Two-lane totally asymmetric simple exclusion process with extended Langmuir kinetics

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Multilane totally asymmetric simple exclusion processes with interactions between the lanes have recently been investigated actively. This paper proposes a two-lane model with extended Langmuir kinetics on a periodic lattice. Both bidirectional and unidirectional flows are investigated. In our model, the hopping, attachment, and detachment rates vary depending on the state of the corresponding site in the other lane. We obtain a theoretical expression for the global density of the system in the steady state from three kinds of mean-field analyses [(1×1)-, (2×1)-, and (2×2)-cluster cases]. We verify that the (2×2)-cluster mean-field analysis reproduces the differences between the two directional flows and approximates well the results of computer simulations for some cases. We observe that (2×1)-cluster mean-field analyses are already good approximations of the simulation results for unidirectional flows; on the other hand, the accuracy of the approximations much improves by (2×2)-cluster one for bidirectional flows. We explain the phenomena in a qualitative manner by a simple analysis of correlations. We expect these findings to give informative suggestions for actual traffic systems.

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I. INTRODUCTION

The asymmetric simple exclusion process (ASEP), which is a stochastic process involving particles on a lattice, has been applied in many fields [1] since it was first proposed by MacDonald and Gibbs [2,3]. A special version of ASEP, in which particles on a lattice can hop unidirectionally, is referred to as a totally asymmetric simple exclusion process (TASEP). Researchers have applied TASEPs to traffic flows of self-driven particles, as in biological transport [4–7], vehicular traffic [8,9], and pedestrian flow [10–12]. Recently, one-lane TASEPs with varying hopping probabilities [13–15], multilane TASEPs with interactions between lanes [16–27], and multilane TASEPs with varying hopping probabilities and interactions between lanes [28,29] have begun to be investigated. For example, in Refs. [13–15], the hopping probability of a particle varies depending on the states of the sites surrounding it, whereas in Refs. [17–19,29], the hopping rate of a particle varies depending on the state of the other lane.

One of the extensions of TASEP, a TASEP with Langmuir kinetics, which we refer to as LK-TASEP in the present paper, has begun to be actively investigated [30–46]. In the original LK-TASEP, a particle attaches (detaches) at a certain rate ω_{A} (ω_{D}) when the targeted site on the lattice is empty (occupied). The LK-TASEP was first proposed in Ref. [30] and was investigated using mean-field theory in Refs. [31,32]. Recently, multilane or/and multispecies LK-TASEPs have been studied [33–38]. In addition, Refs. [39–43] changed the attachment and detachment rates depending on the occupancy of the adjacent sites. We note that the models of Refs. [40–42] are more generalized versions of those discussed in Ref. [31].

In the present paper, we consider a two-lane extended LK-TASEP on a periodic lattice. The proposed model considers the interaction between particles in each lane without lane changing. Specifically, the hopping rate p and the attachment (detachment) rate ω_{A} (ω_{D}) vary depending on the state of the corresponding site in the other lane. We stress that the hopping rule has already been employed in Refs. [17–19]; however, the attachment and detachment rule has not been considered in previous studies. In the present paper, we investigate both unidirectional and bidirectional flows. We conduct computer simulations and perform three kinds of mean-field analyses [(1×1)-, (2×1)-, and (2×2)-cluster cases] to investigate the global density of the system in the steady state. We find that the (2×2)-cluster mean-field analysis captures the differences between unidirectional and bidirectional flows and reproduces the simulation results well, especially for small p.

From a practical point of view, LK-TASEP has been used extensively for analyzing the motions of motor proteins in Refs. [44–46], in which the investigators determined the parameters from experimental data. For applications to traffic flow, a TASEP model with an absorbing lane, which can be classified into the same category as LK-TASEP, has been investigated in Refs. [20] (parking lots) and [21,22] (airport transportation systems). Our proposed model can also be applied to traffic flow; e.g., to crowd dynamics in the situations where multiple lanes are formed in a narrow passage, such as in trains, airplanes, and concert halls. In those situations, we observe two important phenomena. First, we often observe that people decrease their walking

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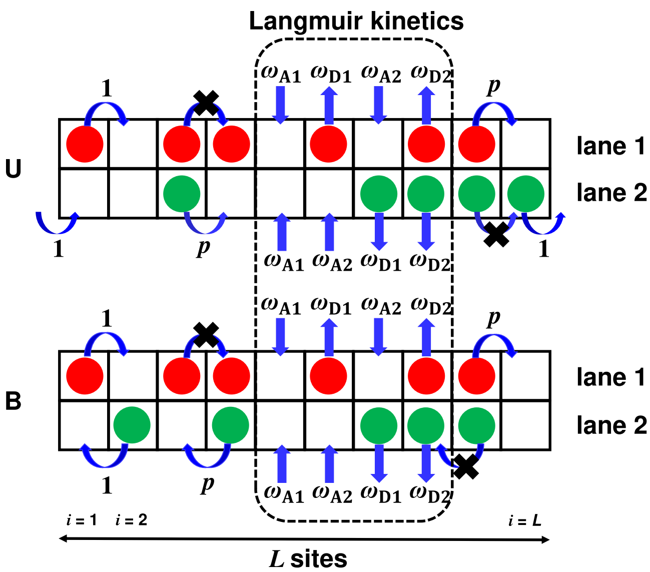

FIG. 1. Schematic illustration of the present model. The upper and lower panels represent unidirectional (U) and bidirectional (B) flows, respectively. Red (green) circles represent particles in lane 1 (2). The left and right boundaries are connected (periodic boundary conditions). The case $L = 10$ is shown.

speed to avoid collisions while walking side by side in unidirectional flows (passing each other in bidirectional flows). Second, decision making during inflow to (outflow from) a passage is influenced by the local state. For example, people tend to hesitate to enter a passage when it is congested locally at the inflow point. The former phenomenon corresponds to a change of the hopping rate, whereas the latter one corresponds to changes of attachment and detachment rates.

This paper is organized as follows. Section II describes the details of our proposed model. In Sec. III the numerical results from mean-field analyses are presented. Section IV compares the results from mean-field analyses and simulation results. Finally, the paper concludes in Sec. V.

II. MODEL

The model consists of two L-site lanes, labeled $i = 1, 2, \dots, L$, as shown in Fig. 1. Each site can either be empty or be occupied by one particle. The state of a site is represented by 1 if a particle occupies that site; otherwise, its state is represented by 0. We employ periodic boundary conditions, i.e., site L and site 1 are connected, and we use random updating. Changing lanes is prohibited in this model. In the present paper, we consider two cases: unidirectional flows (particles in both lanes all hop in the same direction) and bidirectional flows (the particles in the two lanes hop in opposite directions).

Next, we describe the update scheme for the case of unidirectional flows. In the proposed model, the hopping rate and attachment (detachment) rates depend on the occupancy of the corresponding site in the other lane.

For the hopping rate, a particle at site i in one lane hops to site $(i + 1)$ with rate 1 if site i in the other lane is vacant; otherwise, it hops with rate p $(0 \leqslant p \leqslant 1)$.

However, for the attachment and detachment rates, the process can be divided into two patterns; specifically, (1) if site i in one lane is empty and site i in the other lane is empty (occupied), the particle attaches with rate ω_{A1} (ω_{A2}), whereas (2) if site i in one lane is occupied and site i in the other lane is empty (occupied), the particle detaches with rate ω_{D1} (ω_{D2}). For bidirectional flows, only the hopping rule for lane 2 differs from the case of unidirectional flows. In this case, a particle at site i in lane 2 hops to site $(i - 1)$ with rate 1 if the corresponding site in lane1 is vacant; otherwise, it hops with rate p $(0 \leqslant p \leqslant 1)$. We note that this model yields the standard LK-TASEP when $p = 1$, $\omega_{A1} = \omega_{A2}$, and $\omega_{D1} = \omega_{D2}$.

III. MEAN-FIELD ANALYSES

In this section, we investigate the density profile in the steady state using three kinds of mean-field analyses. We hereafter write the probability finding configuration τ as $P(\tau)$. The configuration τ, which is one element of the set S which consists of all possible configurations, contains $(2 \times L)$ figures. The top (bottom) row represents the state of the sites in lane 1 (2). For example, when $L = 5$ and the occupied site numbers in lane 1 are 1, 2, and 3, and those in lane 2 are 4 and 5, τ can be written as

$$\tau = \begin{matrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{matrix}. \tag{1}$$

The master equation for this system can be written as

$$\frac{dP(\tau)}{dt} = \sum_{\tau' \in S} W(\tau' \to \tau) P(\tau') - \sum_{\tau \in S} W(\tau \to \tau') P(\tau), \tag{2}$$

where $W(\tau' \to \tau)$ is the transition weight to go from state τ' to state τ. However, it is very difficult to analyze $(2 \times L)$-site configurations. We therefore consider cluster approximations; specifically, (1×1)-, (2×1)-, and (2×2)-cluster approximations. In the following subsections, we consider both unidirectional and bidirectional flows.

A. (1 × 1)-cluster mean-field analysis

In this subsection, we consider the (1×1)-cluster mean-field analysis, which is identical to the normal mean-field analysis used in ASEP investigations. Translational invariance leads to spatial homogeneity, i.e., the probability is independent of the site number; therefore, we can abbreviate the site number in the following discussions (similarly in Secs. III B and III C).

For the probability $P\binom{1}{*}$, where $*$ represents either 0 or 1, the master equation for unidirectional flows can be written as

$$\frac{dP\binom{1}{*}}{dt} = \left[P\begin{pmatrix} 1 & 0 \\ 0 & \underline{*} \end{pmatrix} + pP\begin{pmatrix} 1 & 0 \\ 1 & \underline{*} \end{pmatrix} + \omega_{A1} P\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix} + \omega_{A2} P\begin{pmatrix} 0 \\ \underline{1} \end{pmatrix}\right] - \left[P\begin{pmatrix} 1 & 0 \\ \underline{0} & * \end{pmatrix} + pP\begin{pmatrix} 1 & 0 \\ \underline{1} & * \end{pmatrix} + \omega_{D1} P\begin{pmatrix} 1 \\ \underline{0} \end{pmatrix} + \omega_{D2} P\begin{pmatrix} 1 \\ \underline{1} \end{pmatrix}\right], \tag{3}$$

where the underlined sites on the right hand corresponds to the sites on the left hand (and similarly hereafter). We note that this equation does not change for bidirectional flows, although that of $P\binom{*}{1}$ changes.

Performing the mean-field analysis, i.e., ignoring higher correlations in Eq. (3), we have

$$\frac{dP\binom{1}{*}}{dt} = \left[P\begin{pmatrix}1\\ *\end{pmatrix}P\begin{pmatrix}0\\ *\end{pmatrix}P\begin{pmatrix}*\\ 0\end{pmatrix} + pP\begin{pmatrix}1\\ *\end{pmatrix}P\begin{pmatrix}0\\ *\end{pmatrix}P\begin{pmatrix}*\\ 1\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}0\\ *\end{pmatrix}P\begin{pmatrix}*\\ 0\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}0\\ *\end{pmatrix}P\begin{pmatrix}*\\ 1\end{pmatrix}\right]$$
$$- \left[P\begin{pmatrix}1\\ *\end{pmatrix}P\begin{pmatrix}0\\ *\end{pmatrix}P\begin{pmatrix}*\\ 0\end{pmatrix} + pP\begin{pmatrix}1\\ *\end{pmatrix}P\begin{pmatrix}0\\ *\end{pmatrix}P\begin{pmatrix}*\\ 1\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1\\ *\end{pmatrix}P\begin{pmatrix}*\\ 0\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1\\ *\end{pmatrix}P\begin{pmatrix}*\\ 1\end{pmatrix}\right], \tag{4}$$

where $P\binom{0}{*} + P\binom{1}{*} = 1$ and $P\binom{*}{0} + P\binom{*}{1} = 1$.

Given the obvious symmetry between lanes 1 and 2, after a long enough time we obtain $\rho = P\binom{1}{*} = P\binom{*}{1}$; therefore, Eq. (4) reduces to

$$\frac{d\rho}{dt} = (\omega_{\mathrm{A1}} + \omega_{\mathrm{D1}} - \omega_{\mathrm{A2}} - \omega_{\mathrm{D2}})\rho^2 + (-2\omega_{\mathrm{A1}} + \omega_{\mathrm{A2}} - \omega_{\mathrm{D1}})\rho + \omega_{\mathrm{A1}}, \tag{5}$$

where all the terms including p disappear. We note that Eq. (5) does not change for bidirectional flows.

Because $\frac{d\rho}{dt} = 0$ in the steady state, we obtain

$$a\rho^2 + b\rho + c = 0, \tag{6}$$

where

$$a = \omega_{\mathrm{A1}} - \omega_{\mathrm{A2}} + \omega_{\mathrm{D1}} - \omega_{\mathrm{D2}}, \tag{7}$$

$$b = -2\omega_{\mathrm{A1}} + \omega_{\mathrm{A2}} - \omega_{\mathrm{D1}}, \tag{8}$$

and

$$c = \omega_{\mathrm{A1}}. \tag{9}$$

For $a \neq 0$, the solution of Eq. (6) can be written as

$$\rho = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{10}$$

where

$$b^2 - 4ac = (\omega_{\mathrm{A2}} - \omega_{\mathrm{D1}})^2 + 4\omega_{\mathrm{A2}}\omega_{\mathrm{D2}} > 0. \tag{11}$$

We discuss the inclusion of Eq. (10) and the exclusion of the other solution of Eq. (6), i.e., $\rho = \frac{-b+\sqrt{b^2-4ac}}{2a}$, in Appendix A.

On the other hand, when $a = 0$, we have

$$\rho = \frac{\omega_{\mathrm{A1}}}{\omega_{\mathrm{A1}} + \omega_{\mathrm{D1}}}. \tag{12}$$

B. (2 × 1)-cluster mean-field analysis

In this subsection, we consider the (2 × 1)-cluster mean-field analysis, where 2(= 2 × 1) sites are regarded as one cluster. We note that this analysis is called simple mean-field method in Refs. [17–19]; however, we do not use this terminology in order to clarify the difference from the analysis in the previous subsection and to avoid misunderstanding.

