Letter

Nonlinear turbulent dynamo induced by fluctuations of the Lorentz force

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The turbulent hydromagnetic dynamo is a process of magnetic field amplification by a chaotic flow of an electrically conducting fluid, responsible for generation of the magnetic fields of planets and stars. Here we demonstrate a curious effect of the Lorentz force, which can act to intensify the magnetic fields, counterintuitively in light of the Lenz law according to which the Lorentz force acts to retard motions and saturate the dynamo-induced magnetic field. However, the net effect of its small-scale fluctuations in a turbulent flow is far from obvious and it is shown that it can lead to amplification rather than saturation of the magnetic energy through creation of negative turbulent diffusivity.

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I. INTRODUCTION

The typical situation for dynamo action to occur in a turbulent flow of an electrically conducting fluid, that is for the magnetic field to be amplified by the small-scale turbulence, is when the flow field exhibits chirality (lack of reflectional symmetry). In natural systems this is achieved by the presence of rapid background rotation. An electromotive force $\boldsymbol{\mathcal{E}}(\langle\mathbf{B}\rangle)$ (EMF) is then generated, which in general depends nonlinearly on the large-scale magnetic field $\langle\mathbf{B}\rangle$, and this leads to amplification of magnetic energy until the growing Lorentz force reacts back upon the flow field, leading to a saturated state [1]. However, at the initial stage of the nonlinear magnetic field evolution the effect of the Lorentz force can be very complex. As mentioned, due to action of the Lorentz force at small scales of the turbulent system, the resulting mean EMF becomes a nonlinear function of large-scale field. In the limit of weak seed field, the EMF can be linearized with respect to $\langle\mathbf{B}\rangle$, which corresponds to the very first evolutional phase of either exponential growth or decay of the magnetic energy. Once the field is amplified, the nonlinear effects start to play a significant role and eventually must lead to saturation. However, it is shown here that before saturation is achieved in the nonlinear evolution the fluctuating component of the Lorentz force can act to create negative turbulent magnetic diffusivity, which clearly accounts for amplification rather than saturation of the magnetic energy. This situation may last until the magnitude of the magnetic field exceeds a certain critical value (estimated by the inverse of the square root of the magnetic Prandtl number) above which the asymptotic structure of the EMF is destroyed. The nonlinear effect of the Lorentz force has been hitherto scarcely considered, but [2] showed numerically that such dynamos, which they termed "essentially nonlinear dynamos," can indeed organize the magnetic field on the scale of the entire system. Even earlier [3] provided evidence, based on numerical simulations, that the mean EMF can be enhanced by growing magnetic field strength, contrary to conventional EMF quenching. In addition [4,5] (see also references therein) reported brief periods of superexponential growth of magnetic energy just before saturation in simulations of rapidly rotating turbulent convection, which also suggests "nonintuitive" action of the Lorentz force.

We study the simplified case of turbulence stirred by a homogeneous, stationary, and isotropic but chiral forcing. This allows one to clearly demonstrate and explain an interesting effect, which was never studied, but numerically observed, of the essentially nonlinear dynamo, that is amplification of the large-scale magnetic field, by the Lorentz force acting at small scales of the MHD turbulence. This a unique, explicit analytic result, as hitherto such a nonlinear evolution of the mean magnetic field under action of the fluctuational Lorentz force has never been studied analytically.

II. MATHEMATICAL FORMULATION

In order to study the effect of the Lorentz force on the large-scale dynamo process induced by the complex flow of an incompressible conducting fluid, we consider the following dynamical equations describing the evolution of the turbulent velocity field of the fluid flow $\mathbf{U}(t,\mathbf{x})$ and the magnetic field $\mathbf{B}(t,\mathbf{x})$:

$$\frac{\partial\mathbf{U}}{\partial t}+(\mathbf{U}\cdot\nabla)\mathbf{U}=\mathbf{f}-\nabla\Pi+(\mathbf{B}\cdot\nabla)\mathbf{B}+\nu\nabla^2\mathbf{U}, \quad (1a)$$

$$\frac{\partial\mathbf{B}}{\partial t}+(\mathbf{U}\cdot\nabla)\mathbf{B}=(\mathbf{B}\cdot\nabla)\mathbf{U}+\eta\nabla^2\mathbf{B}, \quad (1b)$$

