Lévy walks with rests: Long-time analysis

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In this paper we analyze the asymptotic behavior of Lévy walks with rests. Applying recent results in the field of functional convergence of continuous-time random walks we find the corresponding limiting processes. Depending on the parameters of the model, we show that in the limit we can obtain standard Lévy walk or the process describing competition between subdiffusion and Lévy flights. Some other more complicated limit forms are also possible to obtain. Finally we present some numerical results, which confirm our findings.

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I. INTRODUCTION

In recent years Lévy walks (LWs) proved very useful models for anomalous diffusion [1]. They have become an alternative to Brownian diffusive random walk as a process, which underlies random movement with constant velocity. The particle that performs Lévy walks moves with velocity v for a time period, which follows a power law $\psi(t) \propto t^{-(1+\alpha)}$ with $\alpha > 0$. Then it chooses randomly a new direction of the motion [1]. Here we focus on the case $\alpha \in (0, 1)$, which leads to a ballistic regime [2]. Lévy walks were used to describe the dynamics of blinking nanocrystals [3–6]. Other striking and sometimes very beautiful examples of applications include: migration of swarming bacteria [7], light transport in special optical materials (Lévy glass) [8], and foraging patterns of animals [9–12]. More examples are described in a review paper devoted to this model [1], see also Ref. [13].

Lévy walks can be analyzed in a framework of coupled continuous-time random walks (CTRWs). Except from the standard Lévy walk, we can distinguish two other important cases: the so-called wait-first or jump-first models [1]. These are examples of CTRWs, which are closely related to the standard LW. The particles that perform wait-first LW, instead of moving with the constant velocity v for certain random time T, remain motionless for time T and then executes a jump with length equal to $v \cdot T$. As a result trajectories are discontinuous, contrary to the standard LW, see Fig. 1 (left panel). If we linearly interpolate the trajectory of wait-first model, we obtain the trajectory of standard LW, see Fig. 1 (middle panel, solid line). In this case the walker moves with constant velocity (in this paper we will assume for simplicity that $v = 1$) and after a certain random period of time it changes its direction. The CTRW approach to LWs was analyzed in Refs. [14,15]. Although the jump models and the standard LW appear to be very similar, they have very different statistical properties. In Ref. [2] a method to find probability density functions $p(x, t)$ (PDFs) for all these models in ballistic regime was proposed by Froemberg *et al.* For another approach to this problem for the jump models, see also Ref. [16]. It is also worth to mention that PDFs of multidimensional isotropic Lévy walks were found in Refs. [17,18]. Important results related to the ergodic properties of Lévy walks can be found in Refs. [19–21]

Here we analyze the so-called LW with rests. Its trajectories are obtained by adding a random waiting period after each period of ballistic motion, see Fig. 1 (right panel). We underline that the ballistic regime is observed for $\alpha \in (0, 1)$. One usually assumes that the resting periods are chosen from a power-law distribution with some exponent γ. The formal definition of LW with rests will be given in the next section. LW with rests were first introduced and analyzed in Refs. [22,23], where the authors used CTRW to define the corresponding equation for the propagator. Next, this equation was used to analyze the asymptotic behavior of the PDF of the model as well as the asymptotics of the mean-square displacement. One of the most stimulating and important applications of CTRWs with power-law waiting times and jumps can be found in the paper [24], where the authors showed that the dispersal of bank notes and human travel behavior can be described by LW-type of dynamics. In Ref. [25] the authors analyzed the ratio of times a particle spends in flying and resting phases. For a general review on Lévy walks and their generalizations we refer the interested reader to Refs. [1,13]. We also note that recently it has been demonstrated in Ref. [26] that neuronal mRNP transport follows aging LW with truncated power-law run times. That way the authors confirmed that mRNP particles in the analyzed experiment display aging.

In this paper we analyze the asymptotic, long-time behavior of LW with rests. Applying recent results in the field of functional convergence of continuous-time random walks and related models, we derive the corresponding limiting process.

