Algebraic area enumeration of random walks on the honeycomb lattice

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(Received 25 July 2021; revised 3 November 2021; accepted 3 December 2021; published 13 January 2022)

We study the enumeration of closed walks of given length and algebraic area on the honeycomb lattice. Using an irreducible operator realization of honeycomb lattice moves, we map the problem to a Hofstadter-like Hamiltonian and show that the generating function of closed walks maps to the grand partition function of a system of particles with exclusion statistics of order $g = 2$ and an appropriate spectrum, along the lines of a connection previously established by two of the authors. Reinterpreting the results in terms of the standard Hofstadter spectrum calls for a mixture of $g = 1$ (fermion) and $g = 2$ exclusion particles whose properties merit further studies. In this context we also obtain some unexpected Fibonacci sequences within the weights of the combinatorial factors appearing in the counting of walks.

DOI: 10.1103/PhysRevE.105.014112

I. INTRODUCTION

Random walks on lattices emerge in the study of various problems of physical interest. The dynamics of electrons (or quasiparticles) on an atomic lattice can be well approximated by their hopping to the ground-state levels of different atoms in the lattice: Hopping to excited states would introduce extra effective discrete degrees of freedom but such transitions are generally energetically suppressed; likewise, hopping to atoms beyond the few near neighbors of the atom presently binding the electron is also suppressed. As a consequence, the entire dynamical process can be described by a random lattice walk. Percolation processes, Brownian-type diffusion processes, and various other statistical processes can also be modeled as random walks.

The algebraic area enumeration of closed random walks on two-dimensional lattices is a topic with rich mathematical and physical implications. Indeed, it is well known that the algebraic area of a walk introduces in the quantum case an interaction of the particle performing the walk with a constant magnetic field perpendicular to the plane of motion. The algebraic area is defined as the total oriented area spanned by the walk as it traces the lattice. A unit lattice cell enclosed in a counterclockwise (positive) direction has an area $+1$, whereas when enclosed in a clockwise (negative) direction it has an area -1. The total algebraic area is the area enclosed by the walk weighted by its winding number: If the walk winds around more than once, then the area is counted with multiplicity. Figure 1 represents examples of closed random walks on the square, triangular, and honeycomb lattices.

In the case of the square lattice, the algebraic area enumeration is embedded in the quantum dynamics of the Hofstadter model [1] which describes the motion of an electron hopping on a square lattice in a uniform perpendicular magnetic field, with its spin frozen and thus nondynamical. The generating function for the number $C_{2n}(A)$ of closed walks of length $2n$ (necessarily even) enclosing an algebraic area A is given in terms of the trace of the Hofstadter Hamiltonian H_γ,

$$\sum_A C_{2n}(A) Q^A = \operatorname{Tr} H_\gamma^{2n}, \tag{1}$$

where $\gamma = 2\pi\phi/\phi_0$ stands for the flux per plaquette in units of the flux quantum, $Q = e^{i\gamma}$, and H_γ is the Hofstadter Hamiltonian,

$$H_\gamma = u + u^{-1} + v + v^{-1}.$$

The unitary operators u and v are unit magnetic translations (hopping operators) in the x and y directions of the square lattice and satisfy the magnetic translations algebra

$$v\, u = Q\, u\, v \tag{2}$$

due to the perpendicular magnetic field piercing the lattice. Terms contributing to the trace in (1) must involve an equal number of u and u^{-1} and of v and v^{-1}. Such terms represent closed walks, each power of H_γ representing one step. Because of the commutation rules of u and v (2), the power of the total factor of Q for such walks can be seen to be equal to the algebraic area A of the walk, $v^{-1}u^{-1}vu = Q$ corresponding to a walk around an elementary plaquette. In quantum mechanics the trace becomes a sum of the expectation value of H_γ over all quantum states, with an appropriate normalization.

In Ref. [2] the question of enumerating all walks of given length and area was studied, and an explicit algebraic area enumeration was obtained in terms of a sum over compositions (that is, partitions where the order of terms matters) of the integer n which is half the walk length. In Ref. [3] and

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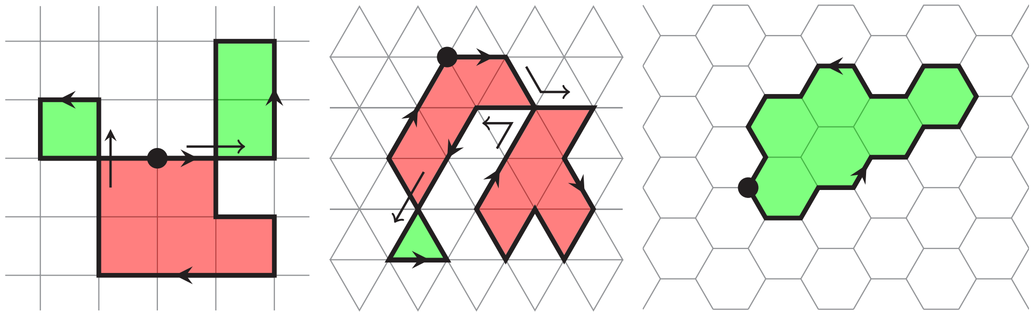

FIG. 1. Closed random walks of length $2n = 20$ on the square, triangular, and honeycomb lattice with algebraic area -2, -12, and 6, respectively.

Ref. [4], an interpretation of this enumeration was given in terms of the statistical mechanics of particles obeying quantum *exclusion statistics* with exclusion parameter g ($g = 0$ for bosons, $g = 1$ for fermions, and higher g means a stronger exclusion beyond Fermi). The square lattice enumeration was found to be governed by $g = 2$ exclusion together with a Hofstadter-induced spectral function $s_k := e^{-\beta\epsilon_k}$ accounting for the one-body quantum spectrum ϵ_k. Other, different types of lattice walks were governed by higher values of g and, in general, other types of spectral functions. Explicit examples of such enumerations were given, in particular for Kreweras-like chiral walks on a triangular lattice [3], corresponding to yet another quantum Hofstadter-like model (chiral and non-Hermitian, though) and $g = 3$ exclusion. This particular chiral model is to be distinguished from the triangular lattice Hofstadter-like model originally proposed in Ref. [5]. Its butterfly structure—among other Hosftadter-like models—has been studied in Ref. [6].

A case of particular physical and mathematical interest is the honeycomb lattice. It arises naturally in the form of graphene and carbon nanotubes, and many of its quantum properties have been extensively studied (see, for example, Refs. [7–9]). The honeycomb lattice is also relevant in graph theory [10] and various physical models [11–13]. The quantum model for a particle hopping on the honeycomb lattice pierced by a perpendicular magnetic field was introduced in Refs. [14,15]. The effect of lattice defects on its spectrum was investigated in Ref. [16] and its butterfly-like spectrum was obtained in Ref. [17].

In this work we address the question of the algebraic area enumeration of closed random walks on the honeycomb lattice: Can this enumeration be explicitly obtained, and does it fall in the category described in Ref. [3] and Ref. [4], i.e., does it correspond to a system of particles with a particular exclusion statistics? We will show that, indeed, the honeycomb enumeration can be interpreted in terms of particles with $g = 2$ exclusion on a single-particle level spectrum identical to the one for the Hofstadter model, i.e., with the same spectral function, but "diluted" by additional zero-energy levels between successive levels, in a "toothcomb" pattern. Alternatively, it can be interpreted in terms of the (undiluted) Hofstadter level spectrum but with a statistical mixture of $g=1$ and $g=2$ exclusion particles. This last system can, in turn, be interpreted as a system of fermions with the possibility that two fermions on neighboring levels can form a bound state of modified energy. The physical properties of these systems and the mapping between their physical observables need further exploration. As a by-product of our analysis we will obtain some unexpected Fibonacci sequences, either for the number of compositions entering the enumeration or for the sum of the coefficients weighting particular compositions, the occurrence of which remains to be better understood.

The paper is structured as follows: In Sec. II we review the Hofstadter model on the square lattice, where the coefficients of the secular determinant of the Hofstadter Hamiltonian [18] are reinterpreted in terms of $g = 2$ exclusion partition functions. The algebraic area enumeration is then obtained in terms of the associated cluster coefficients. In Sec. III we study the honeycomb lattice, establish its correspondence to an exclusion statistics system, and calculate the relevant partition functions and cluster coefficients, arriving at an explicit algebraic area enumeration expression. Some open questions and possible physical applications are exposed under Conclusions.

II. SQUARE LATTICE WALKS ALGEBRAIC AREA ENUMERATION

From now on we consider the flux γ per lattice cell to be rational, i.e., $\phi/\phi_0 = p/q$ with p and q coprime, so $Q = \exp(2i\pi p/q)$.

