

Quantum-parametric-oscillator heat engines in squeezed thermal baths: Foundational theoretical issues

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(Received 8 July 2021; accepted 21 December 2021; published 7 January 2022)

In this paper we examine some foundational issues of a class of quantum engines where the system consists of a single quantum parametric oscillator, operating in an Otto cycle consisting of four stages of two alternating phases: the isentropic phase is detached from any bath (thus a closed system) where the natural frequency of the oscillator is changed from one value to another, and the isothermal phase where the system (now rendered open) is put in contact with one or two squeezed baths of different temperatures, whose nonequilibrium dynamics follows the Hu-Paz-Zhang (HPZ) master equation for quantum Brownian motion. The HPZ equation is an exact non-Markovian equation which preserves the positivity of the density operator and is valid for (1) all temperatures, (2) arbitrary spectral density of the bath, and (3) arbitrary coupling strength between the system and the bath. Taking advantage of these properties we examine some key foundational issues of theories of quantum open and squeezed systems for these two phases of the quantum Otto engines. This includes (1) the non-Markovian regimes for non-Ohmic, low-temperature baths, (2) what to expect in nonadiabatic frequency modulations, (3) strong system-bath coupling, as well as (4) the proper junction conditions between these two phases. Our aim here is not to present ways for attaining higher efficiency but to build a more solid theoretical foundation for quantum engines of continuous variables covering a broader range of parameter spaces that we hope are of use for exploring such possibilities.

DOI: [10.1103/PhysRevE.105.014108](https://doi.org/10.1103/PhysRevE.105.014108)

I. INTRODUCTION

In the broad range of applications of classical thermodynamics the subject of thermal engines and refrigerators certainly stands out, not only in terms of its practical application utility, but also in view of their foundational theoretical values. Thermodynamic laws are often phrased in terms of engine or refrigerator efficiency: “No engine can have a higher efficiency than that of an ideal Carnot engine” (Second Law); “No refrigerator can cool a physical object to the absolute zero temperature” (Third Law). In the new field of quantum thermodynamics [1] the subject of *quantum* engines and refrigerators [2] occupy an equally important place. One can ask, e.g., how would the Third Law be phrased in terms of a quantum refrigerator [3]? Quantum nature takes prominence beyond the validity ranges of classical thermodynamics in *small systems* [4], at *low temperatures*, with *strong coupling*.¹ between the system and its environment [5,6] and in regimes where *non-Markovian* behaviors associated with

quantum memory and correlations (see, e.g., [7–9]) can no longer be ignored. Add to this list the increased weight of *quantum fluctuations* and under *out-of-equilibrium conditions* (see, e.g., [10] and references therein); for instance, there are claims that heat engines may be operated beyond the Carnot limit by exploiting stationary, nonequilibrium reservoirs [11].

The exploration and exploitation of these new elements, new regimes and new processes hold great promises for advancing quantum sciences and engineering involving the thermodynamics of such systems. However, the familiar paradigms, tools and concepts in traditional equilibrium thermodynamics, mean field dynamics, and linear response theories cannot serve these purposes well. Fortunately, two related families of powerful theories, one for open and dissipative quantum systems [12–14] and the other for nonequilibrium quantum fields [15–19], which have been in development over the last three decades, are pretty well established now to meet these new challenges.

a. Otto engines: Quantum engines are a topic of active current interest; e.g., a quantum engine made up of a single calcium ion in a Paul trap operating between a cold reservoir of laser-cooling beam and a hot reservoir of electric field noise [20,21] has received much recent attention. We refer to the lucid reviews by Kosloff and co-workers [2,22] and the references cited therein for a general background to quantum heat engines and refrigerators (see also [11,23–26] before the 2017 review and [27,28] with those in [29] thereafter). In this paper we examine a class of quantum Otto

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¹For a system modeled by a harmonic oscillator of frequency ω coupled to a bath of temperature β^{-1} , low temperature refers to the regime $\beta\omega \gg 1$, and strong coupling refers to the cases when the damping constant γ is of the order of magnitude of ω .

engines with continuous system variables (in contrast to discrete systems such as spins or qubits) interacting with a bath also of continuous variables (in contrast to stochastic actions such as in the collision models). Our system is a quantum parametric oscillator (with time-dependent natural frequency) which in principle can interact with two squeezed baths at different temperatures, operating under fully nonequilibrium conditions. In the simpler setup we shall analyze here, the parametric oscillator enter in one isentropic phase without any bath, and in another isochoric phase where the oscillator's frequency does not change, but it interacts with a fixed valued squeezed thermal bath. Thus, it can also be referred to as an isothermal phase [22].

The classical Otto cycle consists of four stages made up of two alternating phases: the adiabatic (no heat exchange) phase and the isothermal (equal temperature) phase. [We use the same labeling and ordering as Fig. 1(b) of [25].] We shall use the terminology “isentropic” (no heat exchange) for the former, reserving the term adiabatic and nonadiabatic to describe how fast the natural frequency of the system oscillator is modulated and “isothermal” for the latter, as the system will be brought in contact with a constant temperature heat bath. In the isentropic phase a modulation in the frequency of the system oscillator is implemented, but since the system is thermally insulated from its environment and its evolution is unitary, we shall view it as a closed system (i.e., not accounting for an external drive which alters the oscillator frequency). This phase is sometimes called the “power phase.” In the isothermal phases the system can be placed in contact with two baths of different squeezing parameters. (In the simpler setup we'll analyze only one bath is squeezed.) As such the standard processes in open systems occur, like relaxation, dissipation, decoherence, etc. These phases are sometimes called the “relaxation phase” of the cycle, but of course there are more activities than just relaxation.

b. Theoretical background: Many factors are involved in these two phases but for reasons elaborated below we shall focus on a selected few of foundational theoretical interest. By “foundational” we are referring to the theories of open quantum systems and of squeezing quantum systems in the two different phases of the Otto cycle. We first explain the reasons why for the description of the open system phases the commonly invoked Lindblad or the Caldeira-Leggett master equations are inadequate for treating the very low-temperature regimes. We then introduce the route we take here based on the Hu-Paz-Zhang (HPZ) master equation for quantum Brownian motion (QBM). The HPZ equation is an exact non-Markovian equation which preserves the positivity of the density operator and is valid for (1) all temperatures, (2) arbitrary spectral density of the bath, and (3) arbitrary coupling strength between the system and the bath. The non-Ohmic, low-temperature regime is where non-Markovian behavior gets pronounced.

Using this generic model and taking advantage of these attractive properties we shall examine the theoretical foundation of quantum engines based on the quantum Brownian motion in aspects pertaining to (1) the *very low-temperature* regime, (2) the *late-time squeezed thermal state*, correcting some flaws in the literature, (3) the *strong system-bath coupling* in the isothermal phases as well as (4) the *nonadiabatic frequency*

variation in the isentropic phases, and finally (5) the *junction conditions* between the end state of one phase and the start state of another phase.² This entails, specifically, finding the correct late time end state of the isothermal phase and how it joins with the isentropic phase. We emphasize that the aim of this work is not to present ways for attaining higher efficiency in the sense of “you can do this for better results,” but to provide a more solid and broader foundation for different aspects of the theories involved. For example, we shall include non-Markovian behavior, strong coupling, and nonadiabatic drives in our discussions but make clear that at the present level of theoretical understanding, we can work only with weak coupling in the engine design, likewise for adiabatic drives, since they yield higher efficiency. The purposes for our foundational theoretical studies are twofold: to scrutinize the validity of assumptions and approximations in the existing theories so mistakes can be avoided, and to prepare a broader base for future searches into wider parameter spaces for more efficient engines or other quantum thermodynamic applications.

A. Theories based on Markovian master equations

Quantum heat engines operating at sufficiently high temperatures are relatively easy to construct, and the theories supporting them are well understood (see, e.g., [21,24]). The challenge is for low temperatures. Many theoretical work in quantum heat engines invoke the “collision model” for the baths [29]. For instance, the claimed pride of the nice and thorough work of Ref. [25] is the derivation of the efficiency of a quantum Otto engine at *low temperatures*. The authors justified the use of the Lindblad equation with a collision model derivation. At sufficiently high temperatures this is justified, but we have serious reservations in the adequacy of Markovian collision models to describe very low-temperature phenomena. Single-hit collision assumption for the description of the dynamics of gas molecules yields good results primarily for dilute gases at high temperatures.

Markovian processes are aptly described by Markovian master equations, such as the very popular Lindblad equation. The question is, can one still use this model for the opposite regime of very low temperatures or very high density—when correlations build up in collisions and ignoring the memory becomes untenable? We only raise the issue here but save the discussion to a sequel paper [32] where we shall examine the collision models, both Markovian and non-Markovian, and analyze them in relation to the corresponding quantum

²In fact, drawing from classical thermodynamics, the junction condition relates to the problem of continuity of the thermodynamic potentials. In a classical ideal cycle, the system goes from one stage to another being subjected in some sudden fashion to different boundary conditions. However, in reality, this discontinuity is often curbed by the dissipative or nonequilibrium elements present in the cycle. More thorough discussions on this aspects can be found in [30,31]. Such considerations are typically neglected in quantum thermodynamics, thus worth further investigation, in particular, at strong coupling.

Brownian models where a reliable master equation is indeed available for the study of non-Markovian effects.

Another important Markovian master equation is the Caldeira-Leggett (CL) master equation for quantum Brownian motion. Unfortunately both the Lindblad and the CL master equations have problems at low temperatures: Lindblad violates the uncertainty principle while CL loses the positive definiteness of the density matrix. This is an important point which needs some elaboration.

a. Lindblad-GKS and Caldeira-Leggett master equations are problematic at very low temperatures. Many theoretical models of quantum engines use the Lindblad [33] and Gorini-Kossakowski-Sudarshan (GKS) master equation [34] because (1) it is mathematically sound, having a completely positive (CP) dynamical semigroup property, (2) it is commensurate with the Born-Markov master equation [35] often used in quantum optics, and (3) it is consistent with the Caldeira-Leggett (CL) master equation [36] for a Brownian motion model valid with Ohmic bath at high temperatures; note that the CL master equation for the density matrix is not positive-definite [37]. All is well until one explores its low-temperature behavior.