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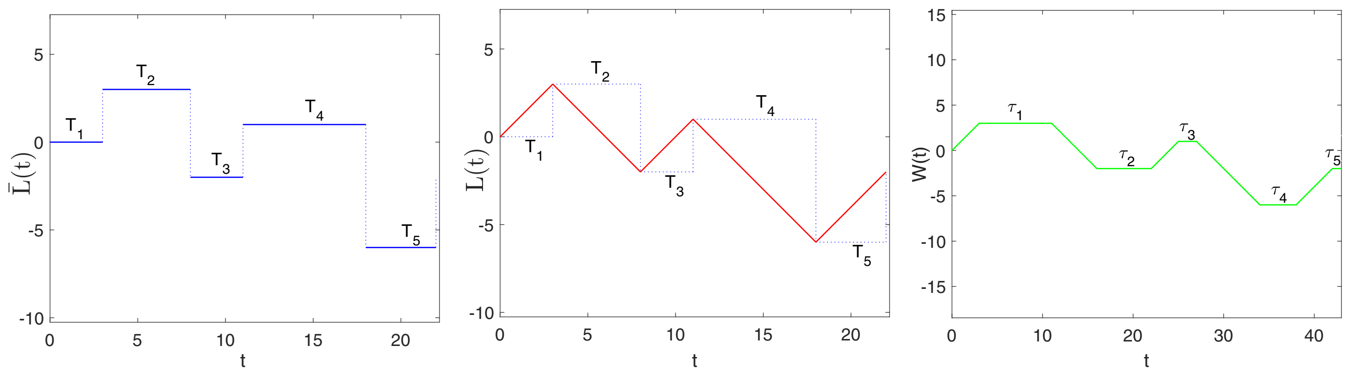

FIG. 1. (Left) Typical trajectory of the wait-first LW; (middle) the corresponding trajectory of the standard LW; (right) the corresponding sample trajectory of LW with rests. Note that the ballistic periods in the middle and right panel are the same.

Depending on the parameters of the model, we show that in the limit we can obtain standard Lévy walk or the process describing competition between subdiffusion and Lévy flights. Some other less popular limiting forms are also possible.

We derive the asymptotic limits of LWs with rests in multidimensional functional setting. Since the general functional convergence of LWs with rests is proved here, it implies that convergence of finite-dimensional distributions holds as well. Previously, only one-dimensional distributions and moments of LWs with rests were analyzed. In this paper we obtain the explicit formulas for the whole limiting processes. This allows us to study multipoint properties of the limits, such as autocorrelation or ergodic coefficients. Moreover, the case when the power-law exponent of the ballistic periods and the power-law exponent of the resting periods coincide is analyzed here in detail. We obtain here the explicit form of the limiting process.

Our results significantly extend the ones presented in Ref. [29], where the standard LW was studied. Compared to Ref. [29], by adding the additional resting periods, we are able to arrive at completely different limiting processes. Our derivation methods are partially based on the ones used in Ref. [29]. However, due to the additional resting periods, also other mathematical tools and methods will be used here. In particular continuity and convergence of more general functionals need to be verified.

II. BASIC DEFINITIONS

Lévy walks can be defined in the framework of CTRW [13]. This approach is used in our paper. It should be added that LWs can also be defined using subordinated Langevin equations, see Refs. [27,28] for the details. Let T_i be a sequence of independent, identically distributed (IID) power-law waiting times such that $\mathbf{P}(T_i > x) \approx Cx^{-\alpha}$ when $x \to \infty$, $\alpha \in (0,1)$, $C > 0$. Here by $f(x) \approx g(x)$ we mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. The jumps are defined as $J_i = V_i T_i$ where V_i govern the direction of the jump. It is assumed that V_i are IID with $\mathbf{P}(V_i = 1) = \mathbf{P}(V_i = -1) = 1/2$. Notice that waiting times T_i and jumps J_i are dependent. This coupling allows the CTRW process to have heavy-tailed jumps—a phenomenon that is often observed in experimental data—but still the process can have a finite mean-square displacement (variance), which makes it a suitable physical model [2].

Note that the parameter α is restricted here to the case $\alpha \in (0,1)$. This is due to the fact that for $\alpha > 1$ the diffusion limit of LW is the well-known and studied α-stable Lévy motion (Lévy flight) [29]. This is in sharp contrast with the case $\alpha \in (0,1)$, for which the diffusion limit is a completely different, much more complicated process [28,29].

The counting process is given by $\bar{N}(t) = \max\{n \geqslant 0 : \sum_{i=1}^{n} T_i \leqslant t\}$. In mathematical and physical literature there appear three different types of Lévy walks [2,29]. Wait-first Lévy walk (also known as undershooting Lévy walk) is defined as:

$$\bar{L}(t) = \sum_{i=1}^{\bar{N}(t)} J_i. \tag{1}$$

The standard Lévy walk is obtained by linear interpolation of $\bar{L}(t)$, see Fig. 1. Formally, it can be defined as

$$L(t) = \sum_{i=1}^{\bar{N}(t)} J_i + \left(t - \sum_{i=1}^{\bar{N}(t)} T_i\right) V_{\bar{N}(t)+1}. \tag{2}$$

$L(t)$ is the most useful process from a physical point of view; it has finite second moments of all orders and continuous trajectories.