A. Hofstadter Hamiltonian

When the magnetic flux is rational the magnetic translations algebra has a finite-dimensional irreducible representation in which u and v are represented by the $q \times q$ matrices [19]

$$u = e^{ik_x}\begin{pmatrix} Q & 0 & 0 & \cdots & 0 & 0 \\ 0 & Q^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & Q^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & Q^q \end{pmatrix},$$

$$v = e^{ik_y}\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (3)$$

(u and v are often referred to as "clock" and "shift" matrices: quantum states correspond to the q "time" positions on a circular clock; u "reads" the time when applied on them, while v "shifts" the time by one unit); k_x and k_y are the quasimomenta (remnants of Bloch momenta) in the x and y directions and are related to the Casimirs of the u, v algebra

$$u^q = e^{iqk_x},\ v^q = e^{iqk_y}.$$

The Casimirs make clear that the relevant range of k_x and k_y is $[0, 2\pi/q]$. Indeed, shifting either k_x or k_y by $2\pi/q$ in (3) amounts to performing a unitary transformation on u, v:

$$k_x \to k_x + \frac{2\pi}{q} \Leftrightarrow u \to v^r u v^{-r},$$
$$k_y \to k_y + \frac{2\pi}{q} \Leftrightarrow v \to u^{-r} v u^r,\ \text{with } rp = 1 \ (\text{mod } q).$$

Since u^q and v^q perform translations by q lattice units in the x or the y direction, and they are set to a phase, this representation corresponds to making the lattice $q \times q$ periodic, with quantum states picking up a phase $e^{iqk_{x,y}}$ on going around each period. Because of this structure, the algebra of u, v is often called the "quantum torus" algebra, and we will refer to it by this name in the sequel.

In this representation the Hofstadter Hamiltonian becomes the $q \times q$ matrix,

$$H_q = \begin{bmatrix} Qe^{ik_x} + Q^{-1}e^{-ik_x} & e^{ik_y} & 0 & \cdots & 0 & e^{-ik_y} \\ e^{-ik_y} & Q^2 e^{ik_x} + Q^{-2}e^{-ik_x} & e^{ik_y} & \cdots & 0 & 0 \\ 0 & e^{-ik_y} & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & e^{ik_y} \\ e^{ik_y} & 0 & 0 & \cdots & e^{-ik_y} & Q^q e^{ik_x} + Q^{-q}e^{-ik_x} \end{bmatrix},$$

whose spectrum follows from the zeros of the secular determinant $\det(1 - zH_q)$, where z denotes the inverse energy. H_q has q eigenvalues, which, on varying k_x and k_y, become q bands. The evolution of these bands as the magnetic flux $2\pi p/q$ takes nearby values but with drastically different q gives rise to the fractal spectral flow known as the "Hofstadter butterfly."

The secular determinant $\det(1 - zH_q)$ has been shown [18] to rewrite as

$$\det(1 - zH_q) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2[\cos(qk_x) + \cos(qk_y)]z^q, \tag{4}$$

where the $Z(n)$'s are given by the nested trigonometric sums

$$Z(n) = \sum_{k_1=0}^{q-2n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_n=0}^{k_{n-1}} 4\sin^2\left[\frac{\pi(k_1 + 2n - 1)p}{q}\right] \times 4\sin^2\left[\frac{\pi(k_2 + 2n - 3)p}{q}\right] \cdots \times 4\sin^2\left[\frac{\pi(k_{n-1} + 3)p}{q}\right] 4\sin^2\left[\frac{\pi(k_n + 1)p}{q}\right] \tag{5}$$

with $Z(0) = 1$.

As we shall see, $Z(n)$ in (5) is at the core of the lattice walks algebraic area enumeration. To recover (5) let us use an alternative form of the Hofstadter Hamiltonian involving a different but equivalent representation of the operators u and v, namely $-uv$ and v (this corresponds to performing a modular transformation on the lattice that leaves it invariant). They still satisfy the same quantum torus algebra

$$v\,(-uv) = Q\,(-uv)\,v,$$

albeit with a different Casimir $(-uv)^q = -e^{iq(k_x + k_y)}$, and lead to the new Hamiltonian

$$H'_q = -uv - (uv)^{-1} + v + v^{-1},$$

i.e.,

$$H'_q = \begin{bmatrix} 0 & (1 - Qe^{ik_x})e^{ik_y} & 0 & \cdots & 0 & (1 - Q^{-q}e^{-ik_x})e^{-ik_y} \\ (1 - Q^{-1}e^{-ik_x})e^{-ik_y} & 0 & (1 - Q^2 e^{ik_x})e^{ik_y} & \cdots & 0 & 0 \\ 0 & (1 - Q^{-2}e^{-ik_x})e^{-ik_y} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (1 - Q^{(q-1)}e^{ik_x})e^{ik_y} \\ (1 - Q^q e^{ik_x})e^{ik_y} & 0 & 0 & \cdots & (1 - Q^{-(q-1)}e^{-ik_x})e^{-ik_y} & 0 \end{bmatrix},$$

or, denoting $\omega(k) = (1 - Q^k e^{ik_x})e^{ik_y}$,

$$H'_q = \begin{bmatrix} 0 & \omega(1) & 0 & \cdots & 0 & \bar{\omega}(q) \\ \bar{\omega}(1) & 0 & \omega(2) & \cdots & 0 & 0 \\ 0 & \bar{\omega}(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega(q-1) \\ \omega(q) & 0 & 0 & \cdots & \bar{\omega}(q-1) & 0 \end{bmatrix}.$$

Its secular determinant is the same as that of H_q given in (4) but for the new Casimirs, that is,

$$\det(1 - zH'_q) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - \left[\prod_{j=1}^{q} \omega(j) + \prod_{j=1}^{q} \bar{\omega}(j)\right] z^q$$
$$= \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2[\cos(qk_y) - \cos(qk_x + qk_y)] z^q. \quad (6)$$

Let us set $\omega(q) = 0$, which makes the cosine term in (6) vanish and the matrix H'_q tridiagonal,

$$H'_q|_{\omega(q)=0} = \begin{bmatrix} 0 & (1-Q^{1-q})e^{ik_y} & 0 & \cdots & 0 & 0 \\ (1-Q^{q-1})e^{-ik_y} & 0 & (1-Q^{2-q})e^{ik_y} & \cdots & 0 & 0 \\ 0 & (1-Q^{q-2})e^{-ik_y} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (1-Q^{-1})e^{ik_y} \\ 0 & 0 & 0 & \cdots & (1-Q)e^{-ik_y} & 0 \end{bmatrix}.$$

This form provides an iterative procedure for calculating the $Z(n)$'s. Putting aside for a moment that $Q = \exp(2i\pi p/q)$ and leaving it as a free parameter, independent of q, we introduce the spectral function

$$s_k = (1 - Q^k)(1 - Q^{-k}),\ \ k = 1, 2, \dots, q. \quad (7)$$

Denoting the secular determinant $\det[1 - zH'_q|_{\omega(q)=0}] = d_q$, its expansion in terms of the first row yields

$$d_q = d_{q-1} - z^2 s_{q-1}\, d_{q-2},\quad q \geqslant 2, \quad (8)$$

where, by convention, $d_0 = d_1 = 1$. Expanding d_q as a polynomial in z and solving the corresponding recursion relation for its coefficients, we obtain (see subsection 1 in the Appendix)

$$Z(n) = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \quad (9)$$

which, on restoring Q to its actual value $\exp(2i\pi p/q)$, i.e., the spectral function s_k to its actual form $s_k = 4\sin^2(\pi kp/q)$, gives (5).

The recursion (8) is at the root of the connection between square lattice walks and $g = 2$ exclusion statistics. Interpreting the spectral function s_k as the Boltzmann factor for a one-body level $e^{-\beta\epsilon_k}$ and $-z^2$ as the fugacity $x = e^{\beta\mu}$, (8) can be interpreted as an expansion of a grand partition function $\mathcal{Z}_{q-1}$—here identified with d_q—of noninteracting particles in $q - 1$ quantum levels $\epsilon_1, \dots, \epsilon_{q-1}$, obeying the exclusion principle that no two particles can occupy adjacent levels, namely

$$\mathcal{Z}_{q-1} = \mathcal{Z}_{q-2} + x s_{q-1}\, \mathcal{Z}_{q-3}$$

in terms of the last level ϵ_{q-1} being empty (first term) or occupied (second term). Then (6) identifies $Z(n)$ as the n-body partition function for particles occupying these $q - 1$ quantum states, with gaps of 2 between successive terms reproducing $g = 2$ exclusion.

