaction networks [4] and citation networks [5], can be abstractly interpreted as complex networks. Based on this, complex networks help one better understand some mechanisms and structural characteristics behind those various systems. Roughly speaking, there are two main streams in the realm of complex networks. The first aims at uncovering generation mechanisms describing how complex systems evolve over time [6]. The goal of the other is to study dynamics taking place on systems and then to understand how the underlying structures affect dynamical behaviors occurring on them [7,8]. Studied examples contain the spread of epidemic disease in the crow [9], the spread of rumor on social networks [10], virus propagation on computer networks [11], and so forth. In general, the examples above may be well described by random walks, which are the widely studied dynamics on corresponding networks [12,13].

It is well known that there is a long history of discussions about random walks on networks in the literature [14–17]. Accordingly, there have been many theoretical assumptions developed for this kind of dynamical behavior on complex networks in the past decades, such as the biased random-walk manner [18] and the maximal entropy random-walk way [19]. Note that in this paper, we will concentrate mainly on discrete-time unbiased random walks. In this case, a walker will hop from its current position u to one in its neighboring set with probability $1/k_u$, where k_u is degree of vertex u. The crucial issue to answer in research of this type is to measure some structural parameters closely associated with random walks. The most fundamental of various parameters is first-passage time that is defined to be the expected time taken by a walker to first reach its destination vertex. Based on this, one has the ability to measure mean first-passage time for a network in question (defined in detail later).

In fact, more effort has been devoted to studying first-passage time for random walks on networks in the past [19–26]. For instance, the sum of first-passage time $F_{u\to v}$ from vertex u to v and first-passage time $F_{v\to u}$ from vertex v to u on network is proved to be two times the product between edge number of network and effective resistance distance $\mathcal{R}_{uv}$ with the help of both spectral technique and electrical network [23]. Similarly, an analog associated with tree networks is derived by means of a combinatorial method [27]. In particular, random walks on a large number of networks with interesting structural characteristics such as scale-free feature and small-world property have been in depth discussed [28,29]. More recently, some tree networks with intriguing

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structural properties (for instance, fractal) have been widely considered as well [29–33]. Nonetheless, almost all reported tree networks are deterministic. As will be stated later, however, the goal of this paper is to study an ensemble of random growth tree networks. It will be clear to the eye that some previous methods suitable for deterministic tree networks cannot be adequately employed to analyze random growth tree networks. Therefore, many other effective techniques need to be developed in order to address this issue. Inspired by this, we study random walks on random growth tree networks built in this paper via establishing new methods. More importantly, the techniques proposed in the subsequent discussions can be used to more conveniently deal with some published deterministic tree networks than those early methods [34,35]. In the meantime, some previous results can be naturally covered by the formulations derived in this paper, as demonstrated later.

The rest of this paper is organized as follows. In Sec. II, we will introduce some helpful notations, including matrix representation of network, Wiener index on network, and graphic operations. Two random growth tree networks $\mathcal{T}(1;t,p)$ and $\mathcal{T}(2;t,p)$ are proposed in Sec. III, and then some fundamental structural properties are discussed in detail. Next, the core of all our work is shown in Sec. IV. Specifically, we analytically obtain the closed-form expressions of mean first-passage time for random walks on the proposed tree networks. Following the theoretical demonstrations above, we perform a great number of experimental simulations, and the results firmly imply that computer simulations are in strong agreement with theoretical analysis. Finally, we conclude with a brief description about this work in Sec. V.

II. BASIC NOTATIONS

A graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is an ordered pair $(\mathcal{V},\mathcal{E})$ consisting of a set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of edges running between vertices. The total number of vertices is denoted by $|\mathcal{V}|$, and $|\mathcal{E}|$ represents the edge number. A connected acyclic graph is called a tree. Equivalently, there is only a path between arbitrary two vertices in a tree. Notation $[a,b]$ indicates a collection of integers $\{a,a+1,\ldots,b\}$. Throughout this paper, all the discussed graphs are simple and connected, that is, without multiedges and loops. Note also that the two terms graph and network are used indistinctly.

Matrix representation of graph [36].

It is conventional to interpret a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ by its adjacency matrix $\mathbf{A}_{\mathcal{G}}=(a_{ij})$ in the following form:

$$a_{ij}=\begin{cases}1, & \text{vertex } i \text{ is adjacent to } j\\ 0, & \text{otherwise.}\end{cases}$$

This thus contains some basic information about a graph itself, such as the degree k_i of vertex i is equal to $k_i=\sum_{j=1}^{|\mathcal{V}|}a_{ij}$. Accordingly, the diagonal matrix, denoted by $\mathbf{D}_{\mathcal{G}}$, may be defined as follows: The ith diagonal entry is k_i, while all nondiagonal elements are zero, i.e., $\mathbf{D}_{\mathcal{G}}=\mathrm{diag}[k_1,k_2,\ldots,k_{|\mathcal{V}|}]$.

Wiener index [37].

Given a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, the distance between vertices u and v $(u,v\in\mathcal{V})$, referred to as d_{uv}, is the number of edges on a shortest path joining u to v. Then the well-known *Wiener index* for a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, denoted by $\mathcal{W}_{\mathcal{G}}$, can be expressed as

$$\mathcal{W}_{\mathcal{G}}=\sum_{u,v\in\mathcal{V}}d_{uv}=\frac{1}{2}\sum_{u\in\mathcal{V}}\sum_{v\in\mathcal{V}}d_{uv}, \tag{1}$$

where u is distinct with v. Note that the distance of the same pair of vertices is only considered one time.

Random walks on network [38].