$$\nabla\cdot\mathbf{U}=0,\qquad\nabla\cdot\mathbf{B}=0, \quad (1c)$$

where $\Pi=p/\rho+B^2/2$ is the total pressure and without loss of generality we assume $\nabla\cdot\mathbf{f}=0$; ν and η denote the viscosity and magnetic diffusivity (proportional to the electrical resistivity) of the fluid, respectively. For the sake of simplicity we have rescaled the magnetic field in the following way: $\mathbf{B}/\sqrt{\mu_0\rho}\to\mathbf{B}$, so that the factor of $1/\sqrt{\mu_0\rho}$ is lost, where ρ

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denotes the fluid's density and μ_0 the magnetic permeability of vacuum.

Next, denoting by angular brackets $\langle\cdot\rangle$ the ensemble mean, let us assume that the forcing $\mathbf{f}$ is homogeneous, stationary, isotropic but chiral (helical) and Gaussian with zero mean, $\langle\mathbf{f}\rangle = 0$, and is fully defined by the following correlation function:

$$\langle \hat{f}_i(\mathbf{k},\omega)\hat{f}_j(\mathbf{k}',\omega')\rangle = \left[\frac{D_0}{k^3}P_{ij}(\mathbf{k}) + i\frac{D_1}{k^5}\epsilon_{ijk}k_k\right]\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'), \quad (2)$$

where D_0 and D_1 are constants, the upper hat denotes a Fourier transform [see (3) below], and $P_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2$ is the projection operator on a plane perpendicular to the wave vector $\mathbf{k}$; δ_{ij} is the Kronecker delta, i.e., the unity matrix, and ϵ_{ijk} is the Levi-Civita symbol. The D_1 term is the helical part of the forcing, introducing chirality into the flow. It can be easily shown that $D_0 > 0$ and $D_1 \leqslant kD_0$ for all k; cf., e.g., Refs. [6,7]. The nonhelical part of the correlation function is inversely proportional to the third power of the wave number. Such a scaling exponent was shown by Yakhot and Orszag [8] to correspond to the Kolmogorov-type turbulence in the absence of a magnetic field. Similar arguments can be put forward to show that the helical part must be inversely proportional to the fifth power of the wave number in order to reproduce the helicity spectrum for an isotropic, homogeneous, and stationary turbulence (cf. Refs. [9], [10], and [7]). Furthermore, we assume that the turbulence is forced only at small scales, i.e., within the wave number band $k > k_\ell = 2\pi/\ell$, where ℓ denotes the size of most energetic turbulent eddies; in other words, the forcing does not possess the large-scale component,

$$\mathbf{f}(\mathbf{x},t) = \int_{k_\ell}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \hat{\mathbf{f}}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \quad (3)$$

We shall consider the situation in which $\mathbf{U}$, $\mathbf{B}$, and p are turbulent fields, spatially homogeneous on scales $\leqslant\ell$. Let us introduce the following standard decomposition for turbulent flows into the mean and fluctuating parts:

$$\mathbf{U} = \langle\mathbf{U}\rangle + \mathbf{u}, \quad \mathbf{B} = \langle\mathbf{B}\rangle + \mathbf{b}, \quad p = \langle p\rangle + p', \quad (4)$$

and assume scale separation between the slowly varying mean and fluctuating quantities. Therefore, we treat the mean fields such as $\langle\mathbf{U}\rangle$, $\langle\mathbf{B}\rangle$, and $\langle p\rangle$ as locally uniform, but varying weakly on scales much greater than ℓ, that is, greatly exceeding the scales of the vigorous background turbulence.