As pointed out by Haake [38], Garraway [39], and many others, *there are serious problems of the Lindblad master equation at very low temperatures.* For example, it leads to a violation of the Heisenberg uncertainty principle. There are proposals to ameliorate these maladies; e.g., in Ref. [40] a term is added to the Born-Markov master equation, which vanishes in the classical limit, bringing the equation to the Lindblad form and satisfying the uncertainty principle. In Ref. [41] a Lindblad-form master equation for *weakly damped* systems accurate for all regimes is derived, which can serve as a replacement for the Bloch-Redfield equation for thermal damping that is completely positive. These amendments target one kind of deficiency but remain restricted by other limitations or approximations.

Thus the focus of attention would be to find a master equation, for the particular class of problems one wants to solve, which is both mathematically correct and physically sound at low temperatures, the regime where non-Markovian behaviors become important.

B. QBM non-Markovian master equation valid for non-Ohmic baths at low temperatures

For systems and baths which can suitably be modeled by quantum harmonic oscillators there is such an equation, the Hu-Paz-Zhang (HPZ) master equation for quantum Brownian motion [42,43]. (An equivalent representation is via the Fokker-Planck equation governing the reduced Wigner function [44].) It is an exact non-Markovian master equation which preserves the positivity of the density operator and is valid for all temperatures, arbitrary spectral density of the bath and arbitrary coupling strength between the system and the bath. This offers a much broader range of utilities such as for strong couplings and for non-Ohmic baths at ultralow temperatures, where non-Markovian behavior are pronounced. Since it has both the sound mathematical (positive definite, unlike the CL equation) and physical properties (respecting the uncertainty principles, unlike the Lindblad) at low temperatures, the HPZ

equation can also serve as a good benchmark where results obtained from other trial models can be compared. We will explore the viability of the popular non-Markovian collision models at low temperatures in our sequel paper [32].

C. Master equation for QBM of parametric oscillators

The Hu-Paz-Zhang master equation [42,43] is for a system of one quantum harmonic oscillator of fixed (time-independent) frequency interacting with a bath of harmonic oscillators also of fixed frequencies. For the quantum engine we are interested, in the isentropic phases, the system oscillator frequency varies with time (parametric oscillators), but without contact with any bath. In the isothermal phases the system oscillator has a fixed frequency but it interacts with a squeezed thermal bath. A more general theory which encompasses both aspects exists. The HPZ master equation for a parametric quantum oscillator in a general squeezed thermal bath valid for all spectral densities, at all temperatures, and for arbitrary coupling strength between the system and its environment was derived by Hu and Matacz [45] using the same method as in the derivation of the original HPZ equation for the Brownian motion of quantum oscillators. It was used to demonstrate the more versatile kinematic approach in the derivation of the Unruh and Hawking effects and going beyond for situations in more general nonequilibrium conditions [46]. Cosmological particle creation as squeezing was discussed in [47] and entropy generation in squeezed quantum open systems in [48]. (For representative works along similar lines of reasoning; see references in, e.g., [49–51].)

The non-Markovian HPZ master equation for parametric oscillators in a squeezed thermal bath [45] is an example of a theory with a broader base and more solid foundation we referred to earlier. Besides non-Ohmic baths at low temperatures, its validity for arbitrary couplings between the system and the bath enables one to examine the coupling strength dependence. We shall show the marked differences between strong and weak coupling. However, for strong coupling there is some arbitrariness in what portion of the interaction energy should be considered as internal energy there may be some intrinsic ambiguity in how heat is defined. While this issue is under investigation [52], we will return to the practical level when evaluating the engine efficiency, and restrict our consideration to only weak coupling between the system and the bath.

Squeezing being the centerpiece in the quantum engine under study, both in the sense of changing the natural frequency of the oscillator in the isentropic phase and for the squeezed thermal bath in the isothermal phase, we give in Sec. II some background description of squeezed states and squeezed thermal baths. We show how a system placed in contact with a squeezed thermal bath evolves and determine the quantum state of the system at late times after equilibration. This is of essential interest to the problem at hand because the end state of this isothermal phase becomes the initial state of the isentropic phase. It then affects the implementation of the junction condition between two phases and the optimization scheme in the latter phase. In Sec. III we describe the Otto cycle. We first treat the simpler *isentropic* phases—no contact with a bath—by the Fokker-Planck-Wigner equation, where

the natural frequency of the harmonic oscillator is allowed to change, both adiabatically (gradual variation) and nonadiabatically (sudden quench). We discuss the cost versus the gain in the performance in both situations. In Sec. IV we move on to the isothermal phase where a squeezed thermal bath is brought into contact, and show the proper treatment of the energy transfer between the oscillator and the bath during the isothermal phase when the correct end state is used. We combine the results from the previous section to calculate the performance efficiency of the Otto quantum engine, in a broader scope of low-temperature and nonadiabaticity effects. We conclude in Sec. V. In the Appendix we provide the coefficients and expressions which enter into the HPZ equation at late times.

II. SQUEEZED STATE FOR THE CLOSED AND OPEN SYSTEMS

In this section we wish to examine more closely the nature and effects of squeezed states in the two separate phases of the Otto cycle. In the isentropic phase the oscillator is not in contact with a heat bath. Although its natural frequency is changed (sometimes the change of frequency is referred to as squeezing), it remains a closed system which follows unitary evolution. In the isothermal phase, the system is in contact with a squeezed thermal bath and thus it constitutes an open system. Of special interest to us is whether there exists a stationary state of the system after it interacts with a squeezed thermal bath for some time, and what state it would be after equilibration. This is needed because that state becomes the initial state of the succeeding isentropic phase in the Otto cycle.

We first review the relevant properties of squeezing. Squeezing manifests itself slightly differently in the context of a closed versus an open systems.

A. Brief overview of squeezing

Suppose we have a free harmonic oscillator of mass m and natural frequency ω , prepared in a squeezed thermal state, whose density matrix $\hat{\rho}_{\text{ST}}^{(x)}$ takes the form

$$\hat{\rho}_{\text{ST}}^{(x)} = \hat{S}(\zeta) \hat{\rho}_{\beta}^{(x)} \hat{S}^{\dagger}(\zeta), \quad (2.1)$$

where $\hat{\rho}_{\beta}^{(x)}$ is the thermal state of the harmonic oscillator, and $\hat{S}(\zeta)$ is the squeezed operator

$$\hat{S}(\zeta) = \exp \left[\frac{1}{2} \zeta^* \hat{a}^2 - \frac{1}{2} \zeta \hat{a}^{\dagger 2} \right]. \quad (2.2)$$

The squeeze parameter $\zeta \in \mathbb{C}$, assumed to be a frequency-independent constant, is usually conveniently written in the polar form $\zeta = \eta e^{i\theta}$ with $\eta \in \mathbb{R}^+$ and $0 \leq \theta < 2\pi$. The creation and annihilation operators satisfy the standard commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$, such that the displacement $\hat{\chi}$ and the corresponding conjugate momentum \hat{p} of the oscillator, obeying the Heisenberg equation $\dot{\hat{\chi}}(t) + \omega^2 \hat{\chi}(t) = 0$, can be expressed as

$$\begin{aligned} \hat{\chi}(t) &= \frac{1}{\sqrt{2m\omega}} (\hat{a} e^{-i\omega t} + \hat{a}^{\dagger} e^{+i\omega t}), \\ \hat{p}(t) &= i\sqrt{\frac{m\omega}{2}} (\hat{a}^{\dagger} e^{+i\omega t} - \hat{a} e^{-i\omega t}). \end{aligned} \quad (2.3)$$

Here the overhead dot denotes the time derivative and $\hbar = 1$ is chosen.

Since the squeezed thermal state is a Gaussian state, its statistical properties are fully described by the first two moments of \hat{a} , \hat{a}^{\dagger} . The higher moments can be obtained by the Wick expansion. Thus we need only

$$\begin{aligned} \langle \hat{a} \rangle_{\text{ST}} &= 0 = \langle \hat{a}^{\dagger} \rangle_{\text{ST}}, \\ \langle \hat{a}^2 \rangle_{\text{ST}} &= -e^{+i\theta} \sinh 2\eta \left(\langle \hat{N} \rangle_{\beta} + \frac{1}{2} \right), \\ \langle \hat{a}^{\dagger} \hat{a} \rangle_{\text{ST}} &= \cosh 2\eta \left(\langle \hat{N} \rangle_{\beta} + \frac{1}{2} \right) - \frac{1}{2}, \end{aligned} \quad (2.4)$$

where the last term carries the average number of particles

$$\langle \hat{N} \rangle_{\text{ST}} = \cosh 2\eta \langle \hat{N} \rangle_{\beta} + \sinh^2 \eta, \quad (2.5)$$

in which the factor $\langle \hat{N} \rangle_{\beta} + \frac{1}{2}$ is often conveniently written as

$$\langle \hat{N} \rangle_{\beta} + \frac{1}{2} = \frac{1}{2} \coth \frac{\beta\omega}{2}, \quad (2.6)$$

and β is the inverse temperature of the thermal state.

The covariance matrix elements $\langle \hat{\chi}^2 \rangle_{\text{ST}}$, $\langle \hat{p}^2 \rangle_{\text{ST}}$, and $\frac{1}{2} \langle \{\hat{\chi}(t), \hat{p}(t)\} \rangle_{\text{ST}}$ characterize the natures of the states of the oscillator. The corresponding density matrix elements can be uniquely constructed from them. For a harmonic oscillator in a squeezed thermal state, its covariance matrix elements are related to the corresponding elements in a thermal state by

$$\langle \hat{\chi}^2(t) \rangle_{\text{ST}} = [\cosh 2\eta - \cos(2\omega t - \theta) \sinh 2\eta] \langle \hat{\chi}^2 \rangle_{\beta}, \quad (2.7)$$

$$\langle \hat{p}^2(t) \rangle_{\text{ST}} = [\cosh 2\eta + \cos(2\omega t - \theta) \sinh 2\eta] \langle \hat{p}^2 \rangle_{\beta}. \quad (2.8)$$

In particular the cross correlation between the canonical variables reveals information about the stationarity of the state

$$\frac{1}{2} \langle \{\hat{\chi}(t), \hat{p}(t)\} \rangle_{\text{ST}} = \sinh 2\eta \sin(2\omega t - \theta) \left(\langle \hat{N} \rangle_{\beta} + \frac{1}{2} \right). \quad (2.9)$$

These elements are time dependent, in contrast to their counterparts in the thermal state. Observe that the factors before $\langle \hat{\chi}^2 \rangle_{\beta}$ and $\langle \hat{p}^2 \rangle_{\beta}$ are positive real numbers, so they can be smaller or larger than unity, depending on the choice of the squeeze angle θ and time t . Namely, one may suppress the dispersion of the χ quadrature at the expense of the momentum uncertainty. Thus one can use the squeeze parameter to tune the coherence of the oscillator such that, in this case, the noise level of the χ quadrature can be much lower than that of the ground state. This trick has been widely applied in quantum optics to bring down the quantum noise in high precision interferometer measurements such as in the Laser Interferometer Gravitational-Wave Observatory (LIGO) [53].