Finally, we define Lévy walk with rests $W(t)$ by adding waiting times τ_i after each ballistic period of $L(t)$, see Fig. 1 (right panel). Let us assume that $\tau_0 = 0$ and τ_i, $i \in \mathbb{N}$, are IID power-law random variables such that $\mathbf{P}(\tau_i > x) \approx C_0 x^{-\gamma}$ when $x \to \infty$, $\gamma \in (0,1)$, $C_0 > 0$. Now, the formal definition of Lévy walk with rests is as follows:

$$W(t) = \sum_{j=1}^{N(t)} T_j V_j + \left(\max\left(t, \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) - \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) \times V_{N(t)+1}, \tag{3}$$

where $N(t) = \max\{k : \sum_{j=1}^{k} (\tau_{j-1} + T_j) \leqslant t\}$. $W(t)$ is the appropriate modification of the standard Lévy walk $L(t)$ defined in (2). In (3) we modify the counting process $N(t)$ in order to include additional waiting times τ_i. Moreover, the modification of the part in large brackets in (3) as compared to (2), stems from the fact that for $W(t)$ the ballistic motion with constant velocity is observed only during the waiting times T_i. During the waiting times τ_i the particle does not move at all. The first sum in (3) tells the walker how to move in

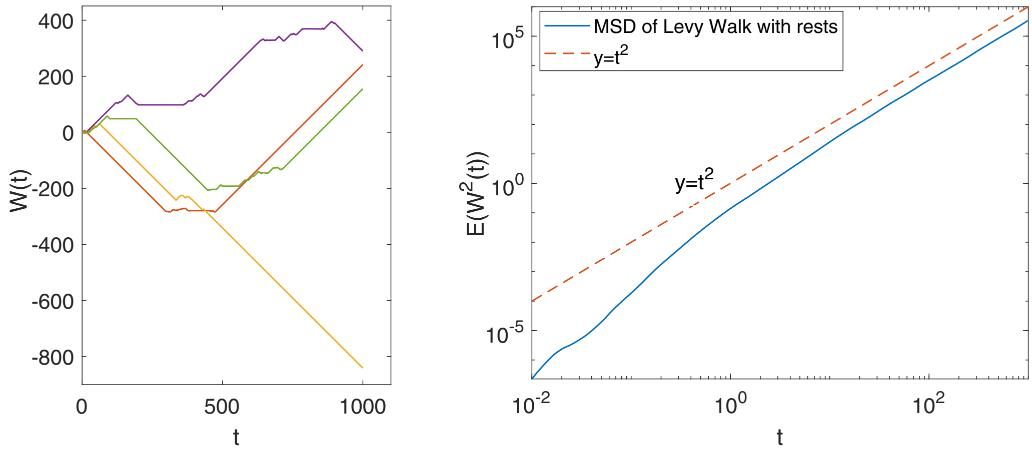

FIG. 2. (Left) Typical trajectories of the LW with rest with $\alpha = 0.3$ and $\gamma = 0.9$; (right) the corresponding mean-square displacement.

the ballistic regime. The expression in large brackets in (3) is either equal to zero or $t - \sum_{j=1}^{N(t)}(T_j + \tau_j)$. If it is equal to zero, then we observe the resting periods of the walker. In the other case, the whole expression (3) coincides with (2) and the walker moves as in the standard Lévy walk. The term $V_{N(t)+1}$ at the end of (3) takes only two values -1 and 1, and is responsible for the direction of the ballistic motion of the walker, upwards or downwards. In what follows we analyze the long-time behavior of $W(t)$.

III. ASYMPTOTIC BEHAVIOR OF LW WITH RESTS

Recall that $\alpha \in (0, 1)$ is the power-law exponent of the ballistic periods T_i, whereas $\gamma \in (0, 1)$ is the power-law exponent of the resting periods τ_i.

A. Case $\alpha < \gamma$

Let us first analyze the case $\alpha < \gamma$. This implies that the ballistic periods will dominate the overall motion of the particle for long times. Thus, heuristically, we have that for long times

$$N(t) \sim \bar{N}(t),$$

$$\sum_{j=1}^{N(t)}(T_j + \tau_j) \sim \sum_{j=1}^{\bar{N}(t)} T_j,$$

$$\max\left(t, \sum_{j=1}^{N(t)}(T_j + \tau_j)\right) - \sum_{j=1}^{N(t)}(T_j + \tau_j) \sim t - \sum_{i=1}^{\bar{N}(t)} T_i.$$