Consider random walks on a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, the *first-passage time* for a walker starting out from source vertex u to visit destination v, defined as $F_{u\to v}$, is equal to the expected time taken by the walker to first reach v. Accordingly, the quantity called *mean first-passage time* for graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ as a whole, denoted by $\langle\mathcal{F}_{\mathcal{G}}\rangle$, can be clearly written as

$$\langle\mathcal{F}_{\mathcal{G}}\rangle=\frac{1}{|\mathcal{V}|(|\mathcal{V}|-1)}\sum_{u\in\mathcal{V}}\sum_{v(\neq u)\in\mathcal{V}}F_{u\to v}. \tag{2}$$

If we let λ_i $(i\in[2,|\mathcal{V}|])$ represent all the nonzero eigenvalues of Laplacian matrix, $\mathbf{L}_{\mathcal{T}}=\mathbf{D}_{\mathcal{T}}-\mathbf{A}_{\mathcal{T}}$, of a tree $\mathcal{T}(\mathcal{V},\mathcal{E})$, then Eq. (2) can be rewritten as

$$\langle\mathcal{F}_{\mathcal{T}}\rangle=2\sum_{i=2}^{|\mathcal{V}|}\frac{1}{\lambda_i}. \tag{3}$$

More generally, the Kirchhoff index $\mathcal{R}_{\mathcal{G}}$ can be defined in terms of the $|\mathcal{V}|-1$ nonzero eigenvalues of Laplacian matrix $\mathbf{L}_{\mathcal{G}}$ for network $\mathcal{G}(\mathcal{V},\mathcal{E})$ as follows:

$$\mathcal{R}_{\mathcal{G}}=2|\mathcal{V}|\sum_{i=2}^{|\mathcal{V}|}\frac{1}{\lambda_i}. \tag{4}$$

Also, the mean effective resistance over all vertex pairs in $\mathcal{G}(\mathcal{V},\mathcal{E})$, also referred to as network criticality, is viewed as

$$\langle\mathcal{R}_{\mathcal{G}}\rangle=\frac{\mathcal{R}_{\mathcal{G}}}{|\mathcal{V}|(|\mathcal{V}|-1)}=\frac{2}{|\mathcal{V}|-1}\sum_{i=2}^{|\mathcal{V}|}\frac{1}{\lambda_i}. \tag{5}$$

This quantity quantifies the robustness of network $\mathcal{G}(\mathcal{V},\mathcal{E})$: The smaller the value $\langle\mathcal{R}_{\mathcal{G}}\rangle$, the more robust the network $\mathcal{G}(\mathcal{V},\mathcal{E})$.

Graphic operations.. For a given edge uv, one can introduce an operation that is viewed as *n-order fractal operation* [39]. Specifically speaking, one inserts a star with n leaves, also called an n-star, into edge uv. An illustrative example is plotted in Fig. 1(a). Analogously, one may connect n new vertices to a given vertex u, resulting in a star with n leaves, which is referred to as *n-order vertex operation* as shown in Fig. 1(b).

It should be mentioned that, in the following, we are particularly interested in the simplest connected graph, namely a tree. In fact, a tree, as a fundamental model, has been used in various applications. For example, the famous T-graph is used to study fractal features encountered in the physics community [39].

III. RANDOM GROWTH TREE NETWORKS $\mathcal{T}(i;t,p)$

In this section, we will introduce two different families of random growth tree networks, $\mathcal{T}(i;t,p)$. Specifically, $\mathcal{T}(1;t,p)$ is built based on an n-order fractal operation, and

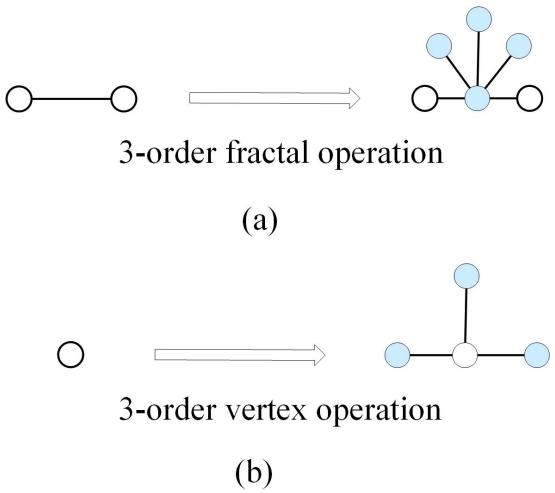


FIG. 1. The diagram of two graphic operations. Panel (a) shows 3-order fractal operation implemented on an edge. The other, i.e., panel (b), describes 3-order vertex operation implemented on a vertex.

$\mathcal{T}(2;t,p)$ is generated using an n-order vertex operation. Parameters t and p denote the time step and a given probability value, respectively. Note that this section is divided into two subsections, one for construction of tree network $\mathcal{T}(1;t,p)$ and the other for tree network $\mathcal{T}(2;t,p)$. At the same time, some fundamental structural properties on both types of tree network will be studied analytically.

A. Tree networks $\mathcal{T}(1;t,p)$

The concrete construction of a random growth tree network $\mathcal{T}(1;t,p)$ is shown in the following algorithm.

Algorithm 1

At $t=0$, an arbitrary tree $\mathcal{T}$ with N vertices is chose as the seed, which is viewed as $\mathcal{T}(1;0,p)$ for convenience.

At $t=1$, new tree $\mathcal{T}(1;1,p)$ is created from $\mathcal{T}(1;0,p)$ by applying either n-order fractal operation on each edge in $\mathcal{T}(1;0,p)$ with probability p or m-order fractal operation on the same edge with probability $1-p$.

At $t\geqslant 2$, the next tree $\mathcal{T}(1;t,p)$ is generated based on tree $\mathcal{T}(1;t-1,p)$ in a similar in spirit manner as stated above.

Obviously, our tree $\mathcal{T}(1;t,p)$ will become deterministic when $m=n$. In particular, if the parameters m and n are supposed to equal 1, then tree $\mathcal{T}(1;t,p)$ is the well-known T-graph [39]. As is known, a T-graph has fractal dimension $d_f=\ln 3/\ln 2$. Analogously, we can find the fractal dimension of tree $\mathcal{T}(1;t,p)$ to equal $\ln[pn+(1-p)m+2]/\ln 2$ with respect to probability.

After t time steps, the total number $|\mathcal{V}(1;t,p)|$ of vertices in tree $\mathcal{T}(1;t,p)$ is given by

$$|\mathcal{V}(1;t,p)|=(N-1)[2+m+p(n-m)]^t+1. \tag{6}$$

B. Tree networks $\mathcal{T}(2;t,p)$

Here we present Algorithm 2 for constructing a random growth tree network $\mathcal{T}(2;t,p)$.

Algorithm 2

At $t=0$, an arbitrary tree $\mathcal{T}$ with N vertices is selected as the seed, which is viewed as $\mathcal{T}(2;0,p)$ for our purposes.

At $t=1$, new tree $\mathcal{T}(2;1,p)$ is obtained from $\mathcal{T}(2;0,p)$ by applying either n-order vertex operation on each vertex in $\mathcal{T}(2;0,p)$ with probability p or m-order vertex operation on the same vertex with probability $1-p$.