For the (2 × 1)-cluster probability, the following two master equations can be derived for the case of unidirectional flows:

$$\frac{dP\binom{0}{0}}{dt} = \left[P\begin{pmatrix}1 & 0\\ \underline{0} & 0\end{pmatrix} + P\begin{pmatrix}1 & 0\\ 0 & \underline{1}\end{pmatrix} + P\begin{pmatrix}0 & 0\\ \underline{1} & 0\end{pmatrix} + P\begin{pmatrix}0 & 1\\ \underline{1} & 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1\\ \underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0\\ \underline{1}\end{pmatrix}\right] - \left[P\begin{pmatrix}1 & 0\\ 0 & \underline{0}\end{pmatrix} + P\begin{pmatrix}0 & 0\\ 1 & \underline{0}\end{pmatrix}\right.$$
$$\left. + 2pP\begin{pmatrix}1 & 0\\ 1 & \underline{0}\end{pmatrix} + 2\omega_{\mathrm{A1}}P\begin{pmatrix}0\\ \underline{0}\end{pmatrix}\right] \tag{13}$$

and

$$\frac{dP\binom{1}{0}}{dt} = \left[P\begin{pmatrix}1 & 0\\ 0 & \underline{0}\end{pmatrix} + pP\begin{pmatrix}1 & 0\\ \underline{1} & 0\end{pmatrix} + pP\begin{pmatrix}1 & 0\\ 1 & \underline{0}\end{pmatrix} + pP\begin{pmatrix}1 & 1\\ 1 & \underline{0}\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}0\\ \underline{0}\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1\\ \underline{1}\end{pmatrix}\right]$$
$$- \left[P\begin{pmatrix}0 & 1\\ 1 & \underline{0}\end{pmatrix} + pP\begin{pmatrix}1 & 1\\ 1 & \underline{0}\end{pmatrix} + P\begin{pmatrix}1 & 0\\ \underline{0} & 0\end{pmatrix} + P\begin{pmatrix}1 & 0\\ \underline{0} & 1\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1\\ \underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1\\ \underline{0}\end{pmatrix}\right]. \tag{14}$$

Again performing the mean-field analysis, i.e., ignoring the higher correlations in Eqs. (13) and (14), we have

$$\frac{dP\binom{0}{0}}{dt} = \left[P\begin{pmatrix}1\\ 0\end{pmatrix}P\begin{pmatrix}0\\ 0\end{pmatrix} + P\begin{pmatrix}1\\ 0\end{pmatrix}P\begin{pmatrix}0\\ 1\end{pmatrix} + P\begin{pmatrix}0\\ 1\end{pmatrix}P\begin{pmatrix}0\\ 0\end{pmatrix} + P\begin{pmatrix}0\\ 1\end{pmatrix}P\begin{pmatrix}1\\ 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1\\ 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0\\ 1\end{pmatrix}\right]$$
$$- \left[P\begin{pmatrix}1\\ 0\end{pmatrix}P\begin{pmatrix}0\\ 0\end{pmatrix} + P\begin{pmatrix}0\\ 1\end{pmatrix}P\begin{pmatrix}0\\ 0\end{pmatrix} + 2pP\begin{pmatrix}1\\ 1\end{pmatrix}P\begin{pmatrix}0\\ 0\end{pmatrix} + 2\omega_{\mathrm{A1}}P\begin{pmatrix}0\\ 0\end{pmatrix}\right]$$
$$= 2P\begin{pmatrix}1\\ 0\end{pmatrix}P\begin{pmatrix}0\\ 1\end{pmatrix} - 2pP\begin{pmatrix}0\\ 0\end{pmatrix}P\begin{pmatrix}1\\ 1\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1\\ 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0\\ 1\end{pmatrix} - 2\omega_{\mathrm{A1}}P\begin{pmatrix}0\\ 0\end{pmatrix} \tag{15}$$

and

$$\begin{aligned}\frac{dP\binom{1}{0}}{dt} &= \left[P\begin{pmatrix}1\\0\end{pmatrix}P\begin{pmatrix}0\\0\end{pmatrix} + pP\begin{pmatrix}1\\1\end{pmatrix}P\begin{pmatrix}0\\0\end{pmatrix} + pP\begin{pmatrix}1\\1\end{pmatrix}P\begin{pmatrix}0\\0\end{pmatrix} + pP\begin{pmatrix}1\\1\end{pmatrix}P\begin{pmatrix}1\\0\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}0\\0\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1\\1\end{pmatrix}\right]\\ &\quad - \left[P\begin{pmatrix}0\\1\end{pmatrix}P\begin{pmatrix}1\\0\end{pmatrix} + pP\begin{pmatrix}1\\1\end{pmatrix}P\begin{pmatrix}1\\0\end{pmatrix} + P\begin{pmatrix}1\\0\end{pmatrix}P\begin{pmatrix}0\\0\end{pmatrix} + P\begin{pmatrix}1\\0\end{pmatrix}P\begin{pmatrix}0\\1\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1\\0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1\\0\end{pmatrix}\right]\\ &= 2pP\begin{pmatrix}0\\0\end{pmatrix}P\begin{pmatrix}1\\1\end{pmatrix} - 2P\begin{pmatrix}1\\0\end{pmatrix}P\begin{pmatrix}0\\1\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}0\\0\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1\\1\end{pmatrix} - \omega_{\mathrm{A2}}P\begin{pmatrix}1\\0\end{pmatrix} - \omega_{\mathrm{D1}}P\begin{pmatrix}1\\0\end{pmatrix}.\end{aligned} \tag{16}$$

We stress here that Eqs. (13) and (14) change for the case of bidirectional flows; however, Eqs. (15) and (16) do not change and thus yield the same results for the mean-field analysis.

Again given the obvious symmetry between lanes 1 and 2 after a long enough time, we obtain

$$P\begin{pmatrix}1\\0\end{pmatrix} = P\begin{pmatrix}0\\1\end{pmatrix}. \tag{17}$$

Moreover, $P\binom{i}{j}$ ($i, j \in \{0, 1\}$) must satisfy the normalization condition:

$$P\begin{pmatrix}0\\0\end{pmatrix} + P\begin{pmatrix}1\\0\end{pmatrix} + P\begin{pmatrix}0\\1\end{pmatrix} + P\begin{pmatrix}1\\1\end{pmatrix} = 1. \tag{18}$$

Because $\frac{d}{dt}P\binom{i}{j} = 0$ in the steady state, we obtain the following expression from Eqs. (15)–(18):

$$A\left[P\begin{pmatrix}1\\0\end{pmatrix}\right]^2 + BP\begin{pmatrix}1\\0\end{pmatrix} + C = 0, \tag{19}$$

where

$$A = 2 + 4p\alpha + 2p\alpha^2, \tag{20}$$

$$\begin{aligned}B &= 4p\alpha\beta + 4p\beta + \omega_{\mathrm{A2}} + \omega_{\mathrm{D1}} + 2\omega_{\mathrm{D2}}\\ &\quad + \omega_{\mathrm{D2}}\alpha - 2p\alpha - \omega_{\mathrm{A1}}\alpha,\end{aligned} \tag{21}$$

$$C = 2p\beta^2 + \omega_{\mathrm{D2}}\beta - 2p\beta - \omega_{\mathrm{A1}}\beta - \omega_{\mathrm{D2}}, \tag{22}$$

$$\alpha = \frac{\omega_{\mathrm{D1}} - \omega_{\mathrm{A2}} - 2\omega_{\mathrm{D2}}}{\omega_{\mathrm{A1}} + \omega_{\mathrm{D2}}}, \tag{23}$$

and

$$\beta = \frac{\omega_{\mathrm{D2}}}{\omega_{\mathrm{A1}} + \omega_{\mathrm{D2}}}. \tag{24}$$

We discuss the detailed derivation of Eq. (19) in Appendix B.

Solving Eq. (19) yields $P\binom{1}{0}$ in the form

$$P\begin{pmatrix}1\\0\end{pmatrix} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{25}$$

We note that because

$$A = 2 + 4p\alpha + 2p\alpha^2 = 2(1-p) + 2p(\alpha+1)^2 > 0 \tag{26}$$

and

$$C = 2p\beta^2 + \omega_{\mathrm{D2}}\beta - 2p\beta - \omega_{\mathrm{A1}}\beta - \omega_{\mathrm{D2}} \tag{27}$$

$$= -\frac{2p\omega_{\mathrm{A1}} - \omega_{\mathrm{D2}}\omega_{\mathrm{A1}} - \omega_{\mathrm{A1}}\omega_{\mathrm{D2}}}{\omega_{\mathrm{A1}} + \omega_{\mathrm{D2}}} < 0, \tag{28}$$

we have

$$B^2 - 4AC > 0. \tag{29}$$

We discuss the exclusion of the other solution of Eq. (19), i.e., $P\binom{1}{0} = \frac{-B-\sqrt{B^2-4AC}}{2A}$ in Appendix C.

Because the density is defined as

$$\rho = P\begin{pmatrix}1\\0\end{pmatrix} + P\begin{pmatrix}1\\1\end{pmatrix}, \tag{30}$$

we finally have

$$\rho = 1 - \beta - \frac{(1+\alpha)(-B + \sqrt{B^2 - 4AC})}{2A}. \tag{31}$$

C. (2 × 2)-cluster mean-field analysis

This subsection presents the (2 × 2)-cluster mean-field analysis, where 4(= 2 × 2) sites are regarded as one cluster. We note that this analysis is called the two-cluster or two-cell mean-field method in Refs. [17–19]; however, as with Sec. III B, we do not use this terminology in order to clarify the difference from the analyses in the last two subsections and to avoid misunderstanding.

Unlike the two previous mean-field analyses, in this case the final results are different for the two directional flows. Therefore, in this subsection we consider the two flows separately.

1. *Unidirectional flows*

The master equation for $P\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ can be expressed as

$$\begin{aligned}\frac{dP\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}}{dt} &= \left[P\begin{pmatrix}0 & 1 & 0\\\underline{0} & \underline{0} & 0\end{pmatrix} + P\begin{pmatrix}0 & 1 & 0\\\underline{0} & \underline{0} & 1\end{pmatrix} + P\begin{pmatrix}0 & 0 & 0\\\underline{0} & \underline{1} & 0\end{pmatrix} + P\begin{pmatrix}0 & 0 & 1\\\underline{0} & \underline{1} & 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1 & 0\\\underline{0} & \underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0 & 0\\\underline{1} & \underline{0}\end{pmatrix}\right.\\ &\quad \left. + \omega_{\mathrm{D1}}P\begin{pmatrix}0 & 1\\\underline{0} & \underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0 & 0\\\underline{0} & \underline{1}\end{pmatrix}\right] - \left[P\begin{pmatrix}1 & 0 & 0\\0 & \underline{0} & \underline{0}\end{pmatrix} + P\begin{pmatrix}0 & 0 & 0\\1 & \underline{0} & \underline{0}\end{pmatrix} + 2pP\begin{pmatrix}1 & 0 & 0\\1 & \underline{0} & \underline{0}\end{pmatrix} + 4\omega_{\mathrm{A1}}P\begin{pmatrix}0 & 0\\\underline{0} & \underline{0}\end{pmatrix}\right].\end{aligned} \tag{32}$$

Utilizing the concept of conditional probability, we can express $P\left(\begin{smallmatrix} i & k & m \\ j & l & n \end{smallmatrix}\right)$ in this mean-field analysis in the form

$$P\begin{pmatrix} i & k & m \\ j & l & n \end{pmatrix} = \frac{P\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right)P\left(\begin{smallmatrix} k & m \\ l & n \end{smallmatrix}\right)}{\sum_{m\in\{0,1\}}\sum_{n\in\{0,1\}} P\left(\begin{smallmatrix} k & m \\ l & n \end{smallmatrix}\right)}, \tag{33}$$

where $i, j, k, l, m, n \in \{0, 1\}$.

Inserting Eq. (33) into Eq. (32) and noting that in the steady state $\frac{d}{dt}P\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right) = 0$, we obtain

$$\begin{aligned}
&\frac{P\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 1 & i \\ 0 & j \end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 1 & i \\ 0 & j \end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 0 & i \\ 1 & j \end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 0 & i \\ 1 & j \end{smallmatrix}\right)} \\
&+ \omega_{D1}P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \omega_{D1}P\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \omega_{D1}P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \omega_{D1}P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} \\
&- \frac{P\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} - \frac{2pP\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}} P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} - 4\omega_{A1}P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0.
\end{aligned} \tag{34}$$

We can obtain the eight other master equations for the (2×2)-cluster probabilities in the steady state similarly, as shown in Appendix D.

In addition, given the obvious symmetry between lanes 1 and 2 after a long enough time, we obtain

$$P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{35}$$

$$P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{36}$$

$$P\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \tag{37}$$

$$P\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = P\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{38}$$

$$P\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{39}$$

and

$$P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = P\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{40}$$

Moreover, $P\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right)$ must satisfy the normalization condition

$$\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}\sum_{k\in\{0,1\}}\sum_{l\in\{0,1\}} P\begin{pmatrix} i & k \\ j & l \end{pmatrix} = 1. \tag{41}$$

From 16 independent Eqs. (34)–(41) and (D9)–(D16), we obtain all the probabilities $P\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right)$.