The above implies that for long times $W(t)$ defined in (3) will behave in the same way as the standard LW $L(t)$ defined in (2). The asymptotic limit of $L(t)$ was already derived in Ref. [29]. Therefore, using Ref. [29], we get the following convergence in distribution for $\alpha < \gamma$

$$\frac{W(nt)}{n} \xrightarrow{d} X(t). \tag{4}$$

Here:

$$X(t) = \begin{cases} L_\alpha^-\left[S_\alpha^{-1}(t)\right] & \text{if } t \in \mathcal{R} \\ L_\alpha^-\left[S_\alpha^{-1}(t)\right] + \frac{t-G(t)}{H(t)-G(t)}\left\{L_\alpha\left[S_\alpha^{-1}(t)\right] - L_\alpha^-\left[S_\alpha^{-1}(t)\right]\right\} & \text{if } t \notin \mathcal{R}, \end{cases} \tag{5}$$

where $L_\alpha(t)$ is the α-stable Lévy motion with Fourier transform $E\exp[izL_\alpha(t)] = e^{-|z|^\alpha}$, $S_\alpha(t)$ is the α-stable subordinator with Laplace transform $E\exp[-zS_\alpha(t)] = e^{-z^\alpha}$. The processes L_α and S_α are strongly dependent. Their instants of jumps and jump lengths coincide. Moreover

$$\mathcal{R} = \{S_\alpha(t) : t \geqslant 0\}, \quad G(t) = S_\alpha^-\left[S_\alpha^{-1}(t)\right],$$
$$H(t) = S_\alpha\left[S_\alpha^{-1}(t)\right].$$

We also use the notation f^- for the left-continuous version of f and f^{-1} for its inverse. The formal proof of the above result is in the Appendix. It should be underlined that the PDF $q(x, t)$ of the limit process $X(t)$ satisfies the following fractional diffusion equation [29]:

$$\frac{1}{2}\left[\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^\alpha + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^\alpha\right]q(x, t)$$
$$= [\delta(x - t) + \delta(x + t)]\frac{t^{-\alpha}}{\Gamma(1 - \alpha)}.$$

Here, the operators $(\frac{\partial}{\partial t} \mp \frac{\partial}{\partial x})^\alpha$ are called fractional material derivatives. They have the Fourier-Laplace symbols $(s \mp ik)^\alpha$. These were introduced by Ref. [30] as a fractional extension of the standard material derivative. The function $\delta(\cdot)$ is the Dirac delta. $X(t)$ is 1-self-similar. The mean-square displacement of $X(t)$ has the ballistic form

$$E(X^2(t)) = t^2(1 - \alpha)/2.$$

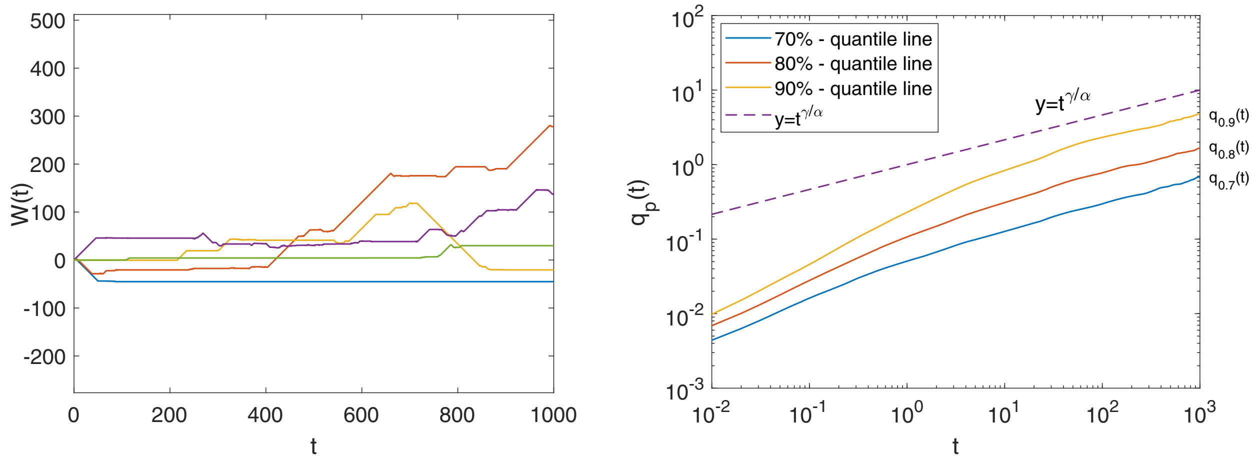

FIG. 3. (Left) Typical trajectories of the LW with rest with $\alpha = 0.9$ and $\gamma = 0.3$; (right) the corresponding quantile lines.

In Fig. 2 (left panel) we observe the typical trajectories of LW with rests for $\alpha < \gamma$. We can see that the ballistic periods dominate the overall motion of the particle. In Fig. 2 (right panel) we observe the asymptotic ballistic behavior of the mean-square displacement of $W(t)$ obtained using Monte Carlo simulations. It confirms our convergence result.

