Quantum counterpart of energy equipartition theorem for a dissipative charged magneto-oscillator: Effect of dissipation, memory, and magnetic field

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(Received 15 June 2021; revised 19 November 2021; accepted 22 November 2021; published 10 December 2021)

In this paper, we formulate and study the quantum counterpart of the energy equipartition theorem for a charged quantum particle moving in a harmonic potential in the presence of a uniform external magnetic field and linearly coupled to a passive quantum heat bath through coordinate variables. The bath is modeled as a collection of independent quantum harmonic oscillators. We derive closed form expressions for the mean kinetic and potential energies of the charged dissipative magneto-oscillator in the forms $E_k = \langle \mathcal{E}_k \rangle$ and $E_p = \langle \mathcal{E}_p \rangle$, respectively, where $\mathcal{E}_k$ and $\mathcal{E}_p$ denote the average kinetic and potential energies of individual thermostat oscillators. The net averaging is twofold; the first one is over the Gibbs canonical state for the thermostat, giving $\mathcal{E}_k$ and $\mathcal{E}_p$, and the second one, denoted by $\langle \cdot \rangle$, is over the frequences ω of the bath oscillators which contribute to E_k and E_p according to probability distributions $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$, respectively. The relationship of the present quantum version of the equipartition theorem with that of the fluctuation-dissipation theorem (within the linear-response theory framework) is also explored. Further, we investigate the influence of the external magnetic field and the effect of different dissipation processes through Gaussian decay and Drude and radiation bath spectral density functions on the typical properties of $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$. Finally, the role of system-bath coupling strength and the memory effect is analyzed in the context of average kinetic and potential energies of the dissipative charged magneto-oscillator.

DOI: 10.1103/PhysRevE.104.064112

I. INTRODUCTION

Various generic phenomena from quantum physics such as wave-particle duality, entanglement of states, decoherence, the Casimir effect, quantum information, etc., demonstrate that the quantum world is remarkably different from its classical counterpart. Still, several new events, properties, and their way of acting need to be explored. In this context, one can mention the quantum counterpart of the energy equipartition theorem for classical systems [1–6], which is still open. The equipartition theorem for classical systems states that the total kinetic energy E_k is shared equally among all the energetically accessible degrees of freedom at thermal equilibrium and is given by $\frac{k_BT}{2}$ per degree of freedom, with T being the temperature of the system in contact with a passive heat bath (k_B is the Boltzmann constant). If one models the heat bath as an infinite collection of harmonic oscillators at temperature T, then one can show that the mean kinetic energy of the bath per degree of freedom is also $\mathcal{E}_k = \frac{k_BT}{2}$ or $E_k = \mathcal{E}_k$; that is, both the system and thermostat have the same average kinetic energy per degree of freedom—which is called energy equipartition. This theorem can be regarded as universal in the sense that it is independent of the number of particles in the system, the nature of the potential force acting on the system, the interacting force working between particles, and the strength of coupling between system and the bath. Although this classical result has been well studied and has been investigated for a long time [7–9], its quantum counterpart has received attention only recently [1–6].

In the literature, one can find various investigations of the energetics of quantum systems [10,11], the free energy of a dissipative quantum oscillator [12–14], the formulation of the quantum Langevin equation [15], and many other novel aspects of quantum Brownian motion [16–23]. However, these studies are not directly related to the quantum counterpart of the energy equipartition theorem. Recently, there have been some advancements in the articulation of the quantum analog of the energy equipartition theorem [1–6]. It has been shown that unlike the classical case, the average energy of an open quantum system can be understood as being the sum of contributions from individual thermostat oscillators (the bath is modeled as an infinite collection of independent harmonic oscillators) distributed over the entire frequency spectrum. The contribution to the mean energy of the system from bath oscillators lying in the frequency range between ω and $\omega + d\omega$ is weighted by a certain probability distribution $\mathcal{P}(\omega)$. Further, this distribution function $\mathcal{P}(\omega)$ is highly sensitive to the microscopic details of the thermostat and coupling between the system and the bath [2]. However, there is still room for novel advancement in this direction. (a) In the present paper, the effect of an external magnetic field and the impact of quantum dissipation via Gaussian decay and Drude and radiation bath spectral density functions on the distribution functions $\mathcal{P}(\omega)$ are carefully studied. (b) We explore the role of the system-bath coupling strength and memory time in the kinetic and potential energies, E_k and E_p, respectively, of the charged magneto-oscillator. (c) In addition, we demonstrate explicitly that this theorem can be derived from the

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fluctuation-dissipation relationship within the framework of linear response theory. (d) Finally, our derivation is based on the Heisenberg picture of the generalized Langevin equation and the derived expressions for $\mathcal{P}(\omega)$, which are highly dependent on the generalized susceptibility of the system, which is an experimentally measurable quantity.

For this purpose, we consider a paradigmatic model of dissipative diamagnetism: a charged particle moving in a harmonic potential in the presence of an external magnetic field and coupled with a heat bath which is modeled as a collection of infinite harmonic oscillators. The magnetic response of a charged quantum particle has wide and important relevance in Landau diamagnetism [24–28], the quantum Hall effect [29,30], atomic physics [31], and two-dimensional electronic systems [32–35], to name just a few. The effect of quantum dissipation due to the coupling with an infinitely large collection of quantum harmonic oscillators was investigated in a series of papers by Ford *et al.* from the point of view of the quantum Langevin equation [36,37]. These authors not only considered the diamagnetic response but also provided a treatment for the free energy expression from which all thermodynamic attributes can be evaluated. By considering such a physically relevant model system, we not only bring this problem to the arena of real three-dimensional systems but also incorporate the effect of external magnetic field, the impact of different forms of quantum dissipation (via several spectral density functions of the bath), and the effect of a higher number of spatial dimensions.

With this motivation, the plan of this paper is as follows. In the next section, we consider the Langevin dynamics of a three-dimensional quantum particle placed in a harmonic trap with eigenfrequency ω_0 and an external magnetic field $\mathbf{B}$. From its equation of motion, we compute the kinetic and potential energies of the oscillator at large times and obtain the corresponding exact expressions for $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$, which correspond to the probability distribution functions associated with the mean kinetic and mean potential energies, respectively, for an arbitrary passive heat bath. Following this, we carefully analyze the behavior of these distribution functions for a few different dissipation mechanisms and bring out the effect of the external magnetic field in Sec. III. In Sec. IV, we represent our average kinetic and potential energies in terms of infinite series and present important insights about the average energy of the quantum oscillator. We end with remarks in the final section.

II. MODEL, METHOD, AND MEASURES

The system of interest is a three-dimensional quantum harmonic oscillator of mass m and electric charge e placed in a uniform magnetic field of strength B and is linearly coupled through coordinate variables with a bosonic heat bath that is composed of an infinite number of quantum harmonic oscillators. Thus, the total Hamiltonian is given by

$$H = \frac{\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2}{2m} + \frac{1}{2}m\omega_0^2\mathbf{r}^2 + \sum_j \left[\frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2\left(\mathbf{q}_j - \frac{c_j}{m_j\omega_j^2}\mathbf{r}\right)^2\right], \tag{1}$$

where the symbols have their usual meanings with $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{r} = (x, y, z)$. The usual commutation relations between coordinates and momenta hold. The resulting equation of motion for the oscillator reads (see, for example, [15] and references therein),

$$m\ddot{\mathbf{r}}(t) + \int_{-\infty}^{t} \mu(t-t')\dot{\mathbf{r}}(t')dt' + m\omega_0^2\mathbf{r}(t) - \frac{e}{c}(\dot{\mathbf{r}} \times \mathbf{B}) = \mathbf{F}(t), \tag{2}$$

where $\mathbf{F}(t)$ is an operator valued random force (or quantum noise) and $\mu(t)$ is the dissipation kernel, which is, by convention, defined to vanish for $t < 0$ in order to respect the causality principle. Typically, the quantum noise is specified by the initial conditions, i.e., $\{\mathbf{q}_j(0), \mathbf{p}_j(0)\}$ on the bath oscillators, making the process non-Markovian. For this particular choice of uniform magnetic field and harmonic confining potential, upon using linear-response theory by considering a c-number generalized perturbing force $\mathbf{f}(t)$ in addition to the random force, one can solve Eq. (2) using a Fourier transform to get [38,39]

$$\tilde{r}_\rho(\omega) = \alpha_{\rho\sigma}(\omega)[\tilde{f}_\sigma + \tilde{F}_\sigma], \tag{3}$$