At $t\geqslant 2$, the next tree $\mathcal{T}(2;t,p)$ is constructed based on tree $\mathcal{T}(2;t-1,p)$ in a similar manner to that mentioned above.

Clearly, our tree $\mathcal{T}(2;t,p)$ will also become deterministic when $m=n$. After t time steps, tree $\mathcal{T}(2;t,p)$ has the total number $|\mathcal{V}(2;t,p)|$ of vertices,

$$|\mathcal{V}(2;t,p)|=N[1+m+p(n-m)]^t. \tag{7}$$

To make further progress, using the concept of cumulative degree distribution, the probability $P_{\rm cum}(k)$ of selecting at random vertex with degree no less than k is given by

$$P_{\rm cum}(k_i\geqslant k)=\frac{\sum_{k_i\geqslant k}N_{k_i}}{|\mathcal{V}(2;t,p)|}\approx[1+m+p(n-m)]^{i-t}, \tag{8}$$

where we denote by N_{k_i} the total number of vertices having degree equal to k_i.

Substituting quantity $k_i=1+(t-i-1)[pn+(1-p)m]$ into the above equation yields

$$P_{\rm cum}(k_i\geqslant k)\approx\exp(\alpha k),\qquad \alpha=\frac{\ln[1+m+p(n-m)]}{pn+(1-p)m}, \tag{9}$$

which suggests that tree $\mathcal{T}(2;t,p)$ follows exponential degree distribution.

IV. MEAN FIRST-PASSAGE TIME ON TREE NETWORKS $\mathcal{T}(i;t,p)$

This section aims at determining the closed-form solutions to parameters $\langle\mathcal{F}_{\mathcal{T}(i;t,p)}\rangle$ on tree networks $\mathcal{T}(i;t,p)$. As will show shortly, we do not employ a typical method based on Laplacian matrix of network to derive the analytical solutions to mean first-passage time $\langle\mathcal{F}_{\mathcal{T}(i;t,p)}\rangle$ and instead make use of a close connection between $\mathcal{W}_{\mathcal{T}}$ and $\langle\mathcal{F}_{\mathcal{T}}\rangle$ on a tree network, i.e., $\langle\mathcal{F}_{\mathcal{T}}\rangle=2\mathcal{W}_{\mathcal{T}}/|\mathcal{T}|$ [27] to obtain what we want. One of the most important reasons for this is that it is not convenient to build up the corresponding matrix representations for both tree networks proposed above. Thus, the method based on Laplacian matrix can be no longer in force. In what follows, the key step in calculation of parameters is to exactly determine the expression of Wiener index $\mathcal{W}_{\mathcal{T}(i;1,p)}$ of tree $\mathcal{T}(i;t,p)$.

A. Determining $\langle\mathcal{F}_{\mathcal{T}(1;t,p)}\rangle$

As mentioned above, we need to derive the analytical formula for the Wiener index $\mathcal{W}_{\mathcal{T}(1;t,p)}$ of tree $\mathcal{T}(1;t,p)$ in order to successfully obtain parameter $\langle\mathcal{F}_{\mathcal{T}(1;t,p)}\rangle$. Below we elaborate on calculation of Wiener index $\mathcal{W}_{\mathcal{T}(1;t,p)}$ according to construction of tree $\mathcal{T}(1;1,p)$ shown in Algorithm 1.

First, let us focus on the quantity $\mathcal{W}_{\mathcal{T}(1;1,p)}$. It is straightforward to observe that for a given tree $\mathcal{T}$ on N vertices, there will exist two different vertex sets after manipulating the procedure defined in Algorithm 1. One set, denoted by $\mathcal{V}_1^*$, consists of all the old vertices of the seed $\mathcal{T}$, and the other,

called $\mathcal{V}_2^*$, contains those newly created vertices. Clearly, the problem of determining the Wiener index on tree $\mathcal{T}(1;1,p)$ is handled by calculating the following three sub-Wiener indices $\mathcal{W}_i^*$: $\mathcal{W}_1^*$ for the sum over distances of all possible vertex pairs in vertex set $\mathcal{V}_1^*$, $\mathcal{W}_2^*$ for vertex set $\mathcal{V}_2^*$, as well as $\mathcal{W}_3^*$ for arbitrary pair of vertices where one vertex is from vertex set $\mathcal{V}_1^*$ and the other is selected from vertex set $\mathcal{V}_2^*$. In what follows, we will show the detailed calculations of indices $\mathcal{W}_i^*$ defined above in stages.

1. Case 1

For an arbitrary pair of vertices u and v in vertex set $\mathcal{V}_1^*$, the distance d_{uv}^* between them turns out to be equal to two times d_{uv}, which is the corresponding distance for the same vertex pair on seed $\mathcal{T}$ in terms of description in Algorithm 1. This thus gives

$$\mathcal{W}_1^* = 2\mathcal{W}_{\mathcal{T}}, \tag{10}$$

where symbol $\mathcal{W}_{\mathcal{T}}$ represents the Wiener index of the seed $\mathcal{T}$.

2. Case 2

For convenience, vertex set $\mathcal{V}_2^*$ may be further divided into two disjoint sets having smaller size with respect to definition of n-order fractal operation, namely $\mathcal{V}_2^* = \mathcal{V}_2^*(1) \cup \mathcal{V}_2^*(2)$. The subset $\mathcal{V}_2^*(1)$ contains the central vertices of all newly added stars using either n-order fractal operation with probability p or m-order fractal operation with probability $1-p$. The leaf vertices introduced by the fractal operations above constitute the subset $\mathcal{V}_2^*(2)$. From now on, let us derive the closed-form solution to index $\mathcal{W}_2^*$.

First, for each edge in tree $\mathcal{T}$, Algorithm 1 produces a star with either n leaf vertices with probability p or m leaf vertices with probability $1-p$. The contribution from these stars to index $\mathcal{W}_2^*$, denoted by $\mathcal{W}_{2,1}^*$, is given by

$$\mathcal{W}_{2,1}^* = [pn(n-1) + (1-p)m(m-1)] \times (N-1). \tag{11}$$

Note that the first term accounts for the sum of distances over all possible leaf vertex pairs in a star according to probability.