The mean induction equation takes the form

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \nabla\times(\langle\mathbf{U}\rangle\times\langle\mathbf{B}\rangle) + \nabla\times\langle\mathbf{u}\times\mathbf{b}\rangle + \eta\nabla^2\langle\mathbf{B}\rangle, \quad (5)$$

and we identify the term $\boldsymbol{\mathcal{E}} = \langle\mathbf{u}\times\mathbf{b}\rangle$ as the large-scale electromotive force (EMF). We adopt the "first-order smoothing approximation" (cf. [6]) in which squares and products of fluctuating quantities in the dynamical equations for $\mathbf{u}$ and $\mathbf{b}$ are ignored. For clarity we are also going to neglect all effects associated with the mean flow $\langle\mathbf{U}\rangle$, thus, e.g., the cross-helicity dynamo or the shear-current effect (cf. [11–13]), in order to concentrate solely on the influence of the Lorentz force on the turbulent magnetic diffusivity (a more detailed calculation involving the mean flow $\langle\mathbf{U}\rangle$ is provided in the Supplemental Material [14]). The first-order smoothing approximation is limiting, but necessary in order to make analytical progress. It effectively corresponds to the so-called weak turbulence regime in which nonlinear interactions of waves influence their evolution only very weakly and such a state can in some cases survive for a long time (cf. [15,16]); however, the theory predicts that eventually turbulence always becomes strong, i.e., the terms nonlinear in the fluctuations start to play a significant role in their dynamics. Moreover, it should be also pointed out that extreme values of the magnetic Prandtl number $\mathrm{Pm} = \nu/\eta$ introduce additionally important differences in the nonlinear evolution of the magnetic and velocity fluctuations. Nevertheless, the fully nonlinear results of numerical simulations mentioned in the Introduction, such as [2–5], suggest that the current theory may be applicable at least qualitatively even beyond the weak turbulence regime.

Introducing a short notation for the gradient of the mean magnetic field

$$\Gamma_{ij} = \frac{\partial\langle B\rangle_i}{\partial x_j}, \quad (6)$$

we write down the equations for the Fourier transforms of the turbulent fluctuations $\hat{\mathbf{u}}$ and $\hat{\mathbf{b}}$ in the form

$$\hat{\mathbf{u}} = \frac{1}{\gamma_u}\hat{\mathbf{f}} - \frac{i\mathbf{k}\cdot\langle\mathbf{B}\rangle}{\gamma_u\gamma_\eta}\boldsymbol{\Gamma}\cdot\hat{\mathbf{u}} + \frac{i\mathbf{k}\cdot\langle\mathbf{B}\rangle}{\gamma_u\gamma_\eta}\mathbf{P}\cdot\boldsymbol{\Gamma}\cdot\hat{\mathbf{u}}, \quad (7a)$$

$$\hat{\mathbf{b}} = i\frac{\mathbf{k}\cdot\langle\mathbf{B}\rangle}{\gamma_\eta}\hat{\mathbf{u}} - \frac{1}{\gamma_\eta}\boldsymbol{\Gamma}\cdot\hat{\mathbf{u}}, \quad (7b)$$

$$\mathbf{k}\cdot\hat{\mathbf{b}} = 0, \quad \mathbf{k}\cdot\hat{\mathbf{u}} = 0, \quad (7c)$$

where

$$\gamma_u = -i\omega + \nu k^2 + \frac{(\mathbf{k}\cdot\langle\mathbf{B}\rangle)^2}{\gamma_\eta}, \quad \gamma_\eta = -i\omega + \eta k^2, \quad (8)$$

and we have eliminated the pressure from the Fourier transformed velocity equation with the use of the projection operator $P_{ij}(\mathbf{k})$ defined below (2). Next, the assumed scale separation between the means and the fluctuations implies that the gradients of means are small and hence will be treated in a perturbational manner. The large scale EMF, on the basis of iterative substitutions for $\hat{u}_j$ and $\hat{b}_j$ from (7a) and (7b) and neglection of higher order terms in the gradient $\boldsymbol{\Gamma}$, can be expressed in the following way:

$$\begin{aligned}\epsilon_{ijk}\langle\hat{u}_j\hat{b}'_k\rangle = {}& i\frac{k'_n\langle B\rangle_n}{\gamma_u\gamma'_u\gamma'_\eta}\epsilon_{ijk}\langle\hat{f}_j\hat{f}'_k\rangle - \frac{\epsilon_{ijk}}{\gamma_u\gamma'_u\gamma'_\eta}\Gamma_{kp}\langle\hat{f}_j\hat{f}'_p\rangle \\ &+ \frac{k'_mk'_n\langle B\rangle_m\langle B\rangle_n}{\gamma_u\gamma'^2_u\gamma'^2_\eta}\epsilon_{ijk}\frac{k'_kk'_s}{k'^2}\Gamma_{sp}\langle\hat{f}_j\hat{f}'_p\rangle \\ &+ \frac{k_mk'_n\langle B\rangle_m\langle B\rangle_n}{\gamma^2_u\gamma'_u\gamma_\eta\gamma'_\eta}\epsilon_{ijk}\frac{k_jk_s}{k^2}\Gamma_{sp}\langle\hat{f}_p\hat{f}'_k\rangle,\end{aligned} \quad (9)$$

where we have used a short notation $\hat{u}'_j = \hat{u}_j(\omega',\mathbf{k}')$.