The mechanical energy of the free oscillator in a squeezed thermal state is

$$\begin{aligned} E_{\text{ST}}^{(c)} &= \frac{1}{2m} \langle \hat{p}^2 \rangle_{\text{ST}} + \frac{m\omega_{\text{R}}^2}{2} \langle \hat{\chi}^2 \rangle_{\text{ST}} \\ &= \left(\langle \hat{N} \rangle_{\text{ST}} + \frac{1}{2} \right) \omega = \cosh 2\eta E_{\beta}^{(c)}, \end{aligned} \quad (2.10)$$

where $E_{\beta}^{(c)}$ is the thermal energy of the oscillator, so it does not depend on the squeeze angle θ , as well as the factor $\sinh 2\eta$. It

is clearly seen that even if one has effectively suppressed the uncertainty in one quadrature, this induces a large uncertainty in the complementary conjugated quadrature, such that the total energy still increases.

Finally we note the Hadamard function of the harmonic oscillator in the squeezed thermal state

$$\begin{aligned}
 G_{H,ST}^{(\chi)}(t, t') &= \frac{1}{2} \text{Tr}[\hat{\rho}_{ST}\{\hat{\chi}(t), \hat{\chi}(t')\}] \\
 &= \frac{1}{4m\omega} \coth \frac{\beta\omega}{2} \{ \cosh 2\eta [e^{-i\omega(t-t')} + e^{+i\omega(t-t')}] \\
 &\quad - \sinh 2\eta [e^{-i\omega(t+t')} e^{+i\theta} + e^{+i\omega(t+t')} e^{-i\theta}] \}
 \end{aligned} \tag{2.11}$$

is not invariant under time translation, and thus the oscillator has nonstationary correlation, while the expression in the brackets next to $\cosh 2\eta$ is stationary.³ Thus it is easily seen that a squeezed thermal state is not a steady state, while a thermal state is.

B. Open system: Squeezed thermal baths

We consider a harmonic oscillator of fixed natural frequency bilinearly coupled to a thermal bath, modeled by a massless quantum scalar field initially in a squeezed thermal state of temperature β^{-1} . The bath thus has a gapless spectrum. Once the oscillator is brought into contact with the bath, both the oscillator and the bath will undergo nonequilibrium evolutions even though the change of the bath in general is negligible due to its overwhelmingly larger number of degrees of freedom. The effects of the coupling are manifested in the forms of the quantum noise from the bath and the damping force on the oscillator. The quantum noise drives the oscillator away from its initial configuration, while the damping counteracts the influence of the driving noise. The latter plays a special role in how squeezing, in the current setting, is passed from the bath to the system it interacts with.

Compared to a thermal bath, the squeezed thermal bath has several remarkable features. As discussed in the previous section, the quantum noise of a squeezed thermal bath is nonstationary. This immediately raises a concern whether the dissipative effect can fully counteract the driving of the nonstationary noise such that the oscillator can relax to a stationary or equilibrium state? We are interested in the similarities and dissimilarities in the behavior of the oscillator in such an equilibrium state compared to that of the oscillator in the closed-system highlighted in the previous section.

Detailed calculations presented elsewhere [54] show that when the oscillator is bilinearly coupled to the squeezed thermal bath, even though the bath gives a nonstationary driving force, the oscillator will eventually approach an equilibrium state. Its correlation function—the Hadamard function—starting off nonstationary in time will become invariant in time

³We call a two-point function $G(t, t')$ “stationary” if it is a function of $t - t'$, i.e., “time-translation-invariant” or “homogeneous in time”; otherwise it is “nonstationary.” We call a function which does not depend on time “static” or time-independent, thus there is an enormous difference between “static” and “nonstationary.”

translation after the relaxation time. It will take the form

$$G_H^{(\chi)}(t, t') = \cosh 2\eta G_{H,\beta}^{(\chi)}(t - t'), \tag{2.12}$$

for $t, t' \gg \gamma^{-1}$ where γ is the damping constant, and η is the squeeze parameter of the bath in its initial configuration. The Hadamard function $G_{H,\beta}^{(\chi)}(t - t')$ gives the correlation of the oscillator when it is coupled to a plain thermal state. In contrast to (2.11), we see that the nonstationary component is absent at late times. Details can be found in [54].

Physically, when the oscillator is coupled to a bath, it behaves like a driven, damped oscillator, instead of the free oscillator in the closed system configuration. It can be shown that although the driving force coming from the bath is nonstationary, the damping will respond in a delicate way. The strength of dissipation depends on both the dissipation kernel, determined by the form of coupling and the property of the bath field, and the state of the oscillator’s motion. Thus when the oscillator is driven by (nonstationary) quantum fluctuations of the bath, the damping will adjust itself to match the driving force in accordance, such that in the end the energy exchange between the oscillator and the bath is balanced, and the system reaches equilibration. This final state, in the weak coupling limit, is essentially a thermal state at an effective temperature β_s^{-1} , satisfying

$$\coth \frac{\beta_s\omega}{2} = \cosh 2\eta \coth \frac{\beta\omega}{2}, \tag{2.13}$$

in contrast to the squeezed thermal state of the free oscillator introduced in the previous section found in the literature. They have distinct features. The latter is clearly nonstationary, thus in conflict with the notion of an equilibrium state. The former is stationary and should be the very state that is often referred to at the end of the isothermal phase of the Otto engine when its working medium is in contact with a squeezed thermal bath. This subtle difference is not clearly noticed or stated in the literature,⁴ and confusion may arise when misidentification occurs. Equally noteworthy is that this behavior, specifically, the *nonstationarity of the two-point function* $G_H^{(\chi)}(t, t')$, will not be restored to the form in (2.11), that is, not reverting to the squeezed thermal state, even in the limit of ultraweak coupling, in stark contrast to the situation when the oscillator interacts with a plain thermal bath. In the latter case, in the weak coupling limit, the dynamical behavior of the oscillator in the final equilibrium state is pretty much the same as that of the oscillator in the canonical thermal state in a closed system description. In other words, if the bath is initially in a squeezed state, the oscillator will not inherit the

⁴A system interacting with a squeezed thermal bath is often *mistaken* to have assumed a *squeezed* thermal state as its end state (e.g., [24,25,56]). This is at odds with the concept of equilibration when the system is fully relaxed, because, as is discussed earlier, the two-point function of the system in such a squeezed thermal state is not stationary. The fully relaxed system in fact ends up in a *thermal state* with a slightly different temperature, which depends on the squeeze parameters of the bath and is given by (2.13). Even if the system is in contact with a squeezed thermal bath for a shorter time, the resultant squeezing is quite limited. See [54] for further details.

state of the bath at the end in the limit of vanishing coupling strength.

In the final equilibrium state, the covariance matrix elements $\langle \hat{\chi}^2(\infty) \rangle$, $\langle \hat{p}^2(\infty) \rangle$ are given by

$$\begin{aligned} \langle \hat{\chi}^2(\infty) \rangle &= \cosh 2\eta \langle \hat{\chi}^2(\infty) \rangle_\beta, \\ \langle \hat{p}^2(\infty) \rangle &= \cosh 2\eta \langle \hat{p}^2(\infty) \rangle_\beta, \quad \frac{1}{2} \langle \{ \hat{\chi}(\infty), \hat{p}(\infty) \} \rangle = 0, \end{aligned} \quad (2.14)$$

where $\langle \hat{\chi}^2(\infty) \rangle_\beta$ and $\langle \hat{p}^2(\infty) \rangle_\beta$ represent the corresponding covariance matrix elements when the bath is initially prepared in a thermal state. The constancy of the first two expressions is distinct from their counterparts in (2.7) in the closed system description. In addition, the elements in (2.14) do not contain the contribution from the nonstationary component, which is typically proportional to $\sinh 2\eta$. In the weak coupling limit, we have

$$\langle \hat{\chi}^2(\infty) \rangle_\beta = \frac{1}{2m\omega} \coth \frac{\beta\omega}{2}, \quad \langle \hat{p}^2(\infty) \rangle_\beta = \frac{m\omega}{2} \coth \frac{\beta\omega}{2}, \quad (2.15)$$

so the forms of $\langle \hat{\chi}^2(\infty) \rangle$ and $\langle \hat{p}^2(\infty) \rangle$ also support the identification of the effective thermal state in (2.13).

The internal energy, defined as the mechanical energy of the oscillator, is then given by

$$E(\infty) = \frac{1}{2m} \langle \hat{p}^2(\infty) \rangle + \frac{m\omega_{\text{R}}^2}{2} \langle \hat{\chi}^2(\infty) \rangle = \cosh 2\eta E_\beta(\infty). \quad (2.16)$$

The expression $E_\beta(\infty)$ is the internal energy of the oscillator if it is initially coupled to an unsqueezed thermal bath at temperature β^{-1} , and, in the ultraweak coupling limit, it assumes the form

$$E_\beta(\infty) = \frac{\omega}{2} \coth \frac{\beta\omega}{2}. \quad (2.17)$$

The results in (2.16) happens to take on the same functional form as the internal energy in (2.10) in the closed system description, even though the states of the system in both formulations have totally different characteristics. This similitude may not come as a surprise because in the weak coupling limits, the squeeze parameter and the temperature of the system are equal to those of the bath, and because the constant energy expression in (2.10) does not depend on the squeeze angle θ or the factor $\sinh 2\eta$. Therefore, even if one inadvertently misidentified the squeezed thermal state, one would still get the correct internal energy expression to compute the heat during the isothermal processes of the Otto engine. This happy end result inconspicuously conceals some oversights or mistreatments in the literature of the nonequilibrium dynamics of quantum systems interacting with a squeezed thermal bath.

In (2.13), since $\cosh 2\eta \geq 1$, the effective system temperature β_{s}^{-1} is always greater than the initial bath temperature β^{-1} . It implies that if the hot thermal bath is initially squeezed, then squeezing can boost the effective temperature of the system at the end of the hot isothermal process. This can enhance the idealized Carnot efficiency.