B. Case $\alpha > \gamma$

In the case $\alpha > \gamma$ the resting periods τ_i will dominate over the waiting times T_i. Therefore we get that for long times

$$N(t) \sim \hat{N}(t),$$

$$\text{where } \hat{N}(t) = \max\left\{k : \sum_{j=1}^{k} \tau_{j-1} \leqslant t\right\},$$

$$\sum_{j=1}^{N(t)}(T_j + \tau_j) \sim \sum_{j=1}^{\hat{N}(t)} \tau_j,$$

$$\max\left(t, \sum_{j=1}^{N(t)}(T_j + \tau_j)\right) - \sum_{j=1}^{N(t)}(T_j + \tau_j) \sim 0.$$

The above implies that for long times the term in large brackets in (3) will disappear. Thus $W(t)$ will behave for long times as a standard CTRW of the form $\sum_{j=1}^{\hat{N}(t)} T_j V_j$. This CTRW process is very well known and studied in the literature [31]. It describes competition between subdiffusion with Lévy flights. The walker jumps according to the random variables $J_i = V_i T_i$ and waits according to the waiting times τ_i. Here J_i and τ_i are independent. The asymptotic limit of this CTRW is also well known [31,32]. Thus we get the following convergence in distribution of $W(t)$ for $\alpha > \gamma$:

$$\frac{W(nt)}{n^{\gamma/\alpha}} \xrightarrow{d} L_\alpha\left[S_\gamma^{-1}(t)\right]. \tag{6}$$

Here $L_\alpha(t)$ is the α-stable Lévy motion with Fourier transform $E\exp[izL_\alpha(t)] = e^{-|z|^\alpha}$. $S_\gamma^{-1}(t)$ is the inverse γ-stable subordinator $S_\gamma^{-1}(t) = \inf\{\tau > 0 : S_\gamma(\tau) > t\}$, where $S_\gamma(t)$ is the γ-stable subordinator with Laplace transform $E\exp[-zS_\gamma(t)] = e^{-z^\gamma}$. Both processes L_α and S_γ are assumed independent here. The formal proof of the above convergence can be found in the Appendix. The PDF $p(x, t)$ of the limit process $L_\alpha[S_\gamma^{-1}(t)]$ satisfies the well-known space-time fractional diffusion equation [31,32]

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\nabla^\alpha p(x,t).$$

Here, the operator ${}_0D_t^{1-\gamma}$ is the fractional derivative of the Riemann-Liouville type and ∇^α is the Riesz fractional derivative.

$L_\alpha[S_\gamma^{-1}(t)]$ does not have finite second moment. However, it is γ/α-self-similar. In Fig. 3 (left panel) we have the typical trajectories of LW with rests for $\alpha > \gamma$. We observe the dominance of the resting periods. In Fig. 2 (right panel) we observe the asymptotic behavior of three quantile lines of $W(t)$ obtained using Monte Carlo simulations. All the quantile lines have the asymptotic form $c \cdot t^{\gamma/\alpha}$, which confirms the γ/α-self-similarity property. Recall that the p-quantile line, $p \in (0, 1)$, for a stochastic process $Y(t)$ is a function $q_p(t)$ given by the relationship $P[Y(t) \leqslant q_p(t)] = p$. A systematic numerical study of the process $L_\alpha[S_\gamma^{-1}(t)]$ can be found in Ref. [33] in the context of the so-called paradoxical diffusion.

C. Case $\alpha = \gamma$

The asymptotic limit in the case $\alpha = \gamma$ is the most technical and least intuitive. However, for completeness we present it here.

After some standard manipulations we get that

$$W(t) = \sum_{j=1}^{N(t)} T_j V_j + \frac{\max\left[t, \sum_{j=1}^{N(t)}(T_j+\tau_j)\right] - \sum_{j=1}^{N(t)}(T_j+\tau_j)}{\sum_{j=1}^{N(t)+1} T_j - \sum_{j=1}^{N(t)} T_j} \times \left(\sum_{j=1}^{N(t)+1} T_j V_j - \sum_{j=1}^{N(t)} T_j V_j\right).$$

Using the above representation we get the following convergence in distribution for $\alpha = \gamma$ (see Appendix for the details)

$$\frac{W(nt)}{n} \xrightarrow{d} Y(t). \tag{7}$$

Here $Y(t)$ has the form

$$Y(t) = L_\alpha^-(S^{-1})(t) + \frac{\max\{t, S^-[S^{-1}(t)]\} - S^-[S^{-1}(t)]}{S_2[S^{-1}(t)[-S_2^-[S^{-1}(t)]} \times \{L_\alpha[S^{-1}(t)] - L_\alpha^-[S^{-1}(t)]\}. \quad (8)$$

Here L_α is the α-stable Lévy motion obtained as the scaling limit of $\sum_{i=1}^{[nt]} V_i T_i$. Next, $S(t) = S_1(t) + S_2(t)$, where S_1 and S_2 are two independent α-stable subordinators. The first one is the scaling limit of $\sum_{i=1}^{[nt]} \tau_i$, the second one is the scaling limit of $\sum_{i=1}^{[nt]} T_i$.