Substituting for the force correlations from (2) into (9) and taking the double Fourier integral over $(\omega,\ \mathbf{k})$ and $(\omega',\ \mathbf{k}')$

$$\mathcal{E}_i = \epsilon_{ijk}\int d^4q\int d^4q'\, e^{i[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}-(\omega+\omega')t]}\langle\hat{u}_j\hat{b}'_k\rangle, \quad (10)$$

where for short we have introduced a four-component vector notation $\mathbf{q} = (\omega, \mathbf{k})$, it is a straightforward task to calculate the mean EMF, which yields (detailed calculation is available in the Supplemental Material [14])

$$\boldsymbol{\mathcal{E}} = \mathcal{I}_1\langle\mathbf{B}\rangle - [\mathcal{I}_2 - \langle B\rangle^2\mathcal{I}_3]\nabla\times\langle\mathbf{B}\rangle - \mathcal{I}_4\nabla\langle B\rangle^2\times\langle\mathbf{B}\rangle, \tag{11}$$

where the integrals $\mathcal{I}_j$ are functions of the mean field $\langle B\rangle$, the viscosity ν, and the magnetic diffusivity η,

$$\mathcal{I}_1 = 4\pi D_1\eta\int_{k_\ell}^{K}dk\int_{-1}^{1}dX\int_{-\infty}^{\infty}d\omega\frac{kX^2}{|\gamma_u|^2|\gamma_\eta|^2}, \tag{12a}$$

$$\mathcal{I}_2 = \pi D_0\eta\int_{k_\ell}^{K}dk\int_{-1}^{1}dX\int_{-\infty}^{\infty}d\omega\frac{k(1+X^2)}{|\gamma_u|^2|\gamma_\eta|^2}, \tag{12b}$$

$$\mathcal{I}_3 = \pi D_0\eta\int_{k_\ell}^{K}dk\int_{-1}^{1}dX\int_{-\infty}^{\infty}d\omega\frac{k^3\mathrm{Re}(\gamma_u\gamma_\eta)}{|\gamma_u|^4|\gamma_\eta|^4}\mathcal{F}(X), \tag{12c}$$

$$\mathcal{I}_4 = \frac{\pi}{2}D_0\eta\int_{k_\ell}^{K}dk\int_{-1}^{1}dX\int_{-\infty}^{\infty}d\omega\frac{k^3\mathrm{Re}(\gamma_u\gamma_\eta)}{|\gamma_u|^4|\gamma_\eta|^4}\mathcal{G}(X), \tag{12d}$$

with $\mathcal{F}(X) = 13X^4 - 10X^2 + 1$ and $\mathcal{G}(X) = 9X^4 - 10X^2 + 1$; since the aim here is to study the effect of the Lorentz force on the turbulent magnetic diffusivity we will not pursue here the analysis of the effect of unalignment between $\nabla\langle B\rangle$ and $\langle\mathbf{B}\rangle$ and hence the last term in (11) will be disregarded as irrelevant (when the field variation along the lines of force is stronger than that in the direction normal to the lines this term is weak, although in general it contributes to the dynamo process in a rather nonobvious way). In the above we have also introduced the upper cutoff K for the Fourier spectra, which in natural systems appears due to enhanced dissipation of energy at small scales; in particular, resistive dissipation of magnetic energy in Kolmogorov-type turbulence introduces cutoff $K = 2\pi(U/\eta)^{3/4}L^{-1/4}$, with L being the size of the entire system and U the large-scale velocity magnitude, hence typically $k_\ell \ll K$.