Finally, the definition of density gives

$$\rho = \sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}\sum_{k\in\{0,1\}} P\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}. \tag{42}$$

2. Bidirectional flows

The master equation for $P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ can be written in the form

$$\begin{aligned}
\frac{dP\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{dt} =& \Bigg[P\begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & \underline{0} \end{pmatrix} + P\begin{pmatrix} 1 & 0 & 0 \\ 0 & \underline{1} & \underline{0} \end{pmatrix} + P\begin{pmatrix} 0 & 1 & 0 \\ \underline{0} & \underline{0} & 0 \end{pmatrix} + P\begin{pmatrix} 0 & 1 & 0 \\ \underline{0} & \underline{0} & 1 \end{pmatrix} + \omega_{D1}P\begin{pmatrix} 1 & 0 \\ \underline{0} & \underline{0} \end{pmatrix} + \omega_{D1}P\begin{pmatrix} 0 & 0 \\ \underline{1} & \underline{0} \end{pmatrix} \\
&+ \omega_{D1}P\begin{pmatrix} 0 & 1 \\ \underline{0} & \underline{0} \end{pmatrix} + \omega_{D1}P\begin{pmatrix} 0 & 0 \\ \underline{0} & \underline{1} \end{pmatrix}\Bigg] - \Bigg[P\begin{pmatrix} 1 & 0 & 0 \\ 0 & \underline{0} & \underline{0} \end{pmatrix} + pP\begin{pmatrix} 1 & 0 & 0 \\ 1 & \underline{0} & \underline{0} \end{pmatrix} + P\begin{pmatrix} 0 & 0 & 0 \\ \underline{0} & \underline{0} & 1 \end{pmatrix} + pP\begin{pmatrix} 0 & 0 & 1 \\ \underline{0} & \underline{0} & 1 \end{pmatrix} \\
&+ 4\omega_{A1}P\begin{pmatrix} 0 & 0 \\ \underline{0} & \underline{0} \end{pmatrix}\Bigg].
\end{aligned} \tag{43}$$

Inserting Eq. (33) into Eq. (43) and noting that in the steady state $\frac{d}{dt}P\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right) = 0$, we obtain

$$\begin{aligned}
&\frac{P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 0 & i \\ 1 & j \end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 0 & i \\ 1 & j \end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 1 & i \\ 0 & j \end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 1 & i \\ 0 & j \end{smallmatrix}\right)} \\
&+ \omega_{\mathrm{D}1}P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \omega_{\mathrm{D}1}P\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \omega_{\mathrm{D}1}P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \omega_{\mathrm{D}1}P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{P\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} \\
&- \frac{pP\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} - \frac{P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} - \frac{pP\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)P\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix} 0 & i \\ 0 & j \end{smallmatrix}\right)} \\
&- 4\omega_{\mathrm{A}1}P\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0. \qquad (44)
\end{aligned}$$

We can again obtain the eight other master equations for the (2×2)-cluster probabilities in the steady state similarly, as shown in Appendix E.

Given the obvious symmetry between lanes 1 and 2 after a long enough time, we obtain

$$P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (45)$$

$$P\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad (46)$$

$$P\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad (47)$$

$$P\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad (48)$$

$$P\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad (49)$$

and

$$P\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \qquad (50)$$

From 16 independent Eqs. (41), (44)–(50), and (E9)–(E16), we obtain all the probabilities $P\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right)$.

Finally, the definition of density gives Eq. (42).

IV. COMPARISON OF NUMERICAL RESULTS WITH MEAN-FIELD ANALYSES AND SIMULATION RESULTS

In this section, we compare numerical results from the three [(1×1)-, (2×1)-, and (2×2)-cluster] mean-field analyses with simulation results. Although analytical solutions can be derived for (1×1)- and (2×1)-cluster mean-field analyses, similar solutions cannot be obtained explicitly for the (2×2)-cluster mean-field analysis (see the last section). We therefore obtain numerical solutions for this case using Newton's iteration method. In this study, we use the function FindRoot, which is based on the Newton's method, in the software package Mathematica 12.0. In all the simulations below, we set $L = 100$ and calculate the steady-state value of ρ for 10^7 time steps after evolving the system for 10^7 time steps, unless otherwise specified.

A. Special case: $\omega_{\mathrm{A}1} = \omega_{\mathrm{A}2}$ and $\omega_{\mathrm{D}1} = \omega_{\mathrm{D}2}$

In this special case, the results of all three mean-field analyses for the two directional flows give us the following equation for the steady-state value of ρ:

$$\rho = \frac{\omega_{\mathrm{A}}}{\omega_{\mathrm{A}} + \omega_{\mathrm{D}}}. \qquad (51)$$

Equation (51) shows that all the mean-field analyses give a result independent of p for this special case.

Figure 2 compares the simulation and mean-field values of ρ as functions of p for various $(\omega_{\mathrm{A}}, \omega_{\mathrm{D}}) \in \{(0.004, 0.008), (0.005, 0.005), (0.008, 0.004)\}$ for (a) unidirectional and (b) bidirectional flows. In both figures, the simulations show very good agreement with our mean-field analyses.

B. General case

In this subsection, we consider the more general case with $\omega_{\mathrm{A}1} \neq \omega_{\mathrm{A}2}$ or $\omega_{\mathrm{D}1} \neq \omega_{\mathrm{D}2}$.

Table I summarizes the three kinds of mean-field analyses [(1×1)-, (2×1)-, and (2×2)-cluster cases], which were discussed in detail in Sec. III. In the (1×1)-cluster mean-field analysis, ρ is a function of ω independent of p; therefore, the influence of p cannot be captured. In contrast, the (2×1)-cluster mean-field analysis gives ρ as a function of both ω and p; however, the function is the same for unidirectional and bidirectional flows. Finally, in the (2×2)-cluster mean-field analysis, ρ is a function of both ω and p, and the function is different for unidirectional and bidirectional flows. From this discussion, we expect that the (1×1)-cluster mean-field analysis to approximate the simulation results well in cases where $p \sim 1$, whereas the (2×1)-cluster mean-field analysis roughly captures the influence of p, and the (2×2)-cluster

TABLE I. Comparison of the three kinds of mean-field analyses.

Mean-field analysis	Direction	p
(1×1)-cluster	Independent	Independent
(2×1)-cluster	Independent	Dependent
(2×2)-cluster	Dependent	Dependent

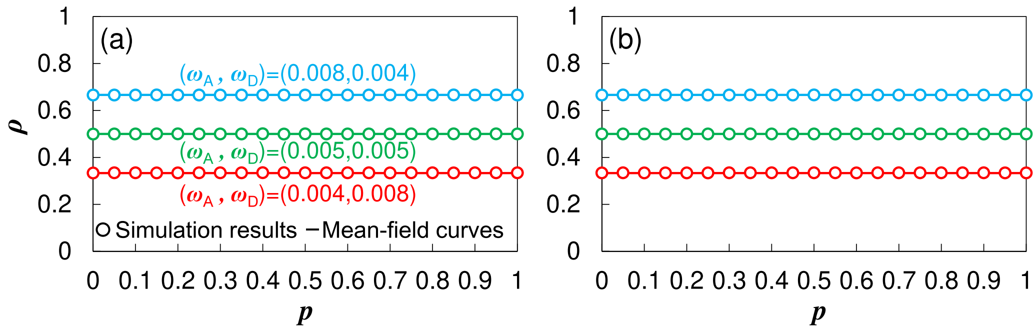

FIG. 2. Simulation (circles) and mean-field values (curves) of ρ for (a) unidirectional and (b) bidirectional flows as functions of p with $(\omega_A, \omega_D) \in \{(0.004, 0.008)$ (red), $(0.005, 0.005)$ (green), $(0.008, 0.004)$ (blue)$\}$.

mean-field analysis reproduces the difference between two directional flows.

We next consider eight fundamental cases; specifically,

$$\begin{aligned}
&\text{(a) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.008, 0.004, 0.004, 0.004).\\
&\text{(b) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.002, 0.004, 0.004, 0.004).\\
&\text{(c) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.008, 0.004, 0.004).\\
&\text{(d) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.002, 0.004, 0.004).\\
&\text{(e) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.008, 0.004).\\
&\text{(f) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.002, 0.004).\\
&\text{(g) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.004, 0.008).\\
&\text{(h) } (\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.004, 0.002).
\end{aligned} \tag{52}$$

These cases enable us to investigate the influence of changes in ω_A or ω_D depending on the state of the corresponding site in the other lane. Specifically, cases (a)–(d) exhibit the influence of ω_A, whereas cases (e)–(h) show that of ω_D. Figure 3 plots the simulation and mean-field curves of ρ for the parameter sets of cases (a)–(h), adding error bars of one standard error for 10 trials, each of whose averaging time is 10^6 time steps. We summarize specific values of the standard errors in Appendix F.

As expected, Fig. 3 shows that (1) the (1×1)-cluster mean-field analysis agrees very well with the simulation results only near $p = 1$, (2) the (2×1)-cluster mean-field results capture well the qualitative change of ρ with (p, ω) for both directions, and (3) the (2×2)-cluster case not only succeeds in reproducing the difference between the two directional flows, but also improves the accuracy of the approximations, mainly for small p. We note that in case of $p = 1$ the deviation of the numerical results from simulations can be the smallest for (1×1)-cluster mean-field analysis, depending on ω, e.g., cases (a) and (f), although the (1×1)-cluster analysis does not capture the p-dependence.

In addition, we observe an interesting phenomenon in Fig. 3. Specifically, there are smaller discrepancies between the simulations and numerical results for the (2×1)- and (2×2)-cluster mean-field analyses for unidirectional flows than for bidirectional flows. In other words, the numerical results from the (2×1)-cluster mean-field analysis are already good approximations for unidirectional flows.

We note that the magnitude relation between the simulation values for the two directional flows can reverse dependent on p for some cases, i.e., Figs. 3(b), 3(e), and 3(g). Considering tiny standard errors, summarized in Table V in Appendix F, this phenomenon is not due to numerical errors; however, even the (2×2)-cluster analysis cannot reproduce it well.

In the following subsections, we discuss in detail the discrepancies between the numerical results for the (2×1)- and (2×2)-cluster mean-field analyses and for (p, ω)-dependence of ρ for the two directional flows.

1. Discrepancies between the numerical results from the (2 × 1)- and (2 × 2)-cluster mean-field analyses

Figure 3 shows that the discrepancies are smaller for unidirectional flows than for bidirectional flows.

To investigate this phenomenon, we first define the following correlation between two adjacent clusters, each of which consists of $2(= 2 \times 1)$ sites:

$$\mu\begin{pmatrix} i & k \\ j & l \end{pmatrix} = P\begin{pmatrix} i & k \\ j & l \end{pmatrix} - P\begin{pmatrix} i \\ j \end{pmatrix} P\begin{pmatrix} k \\ l \end{pmatrix}, \tag{53}$$

where $i, j, k, l \in \{0, 1\}$. From the definition, $|\mu| = 0$ indicates that there is no correlation between the two adjacent clusters, whereas $\mu > 0$ ($\mu < 0$) indicates that the possibility of the spontaneous appearance of two adjacent clusters is larger (smaller) than the possibility under the assumption that clusters appear randomly in the system. This explains why the (2×2)-cluster mean-field analysis improves the approximate accuracy more than the (2×1)-cluster analysis for relatively large $|\mu|$. In contrast, the (2×1)-cluster analysis is already a good approximation for relatively small $|\mu|$. We note that in Appendix G we introduce another measure for analyzing the discrepancies, which is simpler and more qualitative.

Figure 4 compares 16 kinds of μ for unidirectional and bidirectional flows for various $p \in \{0, 0.01, 0.1, 0.2, 0.5, 1\}$, fixing $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ and $\rho = 0.5$, adding error bars of one standard error for 100 (for $p = 0$) and 10 (for $p \in \{0.01, 0.1, 0.2, 0.5, 1\}$) trials. The relatively small error bars indicate an independence of μ on initial conditions. We set $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ to observe the

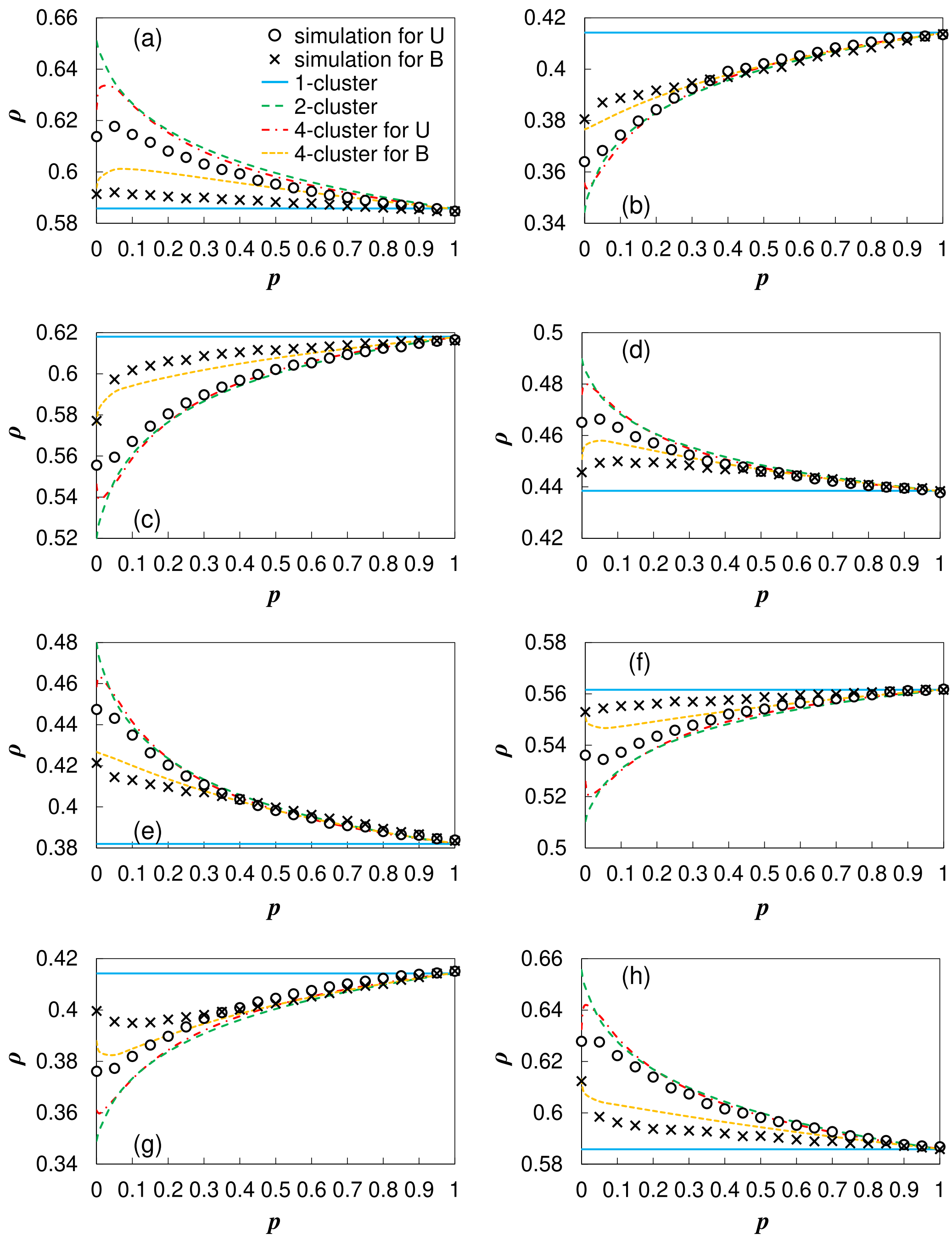

FIG. 3. Values of ρ from simulations (symbols) and mean-field calculations (blue solid, green dashed, red dash-dotted, and orange dotted curves) as functions of p for (a) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.008, 0.004, 0.004, 0.004)$, (b) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.002, 0.004, 0.004, 0.004)$, (c) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.008, 0.004, 0.004)$, (d) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.002, 0.004, 0.004)$, (e) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.008, 0.004)$, (f) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.002, 0.004)$, (g) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.004, 0.008)$, and (h) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.004, 0.002)$. Black circles represent the simulation results for unidirectional flows (U), whereas black crosses represent those for bidirectional flows (B). The blue, green, red, and orange curves represent the (1×1)-cluster, (2×1)-cluster, (2×2)-cluster (U), and (2×2)-cluster (B) mean-field values, respectively. Error bars are almost within the size of the symbols.