The limit process $Y(t)$ is 1-self-similar. It follows from the normalizing factor in the convergence analysis in the Appendix. Therefore, $Y(t)$ scales ballistically, i.e., its mean-square displacement equals

$$E[Y^2(t)] = c \cdot t^2$$

for some appropriate constant $c > 0$. Its realizations display both the ballistic and resting periods. However, it is difficult to find the corresponding generalized diffusion equation governing the PDF of this process.

IV. SUMMARY

In this paper we derived the asymptotic limit for LWs with rests. Applying recent advances in the theory of functional convergence of stochastic processes, we were able to determine the detailed structure of the limiting processes. Depending on the power-law exponents responsible for ballistic periods (α) and resting periods (γ), we showed that if $\alpha < \gamma$ then the limit coincides with standard Lévy walk. For $\alpha > \gamma$ we arrived at the process whose PDF dynamics is described by the well-known space-time fractional diffusion equation. The least intuitive case $\alpha = \gamma$ leads to the process displaying both the ballistic and resting periods. Our findings allow us to compare and verify the theoretical findings with the experimental data, in which LWs with rests are observed. It should be added that for $\alpha, \gamma \in (1, 2)$ the limit process of LW with rests is the α-stable Lévy motion (Lévy flight). It follows from Corollary 4.19. in Ref. [29] and the law of large numbers applied to the resting periods.

We would like to underline that the derived here results for the functional convergence of LWs with rests give the complete picture of the limiting processes. Previously only the one-dimensional asymptotic properties were studied. In this paper we obtained the explicit formulas for the limiting processes, see formulas (5), (6), and (8). This allows us to analyze the whole multidimensional structure of LWs with rests and examine various crucial quantities such as autocorrelation or ergodic coefficients. We would like to add that if we assume that the tails of τ_i and T_i are regularly varying with respective indexes γ and α, then all the obtained asymptotic results will hold.

ACKNOWLEDGMENT

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APPENDIX

In this Appendix we present in the general setting the rigorous proofs of the main results of the paper, related to the asymptotic limits of LWs with rests. Recall the definitions of T_i, τ_i, V_i and $N(t)$ from Sec. II. The LW with rests has the form

$$W(t) \overset{df}{=} \sum_{j=1}^{N(t)} T_j V_j + \left(\max\left(t, \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) - \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) \times V_{N(t)+1}.$$

Then

$$\begin{aligned} W(t) &= \sum_{j=1}^{N(t)} T_j V_j + \left(\max\left(t, \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) - \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) \times \frac{T_{N(t)+1} V_{N(t)+1}}{T_{N(t)+1}} \\ &= \sum_{j=1}^{N(t)} T_j V_j + \frac{\max\left(t, \sum_{j=1}^{N(t)} (T_j + \tau_j)\right) - \sum_{j=1}^{N(t)} (T_j + \tau_j)}{\sum_{j=1}^{N(t)+1} T_j - \sum_{j=1}^{N(t)} T_j} \times \left(\sum_{j=1}^{N(t)+1} T_j V_j - \sum_{j=1}^{N(t)} T_j V_j\right). \end{aligned}$$

Note that the process $W(t)$ has continuous trajectories. To check the asymptotics of $W(t)$ we will implement the methods from Ref. [29].

Let us introduce the necessary notation. Let $D[0, \infty)$ be the space of real functions on $[0, \infty)$, which are right continuous with left limits (r.c.l.l.). Let $D_{u,\uparrow}$ be the subspace of $D[0, \infty)$ of nondecreasing and unbounded functions. For each function $x \in D[0, \infty)$ we define its left continuous version with right limits (l.c.r.l.) x^- as $x^-(t) = x(t-)$ for $t > 0$ and $x^-(0) = x(0)$. For $y \in D_{u,\uparrow}$ we define its inverse y^{-1} as $y^{-1}(t) = \inf\{s : y(s) > t\}$. Clearly y^{-1} is r.c.l.l. and $y^{-1} \in D_{u,\uparrow}$. If $x \in D[0, \infty)$ and $y \in D_{u,\uparrow}$ then $x \circ y$ denotes their superposition, which is r.c.l.l., whereas $x^- \circ y^-$ is l.c.r.l. Its right continuous version is denoted by $(x^- \circ y^-)^+$. It is defined by $(x^- \circ y^-)^+(t) = (x^- \circ y^-)(t+)$. By $\Rightarrow$ we will denote weak convergence in $D[0, \infty)$. In the definitions below we will use the following notation:

$$a \in D[0, \infty), \quad f, c \in D_{u,\uparrow}, \quad e = c^{-1}, \quad \hat{f} = f + c$$

$$x = (a^- \circ e^-)^+, \quad y = a \circ e,$$

$$g = (f^- \circ e^-)^+, \quad h = f \circ e, \quad \hat{g} = (\hat{f}^- \circ e^-)^+.$$