III. NONLINEAR DYNAMO EFFECT INDUCED BY SMALL-SCALE LORENTZ FORCE

The induction equation can now be written in the form

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \nabla\times(\bar{\alpha}\langle\mathbf{B}\rangle) - \nabla\times(\bar{\eta}\nabla\times\langle\mathbf{B}\rangle), \tag{13}$$

where the so-called turbulent α effect and the total magnetic diffusivity (molecular + turbulent) are determined by

$$\bar{\alpha} = \mathcal{I}_1, \quad \bar{\eta} = \eta + \bar{\eta}_1 - \bar{\eta}_2, \tag{14}$$

with

$$\bar{\eta}_1 = \mathcal{I}_2, \quad \bar{\eta}_2 = \langle B\rangle^2\mathcal{I}_3. \tag{15}$$

In the following, we will demonstrate that the contribution to the turbulent magnetic diffusivity $\bar{\eta}_2$, which clearly results from action of the fluctuational Lorentz force, can be positive and thus act to intensify the large-scale dynamo effect at least for some time, until the growing field changes the structure of the turbulent diffusivity.

A. Explicit form of the mean EMF

In order to explicitly calculate the integrals $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ defining the turbulent coefficients $\bar{\alpha}$ and $\bar{\eta}$ we put forward two assumptions, relevant to astrophysical fluid dynamics, in particular the planetary interiors, of a small magnetic Prandtl number $\mathrm{Pm} = \nu/\eta \ll 1$ and strong (but bounded) magnetic fields. More precisely, we assume

$$\forall_{k_\ell\leqslant k\leqslant K}, \quad \mathscr{M}^2(k) = \frac{\langle B\rangle^2}{\nu\eta k^2} \gg 1, \quad \mathrm{Pm}\mathscr{M}^2(k) \ll 1 \tag{16}$$

(for example, in the Earth's liquid core $\langle B\rangle L/\sqrt{\nu\eta} \sim 10^8$ and $\mathrm{Pm} \sim 5\times10^{-7}$; cf. [17]). In that way the magnetic field is bounded, i.e., $1 \ll \mathscr{M} \ll \mathrm{Pm}^{-1/2}$ with $\mathrm{Pm} \ll 1$, so that the field is assumed strong enough for the Lorentz force to significantly influence the dynamics, in particular the turbulent diffusivity, but amplification of the field above a certain threshold value of the order $\mathrm{Pm}^{-1/2}$ will destroy the structure of the EMF obtained below and thus alter the structure of the turbulent magnetic diffusivity. In this asymptotic limit, at leading order we get

$$\bar{\alpha} \approx \frac{4\pi^2 D_1}{k_\ell^2\langle B\rangle^2}, \quad \bar{\eta}_1 \approx \frac{\pi^3 D_0}{3\sqrt{\nu\eta}k_\ell^3\langle B\rangle} > 0, \tag{17}$$

$$\bar{\eta}_2 \approx \frac{\pi^3 D_0\langle B\rangle}{20(\nu\eta)^{3/2}k_\ell^5} = \frac{3}{20}\mathscr{M}^2(k_\ell)\bar{\eta}_1 \gg \bar{\eta}_1. \tag{18}$$

The effect of turbulent diffusion is clearly nonlinear, as the coefficients $\bar{\eta}_1$ and $\bar{\eta}_2$ depend on the magnitude of the magnetic field.

B. Toy model—evolution of energy of a force-free mode

Next, for simplicity let us consider an important class of solutions, which are force free, that is for which the mean Lorentz force vanishes. On the one hand, this allows one to entirely exclude the effect of the large-scale Lorentz force and concentrate solely on the effect of its small-scale component present in the EMF. On the other hand, the force-free modes do not transfer energy to the flow through the Lorentz force; thus their magnetic energy cannot be dissipated by viscosity. We assume, therefore, that the currents flow along the magnetic field lines

$$\langle\mathbf{j}\rangle = \nabla\times\langle\mathbf{B}\rangle = \kappa(\mathbf{x})\langle\mathbf{B}\rangle, \tag{19}$$

where $\langle\mathbf{j}\rangle$ has been rescaled with $\sqrt{\mu_0/\rho}$. This ensures $\langle\mathbf{j}\rangle\times\langle\mathbf{B}\rangle = 0$, i.e., vanishing of the large-scale Lorentz force. By Gauss's law for magnetism, which demands $\nabla\cdot\langle\mathbf{B}\rangle = 0$ (and, by the obvious fact, that $\nabla\cdot\langle\mathbf{j}\rangle = 0$), the function $\kappa(\mathbf{x})$ must satisfy $\langle\mathbf{B}\rangle\cdot\nabla\kappa = 0$; hence $\kappa(\mathbf{x})$ is constant on the field lines. Such states are known to exist and have been intensively investigated, e.g., in seminal works of [18,19] (cf. also a more recent work of [20]).