pure correlations caused by p because Langmuir kinetics reduce the correlations. We note that, strictly speaking, we cannot discuss the case $p = 0$ in the same way as for the cases with $p > 0$ because in the former case, all the particles stop at some point, depending on the initial configurations, due to the blocking effect (which we discuss later). Therefore, Fig. 4(a) is just presented for reference (as are the points at $p = 0$ in Fig. 5).

We observe two important phenomena in Fig. 4. First, for the two directional flows $|\mu|$ approaches to 0 as p increases; i.e., the correlation becomes smaller with larger p. This qualitatively explains that for larger p, the discrepancies between

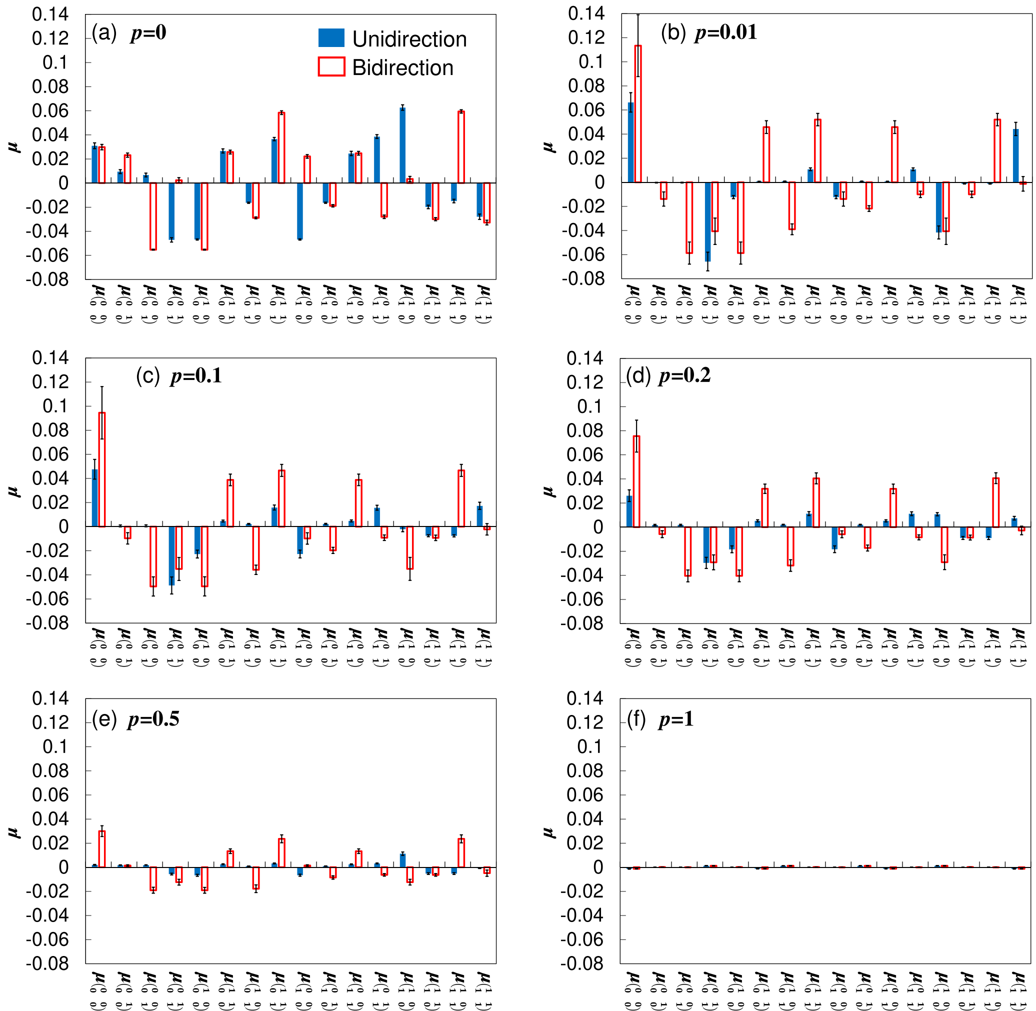

FIG. 4. Calculated values of 16 kinds of μ of unidirectional (blue) and bidirectional (red) flows for various $p \in \{0, 0.01, 0.1, 0.2, 0.5, 1\}$, fixing $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ and $\rho = 0.5$. We note (1) that those values are calculated by averaging 100 different initial configurations for $p = 0$ and (2) that they are calculated for 10^9 time steps after evolving the system with $p = 0.01$ for 10^9 time steps because it takes more time for the system to evolve into the steady state. Error bars represent one standard error. For depicting error bars, the evolving and calculated time is set to 10^8 (for $p = 0.01$) and 10^6 (for $p \in \{0, 0.1, 0.2, 0.5, 1\}$), respectively.

the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses become smaller, which explains the overlap of the (2×1)- and (2×2)-cluster theoretical curves in Fig. 3.

Second, we confirm that in most cases many values of $|\mu|$ for unidirectional flows are smaller than those for bidirectional flows with the same values of p [Figs. 4(b)–4(e)]; i.e., the correlations for unidirectional flows are smaller than those for bidirectional flows. This qualitatively explains that there are smaller discrepancies between the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses for unidirectional flows compared with those for bidirectional flows. In contrast, for $p = 0$ [Fig. 4(a)], some values of $|\mu|$ become large even for unidirectional flows, and therefore, the discrepancies also become large (see Fig. 3).

2. (p, ω)-dependence of ρ

This subsection discusses the (p, ω)-dependence of ρ, observed from Fig. 3, for the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses and for the simulation results.

First, to investigate the influence of p on ρ, we investigate $P\binom{0}{0}$, $P\binom{1}{0} + P\binom{0}{1}$, and $P\binom{1}{1}$, fixing $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ and $\rho = 0.5$, as in Fig. 5. We note that $P\binom{0}{0} = P\binom{1}{1}$ when $\rho = 0.5$.

Figure 5 shows (1) that $P\binom{1}{1}$ and $P\binom{0}{0}$ increase and $P\binom{1}{0} + P\binom{0}{1}$ decreases with smaller p for the two directional flows, (2) that the degrees of those changes are greater for unidirectional flows than for bidirectional flows, excluding the case with $p = 0$ for unidirectional flows, and (3) that $P\binom{1}{1}$ and $P\binom{0}{0}$ decrease and $P\binom{1}{0} + P\binom{0}{1}$ increases abruptly from $p = 0.01$ to $p = 0$ for unidirectional flows.

Three effects can qualitatively explain these phenomena: (1) the trapping, (2) jamming, and (3) blocking effects. First,

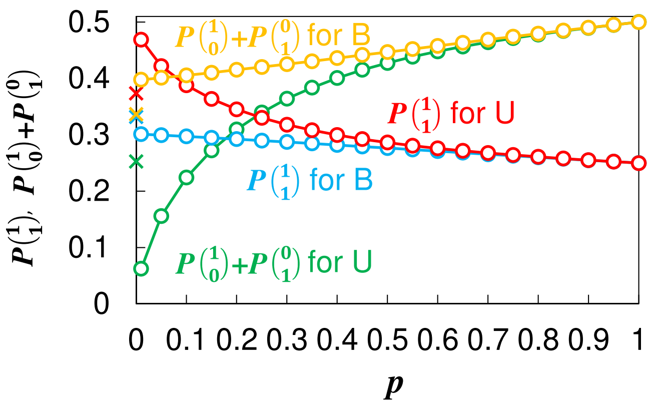

FIG. 5. Simulation results for steady-state values of $P\binom{1}{0}+P\binom{0}{1}$, and $P\binom{1}{1}$ as functions of p, fixing $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ and $\rho = 0.5$. The points between $p = 0$ and $p = 0.05$ are for $p = 0.01$. We note (1) that the values for $p = 0$ are calculated by averaging 100 different initial configurations and (2) that they are calculated for 10^9 time steps after evolving the system with $p = 0.01$ for 10^9 time steps because it takes more time for the system to evolve into the steady state.

in the trapping effect, for small p, particles can become trapped in the state $\binom{1}{1}$, which leads to an increase in $P\binom{1}{1}$ and $P\binom{0}{0}$, and a decrease in $P\binom{1}{0}+P\binom{0}{1}$, as shown in Fig. 6. We stress that the trapping effect works in common between the two directional flows.

Contrarily, in the jamming effect, for small p, the particles following a trapped particle can become involved in a jam, as shown in Fig. 7. For unidirectional flows, this effect works the same as the trapping effect; specifically, $P\binom{1}{1}$ and $P\binom{0}{0}$ increase and $P\binom{1}{0}+P\binom{0}{1}$ decreases with smaller p. In contrast, for bidirectional flows, this effect tends to counter the trapping

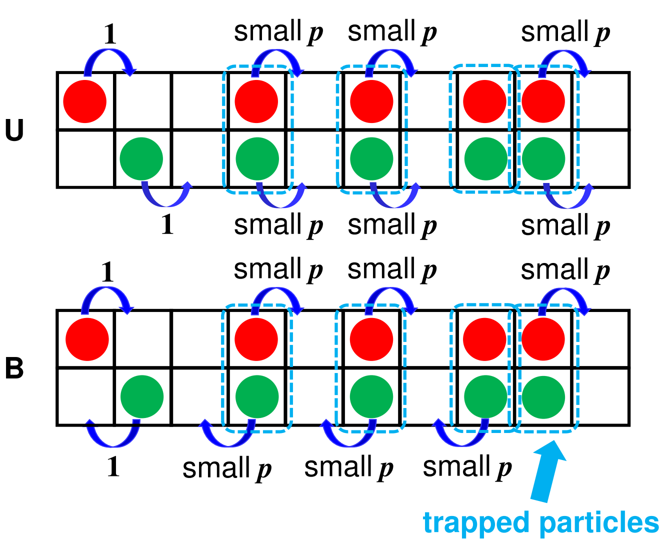

FIG. 6. Schematic illustration of the trapping effect. Langmuir kinetics are not considered here; the case $L = 10$ is shown.

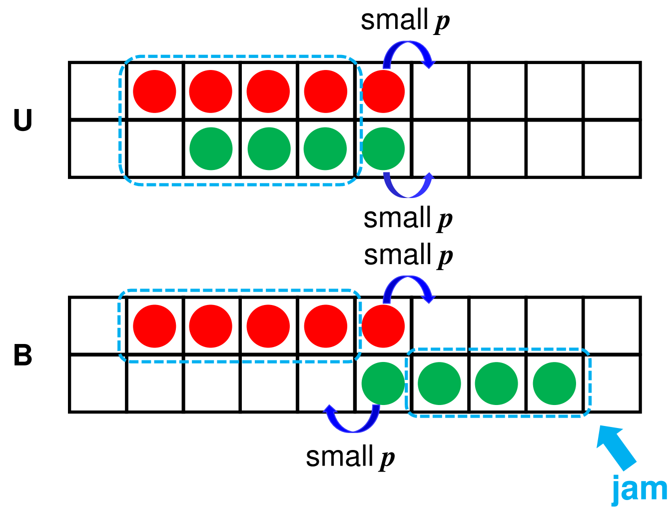

FIG. 7. Schematic illustration of the jamming effect. Langmuir kinetics are not considered here; the case $L = 10$ is shown.

effect; specifically, $P\binom{1}{1}$ and $P\binom{0}{0}$ decrease, and $P\binom{1}{0}+P\binom{0}{1}$ increases with smaller p. We note that the jamming effect is smaller than the trapping effect because jams occur only after many trapped particles appear. Because the jamming effect influences each P oppositely in the two directional flows, the degrees of those changes are greater for unidirectional flows than for bidirectional flows, excluding the case with $p = 0$.

Finally, the blocking effect appears only for $p = 0$, as shown in Fig. 8. Once the particles are trapped in the state $\binom{1}{1}$, this state never changes for $p = 0$ and $\omega_{D2} = 0$; therefore, the isolated particles between the two clusters of $\binom{1}{1}$ must finally make the state $\binom{1}{0}$ or $\binom{0}{1}$. Because this effect counters the trapping and jamming effects, $P\binom{1}{1}$ and $P\binom{0}{0}$ decrease and $P\binom{1}{0}+P\binom{0}{1}$ increases from $p = 0.01$ to $p = 0$ for unidirectional flows. We note (1) that this effect is virtually the same as the jamming effect for bidirectional flows and (2) that the blockages can be dismantled if $\omega_{D2} > 0$.