C. Closed system: Oscillator with varying frequency and unsqueezing

With these details available, we now make some comments on the equivalence between frequency modulation and squeezing, as well as the protocol of unsqueezing. Unsqueezing is understood as a procedure to introduce another squeezing to cancel out the preexisting squeezing in the state of the oscillator, the working medium of the Otto engine. It is often implemented during the isentropic stage that follows the hot squeezed isothermal phase in an attempt to restore the initial state in the isentropic phase back to a thermal state for the purpose of enhancing the overall efficiency. However, as shown in (2.12), (2.14), and (2.16), the final equilibrium state of the isothermal stage is a steady state, essentially a thermal state, stationary in time since the first two moments of the oscillator have the forms given by some canonical thermal state. If this state serves as the initial state of the subsequent isentropic stage, then performing unsqueezing will in fact introduce nonstationarity into the oscillator. Suppose at time $t = 0$, the initial time of an isentropic stage, we carry out an instantaneous unsqueezing by $\zeta^* = \eta^* e^{i\theta^*}$, then according to (2.7) and (2.9), we cannot find a suitable squeeze parameter to remove the η dependence in the covariance matrix elements in (2.14).

A variant of unsqueezing is proposed to carry out at the end of the isentropic phase on account of the possibility that the frequency modulation may introduce squeezing. This is most easily seen in the Heisenberg picture, where the canonical operators $\hat{\chi}(t)$ and $\hat{p}(t)$ can be expressed by

$$\begin{aligned} \hat{\chi}(t) &= d_1(t) \hat{\chi}(0) + \frac{d_2(t)}{m} \hat{p}(0), \\ \hat{p}(t) &= m \dot{d}_1(t) \hat{\chi}(0) + \dot{d}_2(t) \hat{p}(0), \end{aligned} \quad (2.18)$$

where $d_1(t)$ and $d_2(t)$ are a special set of solutions to the classical equation of motion of the parametric oscillator $\ddot{\chi}(t) + \omega^2(t) \chi(t) = 0$, with the initial conditions $d_1(0) = 1$, $\dot{d}_1(0) = 0$ and $d_2(0) = 0$, $\dot{d}_2(0) = 1$. The frequency modulation, in this phase of duration τ , will vary from $\omega(0) = \omega_{\text{H}}$ to $\omega(\tau) = \omega_{\text{L}}$. Then the cross correlation $\langle \{ \hat{\chi}(t), \hat{p}(t) \} \rangle$ is given by

$$\begin{aligned} &\frac{1}{2} \langle \{ \hat{\chi}(t), \hat{p}(t) \} \rangle \\ &= m d_1(t) \dot{d}_1(t) \langle \hat{\chi}^2(0) \rangle + \frac{1}{m} d_2(t) \dot{d}_2(t) \langle \hat{p}^2(0) \rangle \\ &\quad + \frac{1}{2} [d_1(t) \dot{d}_2(t) + \dot{d}_1(t) d_2(t)] \langle \{ \hat{\chi}(0), \hat{p}(0) \} \rangle, \end{aligned} \quad (2.19)$$

where the expectation value is taken with respect to the initial state of the oscillator in the isentropic phase. Since the initial state is stationary, Eq. (2.19) reduces to

$$\begin{aligned} &\frac{1}{2} \langle \{ \hat{\chi}(\tau), \hat{p}(\tau) \} \rangle \\ &= \frac{m}{2} \frac{d}{d\tau} \langle \hat{\chi}^2(\tau) \rangle \\ &= \frac{1}{2} \coth \frac{\beta_{\text{s}} \omega_{\text{H}}}{2} \frac{d}{d\tau} \left[\frac{1}{\omega_{\text{H}}} d_1^2(\tau) + \omega_{\text{H}} d_2^2(\tau) \right], \end{aligned} \quad (2.20)$$

at the end of the isentropic phase. For a harmonic oscillator of fixed frequency ω_{H} , the expression inside the square brackets

is a constant and thus (2.20) gives zero. However, for a general parametric oscillator, (2.20) is not zero, so the end state of the oscillator is not stationary and acquires squeezing. Similarly we can find $\langle \hat{\chi}^2(\tau) \rangle$ and $\langle \hat{p}^2(\tau) \rangle$:

$$\langle \hat{\chi}^2(\tau) \rangle = \frac{1}{2m\omega_H} \coth \frac{\beta_S \omega_H}{2} [d_1^2(\tau) + \omega_H^2 d_2^2(\tau)], \quad (2.21)$$

$$\langle \hat{p}^2(\tau) \rangle = \frac{m\omega_H}{2} \coth \frac{\beta_S \omega_H}{2} \left[\frac{1}{\omega_H^2} \dot{d}_1^2(\tau) + \dot{d}_2^2(\tau) \right], \quad (2.22)$$

for this end state. Then according to (3.9)–(3.11) in [55], for the oscillator with frequency ω in a general Gaussian state, parametrized by the squeezed parameters (η_s, ψ_s) and an

inverse temperature-like parameter ϑ , the covariance matrix elements take on the form

$$\langle \hat{\chi}^2 \rangle = \frac{1}{2m\omega_L} \coth \frac{\vartheta}{2} (\cosh 2\eta_s - \sinh 2\eta_s \cos \psi_s), \quad (2.23)$$

$$\langle \hat{p}^2 \rangle = \frac{m\omega_L}{2} \coth \frac{\vartheta}{2} (\cosh 2\eta_s + \sinh 2\eta_s \cos \psi_s), \quad (2.24)$$

$$\frac{1}{2} \langle \{ \hat{\chi}, \hat{p} \} \rangle = -\frac{1}{2} \coth \frac{\vartheta}{2} \sinh 2\eta_s \sin \psi_s. \quad (2.25)$$

Comparing (2.23)–(2.25) with (2.20)–(2.22), we identify $\vartheta = \beta_S \omega_H$, and the effective squeeze parameter η_s ,

$$\eta_s = \frac{1}{2} \cosh^{-1} \left\{ \frac{1}{2} \left[\frac{\omega_L}{\omega_H} d_1^2(\tau) + \omega_H \omega_L d_2^2(\tau) + \frac{1}{\omega_H \omega_L} \dot{d}_1^2(\tau) + \frac{\omega_H}{\omega_L} \dot{d}_2^2(\tau) \right] \right\}, \quad (2.26)$$

of the oscillator at the end of the isentropic phase. We use (2.25) to find the corresponding squeeze angle ψ_s . These can then be used to unsqueeze the oscillator state. The expression in the curly brackets can serve as a measure of the degree of nonadiabaticity, which is essentially the ratio of the energy at the end of an arbitrary parametric process to its counterpart, the energy for the adiabatic process if the oscillator is initially in the ground state.

We are led to an interesting competition during the nonadiabatic modulation. On one hand, the nonadiabatic frequency change of the parametric oscillator degrades the output work due to unwanted excitations. On the other hand, allowing for unsqueezing, as shown later, can enhance the efficiency at the expense of the agent that performs the action. This will be explored in greater detail in the next section.

III. ISENTROPIC STAGES: FREQUENCY CHANGE IN A CLOSED SYSTEM

We have highlighted the relevant theories for closed and open systems behind the respective isentropic and the isothermal phases of an Otto cycle.⁵ In this section for specificity we first briefly describe the operational protocols of the Otto cycle before entering into the details towards the assessment of engine efficiency.

The Otto cycle consists of the following four stages in two alternating phases (for illustration we use the same labeling as in Fig. 1(b) of [25]):

(i) Isentropic compression phase ($A \mapsto B$): At point A , the oscillator is isolated from a thermal bath, its initial state is a thermal state of temperature β_L^{-1} . The natural frequency of the oscillator is then raised to ω_H from $\omega_L < \omega_H$ by an external agent. In this phase, the oscillator evolves as a closed system,

so there is no heat exchange, and the work⁶ done by the external agent will solely change the system's internal energy. For our theoretical inquiry, we do not limit the isentropic process to be adiabatically slow even though the conventional wisdom tells that the output work from the engine tends to be optimal for an adiabatically slow modulation of the frequency because the excitation to higher energy levels is less likely to occur. Unsqueezing may be introduced at the end of the isentropic phase to reverse the squeezing brought about by the nonadiabatic frequency modulation.

(ii) Hot isothermal phase ($B \mapsto C$): At point B , the oscillator with frequency ω_H is brought back into contact with a hot squeezed thermal bath at temperature β_H^{-1} . The oscillator will undergo nonequilibrium relaxation to an equilibrium state and end up at point C of the cycle. In this phase the energy exchange with the bath, identified as heat, will change the internal energy of the oscillator.

(iii) Isentropic expansion phase ($C \mapsto D$): Here the oscillator frequency is lowered down to ω_L . Since it is isolated from the thermal bath, the oscillator exchanges work only with the outside agent that modulates the frequency. In contrast to the isentropic phase ($A \mapsto B$), we note that in the previous isothermal phase ($B \mapsto C$), the bath is squeezed at the outset, so in principle an unsqueezing is needed at the beginning of the current phase to remove the squeezing of the system during the isothermal phase. However, from our earlier discussions, if the oscillator in the isothermal phase prior to this isentropic phase had undergone full relaxation, then it is not necessary to do so. We only need to carry out unsqueezing at the end of the isothermal phase if the expansion there is nonadiabatic.

(iv) Cold isothermal phase ($D \mapsto A$): At point D , the oscillator of frequency ω_L is placed in contact with a cold thermal bath at temperature β_L^{-1} . The oscillator is then relaxed to its

⁵For the isentropic phase, the system, modeled by a *single* harmonic oscillator, is assumed closed, so the evolution is unitary. Thus its entropy remains constant and the physical process is reversible. In contrast, in a many-body system, we may need an additional quasiequilibrium condition.

⁶Various definitions for work and heat are used in the literature, such as the two-point measurement definition of work, in the context of fluctuation theorems [57]. Here we define work and heat in terms of the changes of the expectation values of energies, not in terms of measured values.

equilibrium state at point A , while heat is exchanged between the oscillator and the bath. In the weak coupling limit, this final state will be a thermal state, not a squeezed thermal state. This completes the cycle.