As previously, for an arbitrary path $P_{uv} := u\eta_1\eta_2\ldots\eta_{d_{uv}-1}v$ with length d_{uv} no less than 2 in seed $\mathcal{T}$, we may suppose that vertices x and y as central vertices of n-star and m-star, respectively, are inserted into two end-edges, i.e., $u\eta_1$ and $\eta_{d_{uv}-1}v$, of path P_{uv} using Algorithm 1. As a result, there is a bijection f_1^* from paths P_{xy}^* to P_{uv}^* in the resulting tree $\mathcal{T}(1;1,p)$ as follows:

$$f_1^*: \quad P_{xy}^* \mapsto P_{uv}^*.$$

This is because path P_{uv}^* reduces to path P_{xy}^* via removing two end-edges ux and yv. Clearly, this leads to the following solution:

$$\mathcal{W}_{2,2}^* = \mathcal{W}_1^* - N(N-1), \tag{12}$$

in which $\mathcal{W}_{2,2}^*$ is the sum over distances between arbitrary two central vertices of such type in tree $\mathcal{T}(1;1,p)$.

Next, for an arbitrary leaf vertex x_i in the n-star and leaf vertex y_j in the m-star, the same reasoning as above yields a $[pm+(1-p)n]^2$-regular mapping f_2^* from aspect of probability, which is as follows:

$$f_2^*: \quad P_{x_iy_j}^* \mapsto P_{xy}^*.$$

Such a mapping f_2^* is established based on a fact that path $P_{x_iy_j}^*$ may be induced to path P_{xy}^* through deleting two edges x_ix and y_jy. In this case, we are able to derive the sum of distances for all possible leaf vertex pairs belonging to newly added stars in tree $\mathcal{T}(1;1,p)$, denoted by $\mathcal{W}_{2,3}^*$, which is given by

$$\mathcal{W}_{2,3}^* = \sum_{i=1}^{N-2} 2[pn+(1-p)m]^2(N-1-i) + [pn+(1-p)m]^2\mathcal{W}_{2,2}^*. \tag{13}$$

Last, we need to consider the distance between the central vertex x in an n-star inserted into edge $u\eta_1$ and a leaf vertex y_i in an m-star added into edge $\eta_{d_{uv}-1}v$. The sum of distances between vertex pairs of this kind is referred to as $\mathcal{W}_{2,4}^*$, which follows

$$\mathcal{W}_{2,4}^* = 2[pn+(1-p)m]\mathcal{W}_{2,2}^* + [pn+(1-p)m](N-1)^2. \tag{14}$$

The correctness of Eq. (14) is based on a $2[pn+(1-p)m]$-regular mapping f_3^*,

$$f_3^*: \quad P_{xy_j}^* \mapsto P_{xy}^*, \quad \text{and,} \quad f_3^*: \quad P_{x_iy}^* \mapsto P_{xy}^*.$$

The first term indicates that with the mapping f_3^*, path $P_{xy_j}^*$ may be reduced to path P_{xy}^* by means of removal of edge y_jy. The similar reasoning is also suitable for the second term.

Taken together, we have

$$\mathcal{W}_2^* = \sum_{i=1}^{4} \mathcal{W}_{2,i}^*.$$

The remainder is to calculate solution of index $\mathcal{W}_3^*$ defined above.

3. Case 3

Along the same research line, we can see that there is always a star added into each internal edge of an arbitrary path $P_{uv} := u(\eta_0)\eta_1\eta_2\ldots\eta_{d_{uv}-1}v(\eta_{d_{uv}})$ with length d_{uv} in seed $\mathcal{T}$. Without loss of generality, we denote by w_i the central vertex of star inserted into internal edge $\eta_{i-1}\eta_i$ ($i \in [1, d_{uv}]$). This thus results in a 2-regular mapping f_4^* as follows:

$$f_4^*: \quad P_{uw_{d_{uv}}}^* \mapsto P_{w_1w_{d_{uv}}}^*, \quad \text{and,} \quad f_4^*: \quad P_{w_1v}^* \mapsto P_{w_1w_{d_{uv}}}^*.$$

It is clear to see that one can obtain path $P_{uw_{d_{uv}}}^*$ from path $P_{w_1w_{d_{uv}}}^*$ by connecting an additional edge uw_1 and similarly for the second term. Therefore, the sum of distances between an arbitrary vertex in set $\mathcal{V}_1^*$ and each vertex in set $\mathcal{V}_2^*(1)$, viewed as $\mathcal{W}_{3,1}^*$, is calculated to equal

$$\mathcal{W}_{3,1}^* = N(N-1) + 2\mathcal{W}_{2,2}^*. \tag{15}$$

Analogously, from the probability point of view, there exists a $[pn+(1-p)m]$-regular mapping f_5^* from path $P_{uw_{d_{uv}}}^*$ to path $P_{uw_{d_{uv}}^i}^*$ in which $w_{d_{uv}}^i$ is a leaf vertex connected to the central vertex $w_{d_{uv}}$. That is,

$$f_5^*: \quad P_{uw_{d_{uv}}}^* \mapsto P_{uw_{d_{uv}}^i}^*.$$

This suggests that the summation $\mathcal{W}_{3,2}^*$ over distances for this type of vertex pairs in which one comes from set $\mathcal{V}_1^*$ and

the other is in set $\mathcal{V}_2^*(2)$ can read

$$\mathcal{W}_{3,2}^* = [pn + (1-p)m]\mathcal{W}_{3,1}^* + [pn + (1-p)m]N(N-1). \tag{16}$$

To summarize, we have

$$\mathcal{W}_3^* = \mathcal{W}_{3,1}^* + \mathcal{W}_{3,2}^*.$$

Putting everything together leads to what we look for, i.e.,

$$\begin{aligned}\mathcal{W}_{\mathcal{T}(1;1,p)} &= \mathcal{W}_1^* + \mathcal{W}_2^* + \mathcal{W}_3^* \\ &= 2[pn + (1-p)m + 2]^2\mathcal{W}_{\mathcal{T}} - [pn + (1-p)m + 2]N(N-1) \\ &\quad + \big\{n(n-1)p + m(m-1)(1-p) - 2[pn + (1-p)m]^2 - [pn + (1-p)m]\big\}(N-1).\end{aligned} \tag{17}$$