We can also confirm the existence of those three effects in Fig. 4. Table II summarizes the kinds of μ, which become

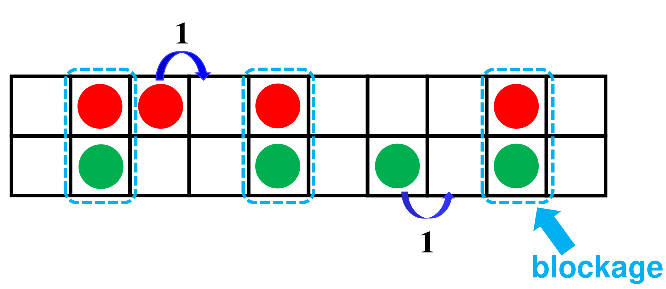

FIG. 8. Schematic illustration of the blocking effect for unidirectional flows. Langmuir kinetics are not considered here; the case $L = 10$ is shown.

TABLE II. Kinds of μ, which generally become larger with smaller p, for each direction (see Fig. 4).

Direction	Corresponding μ
Unidirection	$\mu\begin{pmatrix}0&0\\0&0\end{pmatrix}, \mu\begin{pmatrix}1&1\\0&1\end{pmatrix}, \mu\begin{pmatrix}0&1\\1&1\end{pmatrix}, \mu\begin{pmatrix}1&1\\1&1\end{pmatrix}$
Bidirection	$\mu\begin{pmatrix}0&0\\0&0\end{pmatrix}, \mu\begin{pmatrix}1&1\\0&0\end{pmatrix}, \mu\begin{pmatrix}1&1\\0&1\end{pmatrix}, \mu\begin{pmatrix}0&0\\1&1\end{pmatrix}, \mu\begin{pmatrix}1&0\\1&1\end{pmatrix}$

larger with smaller p. We note that Table II excludes the case $p = 0$ for unidirectional flows because of its singularity.

For unidirectional flows, the trapping and jamming effects explain why all the listed μ values become larger with smaller p. Contrarily, for bidirectional flows, the fact that $\mu\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $\mu\begin{pmatrix}1&0\\1&1\end{pmatrix}$, and $\mu\begin{pmatrix}1&1\\0&1\end{pmatrix}$ become larger with smaller p can be explained by the trapping effect, whereas the fact that $\mu\begin{pmatrix}1&1\\0&0\end{pmatrix}$ and $\mu\begin{pmatrix}0&0\\1&1\end{pmatrix}$ become larger with smaller p can be explained by the jamming effect. Conversely, for $p = 0$, $\mu\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $\mu\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\mu\begin{pmatrix}0&0\\1&1\end{pmatrix}$, and $\mu\begin{pmatrix}1&0\\1&1\end{pmatrix}$ also soar for unidirectional flows, indicating the blocking effect.

On the basis aforementioned discussions, we finally summarize the influences of the three effects on $P\binom{0}{0}$, $P\binom{1}{0} + P\binom{0}{1}$, and $P\binom{1}{1}$ in Table III. We again stress that the results shown in Fig. 5 can be explained from Table III, noting that the trapping effect is stronger than the jamming effect and that the blocking effect only appears for $p = 0$.

Next, we consider the influences of ω on ρ, $P\binom{0}{0}$, $P\binom{1}{0} + P\binom{0}{1}$, and $P\binom{1}{1}$.

From the definition of ω and the state-transition of diagram, as shown in Fig. 9, we can summarize the increase (+) and decrease (−) of each value by the change of ω in Table IV.

TABLE III. Influence of the three effects on $P\binom{0}{0}$, $P\binom{1}{0} + P\binom{0}{1}$, and $P\binom{1}{1}$. The sign + (−) indicates that the corresponding effect causes increases (decreases) in each P. Langmuir kinetics are not considered. We note that ++ (−−), used for unidirectional flows, represents the fact that the effect is larger than that for bidirectional flows.

Direction	Effect	$P\binom{0}{0}$	$P\binom{1}{0} + P\binom{0}{1}$	$P\binom{1}{1}$
Unidirection	Trapping effect	+	−	+
	Jamming effect	+	−	+
	Blocking effect	−	+	−
	Trapping + Jamming	++	−−	++
Bidirection	Trapping effect	+	−	+
	Jamming effect	−	+	−
	Blocking effect	−	+	−
	Trapping + Jamming	+	−	+

$$\boldsymbol{P\binom{0}{0} \underset{\omega_{D1}}{\overset{\omega_{A1}}{\rightleftarrows}} P\binom{1}{0} + P\binom{0}{1} \underset{\omega_{D2}}{\overset{\omega_{A2}}{\rightleftarrows}} P\binom{1}{1}}$$

FIG. 9. State-transition diagram for $P\binom{0}{0}$, $P\binom{1}{0} + P\binom{0}{1}$, and $P\binom{1}{1}$.

Blank cells indicate that the effect is indirect and the sign is not apparent; however, those indirect effects are small enough to be ignored in the following discussion.

On the basis of the aforementioned discussions of (p, ω)-dependence of ρ, we consider the following four phenomena in Fig. 3. We stress that those phenomena can be confirmed not only for the simulation results but also for the numerical results with (2×2)-cluster mean-field analysis.

a. Increase or decrease of ρ, depending on ω, in the region where $p \gtrapprox 0.1$. —Because $P\binom{0}{0}$ and $P\binom{1}{1}$ increase and $P\binom{1}{0} + P\binom{0}{1}$ decreases for smaller p, the effects of ω_{A1} and ω_{D2} (ω_{A2} and ω_{D1}) become larger (smaller). Therefore, considering Fig. 9 and Table IV, ρ increases (decreases) for smaller p for cases (a), (d), (e), and (h) [(b), (c), (f), and (g)] in the region where $p \gtrapprox 0.1$. For example, for case (a), $P\binom{0}{0}$ increases, and the effect of ω_{A1} is enhanced for smaller p, resulting in an increase in ρ for smaller p.

b. Differences in the degree of change between the two directional flows. —Because the effect of p on each P is larger for unidirectional flows than for bidirectional ones (see Fig. 5 and Table III), the degree of the change in ρ with p becomes greater for unidirectional flows than for bidirectional ones.

c. Change of the trend in unidirectional flows for $p = 0$. —For unidirectional flows, due to the blocking effect, the configuration changes drastically from very small $p(\neq 0)$ to $p = 0$, although Langmuir kinetics reduce that effect. This results in a trend change when $p = 0$, the extent of which depends on ω. We note that for unidirectional flows the influence of p on P is large enough so that it is not necessary to consider the influence of ω.

d. Change of the trend in bidirectional flows when p becomes smaller. —Unlike unidirectional flows, we cannot ignore the effect of ω on P for bidirectional flows. Therefore, for cases (a), (d), (f), and (g), where the influence of ω on P counters that of smaller p, the trend changes in the change-easing direction with smaller p. In contrast, for cases (b), (c), (e), and

TABLE IV. Increase (+) and decrease (−) of each value caused by the change in ω ($\Delta\omega > 0$). For example, the first row in the column for ρ indicates that ρ increases with larger ω_{A1}.

Change of ω	ρ	$P\binom{0}{0}$	$P\binom{1}{0} + P\binom{0}{1}$	$P\binom{1}{1}$
$\omega_{A1} \to \omega_{A1} + \Delta\omega$	+	−	+	
$\omega_{A2} \to \omega_{A2} + \Delta\omega$	+		−	+
$\omega_{D1} \to \omega_{D1} + \Delta\omega$	−	+	−	
$\omega_{D2} \to \omega_{D2} + \Delta\omega$	−		+	−

(h), where the influence of ω on P reinforces that of smaller p, the trend changes in the change-accelerating direction with smaller p.

V. CONCLUSION

In the present paper, we have investigated a two-lane extended LK-TASEP on a periodic lattice, where the hopping rate p and the attachment (detachment) rate ω_A (ω_D) vary depending on the state of the corresponding site in the other lane. The proposed model is new in that it introduces a varying rule for the attachment and detachment rate. We have investigated the steady-state global density for unidirectional and bidirectional flows using both computer simulations and mean-field analyses.

We have conducted three kinds of mean-field analyses [(1×1)-, (2×1)-, and (2×2)-cluster cases] and have compared them with simulation results. In the (1×1)-cluster mean-field analysis, the calculated value of ρ is in good agreement with the simulation results only in the region where p is near 1. In contrast, the (2×1)-cluster analysis can reproduce the rough trend of the simulation results for ρ as functions of p, even though it cannot distinguish between unidirectional and bidirectional flows. Finally, the (2×2)-cluster analysis not only reproduces the difference between the two directional flows but also approximates better the simulation results for ρ in case of small p.

We have therefore considered further the discrepancies between the numerical results from (2×1)- and (2×2)-cluster mean-field analyses for unidirectional flows—which are smaller than those for bidirectional flows—by calculating the correlations between two adjacent (2×1) clusters. We have also discussed the (p, ω)-dependence of ρ in terms of three effects (the trapping, jamming, and blocking effects). Those three effects by p and ω determine the trend of ρ.

We again emphasize that, despite its simplicity, the proposed model has a potential for applications to real-world phenomena. For example, for crowd dynamics (traffic flow) in a narrow passage (road), the proposed model can consider the velocity and inflow or outflow of pedestrians (vehicles). In particular, unlike previous models, our model makes it possible to consider of the dependence of the changes of the inflow and outflow on the lane state.

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APPENDIX A: INCLUSION OF EQ. (10) AND EXCLUSION OF THE OTHER SOLUTION OF EQ. (6)

In this Appendix, we discuss the inclusion of Eq. (10) and the exclusion of the other solution of Eq. (6).

As for the inclusion of Eq. (10), we first consider the case where $a > 0$; specifically,

$$a = \omega_{\mathrm{A1}} - \omega_{\mathrm{A2}} + \omega_{\mathrm{D1}} - \omega_{\mathrm{D2}} > 0 \tag{A1}$$

$$\Leftrightarrow \omega_{\mathrm{A1}} + \omega_{\mathrm{D1}} > \omega_{\mathrm{A2}} + \omega_{\mathrm{D2}}. \tag{A2}$$

In this case, we have

$$b = -2\omega_{\mathrm{A1}} + \omega_{\mathrm{A2}} - \omega_{\mathrm{D1}} \tag{A3}$$

$$= -(\omega_{\mathrm{A1}} + \omega_{\mathrm{D1}}) - \omega_{\mathrm{A1}} + \omega_{\mathrm{A2}} \tag{A4}$$

$$< -(\omega_{\mathrm{A2}} + \omega_{\mathrm{D2}}) - \omega_{\mathrm{A1}} + \omega_{\mathrm{A2}} \tag{A5}$$

$$= -\omega_{\mathrm{A1}} - \omega_{\mathrm{D2}}, \tag{A6}$$

resulting in $b < 0$. Noting that $a > 0$, $b < 0$ and $c > 0$, we obtain

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} > \frac{|b| - |b|}{2a} = 0. \tag{A7}$$

On the other hand, because

$$(\sqrt{b^2 - 4ac})^2 - (-b - 2a)^2 \tag{A8}$$

$$= 4a(-c - a - b) = 4a\omega_{\mathrm{D2}} > 0, \tag{A9}$$

we have

$$\sqrt{b^2 - 4ac} > -b - 2a \tag{A10}$$

$$\Leftrightarrow \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 1 \tag{A11}$$

when $-b - 2a \geqslant 0$. In addition, Eq. (A11) obviously holds in the case where $-b - 2a < 0$.

We then consider the case where $a < 0$; specifically,

$$\omega_{\mathrm{A1}} + \omega_{\mathrm{D1}} < \omega_{\mathrm{A2}} + \omega_{\mathrm{D2}}. \tag{A12}$$

Noting that $a < 0$ and $c > 0$, we obtain

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} > \frac{-b - |b|}{2a} \geqslant 0. \tag{A13}$$

On the other hand, because

$$(-b - 2a)^2 - (\sqrt{b^2 - 4ac})^2 \tag{A14}$$

$$= -4a(-c - a - b) = -4a\omega_{\mathrm{D2}} > 0, \tag{A15}$$

we have

$$\sqrt{b^2 - 4ac} < -b - 2a \Leftrightarrow \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 1, \tag{A16}$$

noting

$$-b - 2a = \omega_{\mathrm{A2}} + 2\omega_{\mathrm{D2}} - \omega_{\mathrm{D1}} \tag{A17}$$

$$= \omega_{\mathrm{A2}} + \omega_{\mathrm{D2}} + \omega_{\mathrm{D2}} - \omega_{\mathrm{D1}} \tag{A18}$$

$$> \omega_{\mathrm{A1}} + \omega_{\mathrm{D1}} + \omega_{\mathrm{D2}} - \omega_{\mathrm{D1}} \tag{A19}$$

$$= \omega_{\mathrm{A1}} + \omega_{\mathrm{D2}} > 0. \tag{A20}$$

Therefore, we obtain

$$0 < \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 1. \tag{A21}$$

We go on to discuss the exclusion of the other solution of Eq. (6); specifically,

$$\rho = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{A22}$$

First, we consider the case where $a > 0$; specifically,

$$\omega_{\mathrm{A1}} + \omega_{\mathrm{D1}} > \omega_{\mathrm{A2}} + \omega_{\mathrm{D2}}. \tag{A23}$$

In this case, we have

$$\begin{aligned} b &= -2\omega_{A1} + \omega_{A2} - \omega_{D1} && \text{(A24)}\\ &= -(\omega_{A1} + \omega_{D1}) - \omega_{A1} + \omega_{A2} && \text{(A25)}\\ &< -(\omega_{A2} + \omega_{D2}) - \omega_{A1} + \omega_{A2} && \text{(A26)}\\ &= -\omega_{A1} - \omega_{D2}, && \text{(A27)} \end{aligned}$$

resulting in $b < 0$. Noting that $a > 0$, $b < 0$ and $c > 0$, we have

$$\begin{aligned} &\frac{-b + \sqrt{b^2 - 4ac}}{2a} && \text{(A28)}\\ &> \frac{-b - b}{2a} = -\frac{b}{a} && \text{(A29)}\\ &= \frac{2\omega_{A1} - \omega_{A2} + \omega_{D1}}{\omega_{A1} - \omega_{A2} + \omega_{D1} - \omega_{D2}} && \text{(A30)}\\ &= \frac{\omega_{A1} - \omega_{A2} + \omega_{D1} + \omega_{A1}}{\omega_{A1} - \omega_{A2} + \omega_{D1} - \omega_{D2}} > 1. && \text{(A31)} \end{aligned}$$

Second, we consider the case where $a < 0$. In this case, because

$$-b + \sqrt{b^2 - 4ac} > -b + |b| \geqslant 0, \tag{A32}$$

we have

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0. \tag{A33}$$

Therefore, we obtain

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0 \vee \frac{-b + \sqrt{b^2 - 4ac}}{2a} > 1. \tag{A34}$$

Based on the discussion so far and the fact that ρ lies in the range $0 \leqslant \rho \leqslant 1$, we finally obtain Eq. (10).