Here we look into the oscillator dynamics during the isentropic phases, that is, stages 1 and 3 of the Otto cycle defined above. The oscillator is completely isolated from the thermal bath, but driven by an external agent that modulates its natural frequency $\omega(t)$. Since there is no bath, we consider the parametric oscillator as a closed system. Assuming the oscillator has a mass m , and its canonical variables χ , the displacement, and the conjugate momentum p satisfy the commutation relation $[\hat{\chi}, \hat{p}] = i$. The Hamiltonian of the oscillator then takes the form

$$\hat{H}_s(t) = \frac{\hat{p}^2}{2m} + \frac{m\omega^2(t)}{2} \hat{\chi}^2, \quad (3.1)$$

for $0 \leq t \leq \tau$, where τ is the duration of the process. For example, in the isentropic phase A) the oscillator frequency changes from $\omega(0) = \omega_L$ to $\omega(\tau) = \omega_H$. For simplicity, to capture the essential physics, we assume $\omega^2(t)$ monotonically increases during this stage according to

$$\omega^2(t) = \omega_L^2 + \frac{t}{\tau}(\omega_H^2 - \omega_L^2), \quad 0 \leq t \leq \tau. \quad (3.2)$$

The timescale τ describes how fast the oscillator's natural frequency varies and thus is a measure of the (non)adiabaticity of this process, that is, $\dot{\omega}/\omega^2$. A small τ depicts the qualitative features of sudden quenching while a large value of τ can approach ideal adiabaticity. We will not *a priori* assume that the frequency modulation is adiabatically slow, but for a typical Otto engine to produce a positive net work output, $\omega_H > \omega_C$ is usually required.

We further assume that the initial state of this phase is a generic Gaussian state. Since the working medium of the engine is modeled by a parametric harmonic oscillator throughout this stage, it is guaranteed that the end state is another Gaussian state. Therefore, the Wigner function of the system during the isentropic stage takes the generic form

$$\mathcal{W}(\chi, p, t) = N \exp[\mathcal{A}(t) \chi^2 + \mathcal{B}(t) p^2 + \mathcal{C}(t) \chi p], \quad (3.3)$$

where the normalization factor N is time independent, while the coefficients \mathcal{A} , \mathcal{B} and \mathcal{C} in the exponent are time dependent. The Wigner function satisfies the Wigner-Fokker-Planck equation, e.g. [44],

$$\frac{\partial \mathcal{W}}{\partial t} = -\frac{p}{m} \frac{\partial \mathcal{W}}{\partial \chi} + m\omega^2(t) \chi \frac{\partial \mathcal{W}}{\partial p}, \quad (3.4)$$

so its solution dictates the time evolution of the parametric oscillator. Plugging Eq. (3.3) into Eq. (3.4) gives a coupled set of differential equations for the coefficients \mathcal{A} , \mathcal{B} , and \mathcal{C} ,

$$\begin{aligned} \frac{d}{dt} \mathcal{A}(t) &= m\omega^2(t) \mathcal{C}(t), & \frac{d}{dt} \mathcal{B}(t) &= -\frac{1}{m} \mathcal{C}(t), \\ \frac{d}{dt} \mathcal{C}(t) &= 2 \left[m\omega^2(t) \mathcal{B}(t) - \frac{1}{M} \mathcal{A}(t) \right], \end{aligned} \quad (3.5)$$

with the initial conditions of \mathcal{A} , \mathcal{B} , and \mathcal{C} determined by the initial state of the isentropic phase. Given an arbitrary

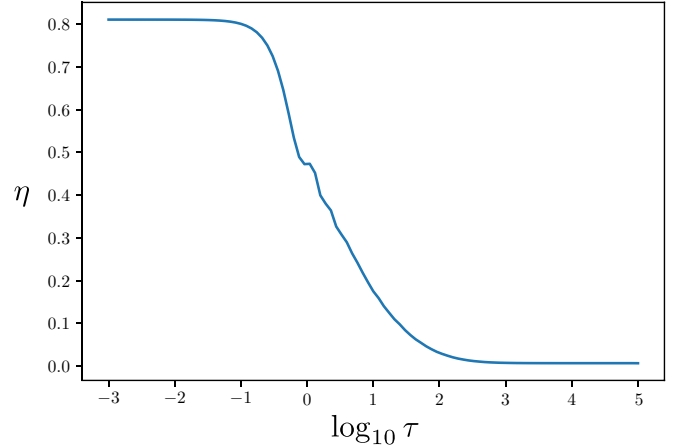


FIG. 1. Here we show how the squeezing of the oscillator at the end of the isentropic stage depends on the speed of its frequency modulation. The parameter τ is the duration of the frequency modulation. The initial state of the isentropic phase is given by the final equilibrium state of the oscillator in the previous isothermal phase where it is in contact with an Ohmic thermal bath of temperature $\beta_L^{-1} = 0.001$. We choose $\omega_L = 1$, $\omega_H = 5\omega_L$, $m = 1$, $\gamma = 2 \times 10^{-5}$, and the cutoff for the bath $\Lambda_L = 1000$.

function $\omega(t)$, it is not easy to find an analytical, closed-form solution for this set of differential equations barring a few exceptions such as (3.2). A rather general analytical treatment is developed in [58], where the unitary time evolution operator in the χ representation can be constructed by a special set of fundamental solutions to the equation of motion of the parametric oscillator. Here we resort to numerical methods to get the solutions of (3.5). For an arbitrary speed of frequency modulation, Fig. 1 shows the variation of the squeezing at the end of frequency modulation with the transition time τ . The sudden change limit results in a squeeze parameter η given by $\frac{1}{2} \ln \omega_H/\omega_L$. On the other hand, the adiabatic limit shows that the squeezing parameter remains the same after the driving. The intermediate regime shows a roughly monotonous behavior between these two regimes.

Since the oscillator is isolated from the bath, the external agent that modifies the oscillator frequency will have energy exchange with the oscillator in the form of work. By energy conservation the work is related to the change of the internal energy of the oscillator. During the isentropic phase $A \mapsto B$, that is, $\omega_L \mapsto \omega_H$, the work is given by the internal energy difference

$$W_{AB} = E_B - E_A, \quad (3.6)$$

where E_B , for example, represents the internal energy of the oscillator at point B . The internal energy is the expectation value of the Hamiltonian \hat{H}_{S_i} for $i = L, H$,

$$\hat{H}_{S_i} = \frac{\hat{p}^2}{2m} + \frac{m\omega_i^2}{2} \hat{\chi}^2. \quad (3.7)$$

We adopt the convention that when W is positive, work is done to the oscillator, but when $W < 0$, the oscillator outputs work. For weak oscillator-bath coupling we immediately have E_A

given by

$$E_A = \frac{\omega_L}{2} \coth \frac{\beta_L \omega_L}{2}. \quad (3.8)$$

In the graduate change (AD) limit, we have

$$W_{AB}^{(AD)} = \left(\frac{\omega_H}{2} - \frac{\omega_L}{2} \right) \coth \frac{\beta_L \omega_L}{2} > 0, \quad (3.9)$$

since the level population does not change with time. On the other hand, in the sudden change (SC) limit, the work is given by

$$W_{AB}^{(SC)} = \frac{\omega_H^2 - \omega_L^2}{4\omega_L} \coth \frac{\beta_L \omega_L}{2} > 0, \quad (3.10)$$

because the state cannot adapt itself fast enough to match the configuration change. We see that typically if we require $\omega_H > \omega_L$, then the external agent needs to input more energy to the oscillator in order to arrive at the same frequency change. This is a consequence of the fact that when the parametric process is nonadiabatic, additional energy goes into (is “wasted”) exciting the oscillator.

Similarly for the other isentropic phase $C \mapsto D$, i.e., $\omega_H \mapsto \omega_L$, the internal energy of the oscillator at C is given by

$$E_C = \cosh 2\eta \times \frac{\omega_H}{2} \coth \frac{\beta_H \omega_H}{2} = \frac{\omega_H}{2} \coth \frac{\beta_S \omega_H}{2}, \quad (3.11)$$

because the hot thermal bath is initially prepared in a squeezed thermal state. Here we would like to remind readers that the state at C is not a squeezed thermal state even though the internal energy of the state takes the form given by (3.11). This will be explained in Sec. IV. After the frequency modulation, the work during this phase is given by

$$W_{CD} = E_D - E_C. \quad (3.12)$$

Assuming no additional unsqueezing, the work would depend on the speed of frequency modulation. In the adiabatic limit W_{CD} is given by

$$W_{CD}^{(AD)} = \cosh 2\eta \left(\frac{\omega_L}{2} - \frac{\omega_H}{2} \right) \coth \frac{\beta_H \omega_H}{2} < 0, \quad (3.13)$$

but in the sudden limit

$$W_{CD}^{(SC)} = \cosh 2\eta \frac{\omega_L^2 - \omega_H^2}{4\omega_H} \coth \frac{\beta_H \omega_H}{2} < 0. \quad (3.14)$$

Clearly the output work is greater in the adiabatic limit when $\omega_H > \omega_L$. For an arbitrary speed of frequency modulation, we can write the internal energy at the end of the parametric process formally as

$$E_D = \cosh \eta_s \times \frac{\omega_H}{2} \coth \frac{\beta_S \omega_H}{2}, \quad (3.15)$$

where the factor $\cosh \eta_s$ generically accounts for the squeezing due to frequency modulation, and η_s is given by (2.26). Note that this is a different squeeze parameter from the one η used for the squeezed bath. If we would like to enhance the engine’s efficiency, we may introduce unsqueezing at the end of the isentropic phase. In so doing, we restore, apart from a rotation, the state of the oscillator to the one according to the adiabatic modulation, thus recovering the result given

by (3.13) for any modulation speed.⁷ This can be viewed as some type of “shortcut to adiabaticity” [59] implemented in the isentropic phase. The implementation is not limited to the specific form of frequency modulation (3.2). From the discussion in Sec. II C, we see it is quite general. We can always identify the suitable ζ_s , in terms of the fundamental solutions $d_{1,2}(\tau)$, to perform unsqueezing given any functional form of $\omega(t)$.