Let $\Re(f) = \{f(t) : t \geqslant 0\}$. Define the mapping $\Phi : D[0,\infty) \times D_{u,\uparrow} \times D_{u,\uparrow} \mapsto D[0,\infty)$ by $\Phi(a, f, c) = w$, where

$$w(t) = \begin{cases} x(t), & \text{if } t \in \Re(f); \\ x(t) + \frac{\max(t-\hat{g}(t),0)}{h(t)-g(t)}(y(t)-x(t)), & \text{if } t \notin \Re(f). \end{cases}$$

Define the processes $A,\ F,\ C,\ \tilde{C},\ \hat{F},\ \tilde{F},\ N$ as

$$A(t) = \sum_{j=1}^{[t]} T_j V_j, \quad F(t) = \sum_{j=1}^{[t]} T_j, \quad C(t) = \sum_{j=1}^{[t]} \tau_j, \quad \tilde{C}(t) = \sum_{j=1}^{[t]} \tau_{j-1},$$

$$\hat{F}(t) = \sum_{j=1}^{[t]} (T_j + \tau_j) \equiv F(t) + C(t), \quad \tilde{F}(t) = \sum_{j=1}^{[t]} (T_j + \tau_{j-1}) \equiv F(t) + \tilde{C}(t),$$

$$N(t) = \max\{k : \tilde{F}(k) \leqslant t\}, \quad \tilde{F}^{-1}(t) = N(t) + 1.$$

Moreover let

$$D \stackrel{df}{=} \tilde{F}, \quad E \stackrel{df}{=} D^{-1}, \quad H \stackrel{df}{=} F \circ E, \quad G \stackrel{df}{=} (F^- \circ E^-)^+, \quad \hat{G} \stackrel{df}{=} (\hat{F}^- \circ E^-)^+$$

$$X \stackrel{df}{=} (A^- \circ E^-)^+, \quad Y \stackrel{df}{=} A \circ E.$$

Then we have

$$E(t) = N(t) + 1, \quad E^-(t+) = N(t), \quad H(t) = F[E(t)] = \sum_{j=1}^{N(t)+1} T_j,$$

$$G(t) = (F^- \circ E^-)(t+) = \sum_{j=1}^{N(t)} T_j, \quad \hat{G}(t) = (\hat{F}^- \circ E^-)(t+) = \sum_{j=1}^{N(t)} (T_j + \tau_j)$$

$$X(t) = (A^- \circ E^-)(t+) = A[N(t)] = \sum_{j=1}^{N(t)} T_j V_j, \quad Y(t) = A[E(t)] = A[N(t)+1] = \sum_{j=1}^{N(t)+1} T_j V_j.$$

$$W(t) = \begin{cases} X(t), & \text{if } t \in \Re(F), \\ X(t) + \frac{\max(t-\hat{G}(t),0)}{H(t)-G(t)}[Y(t) - X(t)], & \text{if } t \notin \Re(F). \end{cases}$$

Let b_n and B_n be positive sequences, such that $b_n \to 0$, $B_n \to 0$ as $n \to \infty$. Define

$$A_n(t) \stackrel{df}{=} B_n A([nt]), \quad W_n(t) \stackrel{df}{=} B_n W(t/b_n),$$

$$C_n(t) \stackrel{df}{=} b_n C([nt]), \quad \tilde{C}_n(t) \stackrel{df}{=} b_n \tilde{C}([nt]),$$

$$F_n(t) \stackrel{df}{=} b_n F([nt]), \quad \hat{F}_n(t) \stackrel{df}{=} b_n \hat{F}([nt]) \equiv F_n(t) + C_n(t), \quad \tilde{F}_n(t) \stackrel{df}{=} b_n \tilde{F}([nt]) \equiv F_n(t) + \tilde{C}_n(t).$$

$$D_n(t) \stackrel{df}{=} \tilde{F}_n(t)$$

$$E_n(t) \stackrel{df}{=} D_n^{-1}(t) \equiv \tilde{F}_n^{-1}(t) \equiv n^{-1}(N(t/b_n) + 1), \quad E_n^-(t) \equiv (\tilde{F}_n^{-1})^-(t) \equiv n^{-1} N(t/b_n),$$

$$G_n(t) \stackrel{df}{=} F_n^-(E_n^-(t+)) \equiv b_n F(N(t/b_n)) \equiv F_n^-(E^-(t+)), \quad \hat{G}_n(t) \stackrel{df}{=} \hat{F}_n^-(E_n^-(t+)) \equiv b_n \hat{F}(N(t/b_n)),$$

$$H_n(t) \stackrel{df}{=} F_n(E_n(t)) = b_n F((N(t/b_n) + 1),$$

$$X_n(t) \stackrel{df}{=} A_n^-(E_n^-(t+)) = B_n A(N(t/b_n)), \quad Y_n(t) \stackrel{df}{=} A_n(E_n(t)) = B_n A(N(t/b_n) + 1).$$