APPENDIX B: DERIVATION OF EQ. (19)

In this Appendix, we discuss the detailed derivation of Eq. (19).

Substituting Eqs. (17) and (18) into Eqs. (15) and (16) and considering the steady state, i.e., $\frac{d}{dt}P\binom{i}{j} = 0$, we obtain the two following equations; specifically,

$$2\left[P\begin{pmatrix}1\\0\end{pmatrix}\right]^2 - 2pP\begin{pmatrix}0\\0\end{pmatrix}\left[1 - 2P\begin{pmatrix}1\\0\end{pmatrix} - P\begin{pmatrix}0\\0\end{pmatrix}\right] + 2\omega_{D1}P\begin{pmatrix}1\\0\end{pmatrix} - 2\omega_{A1}P\begin{pmatrix}0\\0\end{pmatrix} = 0 \tag{B1}$$

and

$$2pP\begin{pmatrix}0\\0\end{pmatrix}\left[1-2P\begin{pmatrix}1\\0\end{pmatrix} - P\begin{pmatrix}0\\0\end{pmatrix}\right] - 2\left[P\begin{pmatrix}1\\0\end{pmatrix}\right]^2 + \omega_{A1}P\begin{pmatrix}0\\0\end{pmatrix} + \omega_{D2}\left[1 - 2P\begin{pmatrix}1\\0\end{pmatrix} - P\begin{pmatrix}0\\0\end{pmatrix}\right] - (\omega_{A2} + \omega_{D1})P\begin{pmatrix}1\\0\end{pmatrix} = 0. \tag{B2}$$

Adding Eqs. (B1) and (B2), we get

$$P\begin{pmatrix}0\\0\end{pmatrix} = \frac{\omega_{D1} - 2\omega_{D2} - \omega_{A2}}{\omega_{A1} + \omega_{D2}}P\begin{pmatrix}1\\0\end{pmatrix} + \frac{\omega_{D2}}{\omega_{A1} + \omega_{D2}}. \tag{B3}$$

Substituting Eq. (B3) into Eq. (B1), Eq. (B1) finally reduces to a quadratic equation of $P\binom{1}{0}$, i.e., Eq. (19).

APPENDIX C: EXCLUSION OF THE OTHER SOLUTION OF EQ. (19)

In this Appendix, we discuss the exclusion of the other solution of Eq. (19); specifically,

$$P\begin{pmatrix}1\\0\end{pmatrix} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{C1}$$

Noting Eq. (26) and (28), we have

$$-B - \sqrt{B^2 - 4AC} < -B - |B| \leqslant 0; \tag{C2}$$

therefore,

$$\frac{-B - \sqrt{B^2 - 4AC}}{2A} < 0. \tag{C3}$$

Because $P\binom{1}{0}$ lies in the range $0 \leqslant P\binom{1}{0} \leqslant 1$, this solution is inappropriate, and because the system must evolve into only one steady state, we obtain Eq. (25).

APPENDIX D: INDEPENDENT MASTER EQUATIONS FOR THE (2 × 2)-CLUSTER MEAN-FIELD ANALYSIS FOR UNIDIRECTIONAL FLOWS OTHER THAN EQ. (34)

In this Appendix, we summarize the independent master equations for (2 × 2)-cluster probabilities for unidirectional flows other than Eq. (34).

For $P\begin{pmatrix}1&0\\1&0\end{pmatrix}$, $P\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $P\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $P\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $P\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $P\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $P\begin{pmatrix}1&1\\1&0\end{pmatrix}$, and $P\begin{pmatrix}1&1\\0&1\end{pmatrix}$, the master equations can be expressed as

$$\begin{aligned} \frac{dP\begin{pmatrix}1&0\\1&0\end{pmatrix}}{dt} = &\left[P\begin{pmatrix}1&0&0\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&0&0\\1&\underline{1}&\underline{0}\end{pmatrix} + P\begin{pmatrix}0&1&0\\1&\underline{0}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&1&0\\1&\underline{0}&\underline{0}\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&0\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&1\end{pmatrix}\right.\\ &\left.+ P\begin{pmatrix}1&0&0\\\underline{1}&\underline{1}&0\end{pmatrix} + P\begin{pmatrix}1&0&1\\\underline{1}&\underline{1}&0\end{pmatrix} + \omega_{A2}P\begin{pmatrix}0&0\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{A2}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{D1}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{D1}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}\right]\\ &- \left[2pP\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix} + 2\omega_{D2}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix} + 2\omega_{A1}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}\right], \end{aligned} \tag{D1}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0&1\\0&\underline{1}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&0&1\\1&\underline{1}&\underline{1}\end{pmatrix}+P\begin{pmatrix}0&1&1\\1&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1&1\\1&\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\\underline{1}&\underline{1}\end{pmatrix}\\&+\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}\Bigg]-\Bigg[2pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&0\end{pmatrix}\\&+pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&1\end{pmatrix}+pP\begin{pmatrix}1&1&1\\\underline{1}&\underline{1}&0\end{pmatrix}+4\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{1}\end{pmatrix}\Bigg],\end{aligned}\tag{D2}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0&0\\0&\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&0&0\\1&\underline{0}&\underline{0}\end{pmatrix}+P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&0\end{pmatrix}+P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&1\end{pmatrix}+P\begin{pmatrix}1&0&0\\\underline{0}&\underline{1}&0\end{pmatrix}+P\begin{pmatrix}1&0&1\\\underline{0}&\underline{1}&0\end{pmatrix}\\&+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}\Bigg]-\Bigg[P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}+P\begin{pmatrix}0&1&0\\1&\underline{0}&\underline{0}\end{pmatrix}\\&+pP\begin{pmatrix}1&1&0\\1&\underline{0}&\underline{0}\end{pmatrix}+2\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}\Bigg],\end{aligned}\tag{D3}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}0&1&0\\\underline{0}&\underline{1}&0\end{pmatrix}+pP\begin{pmatrix}0&1&1\\\underline{0}&\underline{1}&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}0&1\\\underline{1}&\underline{0}\end{pmatrix}\\&+\omega_{\mathrm{D2}}P\begin{pmatrix}0&1\\\underline{0}&\underline{1}\end{pmatrix}\Bigg]-\Bigg[P\begin{pmatrix}1&0&1\\0&\underline{0}&\underline{0}\end{pmatrix}+P\begin{pmatrix}0&0&1\\1&\underline{0}&\underline{0}\end{pmatrix}+2pP\begin{pmatrix}1&0&1\\1&\underline{0}&\underline{0}\end{pmatrix}+P\begin{pmatrix}0&1&0\\\underline{0}&\underline{0}&0\end{pmatrix}+P\begin{pmatrix}0&1&0\\\underline{0}&\underline{0}&1\end{pmatrix}\\&+2\omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}0&1\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\\underline{0}&\underline{0}\end{pmatrix}\Bigg],\end{aligned}\tag{D4}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0&1\\0&\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&0&1\\1&\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&0\end{pmatrix}+pP\begin{pmatrix}1&1&1\\\underline{0}&\underline{1}&0\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}\\&+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}\Bigg]-\Bigg[P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&0\end{pmatrix}+P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&1\end{pmatrix}+P\begin{pmatrix}0&1&1\\1&\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&1&1\\1&\underline{0}&\underline{0}\end{pmatrix}\\&+2\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}+2\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}\Bigg],\end{aligned}\tag{D5}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0&0\\0&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&0&0\\1&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&0\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&1\end{pmatrix}+pP\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{0}&\underline{1}\end{pmatrix}\\&+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}\Bigg]-\Bigg[P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}+P\begin{pmatrix}0&1&0\\1&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1&0\\1&\underline{0}&\underline{1}\end{pmatrix}\\&+P\begin{pmatrix}1&0&0\\0&\underline{1}&\underline{0}\end{pmatrix}+P\begin{pmatrix}1&0&1\\0&\underline{1}&\underline{0}\end{pmatrix}+2\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}+2\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}\Bigg],\end{aligned}\tag{D6}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0&1\\0&\underline{1}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&0&1\\1&\underline{1}&\underline{0}\end{pmatrix}+P\begin{pmatrix}0&1&1\\1&\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&1&1\\1&\underline{0}&\underline{0}\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&0\end{pmatrix}+pP\begin{pmatrix}1&1&1\\\underline{1}&\underline{1}&0\end{pmatrix}\\&+\omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\\underline{1}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{1}\end{pmatrix}\Bigg]-\Bigg[P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&0\end{pmatrix}+P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&1\end{pmatrix}\\&+pP\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}+2\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}\Bigg],\end{aligned}\tag{D7}$$

and

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{dt}=&\Bigg[P\begin{pmatrix}1&0&1\\0&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&0&1\\1&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}\\&+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}\Bigg]-\Bigg[P\begin{pmatrix}0&1&1\\1&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1&1\\1&\underline{0}&\underline{1}\end{pmatrix}+2pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&0\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&1\end{pmatrix}\\&+pP\begin{pmatrix}1&1&1\\\underline{0}&\underline{1}&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}+2\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}\Bigg].\end{aligned}\tag{D8}$$

Inserting Eq. (33) into Eqs. (D1)–(D8) and noting that in the steady state, $\frac{d}{dt}P\left(\begin{smallmatrix}i&k\\j&l\end{smallmatrix}\right)=0$ $(i,j,k,l\in\{0,1\})$, we get

$$\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}\\
&\quad\frac{P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}\\
&\quad+\omega_{\mathrm{A2}}P\begin{pmatrix}0&0\\1&0\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\1&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\1&1\end{pmatrix}-2pP\begin{pmatrix}1&0\\1&0\end{pmatrix}\\
&\quad-2\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\1&0\end{pmatrix}-2\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\1&0\end{pmatrix}=0,
\end{aligned}\tag{D9}$$

$$\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}\\
&+\omega_{\mathrm{A2}}P\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)+\omega_{\mathrm{A2}}P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)+\omega_{\mathrm{A2}}P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)+\omega_{\mathrm{A2}}P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)-\frac{2pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}\\
&-\frac{pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-4\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\1&1\end{pmatrix}=0,
\end{aligned}\tag{D10}$$

$$\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}\\
&\quad+\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\0&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&\quad+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\1&0\end{pmatrix}-P\begin{pmatrix}1&0\\0&0\end{pmatrix}-\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-2\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}\\
&\quad-\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}-\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}=0,
\end{aligned}\tag{D11}$$

$$\begin{aligned}
&P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)+\frac{pP\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\0&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}0&1\\1&0\end{pmatrix}\\
&\quad+\omega_{\mathrm{D2}}P\begin{pmatrix}0&1\\0&1\end{pmatrix}-\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}-\frac{P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}-\frac{2pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}\\
&\quad-\frac{P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-2\omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\0&0\end{pmatrix}-\omega_{\mathrm{D1}}P\begin{pmatrix}0&1\\0&0\end{pmatrix}-\omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\0&0\end{pmatrix}=0,
\end{aligned}\tag{D12}$$

$$\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}\\
&\quad+\omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\0&0\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\1&0\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}-\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}\\
&\quad-\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}
\end{aligned}$$

$$-2\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}-2\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}=0, \tag{D13}$$

$$\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}$$
$$+pP\begin{pmatrix}1&0\\1&0\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\0&1\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\1&1\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}-P\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$-\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}-\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}$$
$$-2\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}-2\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}=0, \tag{D14}$$

$$\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}$$
$$+\frac{pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\1&0\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\1&0\end{pmatrix}$$
$$+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\1&1\end{pmatrix}-pP\begin{pmatrix}1&1\\1&0\end{pmatrix}-\frac{P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-2\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\1&0\end{pmatrix}$$
$$-\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\1&0\end{pmatrix}-\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\1&0\end{pmatrix}=0, \tag{D15}$$

and

$$pP\begin{pmatrix}1&1\\1&0\end{pmatrix}+\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\0&1\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\1&1\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}-\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{2pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}$$
$$-\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}-\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}-2\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}=0, \tag{D16}$$

respectively.

APPENDIX E: INDEPENDENT MASTER EQUATIONS FOR THE (2 × 2)-CLUSTER MEAN-FIELD ANALYSIS FOR BIDIRECTIONAL FLOWS OTHER THAN EQ. (44)

In this Appendix, we summarize the independent master equations for (2 × 2)-cluster probabilities for bidirectional flows other than Eq. (44).