B. Algebraic area enumeration on the square lattice

As already stressed, when $Q = \exp(2i\pi p/q)$ the algebraic area counting (1),

$$\sum_A C_{2n}(A) Q^A = \frac{1}{q} \mathrm{Tr}\, H_q^{2n}, \quad (10)$$

involves a trace over a finite number q of quantum states. To normalize the contribution of each walk to Q^A and reproduce the left-hand side of (10), a factor of $1/q$ must be included in the normalization. Also, when $2n \geqslant q$ the trace involves extra terms arising from the Casimirs k_x, k_y similarly to the cosine terms in (4), corresponding to open walks that close only up to periods (q, q) on the lattice ("umklapp" on the quantum torus). These spurious contributions can be eliminated by integrating the Casimirs k_x and k_y over $[0, 2\pi]$ which makes all factors of e^{iqk_x} and e^{iqk_y} vanish. So the definition of the trace in (10) is

$$\mathrm{Tr}\, H_q^{2n} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dk_x \int_0^{2\pi} dk_y\ \mathrm{tr}\, H_q^{2n},$$

which corresponds to summing over the q bands of the spectrum and over the scattering states labeled by k_x, k_y, in a continuum normalization (we harmlessly extended the range of k_x, k_y to the full interval $[0, 2\pi]$ to simplify the expression).

To relate this trace to the $Z(n)$'s in (5) or, equivalently, in (9), we make use of the fact that $\det(1 - zH_q)$ is interpreted as a grand partition function and the $Z(n)$ as n-body partition functions. These lead to cluster coefficients $b(n)$ defined via the expansion of the grand potential

$$\log\left[\sum_{n=0}^{\infty} Z(n)x^n\right] = \sum_{n=1}^{\infty} b(n)x^n \tag{11}$$

with x the fugacity. Using the identity

$$\log\det(1-zH_q) = \operatorname{tr}\log(1-zH_q) = -\sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{tr}H_q^n,$$

setting $x=-z^2$ in (11), keeping in mind that trivially $\operatorname{tr}H_q^{2n+1}=0$, and comparing the two expressions we reach the conclusion [2,3] that the trace in (10) for $2n$ is nothing but the cluster coefficient $b(n)$ up to a trivial factor

$$\operatorname{Tr}H_q^{2n} = 2n(-1)^{n+1}b(n). \tag{12}$$

The cluster coefficients can be directly calculated in terms of the spectral function. One obtains

$$b(n) = (-1)^{n+1}\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}} c(l_1,l_2,\ldots,l_j)\sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1}, \tag{13}$$

where the $c(l_1,l_2,\ldots,l_j)$'s are labeled by the compositions of the integer n with

$$c(l_1,l_2,\ldots,l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2}\; l_2\frac{\binom{l_2+l_3}{l_2}}{l_2+l_3}\;\cdots\; l_{j-1}\frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}. \tag{14}$$

Further, the trigonometric sums $\frac{1}{q}\sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1}$ can also be computed [2,4]

$$\frac{1}{q}\sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1} = \sum_{A=-\infty}^{\infty}\cos\left(\frac{2A\pi p}{q}\right)\sum_{k_3=-l_3}^{l_3}\sum_{k_4=-l_4}^{l_4}\cdots\sum_{k_j=-l_j}^{l_j}\binom{2l_1}{l_1+A+\sum_{i=3}^{j}(i-2)k_i}\binom{2l_2}{l_2-A-\sum_{i=3}^{j}(i-1)k_i}$$
$$\times\prod_{i=3}^{j}\binom{2l_i}{l_i+k_i}. \tag{15}$$

Using (12), (13), (14), and (15), we deduce the desired algebraic area counting

$$\sum_A C_{2n}(A)Q^A = \frac{1}{q}\operatorname{Tr}H_q^{2n} = 2n\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}} c(l_1,l_2,\ldots,l_j)\frac{1}{q}\sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1},$$

i.e.,

$$C_{2n}(A) = 2n\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}}\frac{\binom{l_1+l_2}{l_1}}{l_1+l_2}\; l_2\frac{\binom{l_2+l_3}{l_2}}{l_2+l_3}\;\cdots\; l_{j-1}\frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}\sum_{k_3=-l_3}^{l_3}\sum_{k_4=-l_4}^{l_4}\cdots\sum_{k_j=-l_j}^{l_j}\binom{2l_1}{l_1+A+\sum_{i=3}^{j}(i-2)k_i}$$
$$\times\binom{2l_2}{l_2-A-\sum_{i=3}^{j}(i-1)k_i}\prod_{i=3}^{j}\binom{2l_i}{l_i+k_i}. \tag{16}$$

We also note that, since

$$\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}} c(l_1,l_2,\ldots,l_j) = \frac{\binom{2n}{n}}{2n},$$

and, when $q\to\infty$ [2,3],

$$\frac{1}{q}\sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1} \to \binom{2(l_1+l_2+\ldots+l_j)}{l_1+l_2+\ldots+l_j}, \tag{17}$$

the overall closed square lattice walks counting

$$2n\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}} c(l_1,l_2,\ldots,l_j)\binom{2(l_1+l_2+\ldots+l_j)}{l_1+l_2+\ldots+l_j}$$
$$=\binom{2n}{n}^2$$

is recovered as it should (see subsection 2 in the Appendix for some enumeration examples).

III. HONEYCOMB LATTICE WALKS ALGEBRAIC AREA ENUMERATION

We saw that the algebraic area enumeration of walks on the square lattice is directly related to the quantum mechanics of an underlying particle system and invokes statistical mechanical concepts that put the calculations and results in a physical context. We plan to follow the same route to obtain an explicit algebraic area enumeration for closed walks on the honeycomb lattice.

A. Honeycomb Hamiltonian

Consider a particle hopping on a honeycomb lattice pierced by a constant magnetic field (see Fig. 2). The honeycomb lattice is bipartite and each individual hop moves the particle to a site of the other part, so unitary operators representing such translations act off-diagonally in the two sublattices. This also means that we can define a unique hopping operator for each of the three orientations of links, irrespective of the direction of the move, since the action of such operators is uniquely determined by the sublattice on which they act. Therefore, we define three operators U, V, and W generating the hops in each direction and such that when the particle hops around a honeycomb cell it picks up a phase Q due to the magnetic field. They satisfy the "honeycomb algebra"

$$U^2 = V^2 = W^2 = 1,\ (UVW)^2 = Q. \tag{18}$$

U, V, and W are both unitary and Hermitian. The Hofstadter-like Hamiltonian follows as

$$H_{\text{honeycomb}} = aU + bV + cW,$$

with $a, b, c \in \mathbb{R}^+$ transition amplitudes. The physical Hilbert space consists of the irreducible representations of the honeycomb algebra. As in the square lattice case, the quasimomenta are encoded in the Casimirs of the algebra.

In the case of a rational flux $Q = \exp(2i\pi p/q)$ with p and q coprime, the irreducible representation of U, V, and W for generic quasimomenta (Casimirs) becomes $2q$-dimensional

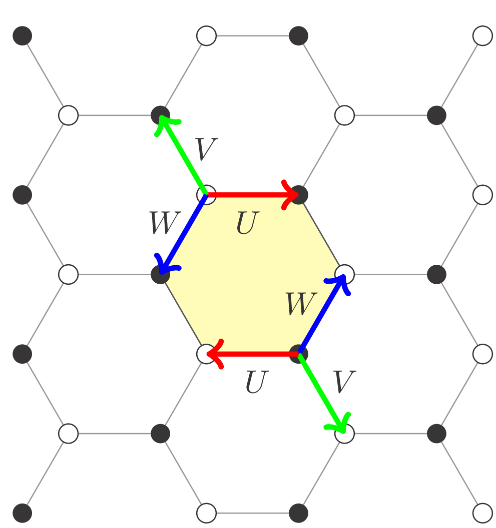

FIG. 2. Hopping operators U, V, and W on the honeycomb lattice with $U^2 = V^2 = W^2 = 1$ and $(UVW)^2 = Q$.

(see subsection 3 in the Appendix) and can be realized as

$$U = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix},$$
$$W = \begin{pmatrix} 0 & Q^{1/2}vu^{-1} \\ Q^{-1/2}uv^{-1} & 0 \end{pmatrix} \tag{19}$$

with u, v given in (3) and $Q^{\pm 1/2}$ understood to stand for $\exp(\pm i\pi p/q)$. The Casimirs of this algebra for rational flux can be written as

$$C_1 := (UV)^q + (VU)^q = -2(-1)^q \cos[q(k_x - k_y)],$$
$$C_2 := (VW)^q + (WV)^q = 2(-1)^p \cos(qk_x),$$
$$C_3 := (WU)^q + (UW)^q = 2(-1)^p \cos[q(k_y - 2k_x)],$$

where the second expressions evaluate them in the specific realization (19).

C_1, C_2, and C_3 are not independent, but satisfy

$$C_1^2 + C_2^2 + C_3^2 + (-1)^q C_1C_2C_3 = 4,$$

leading to two independent Casimirs, as expected for a two-dimensional lattice, encoded in the phases u^q, v^q. From the definitions of U, V, and W in Fig. 2 we see that C_2 generates translations by one plaquette width in the vertical direction (up or down depending on the sublattice), while C_1 and C_3 generate translations in directions at angles $\pm 2\pi/3$ from the vertical. $C_2 = 2(-1)^p \cos(qk_x)$ then indicates that k_x is actually the pseudomomentum in the vertical direction, whereas the values of C_1 and C_3 imply that $3k_x - 2k_y$ is the pseudomomentum in the horizontal direction.