From this we can easily derive the solution to the Wiener index $\mathcal{W}_{\mathcal{T}(1;t,p)}$ in an iterative fashion. Then the analytical solution to mean first-passage time $\langle \mathcal{F}_{\mathcal{T}(1;t,p)} \rangle$ can be expressed as

$$\mathcal{F}_{\mathcal{T}(1;t,p)\rangle} = \frac{2\big(\Lambda_1^t\mathcal{W}_{\mathcal{T}} - \Lambda_2\sum_{i=0}^{t-1}|\mathcal{V}(1;t-1-i,p)|^2\Lambda_1^i + \Lambda_3\sum_{i=0}^{t-1}|\mathcal{V}(1;t-1-i,p)|\Lambda_1^i + \Lambda_4\sum_{i=0}^{t-1}\Lambda_1^i\big)}{(N-1)[2+m+p(n-m)]^t + 1}, \tag{18}$$

where

$$\Lambda_1 = 2[pn + (1-p)m + 2]^2, \tag{19a}$$

$$\Lambda_2 = pn + (1-p)m + 2, \tag{19b}$$

$$\Lambda_3 = 2 + n(n-1)p + m(m-1)(1-p) - 2[pn + (1-p)m]^2, \tag{19c}$$

$$\Lambda_4 = 2[pn + (1-p)m]^2 + pn + (1-p)m - n(n-1)p + m(m-1)(1-p). \tag{19d}$$

As a case study, tree $\mathcal{T}(1;t,p)$ will be induced into a specific iterative model when the seed is limited to a single edge and parameter p is supposed to equal 1. Such a model has been discussed in depth in Ref. [34]. Consequently, the corresponding solution of mean first-passage time is already analytically evaluated using the method based on Laplacian spectra. For our purposes and convenience, this model is thought of as model $\mathcal{A}_t$ hereinafter. After that, from Eq. (18), we can find that the exact solution of mean first-passage time $\langle \mathcal{F}_{\mathcal{A}_t} \rangle$ on model $\mathcal{A}_t$ is given by

$$\langle \mathcal{F}_{\mathcal{A}_t} \rangle = \frac{2(n+2)^t}{(2n^2+7n+6)[(n+2)^t+1]}\big[(n+1)^2(n+2)^{t[1+\ln 2/\ln(n+2)]} + (2n+3)(n+2)^t + n^2 + 3n + 2\big], \tag{20}$$

which is identical with that derived in Ref. [34]. From this one can see that Eq. (18) shows a more general result, which is helpful to understand the underlying structure of tree network of this kind.

B. Determining $\langle \mathcal{F}_{\mathcal{T}(2;t,p)} \rangle$

Now let us focus on tree network $\mathcal{T}(2;t,p)$. As previously, the primary task is to determine the exact formula for the Wiener index of tree $\mathcal{T}(2;t,p)$. This will be answered by the coming calculations.

Given a tree $\mathcal{T}$ on N vertices, we suppose that the Wiener index of tree $\mathcal{T}$ is equal to $\mathcal{W}_{\mathcal{T}}$. Now let us recall the concrete procedure in Algorithm 2. For each pre-existing vertex v in tree $\mathcal{T}$, we find that (1) n new vertices will be connected to vertex v with probability p or (2) m new vertices are linked to the same vertex with probability $1-p$. It is clear to see that vertices of tree $\mathcal{T}(2;1,p)$ can also be classified into two subsets. More concretely, the total vertices in tree $\mathcal{T}$ are grouped into set $\mathcal{V}_1'$ and those newly created vertices using Algorithm 2 in the resulting tree $\mathcal{T}(2;1,p)$ are naturally collected in set $\mathcal{V}_2'$. After that, the analytical solution to Wiener index $\mathcal{W}_{\mathcal{T}(2;1,p)}$ is derived through precisely determining solutions to three quantities $\mathcal{W}_i'$ $(i \in [1,3])$, which are defined in the following:

$\mathcal{W}_1'$: The sum over distances between arbitrary vertex pair in set $\mathcal{V}_1'$.

$\mathcal{W}_2'$: The sum over distances between each vertex in set $\mathcal{V}_1'$ and an arbitrary vertex in set $\mathcal{V}_2'$.

$\mathcal{W}_3'$: The sum over distances between two vertices at random selected from set $\mathcal{V}_2'$.

We are now ready to perform the detailed calculations, which is shown in stages.

1. Case 1

Obviously, there is no influence on distance d_{uv}' between vertices u and v in set $\mathcal{V}_1'$ after applying Algorithm 2. By definition, this immediately suggests that

$$\mathcal{W}_1' = \sum_{u,v\in\mathcal{V}_1'} d_{uv}' = \sum_{u,v\in\mathcal{V}} d_{uv} = \mathcal{W}_{\mathcal{T}}, \tag{21}$$

in which $\mathcal{V}$ and d_{uv} represent the vertex set of seed $\mathcal{T}$ and the distance between vertices u and v in seed $\mathcal{T}$, separately.

2. Case 2

For each vertex u in set $\mathcal{V}_1'$, Algorithm 2 produces either with probability p an n-star whose center is vertex u or with probability $1-p$ an m-star whose center is still vertex u.

According to the probability viewpoint, we can write

$$\mathcal{W}'_{2,1} = [pn(n-1) + (1-p)m(m-1)] \times N. \tag{22}$$

The first term accounts for the sum of distances over all possible leaf vertex pairs in a star with an arbitrary vertex in set $\mathcal{V}'_1$ as central vertex.

Following the computations above, we consider the case of double stars. Without loss of generality, we choose two stars in the resulting tree $\mathcal{T}(2;1,p)$ whose centers are, respectively, vertices u and v. The former star has leaf vertices u_i and the leaf vertices in the latter are labeled utilizing v_j. So far, there is a provable $[pn+(1-p)m]^2$-mapping f'_1 between two paths $P'_{u_iv_j}$ and P'_{uv}, namely,

$$f'_1: \quad P'_{u_iv_j} \mapsto P'_{uv},$$

such that we can have $d'_{u_iv_j} = 2 + d'_{uv}$ and finally find

$$\mathcal{W}'_{2,2} = [pn+(1-p)m]^2\left[\mathcal{W}'_1 + 2\frac{N(N-1)}{2}\right]. \tag{23}$$

Here let index $\mathcal{W}'_{2,2}$ be the sum of distances between two leaf vertices of such type, say, u_i and v_j.