From Eq. (3.9) and Eq. (3.13), we learn that in the adiabatic limit, the external agent pumps less energy to the oscillator during the isentropic compression phase AB , but the oscillator outputs more work to the outside in the isentropic expansion phase CD . The net work in the adiabatic limit is thus

$$W_{TOT}^{(AD)} = W_{AB}^{(AD)} + W_{CD}^{(AD)} = - \left(\frac{\omega_H}{2} - \frac{\omega_L}{2} \right) \times \left(\cosh 2\eta \coth \frac{\beta_H \omega_H}{2} - \coth \frac{\beta_L \omega_L}{2} \right). \quad (3.16)$$

We see that the magnitude of the total work depends on the combinations of η , $\beta_H \omega_H$ and $\beta_L \omega_L$ with the constraints $\eta \geq 0$, $\beta_H < \beta_L$ and $\omega_H > \omega_L$, so the design of the engine is to find an optimal combination to have the maximal output work for a given incoming heat, that is, $W_{TOT} < 0$ and maximal $|W_{TOT}|$. At least when $\beta_H \omega_H < \beta_L \omega_L$, the magnitude of the total work increases with growing η , but independent of θ . Furthermore, since $\cosh 2\eta \geq 1$, we may not want to put additional squeezing in the cold thermal bath if we want to extract maximal total work. Meanwhile the condition $\beta_H \omega_H < \beta_L \omega_L$ implies

$$\frac{\omega_L}{\omega_H} > \frac{\beta_H}{\beta_L}, \quad (3.17)$$

and in turn leads to the well-known fact that the efficiency of the Otto engine is always less than that of the Carnot engine.

The θ -independence implies that the result is independent of which quadrature is squeezed. This is very different from the application of squeezing in LIGO, where a particular quadrature has to be squeezed such that the effects of the corresponding quantum noise are suppressed in order to maintain high coherence. In the context of heat engine, the independence of θ can be traced to the internal energy of the oscillator at the end of the hot isothermal phase (3.11), which is also independent of θ . Physically speaking, when we have suppressed the uncertainty in one quadrature, a large uncertainty in the complementary conjugated quadrature will be induced. Thus, if the elastic energy of the oscillator is suppressed by a

⁷Performing an unsqueezing is a procedure often invoked in the literature with little discussion of the realistic implementations and the cost and effect involved. We don’t have anything intelligent to say about the practical protocols, but wish to point out a theoretical concern. Any squeezing or unsqueezing involves an outside agent, with which the system will engage in energy exchange. Therefore, to be fair the assessment of engine efficiency needs to include this factor into the consideration. And, to be complete, one needs to include the dynamical variables of the outside agent in the model calculations. There must also be a corresponding optimization issue of the energy used in the (un)squeezing process. Further investigations are needed in these aspects [60].

factor $e^{-2\eta}$, the kinetic energy is enhanced by $e^{+2\eta}$ such that the total energy still gets bigger by $e^{+2\eta'} \sim \cosh 2\eta'$, and vice versa.

$$W_{\text{TOT}}^{(\text{SC})} = W_{\text{AB}}^{(\text{SC})} + W_{\text{CD}}^{(\text{SC})} = -\frac{1}{4}(\omega_{\text{H}}^2 - \omega_{\text{L}}^2) \left(\frac{1}{\omega_{\text{H}}} \cosh 2\eta \coth \frac{\beta_{\text{H}}\omega_{\text{H}}}{2} - \frac{1}{\omega_{\text{L}}} \coth \frac{\beta_{\text{L}}\omega_{\text{L}}}{2} \right), \quad (3.18)$$

or, more compactly,

$$W_{\text{TOT}}^{(\text{SC})} = -\frac{1}{4}(\omega_{\text{H}}^2 - \omega_{\text{L}}^2) \left(\frac{1}{\omega_{\text{H}}} \coth \frac{\beta_{\text{S,H}}\omega_{\text{H}}}{2} - \frac{1}{\omega_{\text{L}}} \coth \frac{\beta_{\text{S,L}}\omega_{\text{L}}}{2} \right), \quad (3.19)$$

where $\beta_{\text{S,H}}$ is introduced according to (2.13) and $\beta_{\text{S,L}} = \beta_{\text{L}}$ because there is no squeezing in the low-temperature thermal bath. Here we need a stronger condition

$$\frac{1}{\omega_{\text{H}}} \coth \frac{\beta_{\text{S,H}}\omega_{\text{H}}}{2} > \frac{1}{\omega_{\text{L}}} \coth \frac{\beta_{\text{S,L}}\omega_{\text{L}}}{2} \quad (3.20)$$

to guarantee an output work, rather than $\beta_{\text{H}}\omega_{\text{H}} < \beta_{\text{L}}\omega_{\text{L}}$. In addition we can verify that the magnitude of the output work in the sudden quench case is always smaller than that in the adiabatic limit. Finally, squeezing has the same effect on the output work as in the adiabatic process.

Next we move on to the isothermal phases.

IV. ISOTHERMAL STAGES: SQUEEZED THERMAL BATHS

In an isothermal stage our system of one quantum harmonic oscillator with a fixed frequency is brought into contact with a squeezed thermal bath. We shall allow the system to

In the quench limit, Eqs. (3.10) and (3.14) provide the total work

evolve to relaxation whose timescale is determined by the inverse damping constant γ^{-1} . The existence of a steady state wherein the energy exchange between the oscillator and the bath is in balance and the correlation function of the oscillator becomes stationary, that is, invariant in time translation, is demonstrated in [54]. We rely on the HPZ equation for a Brownian oscillator in a squeezed thermal bath derived in [45] and provide the late time behavior of its coefficient functions in the Appendix. We perform numerical computations from these expressions to obtain the quantities we need for the isothermal phases of the Otto engine.

With the Hamiltonian for an oscillator of mass m and physical (renormalized) frequency ω given by

$$\hat{H}_{\text{r}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} \hat{\chi}^2, \quad (4.1)$$

the Wigner function after the system reaches a steady state takes the generic form

$$\mathcal{W}_{\text{ss}}(\chi, p) = \frac{1}{4\pi\sqrt{ab}} \exp\left[\frac{p^2}{4a} + \frac{\chi^2}{4b}\right], \quad (4.2)$$

where the coefficients b, a are related to the covariance matrix elements $\langle \hat{\chi}^2(\infty) \rangle, \langle \hat{p}^2(\infty) \rangle$ by

$$\langle \hat{\chi}^2(\infty) \rangle = -2b = \cosh 2\eta \frac{2\gamma}{m\pi} \int_0^\Lambda d\kappa \kappa \coth \frac{\beta\kappa}{2} \frac{1}{|(\gamma + i\kappa)^2 + \omega_{\text{r}}^2|^2}, \quad (4.3)$$

$$\langle \hat{p}^2(\infty) \rangle = -2a = \cosh 2\eta \frac{2m\gamma}{\pi} \int_0^\Lambda d\kappa \kappa^3 \coth \frac{\beta\kappa}{2} \frac{1}{|(\gamma + i\kappa)^2 + \omega_{\text{r}}^2|^2}, \quad (4.4)$$

$$\langle \{\hat{\chi}(\infty), \hat{p}(\infty)\} \rangle = 0, \quad (4.5)$$

if the bath is initially in a squeezed thermal state at temperature β^{-1} and squeeze parameter $\zeta = \eta e^{i\theta}$. Here ∞ in the argument of the covariance matrix elements means that these elements are evaluated at the final equilibrium state at $t \gg \gamma^{-1}$. The frequency ω_{r} denotes the resonance frequency, defined by $\omega_{\text{r}}^2 = \omega^2 - \gamma^2$. In the weak oscillator-bath coupling limit ω and ω_{r} are approximately the same.

The integral expression in $\langle \hat{p}^2(\infty) \rangle$ is not well defined, so we introduce a frequency cutoff Λ in the upper limit of the κ -integral to suppress the excessive high-frequency contributions. The cutoff parameter is typically the highest energy scale below which the theory remains valid. In Fig. 2 we show the typical dependences of the covariance matrix elements $\langle \hat{\chi}^2(\infty) \rangle_{\beta}$ and $\langle \hat{p}^2(\infty) \rangle_{\beta}$ at late times when the oscillator is in a plain thermal bath of temperature β^{-1} . They differ from $\langle \hat{\chi}^2(\infty) \rangle$ and $\langle \hat{p}^2(\infty) \rangle$ by a constant factor $\cosh 2\eta$. We mention several salient features: (1) When the bath temperature

is sufficiently high, both covariance matrix elements take on the classical values, which have very weak dependence on γ . It reflects that the quantum nature of the oscillator is heavily suppressed by large thermal fluctuations of the bath. (2) At low temperatures, they deviate from their counterparts in a closed system, with a larger damping constant γ . (3) Since they are rescaled with respect to their closed-system counterparts, the state of the oscillator is slightly squeezed [55] (due to finite oscillator-bath coupling there unlike the squeezing due to frequency modulation or a fixed squeezed thermal bath here). (4) $\langle \hat{p}^2(\infty) \rangle_{\beta}$ has a stronger dependence on γ at low temperatures because typically it contains a cutoff-dependent term like $\gamma \ln(\Lambda/\omega)$, aimed at the ultraviolet behavior of the vacuum fluctuations of the bath field.

We note that both elements in (4.3) and (4.4) are proportional to the factor $\cosh 2\eta$, independent of the squeeze angle θ , so the initial phase information of the bath is lost. The