For an isotropic lattice, $a = b = c = 1$, the honeycomb Hamiltonian reduces to

$$H_{2q} = \begin{pmatrix} 0 & u+v+Q^{1/2}vu^{-1} \\ u^{-1}+v^{-1}+Q^{-1/2}uv^{-1} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}. \tag{20}$$

As expected, it is block off-diagonal. Its square, however, is block-diagonal

$$H_{2q}^2 = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_q & 0 \\ 0 & \tilde{H}_q \end{pmatrix},$$

where $H_q = AA^\dagger$ and $\tilde{H}_q = A^\dagger A$ have identical spectra equal to the square of the honeycomb Hamiltonian spectrum. Denoting

$$\omega(k) = Q^{-k}\big(1 + e^{-ik_x}Q^{\frac{1}{2}-k}\big)e^{-i(k_x-k_y)},$$

H_q can be rewritten as

$$H_q = \begin{bmatrix} 1+\omega(2)\bar{\omega}(2) & \omega(2) & 0 & \cdots & 0 & \bar{\omega}(1) \\ \bar{\omega}(2) & 1+\omega(3)\bar{\omega}(3) & \omega(3) & \cdots & 0 & 0 \\ 0 & \bar{\omega}(3) & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & \omega(q) \\ \omega(1) & 0 & 0 & \cdots & \bar{\omega}(q) & 1+\omega(1)\bar{\omega}(1) \end{bmatrix} \tag{21}$$

with secular determinant

$$\begin{aligned} \det\left(1 - zH_{2q}\right) &= \det(1 - z^2 H_q) \\ &= \sum_{n=0}^{q} (-1)^n Z(n) z^{2n} + \left[(-1)^q \prod_{j=1}^{q} \omega(j)\bar{\omega}(j) - \prod_{j=1}^{q} \omega(j) - \prod_{j=1}^{q} \bar{\omega}(j)\right] z^{2q} \\ &= \sum_{n=0}^{q} (-1)^n Z(n) z^{2n} + 2\{-(-1)^p[\cos(qk_y - 2qk_x) + \cos(qk_x)] + (-1)^q[\cos(qk_y - qk_x) + 1]\} z^{2q}. \end{aligned} \tag{22}$$

B. Honeycomb coefficients $Z(n)$

Our aim is to find for the $Z(n)$ in (22) an expression analogous to the one in (5) or (9) obtained in the Hofstadter case. To this end, we reduce the honeycomb matrix (21) to a tridiagonal form by making both corners $\omega(1)$ and $\bar{\omega}(1)$ vanish, i.e., by setting $e^{-ik_x} = -Q^{\frac{1}{2}}$ so that $\omega(k)$ becomes

$$\omega(k)|_{\omega(1)=0} = -Q^{\frac{1}{2}-k}(1 - Q^{1-k})e^{ik_y},$$

and

$$H_q\Big|_{\omega(1)=0} = \begin{bmatrix} 1+(1-Q^{-1})(1-Q) & -Q^{-\frac{3}{2}}(1-Q^{-1})e^{ik_y} & 0 & \cdots & 0 & 0 \\ -Q^{\frac{3}{2}}(1-Q)e^{-ik_y} & 1+(1-Q^{-2})(1-Q^2) & -Q^{-\frac{5}{2}}(1-Q^{-2})e^{ik_y} & \cdots & 0 & 0 \\ 0 & -Q^{\frac{5}{2}}(1-Q^2)e^{-ik_y} & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & -Q^{\frac{1}{2}-q}(1-Q^{-(q-1)})e^{ik_y} \\ 0 & 0 & 0 & \cdots & -Q^{q-\frac{1}{2}}(1-Q^{q-1})e^{-ik_y} & 1+(1-Q^{-q})(1-Q^q) \end{bmatrix}.$$

This also eliminates the z^{2q} umklapp term in (22), i.e., the secular determinant reduces to

$$\det\left[1 - z^2 H_q|_{\omega(1)=0}\right] = \sum_{n=0}^{q} (-1)^n Z(n) z^{2n}.$$

This has again the suggestive form of a grand partition function, with $Z(n)$ the n-body partition function and $x = -z^2$ the fugacity. However, the analogy with the Hofstadter (square lattice) case is imperfect, since H_q in (21) has a nonvanishing diagonal. As a result, the exclusion statistics connection is not straightforward. Nevertheless, we will proceed as before: We will consider Q as a free parameter and denote $d_q = \det[1 - z^2 H_q|_{\omega(1)=0}]$. Then expanding d_q in terms of its bottom row we obtain the recursion relation

$$\begin{aligned} d_q = &\left\{1 - [1 + (1 - Q^q)(1 - Q^{-q})]z^2\right\} d_{q-1} - z^4(1 - Q^{q-1}) \\ &\times (1 - Q^{-(q-1)}) d_{q-2}, \; q \geqslant 1, \end{aligned}$$

i.e.,

$$d_q = [1 - (1 + s_q)z^2] d_{q-1} - z^4 s_{q-1} d_{q-2}, \tag{23}$$

with $d_0 = 1$, $d_j = 0$ for $j < 0$, and s_k as in (7). From (23) we can iteratively derive the $Z(n)$'s in (22) (see subsection 4 in the Appendix).

The recursion relation (23) is distinct from (8) but still admits a simple $g = 2$ exclusion statistics interpretation. Consider a set of $2q$ energy levels with spectral parameters S_r, $r = 1, 2, \ldots, 2q$ given by

$$S_{2k-1} = 1, \quad S_{2k} = s_k,$$

that is, s_k "diluted" by unit insertions: $1, s_1, 1, s_2, \ldots, 1, s_q$, and consider the grand partition function of $g = 2$ exclusion particles in the above spectrum with fugacity parameter x. Calling $\mathcal{Z}_r$ the truncated grand partition function for levels $S_1, S_2, \ldots, S_r$ and expanding it in terms of the last level r being empty or filled, we obtain the recursion relations

$$\begin{aligned} r = 2k: &\quad \mathcal{Z}_{2k} = \mathcal{Z}_{2k-1} + x s_k \mathcal{Z}_{2k-2}, \\ r = 2k-1: &\quad \mathcal{Z}_{2k-1} = \mathcal{Z}_{2k-2} + x \mathcal{Z}_{2k-3}. \end{aligned}$$

From the $r = 2k$ relation we can express the odd functions $\mathcal{Z}_{2k-1}$ in terms of even ones, $\mathcal{Z}_{2k-1} = \mathcal{Z}_{2k} - x s_k \mathcal{Z}_{2k-2}$. Substituting this expression in the $r = 2k - 1$ relation and rearranging we obtain

$$\mathcal{Z}_{2k} = (1 + x + x s_k)\mathcal{Z}_{2k-2} - x^2 s_{k-1} \mathcal{Z}_{2k-4}.$$

This is identical to the recursion (23) on putting $x = -z^2$ and identifying $\mathcal{Z}_{2k} = d_k$. Moreover, $\mathcal{Z}_{2k}$ satisfies the same initial conditions as d_k, namely $\mathcal{Z}_0 = 1$, $\mathcal{Z}_{2k} = 0$ for $k < 0$. Therefore, $d_q = \mathcal{Z}_{2q}$.

It follows that the expressions for the n-body partition functions $Z(n)$ and the cluster coefficients $b(n)$ are identical to the corresponding expressions (9) and (13) for square lattice walks but now, instead of the spectrum s_k, one has to consider the diluted spectrum S_k, $k = 1, \ldots, 2q$ (but note that $S_{2q} = s_q = 0$, so the levels effectively end at $S_{2q-1} = 1$)

$$Z(n) = \sum_{k_1=1}^{2q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} S_{k_1+2n-2} S_{k_2+2n-4} \cdots S_{k_{n-1}+2} S_{k_n},$$

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \times \sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1}$$

with the same Hofstadter combinatorial factors $c(l_1, l_2, \ldots, l_j)$ given in (14). The new diluted trigonometric sums $\frac{1}{q}\sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1}$ now entering the definition of the $b(n)$'s have to computed. They can be obtained using the same tools [4] as for the usual trigonometric sums (15) (see subsection 5 in the Appendix for an explicit expression).

Finally, following the same steps as in Sec. II B regarding the number $C_{2n}(A)$ of closed random walks of length $2n$ enclosing an algebraic area A on the honeycomb lattice, i.e., considering on the one hand

$$\sum_A C_{2n}(A) Q^A = \frac{1}{2q} \mathrm{Tr}\, H_{2q}^{2n},$$

which is the analog of (10) for the honeycomb Hamiltonian (20) [where the factor $1/q$ is replaced by $1/(2q)$ in view of the normalization over $2q$ states], and on the other hand

$$\mathrm{Tr}\, H_{2q}^{2n} = 2n(-1)^{n+1} b(n),$$

which generalizes (12), the expressions above directly lead to an algebraic area enumeration similar to the square lattice walks enumeration (16).