where

$$\alpha_{\rho\sigma} = \frac{[\lambda(\omega)]^2\delta_{\rho\sigma} - (\omega e/c)^2 B_\rho B_\sigma - i\frac{e\omega}{c}\lambda(\omega)\epsilon_{\rho\sigma\eta}B_\eta}{\det[D(\omega)]}, \tag{4}$$

with

$$\det[D(\omega)] = \lambda(\omega)[\lambda(\omega)^2 - (e\omega/c)^2B^2], \tag{5}$$

$$\lambda(\omega) = \left[m\left(\omega_0^2 - \omega^2\right) - i\omega\tilde{\mu}(\omega)\right], \tag{6}$$

$$\tilde{r}_\rho = \int_{-\infty}^{\infty} dt e^{i\omega t} r_\rho(t), \tag{7}$$

$$\tilde{\mu} = \int_0^{\infty} dt e^{i\omega t} \mu(t), \tag{8}$$

and $\epsilon_{\rho\sigma\eta}$ being the Levi-Civita antisymmetric tensor. The Greek indices stand for three spatial directions (i.e., $\rho, \sigma, \eta = x, y, z$), and the Einstein summation convention is used. Further, the Fourier transform of a dynamical variable is denoted by a tilde. The memory function $\mu(t)$ vanishes for negative times, and the c-number generalized susceptibility tensor $\alpha_{\rho\sigma}(\omega)$ determines the dynamics of such a linear system in a unique way.

We now introduce the symmetrized position autocorrelation function, which can be obtained from the fluctuation-dissipation relationship as follows [38,39]:

$$\begin{aligned}\psi_{\rho\sigma}(t-t') &= \frac{1}{2}\langle r_\rho(t)r_\sigma(t') + r_\sigma(t')r_\rho(t)\rangle \\ &= \frac{\hbar}{\pi}\int_0^\infty \mathrm{Im}[\alpha_{\rho\sigma}^s(\omega + i0^+)]\coth\left(\frac{\hbar\omega}{2k_BT}\right) \\ &\quad \times \cos[\omega(t-t')]d\omega \\ &\quad - \frac{\hbar}{\pi}\int_0^\infty \mathrm{Re}[\alpha_{\rho\sigma}^a(\omega + i0^+)]\coth\left(\frac{\hbar\omega}{2k_BT}\right) \\ &\quad \times \sin[\omega(t-t')]d\omega,\end{aligned} \tag{9}$$

where $\alpha_{\rho\sigma}^s(\omega)$ and $\alpha_{\rho\sigma}^a(\omega)$ are the symmetric and antisymmetric parts of the generalized susceptibility tensor. For definiteness, we now choose the direction of the magnetic field to be the z direction in calculations throughout this paper. Due to the cylindrical symmetry of the system, the only nonzero elements of the generalized susceptibility tensor $\alpha^s_{\rho\sigma}(\omega)$ are $\alpha^s_{xx}(\omega)$, $\alpha^s_{yy}(\omega)$, $\alpha^s_{zz}(\omega)$, $\alpha_{xy}(\omega)$, and $\alpha_{yx}(\omega)$. The nonvanishing elements are given as follows:

$$\alpha^s_{xx}(\omega) = \alpha^s_{yy}(\omega) = \frac{[\lambda(\omega)]^2}{\det D(\omega)}, \quad \alpha^s_{zz}(\omega) = \frac{1}{\lambda(\omega)},$$
$$\alpha_{xy}(\omega) = -\alpha_{yx}(\omega) = -i\omega\frac{e}{c}\frac{B\lambda(\omega)}{\det D(\omega)}. \tag{10}$$

One can obtain the position autocorrelation functions (also called dispersions) of the motions perpendicular and parallel to the magnetic field B and use them to obtain the potential energy of the system for $t = t'$ at long times,

$$\langle x^2\rangle = \langle y^2\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega \mathrm{Im}\big[\alpha^s_{xx}(\omega + i0^+)\big] \coth\left(\frac{\hbar\omega}{2k_BT}\right),$$
$$\langle z^2\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega \mathrm{Im}\big[\alpha^s_{zz}(\omega + i0^+)\big] \coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{11}$$

Next, differentiating Eq. (9) first with respect to t and then with respect to t' and finally setting $t = t'$, one may obtain the expressions

$$\langle \dot{x}^2\rangle = \langle \dot{y}^2\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega\omega^2 \mathrm{Im}\big[\alpha^s_{xx}(\omega + i0^+)\big] \coth\left(\frac{\hbar\omega}{2k_BT}\right),$$
$$\langle \dot{z}^2\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega\omega^2 \mathrm{Im}\big[\alpha^s_{zz}(\omega + i0^+)\big] \coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{12}$$

Exact relations of this kind can be found in Ref. [40], although the notion of energy partition was not discussed there. These expressions will allow us to compute the average kinetic and potential energies at long times for the charged dissipative oscillator placed in an external field, which are respectively given by

$$E_k = \frac{m\langle \dot{\mathbf{r}}^2\rangle}{2} = \frac{m\langle \dot{x}^2\rangle + m\langle \dot{y}^2\rangle + m\langle \dot{z}^2\rangle}{2},$$
$$E_p = \frac{m\omega_0^2\langle \mathbf{r}^2\rangle}{2} = \frac{m\langle x^2\rangle + m\langle y^2\rangle + m\langle z^2\rangle}{2}. \tag{13}$$

Consequently, they can be expressed conveniently as

$$E_k = \int_0^\infty \mathcal{E}_k(\omega)\mathcal{P}_k(\omega)d\omega \tag{14}$$

and

$$E_p = \int_0^\infty \mathcal{E}_p(\omega)\mathcal{P}_p(\omega)d\omega, \tag{15}$$

where $\mathcal{E}_k(\omega)$ and $\mathcal{E}_p(\omega)$ are given by

$$\mathcal{E}_k(\omega) = \mathcal{E}_p(\omega) = \frac{3\hbar\omega}{4}\coth\left(\frac{\hbar\omega}{2k_BT}\right), \tag{16}$$

which are equivalent to the average kinetic and potential energies of the three-dimensional oscillators of the thermostat. Let us note that these average energies [$\mathcal{E}_k(\omega)$ and $\mathcal{E}_p(\omega)$] are obtained by averaging over the canonical ensemble (Gibbsian distribution) of the thermostat so that the mean energy is equal to the total energy of each thermostat oscillator and is given by

$$\mathcal{E}(\omega) = \mathcal{E}_k(\omega) + \mathcal{E}_p(\omega) = \frac{3\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right), \tag{17}$$

which is a well-known result from elementary statistical mechanics. Now, the expressions for the distribution functions $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$ can be straightforwardly expressed in the following forms using Eqs. (11)–(16):

$$\mathcal{P}_k(\omega) = \frac{2m\omega}{3\pi}\mathrm{Im}[\alpha^s_{\rho\rho}(\omega + i0^+)] \tag{18}$$

and

$$\mathcal{P}_p(\omega) = \frac{2m\omega_0^2}{3\omega\pi}\mathrm{Im}[\alpha^s_{\rho\rho}(\omega + i0^+)]. \tag{19}$$

For the present case in which we have taken the magnetic field to be in the z direction, we can write $\mathrm{Im}[\alpha^s_{\rho\rho}(\omega)] = \mathrm{Im}[\alpha_{xx}(\omega)] + \mathrm{Im}[\alpha_{yy}(\omega)] + \mathrm{Im}[\alpha_{zz}(\omega)]$, with

$$\mathrm{Im}[\alpha_{xx}(\omega)] = \mathrm{Im}[\alpha_{yy}(\omega)] = \frac{1}{2m}\left[\frac{\omega\mathrm{Re}[\tilde{\mu}(\omega)]/m}{\{\omega_0^2 - \omega^2 + \omega\omega_c + \omega\mathrm{Im}[\tilde{\mu}(\omega)]/m\}^2 + \{\omega\mathrm{Re}[\tilde{\mu}(\omega)]/m\}^2} + \frac{\omega\mathrm{Re}[\tilde{\mu}(\omega)]/m}{\{\omega_0^2 - \omega^2 - \omega\omega_c + \omega\mathrm{Im}[\tilde{\mu}(\omega)]/m\}^2 + \{\omega\mathrm{Re}[\tilde{\mu}(\omega)]/m\}^2}\right] \tag{20}$$

and

$$\mathrm{Im}[\alpha_{zz}(\omega)] = \frac{1}{m}\left[\frac{\omega\mathrm{Re}[\tilde{\mu}(\omega)]/m}{\{\omega_0^2 - \omega^2 + \omega\mathrm{Im}[\tilde{\mu}(\omega)]/m)\}2 + \{\omega\mathrm{Re}[\tilde{\mu}(\omega)]/m\}^2}\right]. \tag{21}$$