Then we have

$$W_n(t) = \begin{cases} X_n(t), & \text{if } t \in \Re(F_n)\ ; \\ X_n(t) + \frac{\max\left(t/b_n - \sum_{j=1}^{N(t/b_n)} (T_j+\tau_j), 0\right)}{1/b_n[H_n(t) - G_n(t)]}[Y_n(t) - X_n(t)], & \text{if } t \notin \Re(F_n). \end{cases}$$

Moreover

$$W_n(t) = \begin{cases} X_n(t), & \text{if } t \in \Re(F_n)\ ; \\ X_n(t) + \frac{\max(t-\hat{G}_n(t),0)}{H_n(t)-G_n(t)}[Y_n(t) - X_n(t)], & \text{if } t \notin \Re(F_n). \end{cases}$$

Then we get:

Theorem 1. If the following convergence holds

$$(A_n, F_n, C_n) \Rightarrow (\mathbb{A}, \mathbb{F}, \mathbb{C}) \tag{A1}$$

and realizations of the processes $\mathbb{F}$ and $\mathbb{C}$ are strictly increasing, then $W_n \Rightarrow \mathbb{W}$, where $\mathbb{W}(t) = \mathbb{X}(t)$ for $t \in \Re(\mathbb{F})$ and

$$\mathbb{W}(t) = \mathbb{X}(t) + \frac{\max[t - \hat{\mathbb{G}}(t), 0]}{\mathbb{H}(t) - \mathbb{G}(t)} \times [\mathbb{Y}(t) - \mathbb{X}(t)] \tag{A2}$$

for $t \notin \Re(\mathbb{F})$. Here

$$\hat{\mathbb{F}} = \tilde{\mathbb{F}} = \mathbb{D} = \mathbb{F} + \mathbb{C}, \quad \mathbb{E}(t) = \mathbb{D}^{-1}(t).$$

$$\hat{\mathbb{G}}(t) = \hat{\mathbb{F}}^{-}[\mathbb{E}^{-}(t+)], \quad \mathbb{G}(t) = \mathbb{F}^{-}[\mathbb{E}^{-}(t+)], \quad \mathbb{H}(t) = \mathbb{F}^{-}[\mathbb{E}(t)],$$

$$\mathbb{Y}(t) = \mathbb{A}[\mathbb{E}(t)], \quad \mathbb{X}(t) = \mathbb{A}^{-}[\mathbb{E}^{-}(t+)],$$

Proof. We get the result by applying Proposition 4.5 in Ref. [29]. Note that processes W_n and $\mathbb{W}$ can be written as $\Phi(A_n, F_n, C_n)$ and $\Phi(\mathbb{A}, \mathbb{F}, \mathbb{C})$, respectively.

Using the above result applied to LW with rest we get the following:

Lemma.

(i) Let $\alpha = \gamma$, $B_n = b_n = n^{-1/\alpha}$. Then $W_n \Rightarrow \mathbb{W}$, where $\mathbb{W}$ is defined in Theorem 1.

(ii) Let $\alpha < \gamma$, $B_n = b_n = n^{-1/\alpha}$. Then $(F_n, \tilde{F}_n, \hat{F}_n) \Rightarrow (\xi, \xi, \xi)$, and $W_n \Rightarrow \mathbb{W}$, where ξ is the α-stable subordinator and $\mathbb{W}$ is defined in (4).

(iii) Let $\alpha > \gamma$, $B_n = n^{-1/\alpha}$, $b_n = n^{-1/\gamma}$. Then $W_n \Rightarrow \mathbb{W}$, where $\mathbb{W}(t) = \mathbb{A}^{-}(\mathbb{E}^{-})(t+)$.

Proof. Case (i) is straightforward. Case (ii) follows from the fact that

$$(\tilde{F}_n, \hat{F}_n) = (F_n + C_n, F_n + \tilde{C}_n) = [F_n + o(1), F_n + o(1)] \Rightarrow (\xi, \xi).$$

Since in this case the realizations of ξ are strictly increasing, using Corollary 13.6.4 in Ref. [34] we get that

$$\left(\tilde{F}_n^{-1}, \hat{F}_n^{-1}\right) \Rightarrow (\xi^{-1}, \xi^{-1}).$$

Thus we get the convergence $W_n \Rightarrow \mathbb{W}$, with $\mathbb{C}$ equal to zero.

For (iii) we have

$$X_n(t) = B_n \sum_{j=1}^{N(t/b_n)} T_j V_j = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n\frac{1}{n}N(tn^{1/\gamma})} T_j V_j = A_n^{-}(E_n^{-})(t+),$$

which converges to $\mathbb{A}^{-}(\mathbb{E}^{-})(t+)$. This ends the proof. ■

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