In the sequel, we will consider d_q in terms of the original (undiluted) Hofstadter spectrum s_k. In that case, the $g = 2$ exclusion interpretation does not hold anymore and has to be traded for a mixture of $g = 2$ and $g = 1$ statistics, as we are going to show in detail.

C. Modified statistics for the spectral function s_k

If we insist on keeping s_k as the spectral function, then the first few $Z(n)$ can be written in the form

$$Z(1) = +\sum_{i=1}^{q}(1 + s_i),$$

$$Z(2) = +\sum_{i=1}^{q-1}\sum_{j=1}^{i}(1 + s_{i+1})(1 + s_j) - \sum_{i=1}^{q-1} s_i,$$

$$Z(3) = +\sum_{i=1}^{q-2}\sum_{j=1}^{i}\sum_{k=1}^{j}(1 + s_{i+2})(1 + s_{j+1})(1 + s_k) - \sum_{i=1}^{q-2}\sum_{j=1}^{i}(1 + s_{i+2})s_j - \sum_{i=1}^{q-2}\sum_{j=1}^{i} s_{i+1}(1 + s_j),$$

$$Z(4) = +\sum_{i=1}^{q-3}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}(1 + s_{i+3})(1 + s_{j+2})(1 + s_{k+1})(1 + s_l) - \sum_{i=1}^{q-3}\sum_{j=1}^{i}\sum_{k=1}^{j}(1 + s_{i+3})(1 + s_{j+2})s_k$$
$$- \sum_{i=1}^{q-3}\sum_{j=1}^{i}\sum_{k=1}^{j}(1 + s_{i+3})s_{j+1}(1 + s_k) - \sum_{i=1}^{q-3}\sum_{j=1}^{i}\sum_{k=1}^{j} s_{i+2}(1 + s_{j+1})(1 + s_k) + \sum_{i=1}^{q-3}\sum_{j=1}^{i} s_{i+2}s_j,$$

etc.

Studying the above nested sums we can infer some general rules for their structure. The $Z(n)$'s are combinations of all possible nested sums of products of $1 + s_k$ and $-s_k$ distributed over all $k = 1, 2, \ldots, q$ in a natural alphabetical ordering inferred from their nested indices $i, j, k, \ldots, r$ such that

(i) The rightmost factor is either $-s_r$ or $1 + s_r$.

(ii) Any factor multiplying $-s_l$ immediately on its left obeys $g = 2$ exclusion, i.e., $\sum_k \sum_l s_k s_l$ or $-\sum_k \sum_l (1 + s_k)s_l$ where $k - l \geqslant 2$.

(iii) Any factor multiplying $1 + s_l$ immediately on its left obeys $g = 1$ exclusion, i.e., $-\sum_k \sum_l s_k(1 + s_l)$ or $\sum_k \sum_l (1 + s_k)(1 + s_l)$ where $k - l \geqslant 1$.

(iv) The leftmost factor is either $-s_{i+n-2}$ or $1 + s_{i+n-1}$ with summation range $\sum_{i=1}^{q-(n-1)}$.

It follows that products will have n_1 factors $1 + s_l$ and n_2 factors $-s_l$ such that $n_1 + 2n_2 = n$.

These rules admit a simple physical interpretation: Consider a system of one-body levels $k = 1, 2, \ldots, q$ with fermions in level k having Boltzmann factor $1 + s_k$ and two-fermion bound states in levels $k, k + 1$ having Boltzmann factor $-s_k$. Then $Z(n)$ is the n-fermion partition function with all possible bound states. The two-fermion bound states behave effectively as $g = 2$ exclusion particles. The honeycomb lattice secular determinant can, therefore, be described as the grand partition function of a mixture of $g = 1$ and $g = 2$ exclusion particles.

From these rules and the definition (11) we get the $b(n)$'s in terms of single sums of products of s_k (up to terms involving s_q which vanish anyway) with a form a bit more complicated than in the Hofstadter case

$$\begin{aligned}
b(1) &= \sum_{k=1}^{q-1} s_k + \sum_{k=1}^{q} s_k^0,\\
-b(2) &= \frac{1}{2}\sum_{k=1}^{q-1} s_k^2 + 2\sum_{k=1}^{q-1} s_k + \frac{1}{2}\sum_{k=1}^{q} s_k^0,\\
b(3) &= \frac{1}{3}\sum_{k=1}^{q-1} s_k^3 + 2\sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_{k+1}s_k + 3\sum_{k=1}^{q-1} s_k\\
&\quad + \frac{1}{3}\sum_{k=1}^{q} s_k^0,\\
-b(4) &= \frac{1}{4}\sum_{k=1}^{q-1} s_k^4 + 2\sum_{k=1}^{q-1} s_k^3 + \sum_{k=1}^{q-2} s_{k+1}^2 s_k + \sum_{k=1}^{q-2} s_{k+1}s_k^2\\
&\quad + 5\sum_{k=1}^{q-1} s_k^2 + 4\sum_{k=1}^{q-2} s_{k+1}s_k + 4\sum_{k=1}^{q-1} s_k + \frac{1}{4}\sum_{k=1}^{q} s_k^0,
\end{aligned} \tag{24}$$

etc. Note that it is again possible to rewrite these expressions in terms of the Boltzmann factors of fermions $1+s_k$ and bound states $-s_k$; e.g.,

$$\begin{aligned}
b(1) &= \sum_{k=1}^{q}(1+s_k),\\
-b(2) &= \frac{1}{2}\sum_{k=1}^{q}(1+s_k)^2 + \sum_{k=1}^{q-1} s_k,\\
b(3) &= \frac{1}{3}\sum_{k=1}^{q}(1+s_k)^3 + \sum_{k=1}^{q-1} s_k(1+s_k) + \sum_{k=1}^{q-1}(1+s_{k+1})s_k,\\
-b(4) &= \frac{1}{4}\sum_{k=1}^{q}(1+s_k)^4 + \sum_{k=1}^{q-1} s_k(1+s_k)^2\\
&\quad + \sum_{k=1}^{q-1}(1+s_{k+1})s_k(1+s_k) + \sum_{k=1}^{q-1}(1+s_{k+1})^2 s_k\\
&\quad + \sum_{k=1}^{q-2} s_{k+1}s_k + \frac{1}{2}\sum_{k=1}^{q-1} s_k^2,
\end{aligned}$$

etc. The form of these expressions satisfy the physical interpretation discussed before since it identifies them as cluster coefficients of a mixture of $g=1$ fermions and $g=2$ bound states particles. Pure fermionic terms $(1+s_k)^n/n$ are the familiar fermion cluster coefficients, while pure $g=2$ terms (arising only for even n) are the exclusion-2 cluster coefficients found in Ref. [2]. Mixed terms consist of $g=2$ cluster terms, involving factors $-s_k$, with fermions accumulating on levels k and $k+1$ for each such factor with appropriate multiplicities.

Coming back to the algebraic area enumeration we focus on the $b(n)$'s in (24) expressed solely in terms of the s_k's to infer in general that

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leqslant \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j) \times \sum_{k=1}^{q-j} s_{k+j-1}^{l_j}\cdots s_{k+1}^{l_2}s_k^{l_1}. \tag{25}$$

The combinatorial coefficients $c_n(l_1,l_2,\ldots,l_j)$ appearing in (25) are labeled by the compositions of $n'=0,1,2,\ldots,n$ with a number of parts $j\leqslant \min(n',n-n'+1)$ (by convention the unique composition of $n'=0$ has only one part and the trigonometric sum becomes $\sum_{k=1}^{q} s_k^0$). Since the number of compositions of an integer n' with j parts is $\binom{n'-1}{j-1}$, the total number of such compositions is

$$\begin{aligned}
1+\sum_{n'=1}^{n}\sum_{j=1}^{\min(n',n-n'+1)}\binom{n'-1}{j-1} &= 1+\sum_{j=1}^{\lfloor (n+1)/2\rfloor}\sum_{n'=j}^{n-j+1}\binom{n'-1}{j-1}\\
&= \sum_{j=0}^{\lfloor (n+1)/2\rfloor}\binom{n-j+1}{j}\\
&= F_{n+2}.
\end{aligned}$$

Note that the Fibonacci number F_{n+2} is also the number of compositions of $(n+1)$ with only parts 1 and 2.