Here $\tilde{\mu}(\omega) = \mathrm{Re}[\tilde{\mu}(\omega)] + i\mathrm{Im}[\tilde{\mu}(\omega)]$ is the Fourier transform of the dissipation kernel $\mu(t)$. Thus, using Eqs. (20) and (21) for different dissipative environments, one can compute the exact structure of the distribution functions $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$ in the closed form according to which the thermostat oscillators contribute to the average kinetic and potential energies of the system. Although we are naively demanding that $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$ be the probability distributions according to which the thermostat oscillators contribute to the average energy of the dissipative charged magneto-oscillator, the function

$\mathcal{P}_{i=k,p}(\omega)$ is a probability density if and only if it is nonnegative and normalized on the interval $(0,\infty)$. One can further demand from probability theory that a random variable ξ_ω exists for which $\mathcal{P}_{i=k,p}(\omega)$ is the appropriate probability distribution function. In the present problem, this random variable can be interpreted as the eigenfrequency of thermostat oscillators. In the thermodynamic limit for the bath, there are infinitely many oscillators with a continuous spectrum of eigenfrequencies which contribute to E_k and E_p according to $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$, respectively. In the two sections below, we prove that the functions $\mathcal{P}_{i=k,p}(\omega)$ are indeed nonnegative and normalized in the interval $(0,\infty)$.

A. Nonnegativity of $\mathcal{P}_{i=k,p}(\omega)$

Let us first consider the nonnegativity of the distribution functions $\mathcal{P}_{i=k,p}(\omega)$. Following Ref. [38] (see Appendix C therein), we can write the work done by an external c-number arbitrary force $\mathbf{f}(t)$ (besides the magnetic field) as

$$W=\int_{-\infty}^{\infty}dt\,f_\rho(t)\langle v_\rho(t)\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\tilde{f}_\rho(\omega)\langle\tilde{v}_\rho(-\omega)\rangle, \tag{22}$$

where $v_\rho(t)$ is the velocity operator of the particle and the external force $\mathbf{f}(t)$ vanishes in the distant past and future. The second equality of Eq. (22) is obtained using Parseval's theorem. We also know that $\tilde{v}_\rho(\omega)=-i\omega r_\rho(\omega)$, $\tilde{v}_\rho(-\omega)=\tilde{v}^*_\rho(\omega)$ (the reality condition), and $\tilde{r}_\rho(\omega)=\alpha_{\rho\sigma}[\tilde{f}_\sigma+\tilde{F}_\sigma(\omega)]$. Utilizing these relations, we can show that

$$\begin{aligned}W&=\frac{1}{4\pi i}\int_{-\infty}^{\infty}d\omega\omega[\alpha_{\rho\sigma}(\omega)-\alpha^*_{\sigma\rho}(\omega)]\tilde{f}_\sigma(\omega)\tilde{f}^*_\rho(\omega)\\&=\frac{1}{\pi}\int_0^{\infty}d\omega\mathrm{Re}[\tilde{\mu}(\omega)]\sum_\mu|\langle\tilde{v}_\mu(\omega)\rangle|^2,\end{aligned} \tag{23}$$

which implies W is a positive quantity consistent with the second law of thermodynamics. Considering the fact that $\tilde{f}_\rho(\omega)$ is arbitrary and may be chosen to be a real and even function of ω and utilizing $\alpha_{\rho\sigma}(\omega)-\alpha^*_{\sigma\rho}(\omega)=2i\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]+2\mathrm{Im}[\alpha^a_{\rho\sigma}(\omega)]$ [with $\alpha^s_{\rho\sigma}(\omega)$ and $\alpha^a_{\rho\sigma}(\omega)$ being the symmetric and antisymmetric parts of $\alpha_{\rho\sigma}(\omega)$] in Eq. (23), together with noting that $\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]$ is an odd function of ω, while $\mathrm{Im}[\alpha^a_{\rho\sigma}(\omega)]$ is an even function of ω, we finally obtain

$$W=\frac{1}{\pi}\int_0^{\infty}d\omega\omega\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]\tilde{f}_\rho(\omega)\tilde{f}^*_\sigma(\omega). \tag{24}$$

Hence, the positivity condition of W implies the integrand of Eq. (24) must be positive for all ω, i.e., $\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]\tilde{f}_\rho(\omega)\tilde{f}^*_\sigma(\omega)>0\ \ \forall\ \ \omega>0$. As a result, $\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]\tilde{f}_\rho(\omega)$ is positive definite for all $\omega>0$. Finally, we demand that both $\omega\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]\tilde{f}_\rho(\omega)$ and $\frac{\mathrm{Im}[\alpha^s_{\rho\sigma}(\omega)]\tilde{f}_\rho(\omega)}{\omega}$ are positive definite for all $\omega>0$. These results suggest the positivity condition of $\mathcal{P}_{i=k,p}(\omega)$. Next, we move to the proof of the normalization of $\mathcal{P}_{i=k,p}(\omega)$.

B. Normalization of $\mathcal{P}_{i=k,p}(\omega)$

Let us first consider the distribution function corresponding to the potential energy, which is $\mathcal{P}_p(\omega)=\frac{2m\omega_0^2}{3\pi}\frac{\mathrm{Im}[\alpha^s_{\rho\rho}(\omega+i0^+)]}{\omega}$. Hence, we need to prove that $\int_0^\infty d\omega\mathcal{P}_p(\omega)=1$. Now,

$$\int_0^{\infty}d\omega\mathcal{P}_p(\omega)=\frac{m\omega_0^2}{3}\frac{2}{\pi}\int_0^{\infty}d\omega\frac{\mathrm{Im}[\alpha^s_{\rho\rho}(\omega+i0^+)]}{\omega}. \tag{25}$$

Next, we recall the following relation from Ref. [41]:

$$\alpha_{\rho\rho}(i\omega)=\frac{2}{\pi}\int_0^{\infty}dv\frac{v\mathrm{Im}[\alpha_{\rho\rho}(v)]}{v^2+\omega^2}. \tag{26}$$

Setting $\omega=0$ in Eq. (26), we obtain for the symmetric part of the generalized susceptibility tensor

$$\alpha^s_{\rho\rho}(0)=\frac{2}{\pi}\int_0^{\infty}dv\frac{\mathrm{Im}[\alpha^s_{\rho\rho}(v)]}{v}. \tag{27}$$

Combining Eqs. (25) and (27), we get

$$\int_0^{\infty}d\omega\mathcal{P}_p(\omega)=\frac{m\omega_0^2}{3}\alpha^s_{\rho\rho}(0). \tag{28}$$

Thus, we observe that the problem of normalization of $\mathcal{P}_p(\omega)$ is converted to the condition of whether the equality $\alpha^s_{\rho\rho}(0)=\frac{3}{m\omega_0^2}$ holds or not. Now, we know that the symmetric part of the generalized susceptibility tensor for our system is given by

$$\alpha^s_{\rho\rho}(\omega)=\frac{[\lambda(\omega)^2-(e\omega/c)^2B^2]}{\lambda(\omega)[\lambda(\omega)^2-(e\omega/c)^2B^2]}=\frac{1}{\lambda(\omega)}. \tag{29}$$

But since $\lambda(\omega)=m(\omega_0^2-\omega^2)-i\omega m\tilde{\mu}(\omega)$, we have $\alpha^s_{\rho\rho}(0)=\alpha^s_{xx}(0)+\alpha^s_{yy}(0)+\alpha^s_{zz}(0)=\frac{3}{m\omega_0^2}$. This proves that $\mathcal{P}_p(\omega)$ is normalized.