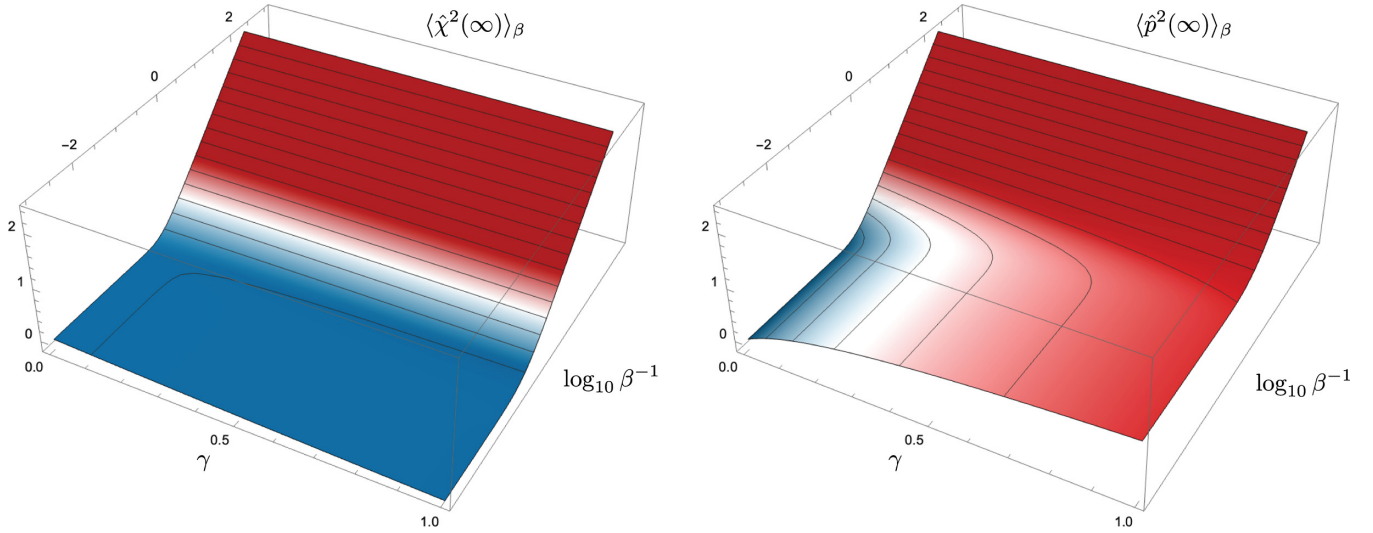


FIG. 2. Dependences of the late-time covariance matrix elements $\langle \hat{x}^2(\infty) \rangle_\beta$ and $\langle \hat{p}^2(\infty) \rangle_\beta$ on the damping constant γ and the initial bath temperature β^{-1} indicating the stark differences between strong and weak coupling, low and high temperature. The former is scaled with respect to $(2m\omega)^{-1}$ and the latter to $m\omega/2$. The variables γ and β^{-1} are in the unit of ω . Here we choose $m = 1 \times \omega$ and the cutoff scale $\Lambda = 1000\omega$.

vanishing of the cross correlation between the canonical variable operators also reflects the stationarity of the final state. These properties show that the final steady state is a thermal state at a temperature β_s^{-1} different from the bath's initial temperature β^{-1} . Essentially in the weak coupling limit, we find

$$\coth \frac{\beta_s \omega}{2} = \cosh 2\eta \coth \frac{\beta \omega}{2}. \quad (4.6)$$

Since $\cosh 2\eta \geq 1$, we always have $\beta_s \leq \beta$ due to the fact that $\coth z$ is a monotonically decreasing function of z . This implies $\beta_s^{-1} \geq \beta^{-1}$, and the squeezing always effectively raises the oscillator temperature, independent of the squeeze angle θ . For a given initial state of the oscillator, more heat from the hot thermal bath goes into the oscillator with a larger squeeze parameter η , following (3.11) and the argument in the previous section. This feature can be useful in boosting the theoretical Carnot efficiency for a fixed cold thermal bath temperature.

Since there is no external agent, the heat exchange with the bath is what changes the internal energy of the oscillator. The

amount of heat injected into the oscillator or released to the bath is given by the difference of the internal energy of the oscillator between its initial and final state of the isothermal process. For the hot isothermal phase ($B \mapsto C$), in the weak coupling limit, we already know the internal energy of the oscillator at C is given by (3.11). Thus the heat exchange in this phase is

$$Q_{\text{IN}} = E_C - E_B, \quad (4.7)$$

where B denotes the final state of the previous isentropic phase. The positive value of heat means that the heat flows into the oscillator. If the isentropic phase ($A \mapsto B$) are adiabatically slow, then E_B is given by

$$E_B = \frac{\omega_H}{2} \coth \frac{\beta_L \omega_L}{2}, \quad (4.8)$$

which gives the corresponding injected heat

$$Q_{\text{IN}} = E_C - E_B = \frac{\omega_H}{2} \left(\cosh 2\eta \coth \frac{\beta_H \omega_H}{2} - \coth \frac{\beta_L \omega_L}{2} \right). \quad (4.9)$$

Thus in the adiabatic limit, the efficiency $\xi^{(\text{AD})}$ of the Otto engine in contact with the hot squeezed bath is given by

$$\xi^{(\text{AD})} = \frac{|W_{\text{TOT}}^{(\text{AD})}|}{Q_{\text{IN}}} = \frac{\left(\frac{\omega_H}{2} - \frac{\omega_L}{2}\right) \left(\cosh 2\eta \coth \frac{\beta_H \omega_H}{2} - \coth \frac{\beta_L \omega_L}{2}\right)}{\frac{\omega_H}{2} \left(\cosh 2\eta \coth \frac{\beta_H \omega_H}{2} - \coth \frac{\beta_L \omega_L}{2}\right)} = 1 - \frac{\omega_L}{\omega_H}. \quad (4.10)$$

This is the optimal efficiency of the Otto engine, and it is independent of the squeezing in the hot thermal bath and independent of the temperatures of the baths.

In comparison, if both isentropic phases undergo a sudden frequency change, then the efficiency $\xi^{(\text{SC})}$ of the engine is given by

$$\xi^{(\text{SC})} = \frac{|W_{\text{TOT}}^{(\text{SC})}|}{Q_{\text{IN}}} = \frac{\frac{1}{4}(\omega_H^2 - \omega_L^2) \left(\frac{1}{\omega_H} \coth \frac{\beta_{S,H} \omega_H}{2} - \frac{1}{\omega_L} \coth \frac{\beta_{S,L} \omega_L}{2}\right)}{\frac{\omega_H}{2} \left(\coth \frac{\beta_{S,H} \omega_H}{2} - \coth \frac{\beta_{S,L} \omega_L}{2}\right)}, \quad (4.11)$$

with the help of (3.19). It shows the dependence of the squeeze parameter and the bath temperatures in a nontrivial way. For example, in the extreme squeezing limit $\eta \rightarrow \infty$, the efficiency $\xi^{(\text{SC})}$ reduces to

$$\lim_{\eta \rightarrow \infty} \xi^{(\text{SC})} = \left(1 - \frac{\omega_L}{\omega_H}\right) \frac{\omega_H + \omega_L}{2\omega_H} < \xi^{(\text{AD})}. \quad (4.12)$$

Between these two limits, numerical means are needed to explore the full parameter space of the efficiency. That is, in general E_B will take the form

$$E_B = \cosh 2\eta_s \frac{\omega_H}{2} \coth \frac{\beta_L \omega_L}{2}, \quad (4.13)$$

with again η_s computed via (2.26) by the numerical approach.

V. CONCLUSION

We have investigated some foundational theoretical issues for the isentropic and the isothermal phases of continuous variables quantum Otto engines, based on the conceptual and technical framework of the Hu-Paz-Zhang master equation which gives a correct treatment of the non-Markovian regime and strong system-bath coupling. We showed that on account of the pathologies of the Lindblad or Caldeira-Leggett equations at very low temperatures, and for strong system-bath coupling, the HPZ equation is more suitable for a wider range of applicability in the development of quantum thermodynamics.

An important issue which is largely ignored in quantum thermal engines theories is the junction conditions. In the isothermal phase, when the oscillator which is the working medium of the quantum Otto engine studied here, interacts with a squeezed thermal bath, the end state ($t \gg \gamma^{-1}$) of the oscillator is *not* a squeezed thermal state even for weak oscillator-bath coupling. This is very different from the popular case when the oscillator is coupled to a plain thermal state. There, in the weak coupling limit, the oscillator does end up in a thermal state having the same temperature as the bath. But this is not true here. The squeezed thermal state is not a stationary state, so it cannot serve as an equilibrium state. The deeper physics behind it, in simple words, is that the dissipation on the oscillator's motion will adapt itself to counter the nonstationary noise from the bath, such that in the end the oscillator still approaches a stationary equilibrium state. The parameters in this end state depend on the initial amplitude of bath squeezing, but not on the initial squeeze phase. Thus the correct end state we obtained is totally different in nature from the end state of the isothermal phase assumed in the literature. As a consequence, a different *junction condition* is needed at the interface of the isothermal and the isentropic phases. This is not only a conceptual issue, it impacts on the implementation of the optimization protocols used in the ensuing isentropic phase to boost the engine efficiency.

In our treatment of the isentropic phase, we did not limit the frequency modulation to be adiabatic. Our formalism is general enough to account for nonadiabaticity, borrowing well-developed ideas and treatments from cosmology and quantum field theory in curved space. Finally, although we have not implemented the strong oscillator-bath coupling to

the current setting of the quantum Otto engine, we have demonstrated the emergent disparity in the behavior of the oscillator when the coupling strength is gradually increased. Strong coupling physics in the realm of the quantum heat engine remains an open issue.

As noted earlier, by scrutinizing the validity of assumptions and approximations in existing theories and in exploring a broader range of parameter spaces in the open quantum systems and squeezed quantum systems, we hope to provide a more solid theoretical foundation for the search for optimal quantum engine performance and more general quantum thermodynamic applications of these systems.

ACKNOWLEDGMENTS

We thank Chris Jarzynski for a careful reading of our completed manuscript, his refined queries, and helpful comments. We also thank an anonymous referee for bringing our attention to uncharted but important aspects of the junction conditions implemented in quantum thermodynamics. O.A. acknowledges a GRA from MCFP in 2021 and summer support from the Joint Quantum Institute in 2020. J.-T.H. is supported by the Ministry of Science and Technology of Taiwan under Grant No. MOST 110-2811-M-008-522.

APPENDIX: LATE-TIME COEFFICIENTS OF HPZ MASTER EQUATION

The HPZ master equation for a bath of squeezed harmonic oscillators is derived in [45] for the convenience of readers following that stream of work [47,48]:

$$\begin{aligned} i \frac{\partial \hat{\rho}}{\partial t} &= [\hat{H}_{\text{ren}}, \hat{\rho}] + i D_{pp}[\hat{\chi}, [\hat{\chi}, \hat{\rho}]] + i D_{xx}[\hat{p}, [\hat{p}, \hat{\rho}]] \\ &\quad + i (D_{px} + D_{xp})[\hat{\chi}, [\hat{p}, \hat{\rho}]] + \Gamma[\hat{\chi}, \{\hat{p}, \hat{\rho}\}], \\ \hat{H}_{\text{ren}} &= \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{\chi}^2}{2}, \quad \omega^2 = \omega_b^2 - \int d\kappa \frac{I(\kappa)}{\kappa}, \end{aligned} \quad (\text{A1})$$

where ω_b is the bare frequency of the system and $I(\omega)$ is the spectral density of the bath.