On the basis of both situations above, the sum of distances over all possible newly generated vertex pairs, denoted by $\mathcal{W}'_2$, is equivalent to

$$\mathcal{W}'_2 = \sum_{u_i\in\mathcal{V}'_2}\sum_{v_j\in\mathcal{V}'_2} d'_{u_iv_j} = \mathcal{W}'_{2,1} + \mathcal{W}'_{2,2}.$$

3. Case 3

The rest of our task is to estimate distance d'_{uv_j} of both vertex u in set $\mathcal{V}'_1$ and vertex v_j in set $\mathcal{V}'_2$. By analogy with determining the aforementioned equations, we take a $2[pn+(1-p)m]$-mapping f'_2 defined as follows:

$$f'_2: \quad P'_{uv_j} \mapsto P'_{uv}, \quad \text{and,} \quad f'_2: \quad P'_{u_iv} \mapsto P'_{uv},$$

in which we abuse symbols u_i and v_j whose definitions are shown in the preceding case. This leads to

$$\mathcal{W}'_3 = \sum_{u\in\mathcal{V}'_1}\sum_{v_j\in\mathcal{V}'_2} d'_{uv_j} = [pn+(1-p)m](2\mathcal{W}'_1 + N^2). \tag{24}$$

Last, using some fundamental arithmetic, Eqs. (21)–(24) together yield what we are seeking. That is, Wiener index $\mathcal{W}_{\mathcal{T}(2;1,p)}$ is given by

$$\begin{aligned}\mathcal{W}_{\mathcal{T}(2;1,p)} &= \mathcal{W}'_1 + \mathcal{W}'_2 + \mathcal{W}'_3\\ &= [pn+(1-p)m+1]^2\mathcal{W}_{\mathcal{T}} + [pn+(1-p)m][pn+(1-p)m+1]N^2\\ &\quad - \{[pn+(1-p)m]^2 - [pn(n-1)+(1-p)m(m-1)]\}N.\end{aligned} \tag{25}$$

With the results of Eq. (25), we can obtain the expression of the Wiener index $\mathcal{W}_{\mathcal{T}(2;t,p)}$ in a recursive manner and further write the mean first-passage time $\langle\mathcal{F}_{\mathcal{T}(2;t,p)}\rangle$ as follows:

$$\begin{aligned}\langle\mathcal{F}_{\mathcal{T}(2;t,p)}\rangle &= \frac{2[pn+(1-p)m+1]^t\mathcal{W}_{\mathcal{T}}}{N} + 2t[pn+(1-p)m+1]^{t-1}\times[pn+(1-p)m]N\\ &\quad - 2\{[pn+(1-p)m]^2 - [pn(n-1)+(1-p)m(m-1)]\}\times\frac{[pn+(1-p)m+1]^t - 1}{[pn+(1-p)m+1][pn+(1-p)m]}.\end{aligned} \tag{26}$$

As mentioned previously, tree $\mathcal{T}(2;t,p)$ will also degenerate into a recursive growth deterministic tree if we let the seed be a single edge and parameter p be equal to 1. For our purposes and convenience, such a deterministic model is viewed as model $\mathcal{B}_t$. Using the intrinsic self-similarity of model $\mathcal{B}_t$ itself, the authors in Ref. [40] have reported the analytical solution to mean first-passage time. At present, from Eq. (26), we can choose appropriate parameter values and then reach the exact solution of mean first-passage time $\langle\mathcal{F}_{\mathcal{B}_t}\rangle$ on model $\mathcal{B}_t$,

$$\langle\mathcal{F}_{\mathcal{B}_t}\rangle = (n+1)^t + 2(n+1)^{t-1}(2nt-1) + \frac{2}{n+1}. \tag{27}$$

Clearly, this suggests that our result is more general. Based on this, one is able to understand a tree network of such type in detail.

C. Scaling relation

Last, we unveil in detail how the underlying structures of the generated tree networks, $\mathcal{T}(1;t,p)$ and $\mathcal{T}(2;t,p)$, affect random walks on them discussed above. The following parameters will allow us to obtain what we want. Note also that the limit behavior of parameters is just considered as below.

According to Eqs. (6), (7), (18), and (26), we can obtain

$$\langle\mathcal{F}_{\mathcal{T}(1;t,p)}\rangle = O(|\mathcal{V}_{\mathcal{T}(1;t,p)}|^{\gamma_1^*}), \tag{28}$$

in which $\gamma_1^* = 1 + \frac{2}{pn+(1-p)m+2}$, and

$$\langle\mathcal{F}_{\mathcal{T}(2;t,p)}\rangle = O(|\mathcal{V}_{\mathcal{T}(2;t,p)}|^{\gamma'_1}\ln|\mathcal{V}_{\mathcal{T}(2;t,p)}|), \tag{29}$$

in which $\gamma'_1 = 1$.

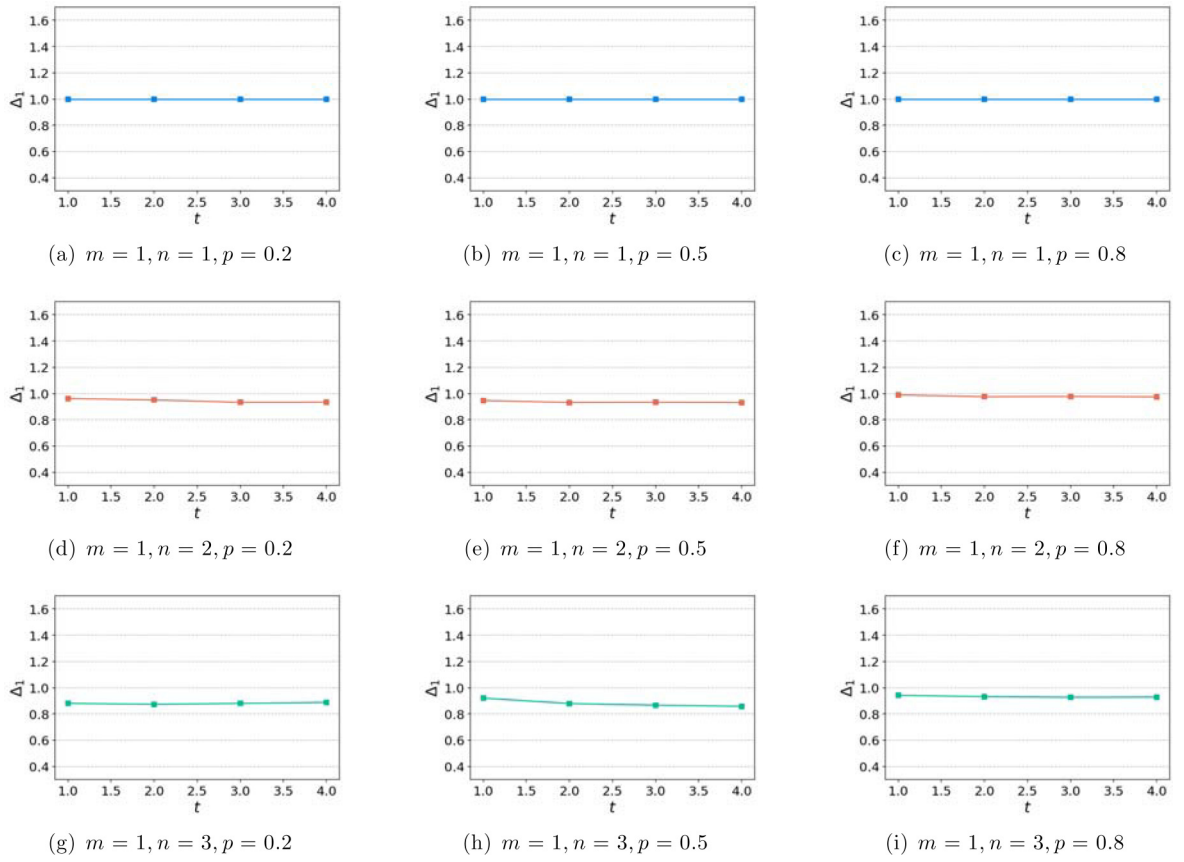
Clearly, the larger power exponent γ_1^* clarifies that, on average, the efficiency of transmitting a pair of information from a source to destination on tree network $\mathcal{T}(1;t,p)$ is lower than that on tree network $\mathcal{T}(2;t,p)$ in a random-walks-based situation.