Next, let us consider $\mathcal{P}_k(\omega)=\frac{m}{3}\frac{2}{\pi}\omega\mathrm{Im}[\alpha^s_{\rho\rho}(\omega+i0^+)]$. Since $\mathrm{Im}[\alpha^s_{\rho\rho}(\omega+i0^+)]$ is an odd function of ω, $\mathcal{P}_k(\omega)$ is an even function of ω. Thus, we can write

$$\mathcal{P}_k(\omega)=\frac{2}{\pi}\int_0^{\infty}dt\,\chi(t)\cos(\omega t)=\tilde{\chi}_{FC}(\omega), \tag{30}$$

which is nothing more than the Fourier cosine transform of the response function $\chi(t)$, while the inverse Fourier cosine transform is defined as

$$\chi(t)=\int_0^{\infty}d\omega\tilde{\chi}_{FC}(\omega)\cos(\omega t). \tag{31}$$

Setting $t=0$ in Eq. (31) we obtain

$$\chi(0)=\int_0^{\infty}d\omega\tilde{\chi}_{FC}(\omega)=\int_0^{\infty}d\omega\mathcal{P}_k(\omega). \tag{32}$$

Now, from the initial value theorem of the Laplace transform [42], we can write

$$\lim_{s\to\infty}s\tilde{\chi}_L(s)=\lim_{t\to0}\chi(t)=\chi(0). \tag{33}$$

Further, taking a Laplace transform of Eq. (2), we can show that

$$\tilde{\chi}(s)=\frac{ms}{\left[ms^2+m\omega_0^2+(\tilde{\mu}(s)+\omega_c)s\right]}, \tag{34}$$

where $\omega_c=eB/mc$ is the cyclotron frequency and $\mathbf{B}=B\hat{z}$. Thus,

$$\chi(0)=\lim_{s\to\infty}s\tilde{\chi}_L(s)=\lim_{s\to\infty}\frac{ms^2}{\left[ms^2+m\omega_0^2+(\tilde{\mu}(s)+\omega_c)s\right]}=1. \tag{35}$$

This proves the normalization of $\mathcal{P}_k(\omega)$. Starting from the quantum Langevin equation of the paradigmatic model of dissipative diamagnetism in the presence of a c-number arbitrary force $\mathbf{f}(t)$, we obtain the distribution functions $\mathcal{P}_{i=k,p}(\omega)$ in the framework of the linear-response theory. In the process we use the fluctuation-dissipation theorem to establish a relationship between the generalized susceptibility tensor $\mathrm{Im}[\alpha^s_{\rho\rho}(\omega)]$ and the probability distribution functions $\mathcal{P}_{i=k,p}(\omega)$. This implies that one can infer different properties of the quantum environment, its coupling to a given quantum system, the effect of dissipation, and the effect of external magnetic field, which characterizes $\mathcal{P}_{i=k,p}(\omega)$, by experimentally measuring relevant quantities (like susceptibility) of the system using the linear response of the system of interest against an applied perturbation.

Note that we have not yet specified the exact functional form of $\tilde{\mu}(\omega)$, which depends on the particular dissipation model being considered. In what follows, we shall consider three different kinds of heat baths, namely, the Gaussian decay and Drude and radiation baths, and investigate their impact on $\mathcal{P}_{i=k,p}(\omega)$ in the following section.

III. IMPACT OF DISSIPATION, MEMORY TIME, AND MAGNETIC FIELD

It is well known from the classical equipartition theorem that the average energy of a system equals $k_BT/2$ per degree of freedom, so that all the degrees of freedom contribute an equal amount to the total average energy, which does not depend on the frequency of the thermostat oscillators. On the other hand, we observe that the average kinetic or potential energy of an open quantum system receives contributions from the bath degrees of freedom such that bath oscillators of various frequencies subscribe to $E_{i=k,p}$ with different probabilities. Naturally, we will be interested to investigate which frequencies contribute to the mean energy of the system more than the others depending on the dissipation mechanism, external magnetic field, and memory time. The influence of different dissipation mechanisms can be investigated via specification of the spectral density $J(\omega)$ of the bath or the dissipation kernel $\mu(t)$. In this section, we examine the properties of the probability distributions $\mathcal{P}_{i=k,p}(\omega)$ for a few different classes of $\mu(t)$.

A. Gaussian decay

We will begin with a dissipation kernel which decays as a Gaussian (see, for example, [2]). To this end, we recall the definition of the bath spectral function characterizing the nature of the dissipative environment, which is related to the dissipation kernel $\mu(t)$ by the cosine transform,

$$\mu(t) = \frac{2}{\pi}\int_0^\infty \frac{J(\omega)}{\omega}\cos(\omega t)d\omega. \tag{36}$$

For the present case, we take

$$J(\omega) = \frac{m\gamma_0\omega}{\pi}e^{-\omega^2/4\omega_{\rm cut}^2}, \tag{37}$$

where γ_0 is the friction coefficient which defines the coupling strength between the system and the thermostat and $\omega_{\rm cut}$ is a cutoff frequency scale. The associated dissipation kernel takes the following form:

$$\mu(t) = \frac{m\gamma_0}{\sqrt{\pi}\,\tau_c}e^{-(t/\tau_c)^2}, \tag{38}$$

with $\tau_c = 1/\omega_{\rm cut}$ being the timescale associated with the cutoff frequency. It should be noted that in the limit $\omega_{\rm cut}\to\infty$, the dissipation kernel reduces to that for the strictly Ohmic or memoryless case. Thus, the larger the value of $\omega_{\rm cut}$ is (or the smaller τ_c is), the closer we get to Ohmic dissipation. There are two other control parameters defined by the confining potential frequency ω_0 and the cyclotron frequency ω_c. Since the imaginary part of the memory kernel vanishes, the distribution functions $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$ admit a rather simple form. If we define $w=\omega/\omega_0$ and other dimensionless parameters $a=\gamma_0/\omega_{\rm cut}$, $\tilde{\omega}_0=\omega_0/\omega_{\rm cut}$, and $\tilde{\omega}_c=\omega_c/\omega_{\rm cut}$, then these distribution functions can be greatly simplified, and they can be expressed in the following dimensionless form:

$$\tilde{P}_k(w) = \frac{3\pi\,\omega_{\rm cut}}{a}\mathcal{P}_k(w\omega_{\rm cut}) = w^2e^{-w^2/4}F_1(w) \tag{39}$$

and

$$\tilde{P}_p(w) = \frac{3\pi\,\omega_{\rm cut}}{a\tilde{\omega}_0^2}\mathcal{P}_p(w\omega_{\rm cut}) = e^{-w^2/4}F_1(w), \tag{40}$$

where the function $F_1(w)$ is given by

$$F_1(w) = \left[\frac{1}{\left(\tilde{\omega}_0^2 - w^2 + \tilde{\omega}_c w + \frac{aw\times\mathrm{erfi}\left[\frac{w}{2}\right]}{2}\right)^2 + \left(\frac{awe^{-\frac{w^2}{4}}}{2}\right)^2} + \frac{1}{\left(\tilde{\omega}_0^2 - w^2 - \tilde{\omega}_c w + \frac{aw\times\mathrm{erfi}\left[\frac{w}{2}\right]}{2}\right)^2 + \left(\frac{awe^{-\frac{w^2}{4}}}{2}\right)^2} + \frac{1}{\left(\tilde{\omega}_0^2 - w^2 + \frac{aw\times\mathrm{erfi}\left[\frac{w}{2}\right]}{2}\right)^2 + \left(\frac{awe^{-\frac{w^2}{4}}}{2}\right)^2}\right]. \tag{41}$$

The distribution functions are plotted in Fig. 1 for different values of the rescaled magnetic field strength $\tilde{\omega}_c$. The plots clearly demonstrate inhomogeneous contributions of thermostat oscillators to $\mathcal{P}_{i=k,p}(\omega)$ and there are most probable values of $\tilde{P}_{i=k,p}(w)$ at which one can find the largest contributions to the kinetic and potential energies of the charged magneto-oscillator. One interesting feature which needs to be highlighted is that as one increases the rescaled magnetic field strength $\tilde{\omega}_c$, one finds an increase in the number of the most probable values (up to a total of three). Clearly, upon increasing $\tilde{\omega}_c$ from 0.1 to 0.3, one can move from a single peak to three distinct peaks. Thus, by tuning the external magnetic field strength, one can control the most probable frequencies $w=w_m$, which make the highest contributions to the kinetic and potential energies of the system. This is a unique feature of the dissipative diamagnetic system.