For $P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, and $P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, the master equations can be expressed as

$$\frac{dP\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{dt}=\left[P\begin{pmatrix}1&0&0\\0&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&0&0\\1&\underline{0}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}0&1&0\\0&\underline{1}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1&0\\0&\underline{1}&\underline{1}\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&0\end{pmatrix}+pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&1\end{pmatrix}\right.$$
$$+P\begin{pmatrix}1&0&0\\\underline{0}&\underline{0}&1\end{pmatrix}+pP\begin{pmatrix}1&0&1\\\underline{0}&\underline{0}&1\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{0}&\underline{1}\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}$$
$$\left.+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}\right]-\left[2P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}+2\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}+2\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix}\right], \tag{E1}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)}{dt} =& \left[P\begin{pmatrix}1&0&1\\0&\underline{1}&\underline{1}\end{pmatrix} + pP\begin{pmatrix}1&0&1\\1&\underline{1}&\underline{1}\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&1\end{pmatrix} + pP\begin{pmatrix}1&1&1\\\underline{1}&\underline{0}&1\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\\underline{1}&\underline{1}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}\right] - \left[pP\begin{pmatrix}1&1&1\\0&\underline{1}&\underline{1}\end{pmatrix} + pP\begin{pmatrix}0&1&1\\0&\underline{1}&\underline{1}\end{pmatrix} + pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&0\end{pmatrix} + pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&1\end{pmatrix}\right.\\ &\left. + 4\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{1}\end{pmatrix}\right],\end{aligned} \tag{E2}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{dt} =& \left[P\begin{pmatrix}1&0&0\\0&\underline{0}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&0&0\\1&\underline{0}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}0&1&0\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&1&0\\\underline{0}&\underline{1}&0\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&1\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&0\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}\right] - \left[P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix} + P\begin{pmatrix}1&0&0\\\underline{0}&\underline{0}&1\end{pmatrix}\right.\\ &\left. + pP\begin{pmatrix}1&0&1\\\underline{0}&\underline{0}&1\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix} + 2\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}\right],\end{aligned} \tag{E3}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{dt} =& \left[P\begin{pmatrix}0&0\\\underline{0}&\underline{1}\end{pmatrix} + P\begin{pmatrix}0&1&0\\\underline{1}&\underline{0}&0\end{pmatrix} + P\begin{pmatrix}0&1&0\\\underline{1}&\underline{0}&1\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0&1\\\underline{1}&\underline{0}\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{D1}}P\begin{pmatrix}0&0\\\underline{1}&\underline{1}\end{pmatrix}\right] - \left[P\begin{pmatrix}0&0&0\\0&\underline{1}&\underline{0}\end{pmatrix} + P\begin{pmatrix}1&0&0\\0&\underline{1}&\underline{0}\end{pmatrix} + P\begin{pmatrix}1&0&0\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&0&0\\1&\underline{1}&\underline{0}\end{pmatrix} + P\begin{pmatrix}0&0&0\\\underline{1}&\underline{0}&1\end{pmatrix}\right.\\ &\left. + pP\begin{pmatrix}0&0&1\\\underline{1}&\underline{0}&1\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}0&0\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}0&0\\\underline{1}&\underline{0}\end{pmatrix} + 2\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\\underline{1}&\underline{0}\end{pmatrix}\right],\end{aligned} \tag{E4}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)}{dt} =& \left[P\begin{pmatrix}1&0&1\\0&\underline{0}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&0&1\\1&\underline{0}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}0&1&1\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&1&1\\0&\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}0&1\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix}\right] - \left[P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&0\end{pmatrix} + 2P\begin{pmatrix}1&1&0\\\underline{0}&\underline{0}&1\end{pmatrix} + pP\begin{pmatrix}1&1&1\\\underline{0}&\underline{0}&1\end{pmatrix} + 2\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}\right.\\ &\left. + 2\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix}\right],\end{aligned} \tag{E5}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{dt} =& \left[P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix} + P\begin{pmatrix}1&0&0\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&0&0\\1&\underline{1}&\underline{0}\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&0\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&1\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}0&0\\\underline{1}&\underline{0}\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}\right] - \left[pP\begin{pmatrix}1&0\\1&0\end{pmatrix} + pP\begin{pmatrix}0&1&0\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&1&0\\0&\underline{1}&\underline{0}\end{pmatrix}\right.\\ &\left. + P\begin{pmatrix}1&0&0\\1&\underline{0}&\underline{1}\end{pmatrix} + pP\begin{pmatrix}1&0&1\\\underline{1}&\underline{0}&1\end{pmatrix} + 2\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix} + 2\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}\right],\end{aligned} \tag{E6}$$

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{dt} =& \left[pP\begin{pmatrix}1&1\\\underline{0}&\underline{1}\end{pmatrix} + P\begin{pmatrix}1&0&1\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&0&1\\1&\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{0}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{1}\end{pmatrix}\right] - \left[pP\begin{pmatrix}0&1&1\\0&\underline{1}&\underline{0}\end{pmatrix} + pP\begin{pmatrix}1&1&1\\0&\underline{1}&\underline{0}\end{pmatrix} + P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&0\end{pmatrix} + 2P\begin{pmatrix}1&1&0\\\underline{1}&\underline{0}&1\end{pmatrix} + pP\begin{pmatrix}1&1&1\\\underline{1}&\underline{0}&1\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix} + 2\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{0}\end{pmatrix}\right],\end{aligned} \tag{E7}$$

and

$$\begin{aligned}\frac{dP\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{dt} =& \left[P\begin{pmatrix}1&0&0\\0&\underline{1}&\underline{1}\end{pmatrix} + pP\begin{pmatrix}1&0&0\\1&\underline{1}&\underline{1}\end{pmatrix} + P\begin{pmatrix}1&0&0\\\underline{1}&\underline{0}&1\end{pmatrix} + pP\begin{pmatrix}1&0&1\\\underline{1}&\underline{0}&1\end{pmatrix} + pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&0\end{pmatrix} + pP\begin{pmatrix}1&1&0\\\underline{1}&\underline{1}&1\end{pmatrix}\right.\\ &\left. + \omega_{\mathrm{A2}}P\begin{pmatrix}0&0\\\underline{1}&\underline{1}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{0}&\underline{1}\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\\underline{1}&\underline{0}\end{pmatrix} + \omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\\underline{1}&\underline{1}\end{pmatrix}\right] - \left[pP\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix} + pP\begin{pmatrix}0&1&0\\0&\underline{1}&\underline{1}\end{pmatrix}\right.\\ &\left. + pP\begin{pmatrix}1&1&0\\0&\underline{1}&\underline{1}\end{pmatrix} + 2\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\\underline{1}&\underline{1}\end{pmatrix}\right].\end{aligned} \tag{E8}$$

Inserting Eq. (33) into Eqs. (E1)–(E8) and noting that in the steady state $\frac{d}{dt}P\left(\begin{smallmatrix}i&k\\j&l\end{smallmatrix}\right)=0$ $(i, j, k, l \in \{0, 1\})$, we get

$$
\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}\\
&+\frac{pP\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}\\
&+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\0&1\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\1&1\end{pmatrix}+\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}-2P\begin{pmatrix}1&0\\0&1\end{pmatrix}-2\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}-2\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}=0,
\end{aligned}
\tag{E9}
$$

$$
\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}\\
&+\omega_{\mathrm{A2}}P\begin{pmatrix}0&1\\1&1\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\1&1\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\0&1\end{pmatrix}+\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\1&0\end{pmatrix}-\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}\\
&-\frac{pP\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}-4\omega_{\mathrm{D2}}P\begin{pmatrix}1&1\\1&1\end{pmatrix}=0,
\end{aligned}
\tag{E10}
$$

$$
\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}\\
&+\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\0&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\1&0\end{pmatrix}-P\begin{pmatrix}1&0\\0&0\end{pmatrix}-\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}\\
&-2\omega_{\mathrm{A1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}-\omega_{\mathrm{A2}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}-\omega_{\mathrm{D1}}P\begin{pmatrix}1&0\\0&0\end{pmatrix}=0,
\end{aligned}
\tag{E11}
$$

$$
\begin{aligned}
&P\begin{pmatrix}0&0\\0&1\end{pmatrix}+\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\frac{P\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}+\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\0&0\end{pmatrix}+\omega_{\mathrm{D2}}P\begin{pmatrix}1&0\\1&0\end{pmatrix}+\omega_{\mathrm{D1}}P\begin{pmatrix}0&1\\1&0\end{pmatrix}\\
&+\omega_{\mathrm{D1}}P\begin{pmatrix}0&0\\1&1\end{pmatrix}-\frac{P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}-\frac{2P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\1&j\end{smallmatrix}\right)}\\
&-\frac{P\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}-\omega_{\mathrm{D1}}P\begin{pmatrix}0&0\\1&0\end{pmatrix}-\omega_{\mathrm{A2}}P\begin{pmatrix}0&0\\1&0\end{pmatrix}-2\omega_{\mathrm{A1}}P\begin{pmatrix}0&0\\1&0\end{pmatrix}=0,
\end{aligned}
\tag{E12}
$$

$$
\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0&i\\0&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}+\frac{pP\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\1&j\end{smallmatrix}\right)}\\
&+\omega_{\mathrm{A1}}P\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)+\omega_{\mathrm{D2}}P\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)+\omega_{\mathrm{A1}}P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)+\omega_{\mathrm{D2}}P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)-\frac{P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}\\
&-\frac{2P\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-\frac{pP\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1&i\\0&j\end{smallmatrix}\right)}-2\omega_{\mathrm{D1}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}-2\omega_{\mathrm{A2}}P\begin{pmatrix}1&1\\0&0\end{pmatrix}=0,
\end{aligned}
\tag{E13}
$$

$$
\begin{aligned}
&P\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{P\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 1 & j\end{smallmatrix}\right)} + \frac{pP\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 1 & j\end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 0 & j\end{smallmatrix}\right)}\\
&\quad + \frac{P\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 0 & j\end{smallmatrix}\right)} + \omega_{\mathrm{A2}}P\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + \omega_{\mathrm{D1}}P\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} - pP\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\\
&\quad - \frac{pP\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)} - \frac{pP\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)} - \frac{P\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 0 & j\end{smallmatrix}\right)} - \frac{pP\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 0 & j\end{smallmatrix}\right)}\\
&\quad - 2\omega_{\mathrm{A1}}P\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} - 2\omega_{\mathrm{D2}}P\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = 0,
\end{aligned} \tag{E14}
$$

$$
\begin{aligned}
&pP\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} + \frac{P\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 1 & j\end{smallmatrix}\right)} + \frac{pP\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 1 & j\end{smallmatrix}\right)} + \omega_{\mathrm{A2}}P\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\\
&+ \omega_{\mathrm{D2}}P\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \frac{2P\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 0 & j\end{smallmatrix}\right)} - \frac{pP\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 0 & j\end{smallmatrix}\right)} - \frac{pP\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)}\\
&- \frac{pP\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)} - \frac{P\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 0 & j\end{smallmatrix}\right)} - 2\omega_{\mathrm{D2}}P\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} - \omega_{\mathrm{D1}}P\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} - \omega_{\mathrm{A2}}P\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = 0,
\end{aligned} \tag{E15}
$$

and

$$
\begin{aligned}
&\frac{P\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 0\\ 1 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 1 & j\end{smallmatrix}\right)} + \frac{pP\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 0\\ 1 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 1 & j\end{smallmatrix}\right)} + \frac{P\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 0 & j\end{smallmatrix}\right)} + \frac{pP\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)P\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}0 & i\\ 0 & j\end{smallmatrix}\right)}\\
&\quad + \frac{pP\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)} + \frac{pP\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)} + \omega_{\mathrm{A2}}P\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix} + \omega_{\mathrm{A2}}P\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \omega_{\mathrm{A1}}P\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\\
&\quad + \omega_{\mathrm{D2}}P\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - pP\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} - \frac{pP\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)} - \frac{pP\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)P\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)}{\sum_{i\in\{0,1\}}\sum_{j\in\{0,1\}}P\left(\begin{smallmatrix}1 & i\\ 1 & j\end{smallmatrix}\right)}\\
&\quad - 2\omega_{\mathrm{D2}}P\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} - \omega_{\mathrm{D1}}P\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} - \omega_{\mathrm{A2}}P\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = 0,
\end{aligned} \tag{E16}
$$

respectively.

APPENDIX F: STANDARD ERRORS OF THE SIMULATION RESULTS IN FIG. 3

In this Appendix, we summarize the standard errors of the simulation results in Fig. 3. The standard errors are calculated over 10 trials, each of whose averaging time is 10^6 time steps and initial configuration is varied. Although error bars are added in Fig. 3, they are almost within the size of the symbols due to their tiny size. Therefore, we show the specific values of the standard errors in Table V. We observe that all the values are less than 10^{-3}.

APPENDIX G: CLUSTER MEASURE FOR ANALYZING THE DISCREPANCIES IN FIG. 3

In this Appendix, we discuss an another approach analyzing the discrepancies between the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses (see Fig. 3).

We here introduce a new measure m, a revised version of one proposed in Ref. [47]; specifically,

$$m = \frac{N - e}{N - 1}. \tag{G1}$$

N is defined as the number of all the particles in the system, and e is as

$$e = \sum_{j=1}^{N} a_j, \tag{G2}$$

where

$$a_j = \begin{cases} 1 & \text{(if the } j\text{th particle is unblocked)} \\ 0 & \text{(otherwise)}. \end{cases} \tag{G3}$$

From the definition, $m = 0$, i.e., $e = N$, indicates that all the particles are unblocked (there is no cluster), whereas $m = 1$, i.e., $e = 1$, does that all the particles make one long cluster.

TABLE V. Standard errors of the simulation results of ρ for unidirectional flows (left panel) and bidirectional flows (right panel). The parameters are set as (a) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.008, 0.004, 0.004, 0.004)$, (b) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.002, 0.004, 0.004, 0.004)$, (c) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.008, 0.004, 0.004)$, (d) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.002, 0.004, 0.004)$, (e) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.008, 0.004)$, (f) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.002, 0.004)$, (g) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.004, 0.008)$, and (h) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.004, 0.004, 0.004, 0.002)$.