The master equation is derived using the influence functional formalism and the evolution operator can be expressed with a function of initial and final values of Δ , Σ functions of forward and backward classical trajectories multiplied by a path-independent factor. The classical trajectories of Δ and Σ functions can be reduced to a linear combination of initial and final values with coefficients u_1, u_2, v_1, v_2 satisfying appropriate boundary conditions. For an Ohmic bath with spectral density

$$I(\kappa) = \frac{2m\gamma\kappa \Theta(\Lambda - \kappa)}{\pi}, \quad (\text{A2})$$

where Λ is a cutoff frequency much larger than all relevant frequency scales of the system so that the dissipation is time local but the physical frequency remains positive, these functions are given by

$$\omega_r^2 = \omega^2 - \gamma^2, \quad (\text{A3})$$

$$u_1(s, t) = -\frac{\sin[\omega_r(s-t)] e^{-\gamma s}}{\sin(\omega_r t)}, \quad u_2(s, t) = \frac{\sin(\omega_r s) e^{-\gamma(s-t)}}{\sin(\omega_r t)}, \quad (\text{A4})$$

$$v_1(s, t) = u_2(t-s, t), \quad v_2(s, t) = u_1(t-s, t). \quad (\text{A5})$$

Reference [45] defines functions $b_1(t), \dots, b_4(t)$ and $a_{ij}(t)$, where i and j take the values of 1 and 2, in terms of u and v to be used in the expressions of the master equation coefficients:

$$b_1(t) = -b_4(t) = m[\omega_r \cot(\omega_r t) - \gamma], \quad (\text{A6})$$

$$b_2(t) = \frac{m\omega_r e^{\gamma t}}{\sin(\omega_r t)}, \quad (\text{A7})$$

$$b_3(t) = -\frac{m\omega_r e^{-\gamma t}}{\sin(\omega_r t)}, \quad (\text{A8})$$

$$a_{ij}(t) = \frac{1}{1 + \delta_{ij}} \int_0^t ds \int_0^t ds' v_i(s) v_j(s') v_j(s'), \quad (\text{A9})$$

$$v(s, s') = \int_0^\infty d\kappa I(\kappa) \coth \frac{\beta\kappa}{2} \{ \cosh 2\eta \cos[\kappa(s-s')] - \sinh 2\eta \cos[2\theta - \kappa(s+s')] \}. \quad (\text{A10})$$

Notice that we set all of the bath oscillators with the same squeezing parameters η and θ and temperature β^{-1} for simplicity.

At this point, we need to give an expression for a_{ij} to evaluate the coefficients of the master equation. However, the integral over spectral density in the expression of $v(s, s')$ cannot be solved analytically, so we will introduce its Laplace transform $\tilde{v}(\sigma, \sigma')$ to be able to deal with the time integrals in the expression of a_{ij} first and then take the integral over the spectral density numerically. It is straightforward to check that any straight line in the complex plane with fixed real part such that $\text{Re}(\sigma), \text{Re}(\sigma') > 0$ is in the region of convergence once we impose $v_1(t < 0) = v_2(t < 0) = 0$, and we stick to this choice for our calculation. The Laplace transform of $v(s, s')$ is given by

$$\tilde{v}(\sigma, \sigma') = \int_0^\infty d\kappa \frac{S(\kappa)}{(\sigma^2 + \kappa^2)(\sigma'^2 + \kappa^2)} \{ \cosh 2\eta \times (\sigma\sigma' + \kappa^2) - \sinh 2\eta [(\sigma\sigma' - \kappa^2) \cos 2\theta + \kappa(\sigma + \sigma') \sin 2\theta] \}, \quad (\text{A11})$$

where $S(\kappa) = \frac{2\gamma m\kappa \Theta(\Lambda - \kappa)}{\pi} \coth \frac{\beta\kappa}{2}$. Using this expression, we can bring the expression of a_{ij} to the form

$$a_{ij}(t) = -\frac{1}{4\pi^2(1 + \delta_{ij})} \int_{\epsilon - i\infty}^{\epsilon + i\infty} d\sigma \int_{\epsilon - i\infty}^{\epsilon + i\infty} d\sigma' \int_0^t ds \int_0^t ds' v_i(s) \tilde{v}(\sigma, \sigma') v_j(s') e^{\sigma s + \sigma' s'}, \quad (\text{A12})$$

where ϵ is an arbitrary real number strictly greater than zero. Plugging the expressions for $v_1(t), v_2(t)$ into Eq. (A12) and taking the integrals over time, we obtain the following expressions for a_{ij} :

$$a_{11}(t) = -\frac{1}{8\pi^2 \sin^2(\omega_r t)} \int_{\epsilon - i\infty}^{\epsilon + i\infty} d\sigma \int_{\epsilon - i\infty}^{\epsilon + i\infty} d\sigma' \tilde{v}(\sigma, \sigma') \frac{\omega_r e^{(\gamma + \sigma)t} - (\gamma + \sigma) \sin(\omega_r t) - \omega_r \cos(\omega_r t)}{(\gamma + \sigma)^2 + \omega_r^2} \times \frac{\omega_r e^{(\gamma + \sigma')t} - (\gamma + \sigma') \sin(\omega_r t) - \omega_r \cos(\omega_r t)}{(\gamma + \sigma')^2 + \omega_r^2}, \quad (\text{A13})$$

$$a_{12}(t) = -\frac{\dot{a}_{11}}{\dot{v}_1(t)} = \frac{\dot{a}_{11} \sin(\omega_r t)}{\omega_r e^{\gamma t}}, \quad \text{from Eq. (D5) of Ref. [45]}, \quad (\text{A14})$$

$$a_{22}(t) = -\frac{1}{8\pi^2 \sin^2(\omega_r t)} \int_{\epsilon - i\infty}^{\epsilon + i\infty} d\sigma \int_{\epsilon - i\infty}^{\epsilon + i\infty} d\sigma' \tilde{v}(\sigma, \sigma') \frac{\omega_r e^{-\gamma t} + [(\gamma + \sigma) \sin(\omega_r t) - \omega_r \cos(\omega_r t)] e^{\sigma t}}{(\gamma + \sigma)^2 + \omega_r^2} \times \frac{\omega_r e^{-\gamma t} + [(\gamma + \sigma') \sin(\omega_r t) - \omega_r \cos(\omega_r t)] e^{\sigma' t}}{(\gamma + \sigma')^2 + \omega_r^2}. \quad (\text{A15})$$

These integrals are evaluated using the residue theorem and an appropriate closed integration contour including the line from $\epsilon - i\infty$ to $\epsilon - i\infty$, and the results are given in Eqs. (A16), (A17), and (A18) with the exponentially decaying terms omitted due to late-time assumption:

$$a_{11}(t \gg 1/\gamma) = \int_0^\infty d\kappa \frac{S(\kappa) \omega_r^2 e^{2\gamma t}}{2 \sin^2(\omega_r t)} \left(\frac{\cosh 2\eta}{|(\gamma + i\kappa)^2 + \omega_r^2|^2} - \sinh 2\eta \text{Re} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right), \quad (\text{A16})$$

$$a_{22}(t \gg 1/\gamma) = \int_0^\infty d\kappa \frac{S(\kappa)}{2} \left(\frac{\cosh 2\eta [(\gamma - \omega_r \cot \omega_r t)^2 + \kappa^2]}{|(\gamma + i\kappa)^2 + \omega_r^2|^2} - \sinh 2\eta [(\gamma - \omega_r \cot \omega_r t)^2 - \kappa^2] \text{Re} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right) - 2\kappa(\gamma - \omega_r \cot \omega_r t) \text{Im} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\}, \quad (\text{A17})$$

$$a_{12}(t \gg 1/\gamma) = \int_0^\infty d\kappa \frac{S(\kappa) \omega_r e^{\gamma t}}{\sin(\omega_r t)} \left[(\gamma - \omega_r \cot \omega_r t) \left(\frac{\cosh 2\eta}{|(\gamma + i\kappa)^2 + \omega_r^2|^2} - \sinh 2\eta \text{Re} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right) \right] + \kappa \sinh 2\eta \text{Im} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\}. \quad (\text{A18})$$

Having explicit expressions for b_1, \dots, b_4 and a_{ij} , and using Eqs. (D8) to (D11) from Ref. [45], the late-time coefficients of Eq. (A1) are the following. The integration $\int_0^\infty d\kappa S(\kappa) \dots$ is omitted but implied in the expressions of D_{pp} and D_{px} for a cleaner notation:

$$\Gamma = \gamma, \quad D_{\chi\chi} = 0, \quad (\text{A19})$$

$$\begin{aligned} -D_{\chi p} = -D_{px} &= \frac{(\kappa^2 - \omega^2) \cosh 2\eta}{2m|(\gamma + i\kappa)^2 + \omega_r^2|^2} \\ &+ \frac{\sinh 2\eta}{2m} \left((3\kappa^2 + \omega^2) \operatorname{Re} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right. \\ &\left. - 2\kappa\gamma \operatorname{Im} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right), \quad (\text{A20}) \end{aligned}$$

$$\begin{aligned} D_{pp} &= \frac{-2\gamma\kappa^2 \cosh 2\eta}{|(\gamma + i\kappa)^2 + \omega_r^2|^2} \\ &+ \sinh 2\eta \left((\kappa^2 - \omega^2)\kappa \operatorname{Im} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right. \\ &\left. - 6\gamma\kappa \operatorname{Re} \left\{ \frac{e^{2i(\kappa t - \theta)}}{[(\gamma + i\kappa)^2 + \omega_r^2]^2} \right\} \right). \quad (\text{A21}) \end{aligned}$$

Before proceeding to find the steady state of Eq. (A1), we can make a further approximation for $D_{\chi p}$ and D_{pp} . For a large enough time t , we can assume the rotating terms of the integrand behave smoothly enough with changes of κ in the order of $1/t$ so that the integral over bath spectrum becomes negligible. Eliminating these terms give the following expressions⁸ for $D_{\chi p}$ and D_{pp} :

$$D_{\chi p} = \frac{(\omega^2 - \kappa^2) \cosh 2\eta}{2m|(\gamma + i\kappa)^2 + \omega_r^2|^2}, \quad D_{pp} = -\frac{2\gamma\kappa^2 \cosh 2\eta}{|(\gamma + i\kappa)^2 + \omega_r^2|^2}. \quad (\text{A22})$$

⁸Notice that the signs of $D_{\chi p}$, D_{px} terms differ in Eqs. (3.4) and (D6) of Ref. [45]. Equations (A19) and (A22) refer to the coefficients of Eq. (A1).

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