We also present $\tilde{P}_{i=k,p}(w)$ as a function of the rescaled frequency w of the thermostat oscillators in Fig. 2 for selected values of the dimensionless parameter $a=\gamma_0/\omega_{\rm cut}$. We observe that the variation of a impacts both $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$. As we decrease $\gamma_0/\omega_{\rm cut}=\tau_c/\tau_\nu$, both probability distributions $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$ develop sharper peaks around the most probable frequencies. Here, $\tau_\nu = 1/\gamma_0$ is the dissipation timescale

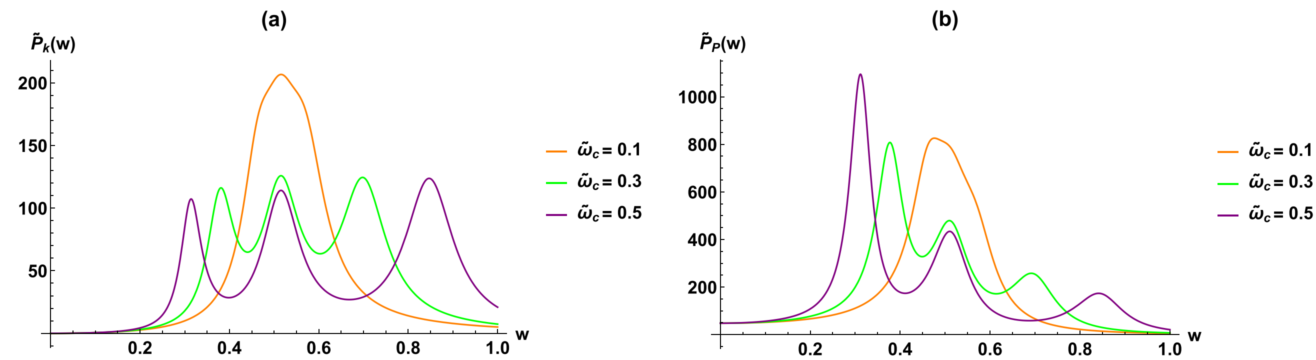

FIG. 1. Variation of (a) $\tilde{P}_k(w)$ and (b) $\tilde{P}_p(w)$ as a function of rescaled thermostat oscillator frequencies w for a bath whose dissipation kernel decays as a Gaussian with selected values of $\tilde{\omega}_c$ while keeping $\tilde{\omega}_0 = 0.5$ and $a = 0.2$.

(classically, the time between collisions in Brownian motion). This means that the smaller the cutoff timescale of the bath gets with respect to the dissipation timescale, the greater the contributions to the mean kinetic and potential energies from the most probable frequencies are, while the other frequencies become less important.

B. Drude dissipation

As a second example we consider the Drude dissipation mechanism, which is described by the following form of the bath spectral function:

$$J(\omega) = m\gamma_0 \frac{\omega}{1 + (\omega/\omega_{\rm cut})^2}, \tag{42}$$

where γ_0 is a constant with appropriate dimensions and $\omega_{\rm cut}$ is a frequency scale characteristic of the specific thermostat. Consequently, from Eq. (36), $\mu(t)$ takes the following form:

$$\mu(t) = \frac{m\gamma_0}{\tau_c} e^{-(t/\tau_c)}, \tag{43}$$

where $\tau_c = 1/\omega_{\rm cut}$ has dimensions of time and is a characteristic timescale associated with the bath. One may note that the memory function contains two nonnegative parameters, γ_0 and τ_c. The first one signifies the coupling strength between the system and the bath, while the latter provides a measure of the timescale over which the open system demonstrates dissipation memory or a non-Markovian nature. It is now straightforward to compute the Fourier transform of $\mu(t)$, which reads

$$\tilde{\mu}(\omega) = \frac{m\gamma_0\omega_{\rm cut}^2}{\omega_{\rm cut}^2 + \omega^2} + i\frac{m\gamma_0\omega_{\rm cut}\omega}{\omega_{\rm cut}^2 + \omega^2}. \tag{44}$$

With these expressions, we can find the expressions for $\mathcal{P}_k(\omega)$ and $\mathcal{P}_p(\omega)$ now. Upon defining $w = \omega/\omega_{\rm cut}$ and the dimensionless parameters $a = \gamma_0/\omega_{\rm cut}$, $\tilde{\omega}_0 = \omega_0/\omega_{\rm cut}$, and $\tilde{\omega}_c = \omega_c/\omega_{\rm cut}$, the expressions for the distribution functions can be made dimensionless as

$$\tilde{P}_k(w) = \frac{3\pi\omega_{\rm cut}}{2a}\mathcal{P}_k(w\omega_{\rm cut}) = \left(\frac{w^2}{1+w^2}\right)F_2(w) \tag{45}$$

and

$$\tilde{P}_p(w) = \frac{3\pi\omega_{\rm cut}}{2a\tilde{\omega}_0^2}\mathcal{P}_p(w\omega_{\rm cut}) = \left(\frac{1}{1+w^2}\right)F_2(w), \tag{46}$$

where $F_2(w)$ is given by

$$\begin{aligned} F_2(w) = {} & \frac{1}{\left(\tilde{\omega}_0^2 - w^2 + \tilde{\omega}_c w + \frac{aw^2}{1+w^2}\right)^2 + \left(\frac{aw}{1+w^2}\right)^2} \\ & + \frac{1}{\left(\tilde{\omega}_0^2 - w^2 - \tilde{\omega}_c w + \frac{aw^2}{1+w^2}\right)^2 + \left(\frac{aw}{1+w^2}\right)^2} \\ & + \frac{1}{\left(\tilde{\omega}_0^2 - w^2 + \frac{aw^2}{1+w^2}\right)^2 + \left(\frac{aw}{1+w^2}\right)^2}. \end{aligned} \tag{47}$$

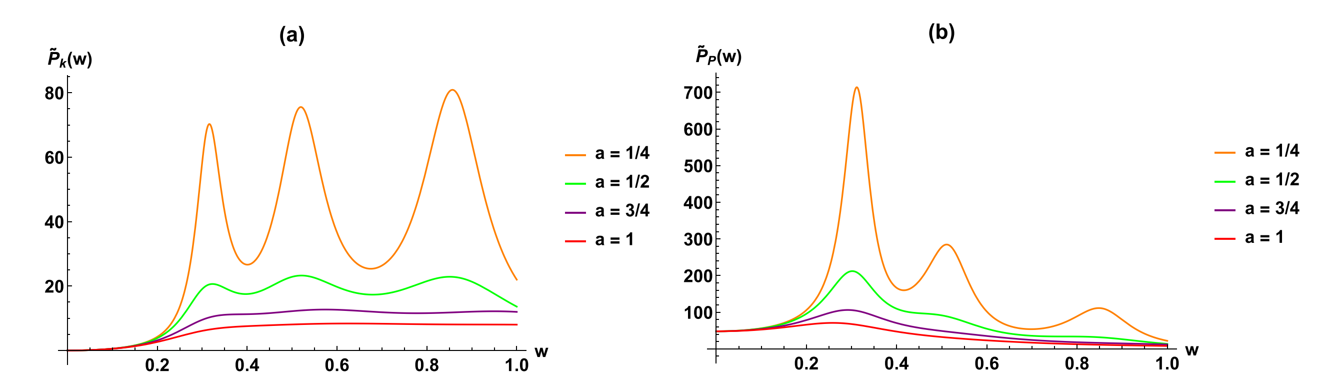

FIG. 2. Variation of (a) $\tilde{P}_k(w)$ and (b) $\tilde{P}_p(w)$ as a function of rescaled thermostat oscillator frequencies w for a bath whose dissipation kernel decays as a Gaussian with selected values of a while keeping $\tilde{\omega}_c = 0.5$ and $\tilde{\omega}_0 = 0.5$.

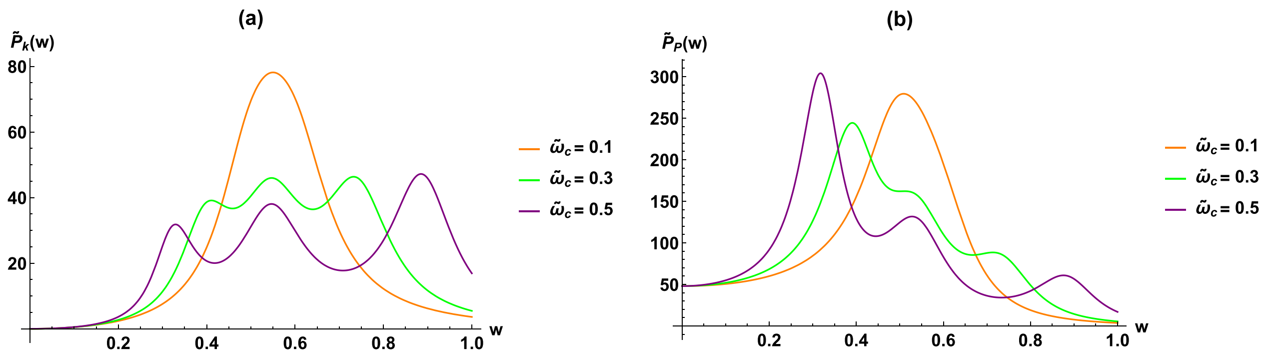

FIG. 3. Variation of (a) $\tilde{P}_k(w)$ and (b) $\tilde{P}_p(w)$ as a function of rescaled thermostat oscillator frequencies w for Drude dissipation with selected values of $\tilde{\omega}_c$ while keeping $\tilde{\omega}_0 = 0.5$ and $a = 0.2$.