Unidirectional flows

p	Case (a)	Case (b)	Case (c)	Case (d)	Case (e)	Case (f)	Case (g)	Case (h)
0	0.00021	0.00023	0.00028	0.00027	0.00033	0.00027	0.00031	0.00028
0.05	0.00016	0.0005	0.00037	0.00036	0.00031	0.00024	0.00021	0.00022
0.1	0.00018	0.00035	0.00026	0.00028	0.0003	0.00018	0.00031	0.00026
0.15	0.00019	0.00026	0.00036	0.00028	0.00029	0.00023	0.00022	0.00034
0.2	0.00018	0.00041	0.00025	0.00029	0.00034	0.00032	0.0002	0.00037
0.25	0.00025	0.00025	0.00024	0.00027	0.00038	0.00027	0.00025	0.00038
0.3	0.00022	0.00029	0.00025	0.00025	0.00031	0.00019	0.00027	0.00039
0.35	0.00027	0.00018	0.0003	0.00029	0.00019	0.00024	0.00013	0.00027
0.4	0.00023	0.00038	0.00027	0.00027	0.00033	0.00026	0.00024	0.00047
0.45	0.0003	0.00043	0.00026	0.0003	0.00036	0.00032	0.00034	0.00053
0.5	0.00021	0.00041	0.00033	0.00026	0.00026	0.00029	0.00015	0.00037
0.55	0.00037	0.0004	0.0004	0.00026	0.00038	0.0002	0.00015	0.00029
0.6	0.0003	0.00029	0.00041	0.00026	0.00024	0.00026	0.00015	0.00031
0.65	0.00022	0.00031	0.0004	0.00028	0.00024	0.00021	0.00024	0.00046
0.7	0.00018	0.00043	0.00037	0.00031	0.00032	0.00033	0.00022	0.00033
0.75	0.00021	0.00031	0.00028	0.0003	0.00035	0.00019	0.00022	0.00037
0.8	0.00024	0.00046	0.00039	0.0003	0.00024	0.00029	0.00016	0.00047
0.85	0.0002	0.00047	0.00021	0.00025	0.00021	0.00024	0.00029	0.00044
0.9	0.00033	0.00034	0.00024	0.00033	0.0003	0.0002	0.00023	0.00031
0.95	0.00025	0.00036	0.00033	0.00024	0.00022	0.00023	0.00018	0.00032
1	0.00024	0.00032	0.00019	0.00023	0.00028	0.00022	0.00026	0.00028

Bidirectional flows

p	Case (a)	Case (b)	Case (c)	Case (d)	Case (e)	Case (f)	Case (g)	Case (h)
0	0.0002	0.00033	0.0003	0.00019	0.00041	0.00022	0.00024	0.0004
0.05	0.00022	0.00032	0.00029	0.00023	0.00041	0.00042	0.00014	0.00031
0.1	0.00023	0.00048	0.00034	0.00023	0.00026	0.0003	0.00031	0.00029
0.15	0.00014	0.00045	0.00029	0.00024	0.00024	0.00028	0.00022	0.00039
0.2	0.00021	0.00054	0.00033	0.00032	0.00046	0.00024	0.00028	0.00041
0.25	0.00019	0.00038	0.00046	0.00023	0.00019	0.00026	0.00028	0.00031
0.3	0.00016	0.00053	0.00042	0.00023	0.00031	0.00031	0.00014	0.00028
0.35	0.00029	0.00039	0.00026	0.00031	0.00028	0.00029	0.00035	0.00042
0.4	0.00019	0.00038	0.00028	0.00029	0.00026	0.0004	0.00021	0.00034
0.45	0.00011	0.00031	0.00022	0.00024	0.00033	0.00021	0.00021	0.00037
0.5	0.00026	0.00039	0.00026	0.00021	0.00034	0.00036	0.00029	0.00035
0.55	0.00026	0.00052	0.00036	0.00022	0.00023	0.00012	0.00026	0.00037
0.6	0.0002	0.00029	0.00022	0.00019	0.00035	0.00037	0.00021	0.00031
0.65	0.00023	0.00044	0.00025	0.00024	0.00025	0.00035	0.00024	0.00019
0.7	0.00025	0.00032	0.00026	0.00029	0.00046	0.00033	0.00025	0.00032
0.75	0.00024	0.00035	0.00031	0.00031	0.00026	0.00035	0.00017	0.00041
0.8	0.00026	0.00046	0.00026	0.00031	0.00027	0.0003	0.00039	0.00037
0.85	0.00027	0.00031	0.00028	0.00025	0.00035	0.00019	0.00038	0.00024
0.9	0.00026	0.0004	0.00025	0.00015	0.00027	0.00026	0.00027	0.00038
0.95	0.00024	0.00039	0.00033	0.00016	0.00027	0.00027	0.00018	0.00033
1	0.00029	0.00056	0.00038	0.00027	0.00033	0.00036	0.0002	0.00035

Therefore, we can interpret that large values of m indicate the large correlations between the sites.

Figure 10 plots $\frac{m}{\rho}$ for unidirectional and bidirectional flows at lane 1 as functions of p, fixing (a) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ and $\rho = 0.5$, and (b) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.008, 0.004, 0.004, 0.004)$. We stress that m generally increases as ρ increases; therefore, we use $\frac{m}{\rho}$ in order to compare the cases with different steady-state values of ρ.

We observe three important phenomena in Fig. 10. First, for the two directional flows m decreases as p increases except for $p = 0$; i.e., the correlation becomes smaller with larger p. This qualitatively explains that for larger p, the discrepancies between the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses become smaller, which also explains the overlap of the (2×1)- and (2×2)-cluster theoretical curves in Fig. 3. We emphasize that we cannot discuss the case $p = 0$ in the same way as for the case $p > 0$ (see Sec. IV B 2).

Second, we confirm that almost all the values of m for unidirectional flows are smaller than those for bidirectional flows with the same values of p; i.e., the correlations for unidirectional flows are smaller than those for bidirectional flows. This qualitatively explains that there are smaller discrepancies between the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses for unidirectional flows compared with those for bidirectional flows.

Finally, comparing Fig. 10(a) and 10(b), $\frac{m}{\rho}$ becomes smaller when Langmuir kinetics are added to the model. This clearly indicates that Langmuir kinetics reduce the correlations. Therefore, we set $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ to observe the pure correlation caused by p in Sec. IV B 1.

As the above discussions, this approach can also grasp the discrepancies between the numerical results from the (2×1)- and (2×2)-cluster mean-field analyses; however, we cannot discuss the details as in Sec. IV B 2, including that of the case $p = 0$.

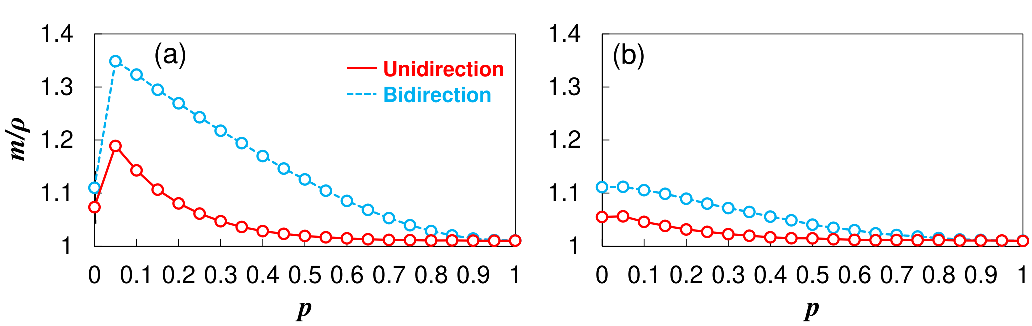

FIG. 10. Calculated values of $\frac{m}{\rho}$ for unidirectional (red solid) and bidirectional (blue dashed) flows at lane 1 as functions of p, fixing (a) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0, 0, 0, 0)$ and $\rho = 0.5$, and (b) $(\omega_{A1}, \omega_{A2}, \omega_{D1}, \omega_{D2}) = (0.008, 0.004, 0.004, 0.004)$. We note that for each trial m is calculated for 10^6 time steps after evolving the system for 10^6 time steps, and for each p the averages and standard errors are calculated over 10 trials. Error bars, representing one standard error, are almost the size of the symbols except for the case $p = 0$ for (a).

[1] A. Schadschneider, D. Chowdhury, and K. Nishinari, *Stochastic Transport in Complex Systems: From Molecules to Vehicles* (Elsevier, Amsterdam, 2010).
[2] C. T. MacDonald, J. H. Gibbs, and A. C. Pipkin, Biopolymers **6**, 1 (1968).
[3] C. T. MacDonald and J. H. Gibbs, Biopolymers **7**, 707 (1969).
[4] T. Chou, K. Mallick, and R. Zia, Rep. Prog. Phys. **74**, 116601 (2011).
[5] C. Appert-Rolland, M. Ebbinghaus, and L. Santen, Phys. Rep. **593**, 1 (2015).
[6] S. Ghosh, A. Dutta, S. Patra, J. Sato, K. Nishinari, and D. Chowdhury, Phys. Rev. E **99**, 052122 (2019).
[7] Y. Zarai, M. Margaliot, and T. Tuller, PLoS ONE **12**, e0182178 (2017).
[8] D. Helbing, Rev. Mod. Phys. **73**, 1067 (2001).
[9] H. Yamamoto, D. Yanagisawa, and K. Nishinari, J. Stat. Mech. (2017) 043204.
[10] C. Arita and A. Schadschneider, Math. Mod Meth. **25**, 401 (2015).
[11] H. Yamamoto, D. Yanagisawa, and K. Nishinari, Phys. Rev. E **100**, 042106 (2019).
[12] D. Yanagisawa, Collect. Dyn. **1**, 1 (2016).
[13] Q. Y. Hao, R. Jiang, M. B. Hu, and Q. S. Wu, Phys. Rev. E **82**, 022103 (2010).
[14] Q. Y. Hao, R. Jiang, M. B. Hu, B. Jia, and W. X. Wang, Sci. Rep. **6**, 19652 (2016).
[15] A. Gabel, P. L. Krapivsky, and S. Redner, Phys. Rev. Lett. **105**, 210603 (2010).
[16] C. Lin, G. Steinberg, and P. Ashwin, J. Stat. Mech. Theory (2011) P09027.
[17] R. Jiang, K. Nishinari, M. B. Hu, Y. H. Wu, and Q. S. Wu, J. Stat. Phys. **136**, 73 (2009).
[18] Q. Y. Hao, R. Jiang, C. Y. Wu, N. Guo, B. B. Liu, and Y. Zhang, Phys. Rev. E **98**, 062111 (2018).
[19] Q. Y. Hao, R. Jiang, M. B. Hu, Y. Zhang, C. Y. Wu, and N. Guo, Phys. Rev. E **100**, 032133 (2019).
[20] T. Ezaki and K. Nishinari, Phys. Rev. E **84**, 061149 (2011).
[21] S. Tsuzuki, D. Yanagisawa, and K. Nishinari, Phys. Rev. E **97**, 042117 (2018).
[22] S. Tsuzuki, D. Yanagisawa, and K. Nishinari, Phys. Rev. E **98**, 042102 (2018).
[23] C. Schiffmann, C. Appert-Rolland, and L. Santen, J. Stat. Mech. (2010) P06002.
[24] R. J. Harris and R. Stinchcombe, Physica A **354**, 582 (2005).
[25] T. Reichenbach, E. Frey, and T. Franosch, New J. Phys. **9**, 159 (2007).
[26] M. Evans, Y. Kafri, K. Sugden, and J. Tailleur, J. Stat. Mech. (2011) P06009.
[27] S. Klein, C. Appert-Rolland, and M. R. Evans, J. Stat. Mech. (2016) 093206.
[28] R. Juhász, J. Stat. Mech. (2010) P03010.
[29] V. Yadav, R. Singh, and S. Mukherji, J. Stat. Mech. (2012) P04004.
[30] A. Parmeggiani, T. Franosch, and E. Frey, Phys. Rev. Lett. **90**, 086601 (2003).
[31] M. R. Evans, R. Juhász, and L. Santen, Phys. Rev. E **68**, 026117 (2003).
[32] A. Parmeggiani, T. Franosch, and E. Frey, Phys. Rev. E **70**, 046101 (2004).
[33] S. Klumpp and R. Lipowsky, Europhys. Lett. **66**, 90 (2004).
[34] R. Wang, M. Liu, and R. Jiang, Physica A **387**, 457 (2008).
[35] M. Ebbinghaus, C. Appert-Rolland, and L. Santen, J. Stat. Mech. (2011) P07004.
[36] I. Dhiman and A. K. Gupta, Europhys. Lett. **107**, 20007 (2014).
[37] N. Sharma and A. Gupta, J. Stat. Mech. Theory (2017) 043211.
[38] S. Garg and I. Dhiman, Physica A **545**, 123356 (2020).
[39] H. D. Vuijk, R. Rens, M. Vahabi, F. C. MacKintosh, and A. Sharma, Phys. Rev. E **91**, 032143 (2015).
[40] S. Ichiki, J. Sato, and K. Nishinari, Eur. Phys. J. B **89**, 135 (2016).
[41] S. Ichiki, J. Sato, and K. Nishinari, J. Phys. Soc. Jpn. **85**, 044001 (2016).
[42] D. Yanagisawa and S. Ichiki, *International Conference on Cellular Automata, ACRI 2016*, edited by S. El Yacoubi, J. Wąs, and S. Bandini, Lecture Notes in Computer Science, Vol. 9863 (Springer, Cham, 2016), pp. 405–412.
[43] T. Midha, A. B. Kolomeisky, and A. K. Gupta, Phys. Rev. E **98**, 042119 (2018).
[44] K. Nishinari, Y. Okada, A. Schadschneider, and D. Chowdhury, Phys. Rev. Lett. **95**, 118101 (2005).
[45] D. M. Miedema, V. S. Kushwaha, D. V. Denisov, S. Acar, B. Nienhuis, E. J. G. Peterman, and P. Schall, Phys. Rev. X **7**, 041037 (2017).
[46] V. S. Kushwaha, S. Acar, D. M. Miedema, D. V. Denisov, P. Schall, and E. J. Peterman, PLoS ONE **15**, e0228930 (2020).
[47] R. J. Concannon, Ph.D. thesis, University of Edinburgh, 2014.