We obtain for the $c_n(l_1,l_2,\ldots,l_j)$'s

$$c_n(l_1,l_2,\ldots,l_j) = \frac{1}{l_1 l_2\ldots l_j}\sum_{m_1=0}^{\min(l_1,l_2)}\sum_{m_2=0}^{\min(l_2,l_3)}\cdots\sum_{m_{j-1}=0}^{\min(l_{j-1},l_j)} \times\left(\prod_{i=1}^{j-1} m_i\binom{l_i}{m_i}\binom{l_{i+1}}{m_i}\right) \times \binom{n+\sum_{i=1}^{j} l_i-\sum_{i=1}^{j-1} m_i - 1}{2\sum_{i=1}^{j} l_i - 1}, \tag{26}$$

and also note that by ignoring the n-dependent binomial $\binom{n+\sum_{i=1}^{j} l_i-\sum_{i=1}^{j-1} m_i-1}{2\sum_{i=1}^{j} l_i-1}$ in the sums (26) one recovers the $c(l_1,l_2,\ldots,l_j)$ in (14), that is, thanks to the identity

$$\frac{1}{l_1 l_2}\sum_{m=0}^{\min(l_1,l_2)} m\binom{l_1}{m}\binom{l_2}{m} = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2},$$

one has

$$\frac{1}{l_1 l_2\ldots l_j}\sum_{m_1=0}^{\min(l_1,l_2)}\sum_{m_2=0}^{\min(l_2,l_3)}\cdots\sum_{m_{j-1}=0}^{\min(l_{j-1},l_j)}\prod_{i=1}^{j-1} m_i\binom{l_i}{m_i}\binom{l_{i+1}}{m_i} = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2}\; l_2\frac{\binom{l_2+l_3}{l_2}}{l_2+l_3}\;\cdots\; l_{j-1}\frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}.$$

We find

$$n\sum_{l=0}^{n} c_n(l) = F_{2n+1}+F_{2n-1}-1,$$

where again a Fibonacci counting appears, and

$$n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n' \\ j\leqslant \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j) = \binom{n}{n'}^2,$$

from which we infer

$$n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leqslant \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j) = \binom{2n}{n}.$$

Last, again using (17), the counting of closed honeycomb lattice walks of length $2n$ is, as it should, recovered as

$$n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leqslant \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j)\binom{2(l_1+l_2+\ldots+l_j)}{l_1+l_2+\ldots+l_j} = \sum_{n'=0}^{n}\left\{n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n' \\ j\leqslant \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j)\binom{2(l_1+l_2+\ldots+l_j)}{l_1+l_2+\ldots+l_j}\right\}$$
$$= \sum_{n'=0}^{n}\binom{n}{n'}^2\binom{2n'}{n'}.$$

D. Algebraic area enumeration on the honeycomb lattice

Remembering that the spectrum of H_q is the square of that of the honeycomb Hamiltonian H_{2q}, the generating function for the number $C_{2n}(A)$ of closed walks of length $2n$ enclosing an algebraic area A can as well be given in terms of the trace of H_q^n weighted by $1/q$, i.e.,

$$\sum_A C_{2n}(A)Q^A = \frac{1}{q}\mathrm{Tr}\, H_q^n,$$

where now, following again the steps of Sec. II B,

$$\mathrm{Tr}\, H_q^n = (-1)^{n+1} nb(n).$$

We arrive at the conclusion that on the honeycomb lattice the $C_{2n}(A)$'s are

$$C_{2n}(A) = n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\leqslant \min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j) \sum_{k_3=-l_3}^{l_3}\sum_{k_4=-l_4}^{l_4}\cdots\sum_{k_j=-l_j}^{l_j}\binom{2l_1}{l_1+A+\sum_{i=3}^{j}(i-2)k_i}\binom{2l_2}{l_2-A-\sum_{i=3}^{j}(i-1)k_i}$$
$$\times\prod_{i=3}^{j}\binom{2l_i}{l_i+k_i}$$

with the $c_n(l_1,l_2,\ldots,l_j)$'s given in (26) and the algebraic area bounded[1] by $\lfloor (n^2+3)/12\rfloor$.

A few examples of $\frac{1}{q}\mathrm{Tr}\, H_q^n$ are listed below, and the corresponding $C_{2n}(A)$ are listed in Table I:

$$\frac{1}{q}\mathrm{Tr}\, H_q = 3,$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^2 = 15,$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^3 = 3\left(29+2\cos\frac{2\pi p}{q}\right),$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^4 = 3\left(181+32\cos\frac{2\pi p}{q}\right),$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^5 = 3\left(1181+360\cos\frac{2\pi p}{q}+10\cos\frac{4\pi p}{q}\right),$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^6 = 3\left(7953+3520\cos\frac{2\pi p}{q}+242\cos\frac{4\pi p}{q}+8\cos\frac{6\pi p}{q}\right),$$

[1]The sequence OEIS A135711 states that the minimal perimeter of a polyhex with A cells is $2\lceil\sqrt{12A-3}\rceil$. The maximum A for walks of length $2n$ is then $\lfloor (n^2+3)/12\rfloor$.

TABLE I. $C_{2n}(A)$ up to $2n = 14$ for honeycomb lattice walks of length $2n$.

	$2n = 2$	4	6	8	10	12	14
$A = 0$	3	15	87	543	3543	23 859	164 769
±1			6	96	1080	10 560	96 096
±2					30	726	11 130
±3						24	798
±4							42
Total counting	3	15	93	639	4653	35 169	272 835

$$\frac{1}{q}\mathrm{Tr}\,H_q^7 = 3\left(54\,923 + 32\,032\cos\frac{2\pi p}{q} + 3710\cos\frac{4\pi p}{q} + 266\cos\frac{6\pi p}{q} + 14\cos\frac{8\pi p}{q}\right).$$

IV. CONCLUSIONS

We demonstrated that the area counting of honeycomb walks derives from an exclusion statistics $g = 2$ system with a "diluted Hofstadter" spectrum or, equivalently, from a mixture of $g = 2$ and $g = 1$ statistics. This fact calls for a more detailed justification: In previous work [3,4], two of the authors had shown that lattice walks that map to exclusion statistics are of the general form

$$H = f(u)\,v + v^{1-g}\,g(u)$$

with u, v the quantum torus matrices and $f(u), g(u)$ scalar functions. The honeycomb Hamiltonian is apparently not of this form. However, the expression of a walk in terms of a Hamiltonian is not unique: Alternative versions corresponding to modular transformations on the lattice, or, equivalently, alternative realizations of the quantum torus algebra, can exist. We expect that an alternative realization of the honeycomb Hamiltonian H_{2q} that makes its connection to $g = 2$ statistics and the diluted spectral function S_k manifest does exist and is related to the form given in Sec. III A by a unitary transformation. The identification of this transformation and the alternative form of H_{2q} is an interesting open question.

Further, the anisotropic honeycomb Hamiltonian with general transition amplitudes a, b, and c is of physical, but also mathematical, interest. The corresponding generating function of lattice walks would depend on these parameters and would "count" the number of moves in the three different lattice directions U, V, and W separately. The calculation of this generalized generating function through traces of powers of the Hamiltonian appears to be within reach using the methods and techniques of this paper and constitutes a subject for further investigation.

There are various physical systems for which the analysis and results of this paper may be relevant, the quantum mechanics of an actual particle hopping on the honeycomb lattice sites being the most immediate example. It should be noted that the original formulation of such a system involves a wave function with values on each site of the full lattice and six hopping operators, one for each of the three direction in each sublattice. Our formulation in terms of three Hermitian operators U, V, and W and a finite Hilbert space is a lot more economical, and yet equivalent to the original one: It reduces the system to specific Bloch sectors, encoded in the Casimirs of the algebra of the operators, and uses a unique operator for the hopping in each direction, which acts off-diagonally in the two sublattices. It therefore reduces the problem of identifying quantum states to its bare bones.

The case of a particle on the honeycomb lattice with no magnetic field is well studied; less so the one with a magnetic field and a "butterfly" spectrum. The calculation of propagators in this case, and in particular of the propagator for identical initial and final lattice points, is of physical interest, since its value would indicate the rate of diffusion of a quantum mechanical particle initially on a single site. The calculation of this propagator in the path integral formulation involves sums of quantities precisely of the form calculated above. For a continuous time system a particular scaling limit has to be taken, distinct from the continuum scaling that would lead to a particle on a plane. The calculation of this and similar propagators using results in the present work remains an open question.

The relation of the area counting problem to the quantum dynamics of a charged particle on the lattice was exploited in Ref. [20] to calculate moments of the area distribution of walks on a square lattice using propagator techniques. A similar calculation for the case of the honeycomb lattice would be of interest. Moments of the area can also be computed from the results in this paper, either using the explicit area counting, or performing an expansion in $1/q$ of the Q-dependent area generating function. Both calculations are nontrivial. These and related issues are interesting topics for future research.

TABLE II. $C_{2n}(A)$ up to $2n = 10$ for square lattice walks of length $2n$.