The effect of an external magnetic field is quite remarkable on both $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$, which can be seen from Fig. 3. As a matter of fact, for a nonzero magnetic field, the distribution functions peak at up to three different frequencies. This is in sharp contrast to the case in zero magnetic field (see, for example, [2,3]), where both $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$ peak at one single frequency. We strongly emphasize the fact that just like in the previous example, the existence of up to three peaks in the distribution functions is not merely because the oscillator is in three dimensions but is rather an effect due to the applied magnetic field. More precisely, they come from the poles of the susceptibility tensor.

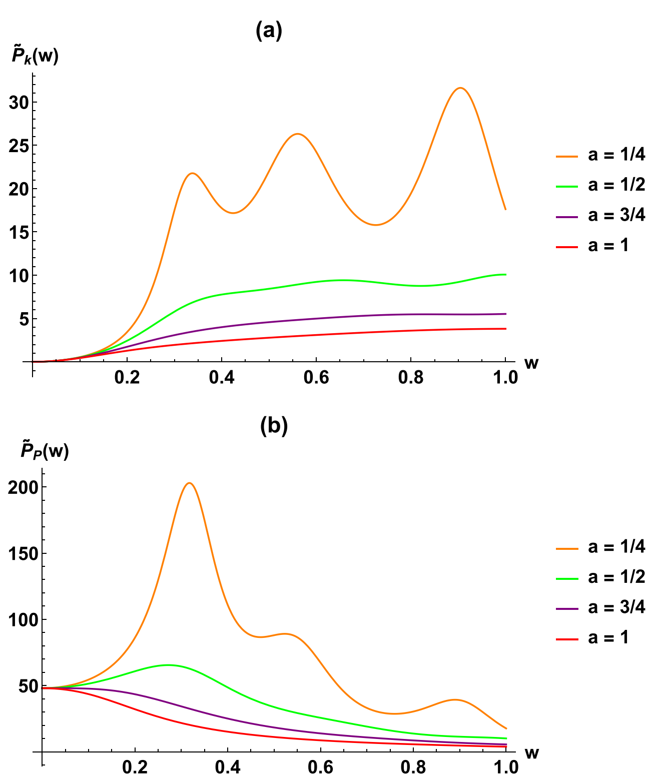

FIG. 4. Variation of (a) $\tilde{P}_k(w)$ and (b) $\tilde{P}_p(w)$ as a function of rescaled thermostat oscillator frequencies w for Drude dissipation with selected values of a while keeping $\tilde{\omega}_0 = 0.5$ and $\tilde{\omega}_c = 0.5$.

In Fig. 4, we demonstrate the distribution functions $\tilde{P}_{i=k,p}(w)$ as a function of rescaled frequency w for selected values of a. The dimensionless ratio $a = \gamma_0/\omega_{\rm cut} = \tau_c/\tau_\nu$ directly compares the dissipation timescale τ_ν with the bath's characteristic memory timescale τ_c and has remarkable control over the overall behavior of the distribution functions $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$. We may observe that for small values of the parameter a, or, equivalently, when τ_c is small with respect to τ_ν, the distribution $\tilde{P}_k(w)$ notably peaks around the three most probable values. The contribution of higher frequency modes is larger in $\tilde{P}_k(w)$. As a consequence we may infer that rapidly vibrating thermostat oscillators contribute significantly to the kinetic energy of the particle. It may further be verified (like for the previous example) that for $\tau_c \ll \tau_\nu$, the probability distribution function approaches three δ function peaks at the three most probable (rescaled) frequencies $w = \mathrm{w}_m$. The situation is quite different for large memory times where τ_c is greater than or equal to τ_ν (or large values of a) where the distribution functions are flattened. This implies that a much wider window of oscillator frequencies contributes to E_k in a similar way. The basic features for $\tilde{P}_p(w)$ are the same as those of $\tilde{P}_k(w)$. The only difference is observed for small memory times for which the low frequency bath oscillators contribute notably to the average potential energy E_p.

In general, we observe that the thermostat oscillators contribute to E_k as well as to E_p in an inhomogeneous way. Three optimal thermostat oscillator frequencies $w = w_m$ make the largest contributions to E_k. A similar observation exists for E_p, but in general these optimal frequencies are different from those of the kinetic energy. The values of w_m for E_k and E_p are highly dependent on the system parameters $\tilde{\omega}_c$, $\tilde{\omega}_0$, and a.

C. Radiation bath

We now consider the third and final bath spectrum, which is a radiation bath [12,13] for which we take the following expression for $\mu(t)$:

$$\mu(t) = \frac{2e^2\Omega^2}{3c^3}[2\delta(t) - \Omega e^{-\Omega t}], \tag{48}$$

and correspondingly, the expression for $\tilde{\mu}(\omega)$ reads

$$\tilde{\mu}(\omega) = \frac{2e^2\omega\Omega^2}{3c^3(\omega + i\Omega)} = \frac{M\omega\Omega}{(\omega + i\Omega)}, \tag{49}$$

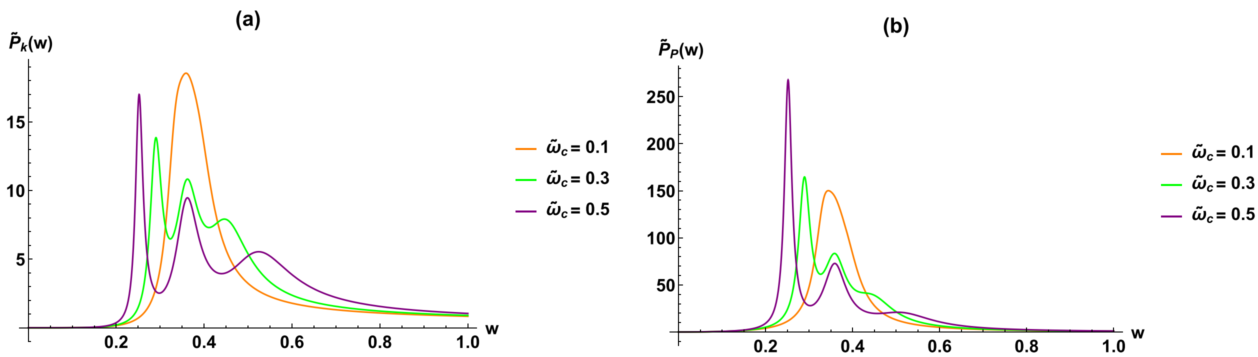

FIG. 5. Variation of (a) $\tilde{P}_k(w)$ and (b) $\tilde{P}_p(w)$ as a function of rescaled thermostat oscillator frequencies w for a radiation bath with selected values of $\tilde{\omega}_c$ while keeping $\tilde{\omega}_0 = 0.5$.

where Ω plays the role of a cutoff frequency scale characteristic of the bath and M is a rescaled mass given by $M \approx \frac{2e^2\Omega}{3c^3}$ in the large cutoff limit ($\Omega \to \infty$). Subsequently, the real and imaginary parts of $\tilde{\mu}(\omega)$ are given by

$$\mathrm{Re}[\tilde{\mu}(\omega)] = \frac{M\omega^2\Omega}{(\omega^2+\Omega^2)}, \quad \mathrm{Im}[\tilde{\mu}(\omega)] = -\frac{M\omega\Omega^2}{(\omega^2+\Omega^2)}. \tag{50}$$

Now, substituting these into Eqs. (20) and (21) and upon defining $w = \omega/\Omega$, $\tilde{\omega}_0 = \omega_0/\Omega$, and $\tilde{\omega}_c = \omega_c/\Omega$, we can derive the following dimensionless forms of the distribution functions:

$$\tilde{P}_k(w) = \frac{3\pi\Omega}{2}\mathcal{P}_k(w\Omega) = \left(\frac{w^4}{1+w^2}\right)F_3(w) \tag{51}$$

and

$$\tilde{P}_p(w) = \frac{3\pi\Omega}{2\tilde{\omega}_0^2}\mathcal{P}_p(w\Omega) = \left(\frac{w^2}{1+w^2}\right)F_3(w), \tag{52}$$

where

$$\begin{aligned} F_3(w) = &\frac{1}{\left(\tilde{\omega}_0^2 - w^2 + \tilde{\omega}_c w - \frac{w^2}{1+w^2}\right)^2 + \left(\frac{w^3}{1+w^2}\right)^2} \\ &+ \frac{1}{\left(\tilde{\omega}_0^2 - w^2 - \tilde{\omega}_c w - \frac{w^2}{1+w^2}\right)^2 + \left(\frac{w^3}{1+w^2}\right)^2} \\ &+ \frac{1}{\left(\tilde{\omega}_0^2 - w^2 - \frac{w^2}{1+w^2}\right)^2 + (\frac{w^3}{1+w^2})^2}. \end{aligned} \tag{53}$$

The dimensionless distribution functions $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$ are plotted in Fig. 5 for different values of $\tilde{\omega}_c$. Once again, we can notice that the increase in magnetic field changes the single peaked $\tilde{P}_{i=k,p}(w)$ into a trimodal distribution. In fact, as we increase the magnetic field strength, the low frequency or slowly vibrating thermostat oscillators contribute significantly to the kinetic and potential energies of the dissipative magneto-oscillator. On the other hand, we plot $\tilde{P}_k(w)$ and $\tilde{P}_p(w)$ in Fig. 6 for selected values of $\tilde{\omega}_0$. The resulting distribution functions are typically trimodal in nature. As we increase the values of $\tilde{\omega}_0$ (in other words, the confining potential frequency), the trimodal distributions move towards the high frequency regime; that is, the high frequency bath oscillator contribution to the average kinetic as well as potential energy increases.