In addition, the solutions to Kirchhoff index on two tree networks $\mathcal{T}(i;t,p)$ can read

$$\mathcal{R}_{\mathcal{T}(1;t,p)} = O(|\mathcal{V}_{\mathcal{T}(1;t,p)}|^{\gamma_2^*}), \tag{30}$$

where $\gamma_2^* = 2 + \frac{2}{pn+(1-p)m+2}$, and

$$\mathcal{R}_{\mathcal{T}(2;t,p)} = O(|\mathcal{V}_{\mathcal{T}(2;t,p)}|^{\gamma'_2}\ln|\mathcal{V}_{\mathcal{T}(2;t,p)}|), \tag{31}$$



(a) $m = 1, n = 1, p = 0.2$ (b) $m = 1, n = 1, p = 0.5$ (c) $m = 1, n = 1, p = 0.8$

(d) $m = 1, n = 2, p = 0.2$ (e) $m = 1, n = 2, p = 0.5$ (f) $m = 1, n = 2, p = 0.8$

(g) $m = 1, n = 3, p = 0.2$ (h) $m = 1, n = 3, p = 0.5$ (i) $m = 1, n = 3, p = 0.8$

FIG. 2. The diagram of parameter Δ_1 as a function of both variables, p and t. From left to right, three panels in the first line illustrate case of $n = m = 1$ when varying parameter p. Similarly, those panels in the second and last lines are correlated to case of both $n = 2$ and $m = 1$ and case of both $n = 3$ and $m = 1$, respectively.

where $\gamma_2' = 2$. Here we have made use of Eqs. (3) and (4). From this, it is straightforward to find that the expressions of network criticality on both tree networks are as follows:

$$\langle \mathcal{R}_{\mathcal{T}(1;t,p)} \rangle = O(|\mathcal{V}_{\mathcal{T}(1;t,p)}|^{\gamma_2^* - 2}), \tag{32a}$$

$$\langle \mathcal{R}_{\mathcal{T}(2;t,p)} \rangle = O(\ln |\mathcal{V}_{\mathcal{T}(2;t,p)}|). \tag{32b}$$

This implies that tree network $\mathcal{T}(2;t,p)$ is more robust than $\mathcal{T}(1;t,p)$.

In other words, the parameters mentioned above, mean first-passage time and network criticality, together indicate that the topological structure corresponding to tree network $\mathcal{T}(2;t,p)$ is more optimal than tree network $\mathcal{T}(1;t,p)$ when distributing information in a random-walk manner. Once again, our results show that the underlying structure on network plays a key role in discussion about random walks.

D. Simulations

Despite the completeness of theoretical analysis, we want to uncover how varying values for different parameters including p, n, and m affects the convergence of experimental simulations to analytical solutions. Therefore, we conduct extensive computer simulations. Note that while an arbitrary tree may be selected as the input of two algorithms introduced in Sec. III, we will be concerned with the simplest case where the seed is a single edge so as to conveniently understand the behaviors of simulation results in the large-t limit. As a result, the previously supposed parameters $\mathcal{W}_\mathcal{T}$ and N in all the derived formulas are now equal to 1 and 2, respectively.

First, we verify the correctness of Eq. (18). For convenience, we define a symbol Δ_1

$$\Delta_1 = \frac{\overline{\langle \mathcal{F}_{\mathcal{T}(1;t,p)} \rangle}}{\langle \mathcal{F}_{\mathcal{T}(1;t,p)} \rangle}, \tag{33}$$

which is used to represent ratio of the numerical solution $\overline{\langle \mathcal{F}_{\mathcal{T}(1;t,p)} \rangle}$, which is viewed as the averaged value over mean first-passage times on tree networks $\mathcal{T}(1;t,p)$ after running computer simulations 100 times to the theoretical result in Eq. (18). Specifically, in view of parameters m and n, the following three cases will be considered, i.e., (1) $m = 1$ and $n = 1$, (2) $m = 1$ and $n = 2$, as well as (3) $m = 1$ and $n = 3$. Refer to Fig. 2 for more information.