On defining the energy of such a force-free mode $E_m = \langle B\rangle^2/2$, by the use of (13), (14), and (15) one obtains

$$\frac{\partial E_m}{\partial t} = 2[\bar{\alpha}\kappa(\mathbf{x}) + 2\bar{\eta}_2 E_m \kappa(\mathbf{x})^2]E_m - 2(\eta + \bar{\eta}_1)\kappa(\mathbf{x})^2 E_m. \tag{20}$$

In the asymptotic limit defined by (16) we have $\bar{\eta}_2 \gg \bar{\eta}_1 > 0$; thus the entire term $2\bar{\eta}_2 E_m \kappa(\mathbf{x})^2$ which results from action of the Lorentz force at small scales is positive definite and clearly contributes to amplification of the magnetic energy, with decay possible through the effects of $\bar{\eta}_1$ and the molecular diffusion.

C. Discussion

As said the mean field is required to be strong, but should not exceed a certain threshold value, above which the asymptotics ceases to be valid. Once the magnitude of the field becomes too large for the second relation in (16) to be satisfied the effect of the Lorentz force becomes more complex [terms neglected in the asymptotic limit (16) become large] and then saturation of the magnetic energy becomes possible. More importantly, however, the rate of energy enhancement by negative diffusion strongly depends on the wavelength of magnetic modes and shorter wavelengths are amplified more vigorously. It follows that once a critical magnitude of the mean magnetic field is reached $\mathscr{M} \gg 1$ and the Lorentz force becomes dynamically important the negative diffusion effects appear and tend to destroy the dynamical structure with scale separation between the mean and fluctuating quantities, i.e., scale separation tends to disappear.

IV. CONCLUSIONS

Naturally the effect of the Lorentz force in hydro-magnetic dynamos is to saturate the magnetic energy once the magnitude of the growing magnetic field starts to exceed a certain threshold value. However, it was shown here that, in the initial period of evolution of a mean magnetic field, the small-scale Lorentz force in a turbulent flow can participate in amplification of magnetic energy through creation of a negative contribution to the turbulent magnetic resistivity. This can only happen once the field becomes strong enough for the Lorentz force to become dynamically important, but also weaker than $k_\ell \eta$; cf. (16). Only later, after the field reaches a certain critical amplitude, can the Lorentz force act to saturate the energy. This is an important, nonlinear effect, which has been explained here via analytic methods. The results are also interesting because they add yet another example to the class of magnetohydrodynamic effects which prompt caution for detail application of the famous Lenz law; cf. also [21–23].

However, it must also be emphasized that, although negative diffusion effects have been reported in physical situations, in particular an enhancement of energy temporarily present below a certain threshold value (cf. [24–28]), such a situation is peculiar, typically short lived, and requires great care in interpretation. In the case at hand the negative, nonlinear turbulent diffusion, which acts on the mean magnetic field, enhances the smaller scales more rapidly, thus aiming to destroy the dynamical structure involving a clear scale separation between the means and the turbulent fluctuations. Such a situation could happen in the evolution long before the asymptotic limit (16) ceases to be valid due to amplification of the magnetic field magnitude, and could also lead to suppression of the negative diffusion effect. In this sense the action of the fluctuational Lorentz force may be the crucial factor responsible for lack of scale separation in strongly developed, low-Pm magnetohydrodynamic turbulence. The presence of negative diffusion in the evolution of magnetic energy is therefore expected to be rather short lived, but dynamically significant as it leads to wavelength-dependent amplification of the magnetic energy.

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Correction: Minor errors in Eqs. (12c) and (12d) as well as in an equation in the second sentence below Eq. (16) have been fixed.