	$2n = 2$	4	6	8	10
$A = 0$	4	28	232	2156	21 944
±1		8	144	2016	26 320
±2			24	616	11 080
±3				96	3120
±4				16	840
±5					160
±6					40
Total counting	4	36	400	4900	63 504

ACKNOWLEDGMENTS

L.G. acknowledges the financial support of China Scholarship Council (No. 202009110129). A.P. acknowledges the hospitality of LPTMS, Université Paris-Saclay, during the early phases of this work.

APPENDIX

1. $Z(n)$ for square lattice walks

We denote $Z(n)$ as $Z_q(n)$ to include its dependence on q.

Substituting $d_q = \sum_{n=0}^{\lfloor q/2\rfloor}(-1)^n Z_q(n) z^{2n}$ into (8) and equating the coefficient of z^{2n} on both sides, we get

$$\begin{aligned}Z_q(n) &= Z_{q-1}(n) + s_{q-1}Z_{q-2}(n-1)\\ &= Z_{q-2}(n) + s_{q-2}Z_{q-3}(n-1) + s_{q-1}Z_{q-2}(n-1)\end{aligned}$$

$$= \cdots$$
$$= Z_1(n) + \sum_{m=0}^{q-2} s_{m+1} Z_m(n-1).$$

Since $Z_m(n-1) = 0$ for $n-1 > \lfloor m/2 \rfloor$, i.e., $m < 2n-2$, we obtain

$$Z_q(n) = \sum_{m=2n-2}^{q-2} s_{m+1} Z_m(n-1)$$

with $Z_q(0) = 1$. Thus,

$$Z_q(1) = \sum_{m=0}^{q-2} s_{m+1} Z_m(0) = \sum_{k_1=1}^{q-1} s_{k_1},$$

$$Z_q(2) = \sum_{m=2}^{q-2} s_{m+1} Z_m(1)$$
$$= \sum_{m=2}^{q-2} \sum_{k_1=1}^{m-1} s_{m+1} s_{k_1} = \sum_{k_1=1}^{q-3} \sum_{k_2=1}^{k_1} s_{k_1+2} s_{k_2},$$

$$Z_q(3) = \sum_{m=4}^{q-2} s_{m+1} Z_m(2)$$
$$= \sum_{m=4}^{q-2} \sum_{k_1=1}^{m-3} \sum_{k_2=1}^{k_1} s_{m+1} s_{k_1+2} s_{k_2}$$
$$= \sum_{k_1=1}^{q-5} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} s_{k_1+4} s_{k_2+2} s_{k_3},$$

etc. Formula (9) can then be proven by induction, where we check

$$Z_q(n+1) = \sum_{m=2n}^{q-2} s_{m+1} Z_m(n)$$
$$= \sum_{m=2n}^{q-2} \sum_{k_1=1}^{m-2n+1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \cdots \sum_{k_n=1}^{k_{n-1}} s_{m+1} s_{k_1+2n-2} \cdots s_{k_{n-1}+2} s_{k_n}$$
$$= \sum_{k_1=1}^{q-2n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \sum_{k_4=1}^{k_3} \cdots \sum_{k_{n+1}=1}^{k_n} s_{k_1+2n} s_{k_2+2n-2} \cdots s_{k_n+2} s_{k_{n+1}}.$$

2. Examples of algebraic area enumeration of random walks on the square lattice

A few examples of $\frac{1}{q}\mathrm{Tr}\, H_q^{2n}$ and the corresponding $C_{2n}(A)$'s are listed below and in Table II.

$$\frac{1}{q}\mathrm{Tr}\, H_q^2 = 4,$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^4 = 4\left(7 + 2\cos\frac{2\pi p}{q}\right),$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^6 = 4\left(58 + 36\cos\frac{2\pi p}{q} + 6\cos\frac{4\pi p}{q}\right),$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^8 = 4\left(539 + 504\cos\frac{2\pi p}{q} + 154\cos\frac{4\pi p}{q} + 24\cos\frac{6\pi p}{q} + 4\cos\frac{8\pi p}{q}\right),$$
$$\frac{1}{q}\mathrm{Tr}\, H_q^{10} = 4\left(5486 + 6580\cos\frac{2\pi p}{q} + 2770\cos\frac{4\pi p}{q} + 780\cos\frac{6\pi p}{q} + 210\cos\frac{8\pi p}{q} + 40\cos\frac{10\pi p}{q} + 10\cos\frac{12\pi p}{q}\right).$$

3. Irreducible representations of the honeycomb algebra

Define three new operators u, v, and σ as

$$\sigma = Q^{-1/2}UVW,\ u = U\sigma,\ v = V\sigma$$
$$\Rightarrow \quad U = u\sigma,\ V = v\sigma,\ W = Q^{1/2}v\sigma u. \tag{A1}$$

From the honeycomb algebra (18) we see that σ, u, and v are all unitary and satisfy

$$vu = Q\,uv,\ \ u\sigma = \sigma u^{-1},\ \ v\sigma = \sigma v^{-1},\ \ \sigma^2 = 1. \tag{A2}$$

Since U, V, and W can be uniquely expressed in terms of σ, u, and v, it is sufficient to derive the irreducible representation ("irrep" for short) of u, v, and σ.

Operators u and v satisfy the quantum torus algebra and have a q-dimensional irrep if $Q = \exp(2i\pi p/q)$. However, σ can be embedded within this irrep only for specific values of the Casimirs $u^q = e^{i\phi}$ and $v^q = e^{i\theta}$. Indeed, assuming σ acts within this irrep,

$$e^{i\phi} = u^q = \sigma u^q \sigma = (\sigma u \sigma)^q = u^{-q} = e^{-i\phi} \quad \Rightarrow \quad e^{i\phi} = e^{-i\phi}.$$

So ϕ can only be 0 or π (mod 2π) and similarly for θ. For $\theta, \phi \in \{0, \pi\}$ we can show that the irrep of (A2) is unique up to unitary transformations, and up to the algebra automorphism $\sigma \to -\sigma$, and is given by the action on basis states $|n\rangle$

$$\begin{aligned} u|n\rangle &= e^{i(\phi+2\pi pn)/q}|n\rangle, \qquad n = 0, 1, \ldots, q-1, \\ v|n\rangle &= e^{i\theta/q}|n-1\rangle, \qquad |-1\rangle \equiv |q-1\rangle, \\ \sigma|n\rangle &= e^{i\theta(2n-r)/q}|r-n\rangle, \quad rp + \phi/\pi = 0 \ (\text{mod } q). \quad \text{(A3)} \end{aligned}$$

The "pivot" r in the inversion action of σ is $r = 0$, if $\phi = 0$, or the primary solution of the Diophantine equation $kq - rp = 1$, if $\phi = \pi$. The momenta $qk_x = \phi$ and $qk_y = \theta$ in this irrep are quantized as

$$k_x = \frac{\pi n_x}{q}, \ k_y = \frac{\pi n_y}{q}, \ \ n_x, n_y \in \mathbb{Z}.$$

For either θ or $\phi \notin \{0, \pi\}$ the irrep of (A2) must decompose into more than one q-dimensional irreps of the quantum torus algebra u, v with σ mixing the irreps. The minimal irrep of the full algebra (A2) involves 2 irreps of the torus algebra, all other situations being reducible. Representing all operators in block diagonal form in the space of the two irreps u_i, v_i, $i = 1, 2$, with Casimirs $u_i^q = e^{i\phi_i}$, $v_i^q = e^{i\theta_i}$,

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \ v = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \ \sigma = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix},$$

and implementing the relations $\sigma u^q \sigma = u^{-q}$, $\sigma v^q \sigma = v^{-q}$ leads to

$$\begin{aligned} &\left(e^{i\phi_1} - e^{-i\phi_1}\right)A = \left(e^{i\phi_2} - e^{-i\phi_2}\right)C = \left(e^{i\phi_1} - e^{-i\phi_2}\right)B = 0, \\ &\left(e^{i\theta_1} - e^{-i\theta_1}\right)A = \left(e^{i\theta_2} - e^{-i\theta_2}\right)C = \left(e^{i\theta_1} - e^{-i\theta_2}\right)B = 0. \end{aligned}$$

If all of $\phi_1, \phi_2, \theta_1, \theta_2$ are 0 or π, then the representation is reducible, as we will soon demonstrate. If ϕ_1, θ_1 are 0 or π, but not both of ϕ_2, θ_2 are, then the above relations imply $C = B = 0$ and thus $\sigma^2 = 1$ is impossible, and similarly if ϕ_2, θ_2 are both 0 or π. Therefore, both ϕ_1, θ_1 and ϕ_2, θ_2 must have at least one angle $\neq 0, \pi$. The above relations then imply $A = C = 0$, and $\sigma^2 = 1$ implies $B^\dagger B = 1$. The last equalities above, then, require $\phi_1 = -\phi_2$, $\theta_1 = -\theta_2$. Further, a unitary transformation

$$S = \begin{pmatrix} B^\dagger & 0 \\ 0 & 1 \end{pmatrix}, \ u \to SuS^{-1}, v \to SvS^{-1}, \sigma \to S\sigma S^{-1}$$

eliminates B in σ, and $\sigma u \sigma = u^{-1}$, $\sigma v \sigma = v^{-1}$ imply $u_1 = u_2^{-1}, v_1 = v_2^{-1}$. Altogether, the irrep of the honeycomb algebra for two arbitrary Casimirs $\phi = \phi_1 = -\phi_2$, $\theta = \theta_1 = -\theta_2$, is given by the $2q$-dimensional matrices