From Fig. 2, one can easily see that in Figs. 2(a)– 2(c), numerical simulations align with the theoretical analysis. This is mainly because, in this setting, tree $\mathcal{T}(1;t,p)$ is in fact deterministic. In other panels, numerical simulations always fluctuate around the theoretical results, which implies that the randomness introduced into the development of growth

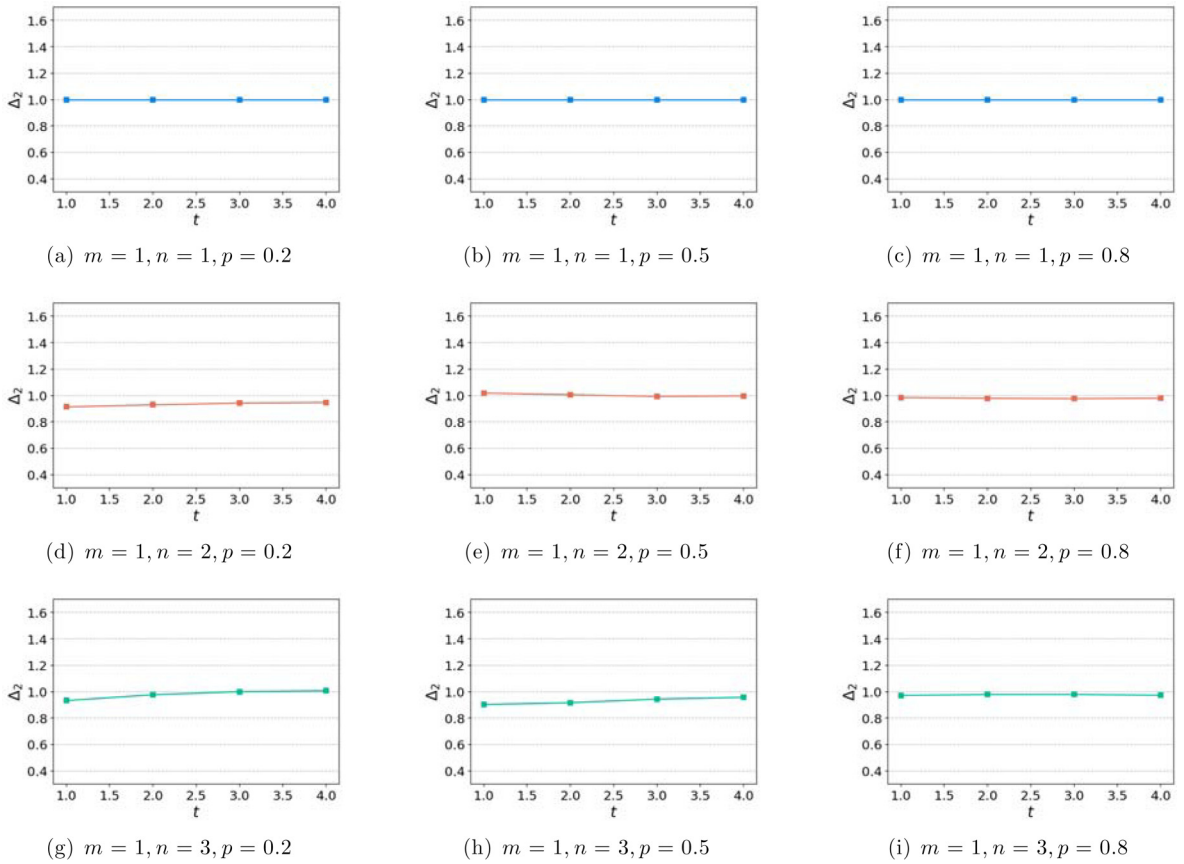


FIG. 3. The diagram of parameter Δ_2 in Eq. (34). From left to right, three panels in the first line illustrate case of $n = m = 1$ where p is equal to 0.2, 0.5, or 0.8. Analogously, those panels in the second and last lines, respectively, show the case of both $n = 2$ and $m = 1$ and the case of both $n = 3$ and $m = 1$.

tree networks $\mathcal{T}(1;t,p)$ has a certain effect on mean first-passage time for random walks on them. Nevertheless, it is clear to the eye that empirical analysis is in good agreement with the theoretical results. This confirms that the analytical solution derived in Eq. (18) holds true in the large-graph-size limit.

Now let us divert our attention to numerical simulations of mean first-passage time $\langle \mathcal{F}_{\mathcal{T}(2;t,p)} \rangle$. Specifically speaking, the following presentation shows how different combinations among parameters m, n, and p impact mean first-passage time for random walks on tree networks $\mathcal{T}(2;t,p)$. As shown in Eq. (26), the analytical solution to quantity $\langle \mathcal{F}_{\mathcal{T}(2;t,p)} \rangle$ is a power function of variable t. As previously, we also need to define an available parameter as follows:

$$\Delta_2 = \frac{\overline{\langle \mathcal{F}_{\mathcal{T}(2;t,p)} \rangle}}{\langle \mathcal{F}_{\mathcal{T}(2;t,p)} \rangle}. \tag{34}$$

Here symbol $\overline{\langle \mathcal{F}_{\mathcal{T}(2;t,p)} \rangle}$ denotes the experimental value for mean first-passage time averaged over 100 runs given a pair of parameters n and m in tree networks $\mathcal{T}(2;t,p)$. Analogously, we have the ability to consider other cases. For brevity, we, however, omit other comparisons between analytical formulas and experimental simulations. More details are shown in Fig. 3.

As Fig. 3 makes clear, it is straightforward to see that network $\mathcal{T}(2;t,p)$ will become deterministic when we let parameters n and m be equal to the same value. In this case, the ratio Δ_1 should be constant 1. Figures 3(a)– 3(c) firmly verify this assertion, implying that the formula developed in Eq. (26) is valid. In addition, the remaining panels, i.e., Figures 3(d)–3(i), consistently show that the randomness controlled by probability parameter p leads to some fluctuations that are in the reasonable range. At the same time, the first several growth steps have a significantly critical influence on the form of fluctuation in the large-graph-size limit as plotted here. In a nutshell, computer simulations are perfectly consistent with the analytical results.

V. CONCLUSION

In conclusion, we study random walks on two families of random growth tree networks $\mathcal{T}(1;t,p)$ and $\mathcal{T}(2;t,p)$ in more detail. Especially, we derive the analytical solutions to mean first-passage time for random walks on both tree networks $\mathcal{T}(i;t,p)$. The results suggest that, by adopting some specified parameters, the previous works focusing mainly on deterministic tree networks can be clearly seen as the specific cases of the derived formulas. More importantly, the results also suggest that the underlying structure on network has a remarkable influence on random walks. The

fractal-growth manner makes the topological structure of a tree network looser than the nonfractal one and thus increases drastically the mean first-passage time in the large-graph limit.

Besides correctness and convenience of the proposed formulations, it is worth stressing that there are a number of interesting tree networks that have various applications in the real world that were not probed in this paper. Thus, this will be our future work.

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