IV. AVERAGE KINETIC AND POTENTIAL ENERGIES AS AN INFINITE SERIES

We have already demonstrated that the average kinetic and potential energies of the charged dissipative magneto-oscillator can be represented as follows:

$$\begin{aligned} E_k &= \langle \mathcal{E}_k \rangle = \int_0^\infty d\omega \mathcal{E}_k(\omega)\mathcal{P}_k(\omega), \\ E_p &= \langle \mathcal{E}_p \rangle = \int_0^\infty d\omega \mathcal{E}_p(\omega)\mathcal{P}_p(\omega). \end{aligned} \tag{54}$$

However, it is quite difficult and nontrivial to infer the dependence of the average kinetic energy and potential energy on the system parameters from Eqs. (54). We can represent Eqs. (54) in the form of a series with the result being physically more intuitive. For instance, after performing the contour integrations of Eqs. (11), one can obtain the equilibrium position dispersion for the Drude model, and the average potential energy can be written in terms of the bosonic Matsubara frequencies $\nu_n = \frac{2\pi n}{\hbar\beta}$ (with $\beta = 1/k_BT$) as

$$E_p = \frac{3}{2\beta} + \frac{2\omega_0^2}{\beta}\sum_{n=1}^{\infty}\frac{A_n}{A_n^2 + (\omega_c\nu_n)^2} + \frac{\omega_0^2}{\beta}\sum_{n=1}^{\infty}\frac{1}{A_n}, \tag{55}$$

where $A_n = \nu_n^2 + \omega_0^2 + \frac{\nu_n\gamma_0\omega_{\mathrm{cut}}}{\nu_n+\omega_{\mathrm{cut}}}$ and $n = 1, 2, \ldots$. Thus, we are able to represent the average potential energy in terms of an infinite series, and some conclusions about E_p can be inferred from this form. The first term in Eq. (55) is the classical contribution, whereas the other two terms are of quantum mechanical origin. Using the fact that $\gamma(\nu_n) > 0$ and $A_n > 0$ for $n \geqslant 1$, all the terms under the sum are nonnegative, and hence, one can say that $3k_BT/2$ is the lower bound of the mean potential energy E_p. Therefore, the potential energy of a quantum charged dissipative magneto-oscillator is always greater than that of its classical counterpart. We observe that the terms under the sum are rational functions of five characteristic energies: k_BT, $\hbar\omega_c$, $\hbar\omega_0$, $\hbar\gamma_0$, and $\hbar\omega_{\mathrm{cut}}$. One can find that the numerator and denominator under the sum are the products of the energy to the power of four terms, except the energy term related to magnetic field $\hbar\omega_c$. Thus, it is easy to show that each term under the sum is a nonincreasing function with respect to magnetic field because it occurs only in the denominator. On the other hand, it can be demonstrated that partial

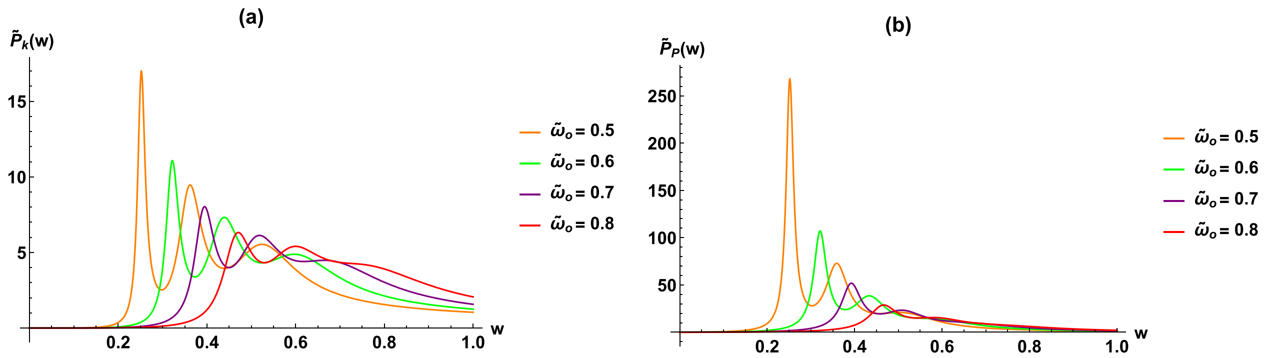

FIG. 6. Variation of (a) $\tilde{P}_k(w)$ and (b) $\tilde{P}_p(w)$ as a function of rescaled thermostat oscillator frequencies w for a radiation bath for selected values of $\tilde{\omega}_0$ while keeping $\tilde{\omega}_c = 0.5$.

derivatives of the terms under the sum with respect to γ_0, ω_{cut}, and ω_0 are nonnegative, and it follows that all terms are nondecreasing with respect to these variables. Thus, one may conclude that E_p is a nondecreasing function of γ_0, ω_{cut}, and ω_0 and a nonincreasing function of the magnetic field $\mathbf{B}$. All these properties are inferred straightforwardly from Eq. (55). We plot the dimensionless potential energy $\tilde{E}_p = \beta E_p$ as a function of the rescaled magnetic field $\tilde{\omega}_c = \omega_c/\omega_{\text{cut}}$ in Fig. 7. The dependence of the mean potential energy of the oscillator on the external magnetic field is clearly demonstrated. As anticipated, the mean potential energy of the oscillator is a nonincreasing function of ω_c.

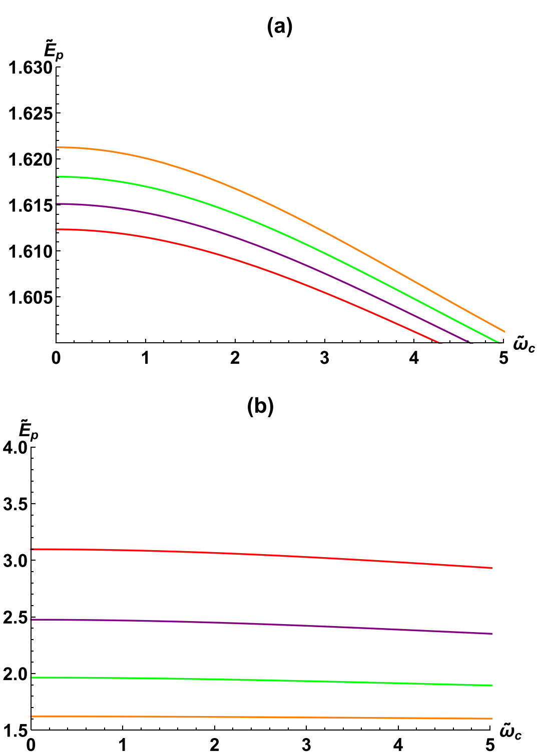

FIG. 7. Variation of $\tilde{E}_p = \beta E_p$ as a function of the rescaled magnetic field $\tilde{\omega}_c = \omega_c/\omega_{\text{cut}}$ with (a) $\omega_0/\omega_{\text{cut}} = 1$, $\beta\hbar\omega_{\text{cut}} = 1$ for different values of $a = \gamma_0/\omega_{\text{cut}}$ and (b) $\gamma_0/\omega_{\text{cut}} = 1$, $\beta\hbar\omega_{\text{cut}} = 1$ for different values of $\tilde{\omega}_0 = \omega_0/\omega_{\text{cut}}$.