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix}, \ v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix}, \ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{(A4)}$$

where u_o and v_o are the basic q-dimensional quantum torus irreps with Casimirs $e^{i\phi}$ and $e^{i\theta}$. Finally, from (A1) we obtain the corresponding irreducible forms for U, V, and W

$$U = \begin{pmatrix} 0 & u_o \\ u_o^{-1} & 0 \end{pmatrix}, \ V = \begin{pmatrix} 0 & v_o \\ v_o^{-1} & 0 \end{pmatrix},$$
$$W = Q^{1/2}\begin{pmatrix} 0 & v_o u_o^{-1} \\ v_o^{-1} u_o & 0 \end{pmatrix}.$$

We conclude with a demonstration that the above representation becomes reducible if $\phi, \theta \in \{0, \pi\}$. In that case, as we demonstrated before in (A3), there is a $q \times q$ matrix σ_o (to be distinguished from the $2q \times 2q$ matrix σ in (A4) above) satisfying (A2) for the matrices u_o and v_o. Performing the unitary transformation

$$S_o = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sigma_o \\ \sigma_o & 1 \end{pmatrix}$$

on all matrices, and using $\sigma_o u_o \sigma_o = u_o^{-1}$, $\sigma_o v_o \sigma_o = v_o^{-1}$, we obtain

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix}, \ v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix}, \ \sigma = \begin{pmatrix} \sigma_o & 0 \\ 0 & -\sigma_o \end{pmatrix},$$

or

$$U = \begin{pmatrix} u_o\sigma_o & 0 \\ 0 & -\sigma_o u_o \end{pmatrix}, \ V = \begin{pmatrix} v_o\sigma_o & 0 \\ 0 & -\sigma_o v_o \end{pmatrix},$$
$$W = Q^{1/2}\begin{pmatrix} v_o u_o^{-1}\sigma_o & 0 \\ 0 & -\sigma_o v_o u_o^{-1} \end{pmatrix}$$

reducing to the direct sum of two q-dimensional irreps.

4. $Z(n)$ for honeycomb lattice walks

We denote $Z(n)$ as $Z_q(n)$ to include its dependence on q.

Substituting $d_q = \sum_{n=0}^{q}(-1)^n Z_q(n) z^{2n}$ into (23) and equating the coefficient of z^{2n} on both sides, we get

$$\begin{aligned} Z_q(n) &= Z_{q-1}(n) + (1+s_q)Z_{q-1}(n-1) - s_{q-1}Z_{q-2}(n-2) \\ &= Z_{q-2}(n) + (1+s_{q-1})Z_{q-2}(n-1) + (1+s_q) \\ &\quad \times Z_{q-1}(n-1) - s_{q-2}Z_{q-3}(n-2) - s_{q-1} \\ &\quad \times Z_{q-2}(n-2) \\ &= \cdots \\ &= Z_1(n) + \sum_{m=1}^{q-1}(1+s_{m+1})Z_m(n-1) \\ &\quad - \sum_{m=0}^{q-2} s_{m+1}Z_m(n-2). \end{aligned}$$

Since $Z_m(n) = 0$ for $n > m$, we obtain

$$Z_q(n) = \sum_{m=n-1}^{q-1}(1+s_{m+1})Z_m(n-1) - \sum_{m=n-2}^{q-2} s_{m+1}Z_m(n-2)$$

with $Z_q(0) = 1$ and $Z_q(j) = 0$ for $j < 0$. Thus,

$$\begin{aligned} Z_q(1) &= \sum_{m=0}^{q-1}(1+s_{m+1})Z_m(0) \\ &= \sum_{k_1=1}^{q}(1+s_{k_1}), \end{aligned}$$

$$\begin{aligned} Z_q(2) &= \sum_{m=1}^{q-1}(1+s_{m+1})Z_m(1) - \sum_{m=0}^{q-2} s_{m+1}Z_m(0) \\ &= \sum_{m=1}^{q-1}\sum_{k_1=1}^{m}(1+s_{m+1})(1+s_{k_1}) - \sum_{m=0}^{q-2} s_{m+1} \end{aligned}$$

$$= \sum_{k_1=1}^{q-1}\sum_{k_2=1}^{k_1}(1+s_{k_1+1})(1+s_{k_2}) - \sum_{k_1=1}^{q-1} s_{k_1},$$

$$Z_q(3) = \sum_{m=2}^{q-1}(1+s_{m+1})Z_m(2) - \sum_{m=1}^{q-2} s_{m+1}Z_m(1)$$

$$= \sum_{m=2}^{q-1}\sum_{k_1=1}^{m-1}\sum_{k_2=1}^{k_1}(1+s_{m+1})(1+s_{k_1+1})(1+s_{k_2})$$

$$- \sum_{m=2}^{q-1}\sum_{k_1=1}^{m-1}(1+s_{m+1})s_{k_1} - \sum_{m=1}^{q-2}\sum_{k_1=1}^{m} s_{m+1}(1+s_{k_1})$$

$$= \sum_{k_1=1}^{q-2}\sum_{k_2=1}^{k_1}\sum_{k_3=1}^{k_2}(1+s_{k_1+2})(1+s_{k_2+1})(1+s_{k_3})$$

$$- \sum_{k_1=1}^{q-2}\sum_{k_2=1}^{k_1}(1+s_{k_1+2})s_{k_2} - \sum_{k_1=1}^{q-2}\sum_{k_2=1}^{k_1} s_{k_1+1}(1+s_{k_2}).$$

Likewise $Z_q(5)$ would read

$$Z_q(5) = +\sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\sum_{m=1}^{l}(1+s_{i+4})(1+s_{j+3})(1+s_{k+2})(1+s_{l+1})(1+s_m)$$
$$- \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}(1+s_{i+4})(1+s_{j+3})(1+s_{k+2})s_l - \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}(1+s_{i+4})(1+s_{j+3})s_{k+1}(1+s_l)$$
$$- \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}(1+s_{i+4})s_{j+2}(1+s_{k+1})(1+s_l) - \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}s_{i+3}(1+s_{j+2})(1+s_{k+1})(1+s_l)$$
$$+ \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}(1+s_{i+4})s_{j+2}s_k + \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}s_{i+3}(1+s_{j+2})s_k + \sum_{i=1}^{q-4}\sum_{j=1}^{i}\sum_{k=1}^{j}s_{i+3}s_{j+1}(1+s_k).$$

5. Diluted trigonometric sums

The diluted trigonometric sums $\sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j}\cdots S_{k+1}^{l_2}S_k^{l_1}$ can be computed and read

$$\frac{1}{q}\sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j}\cdots S_{k+1}^{l_2}S_k^{l_1}$$
$$= \sum_{A=-\infty}^{\infty}\cos\left(\frac{2A\pi p}{q}\right)\Bigg[\sum_{k_5=-l_5}^{l_5}\sum_{k_7=-l_7}^{l_7}\cdots\sum_{k_{2\lfloor(j-1)/2\rfloor+1}=-l_{2\lfloor(j-1)/2\rfloor+1}}^{l_{2\lfloor(j-1)/2\rfloor+1}}\binom{2l_1}{l_1+A+\sum_{i=5,\, i\text{ odd}}^{2\lfloor(j-1)/2\rfloor+1}(i-3)k_i/2}$$
$$\times\binom{2l_3}{l_3-A-\sum_{i=5,\, i\text{ odd}}^{2\lfloor(j-1)/2\rfloor+1}(i-1)k_i/2}\prod_{i=5,\, i\text{ odd}}^{2\lfloor(j-1)/2\rfloor+1}\binom{2l_i}{l_i+k_i}$$
$$+\sum_{k_6=-l_6}^{l_6}\sum_{k_8=-l_8}^{l_8}\cdots\sum_{k_{2\lfloor j/2\rfloor}=-l_{2\lfloor j/2\rfloor}}^{l_{2\lfloor j/2\rfloor}}\binom{2l_2}{l_2+A+\sum_{i=6,\, i\text{ even}}^{2\lfloor j/2\rfloor}(i-4)k_i/2}\binom{2l_4}{l_4-A-\sum_{i=6,\, i\text{ even}}^{2\lfloor j/2\rfloor}(i-2)k_i/2}\prod_{i=6,\, i\text{ even}}^{2\lfloor j/2\rfloor}\binom{2l_i}{l_i+k_i}\Bigg].$$

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