Let us now move to the kinetic energy. The kinetic energy of the particle can be represented in a series form by completing the contour integrations of Eq. (12), thus giving

$$E_k = \frac{3}{2\beta} + \frac{2}{\beta}\sum_{n=1}^{\infty}\frac{A_n \times B_n + (\omega_c \nu_n)^2}{A_n^2 + (\omega_c \nu_n)^2} + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{B_n}{A_n}, \tag{56}$$

where $B_n = \omega_0^2 + \frac{\nu_n \gamma_0 \omega_{\text{cut}}}{\nu_n + \omega_{\text{cut}}}$. Since $\gamma(\nu_n) > 0$, $A_n > 0$, and $B_n > 0$ for $n \geqslant 1$, all terms under the sum are nonnegative. Therefore, the lower bound of the kinetic energy is $3k_BT/2$, and the kinetic energy of the quantum charged magneto-oscillator is always greater than that of the classical version. Once again, kinetic energy is a function of the five characteristic energies discussed above for the potential energy. But there is a major difference between the potential energy and kinetic energy. One may observe that all five characteristic energies appear both in the numerator and denominator under the sum for the kinetic energy. Thus, it can easily be shown that each term under the sum is a nondecreasing function with respect to all five parameters (ω_0, ω_c, γ_0, ω_{cut}, T) related to the system under consideration. To demonstrate the effect of the external magnetic field, we plot the dimensionless kinetic energy $\tilde{E}_k = \beta E_k$ versus the rescaled magnetic field $\tilde{\omega}_c = \omega_c/\omega_{\text{cut}}$ in Fig. 8. The fact that E_k is strongly influenced by the external magnetic field and that it is a nondecreasing function of ω_c is clear from the plots.

V. CONCLUSIONS

Considering a paradigmatic model of dissipative diamagnetism, we formulated and investigated the quantum counterpart of the classical equipartition theorem. Our model system is quite well studied and is close to the realistic three-dimensional dissipative diamagnetism [43,44]. However, unlike most of the previous studies [1–6], we greatly emphasized linear-response theory and the fluctuation-dissipation theorem for this archetype model of dissipative diamagnetism which is formulated in terms of the generalized quantum Langevin equation for a charged oscillator moving under an external magnetic field and interacting with a large number of independent oscillators that form a thermal reservoir. As a result, our investigation revealed that the quantum probability distribution functions related to the average kinetic and potential

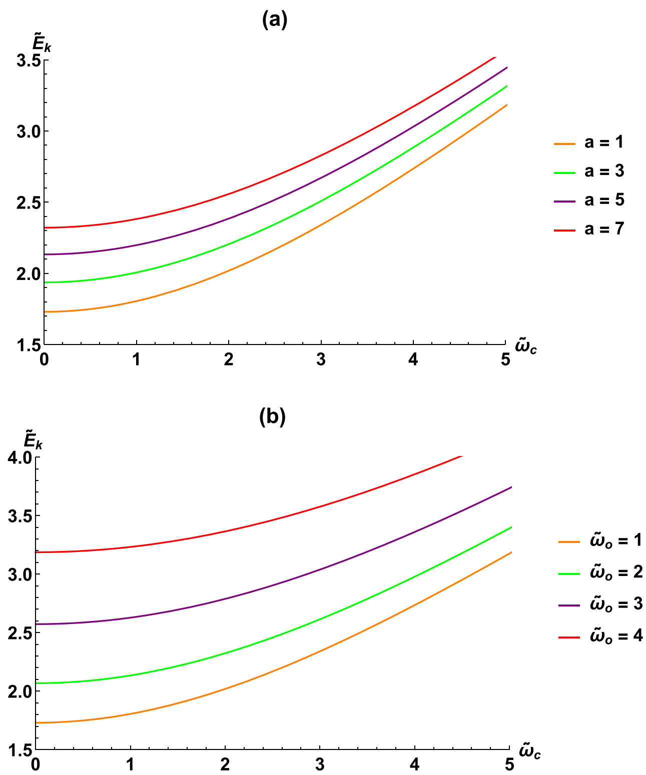

FIG. 8. Variation of $\tilde{E}_k = \beta E_k$ as a function of the rescaled magnetic field $\tilde{\omega}_c = \omega_c/\omega_{\rm cut}$ with (a) $\omega_0/\omega_{\rm cut} = 1$, $\beta\hbar\omega_{\rm cut} = 1$ for different values of $a = \gamma_0/\omega_{\rm cut}$ and (b) $\gamma_0/\omega_{\rm cut} = 1$, $\beta\hbar\omega_{\rm cut} = 1$ for different values of $\tilde{\omega}_0 = \omega_0/\omega_{\rm cut}$.

energies are closely related to the generalized susceptibility tensor, which is an experimentally measurable quantity. Thus, our method opens up a way to study different aspects of open quantum systems in an experimental setting. The quantum probability distributions related to the quantum equipartition theorem characterize the properties of the quantum environment and its coupling to a given quantum system, and hence, they may be experimentally surmised from the measurement of the linear response of the system to an applied perturbation, for instance, electrical or magnetic. In particular, our model system opens a pathway to investigate the influence of various dissipation mechanisms, external magnetic field, the confining potential strength, and the memory time on the average kinetic energy E_k and potential energy E_p of the charged dissipative magneto-oscillator in three dimensions. In this respect, we have reinvestigated the recently formulated quantum law for the partition of energy for dissipative diamagnetism in three dimensions. We have shown that mean kinetic energy E_k and the mean potential energy E_p of the dissipative charged magneto-oscillator can be expressed as $E_{i=k,p} = \langle \mathcal{E}_{i=k,p}(\omega)\rangle$, where (besides Gibbsian state distribution of thermostat oscillators) a second averaging over the frequencies ω of the bath oscillators is performed according to the probability distributions $\mathcal{P}_{i=k,p}(\omega)$. The latter one is strongly influenced by the dissipation kernel $\mu(t)$, external magnetic field ω_c, confining potential ω_0, and memory time τ_c.

Our primary focus has been to demonstrate the influence of the form of the dissipation function (via the spectral density function) and magnetic field on the characteristic features of the probability density $\mathcal{P}_{i=k,p}(\omega)$. For this purpose we considered three (Gaussian decay, Drude, and radiation) different dissipation mechanisms which are relevant for the dissipative diamagnetism. By considering variations of the external magnetic field, we have observed several basic features of $\mathcal{P}_{i=k,p}(\omega)$ for all three dissipation mechanisms, and they can be summarized as follows: (i) For a sufficiently low magnetic field regime one can always find a bell-shaped probability distribution $\mathcal{P}_{i=k,p}(\omega)$. This implies that there is an optimal oscillator frequency which makes the highest contribution to the mean kinetic or potential energy of the dissipative charged magneto-oscillator. Although the values of this optimum frequency are highly sensitive to the parameters of the relevant system, they are different for the kinetic energy and the potential energy distributions. (ii) As we increase the magnetic field, the single peaked distributions divide into trimodal distributions. With the increase in magnetic field, the trimodal distribution $\mathcal{P}_p(\omega)$ shifts towards the low frequency regime.

We have also investigated the influence of the coupling strength γ_0 on the shape of the distribution $\mathcal{P}_{i=k,p}(\omega)$. For large values of γ_0 or strong dissipation, the probability distribution functions $\mathcal{P}_{i=k,p}(\omega)$ are usually flat in nature, which implies all thermostat oscillators contribute in a rather homogeneous way to the respective energies of the charged magneto-oscillator. However, in the weak dissipation case (γ_0 small) the distribution functions are noticeably peaked around the three most probable values. The high frequency thermostat oscillators contribute a much higher amount of energy to E_k, but the major contributions to E_p come from the slowly vibrating thermostat oscillators for this weak dissipation case. We considered the effect of memory time τ_c on the shape of the relevant probability distribution functions. In general, we find that if the memory time τ_c is short compared to the dissipation timescale τ_ν, the probability densities show three pronounced peaks, whereas for a large τ_c the distribution functions are almost flat.

To summarize, it should be noted that we have connected the problem of the quantum law for energy partition with a realistic three-dimensional system of dissipative diamagnetism and related the probability distribution functions $\mathcal{P}_{i=k,p}(\omega)$ to an experimentally measurable quantity, i.e., the generalized susceptibility tensor. As a result our method is a conceptually simple yet very powerful tool for the analysis of quantum open systems. We hope that our work will stimulate further successful applications in this active area and that the present method will open new doorways of experimental verification of this quantum equipartition theorem.

ACKNOWLEDGMENTS

J.K. acknowledges the financial support received from IIT Bhubaneswar in the form of an Institute Research Fellowship. The work of A.G. is supported by the Ministry of Human Resource Development, Government of India, in the form of a Prime Minister's Research Fellowship. M.B. gratefully acknowledges financial support from Department of Science and Technology (DST), India, under the Core Grant (Project No. CRG/2020//001768). The authors are grateful to the anonymous referees for their valuable comments, which led to an improvement of the article.

[1] P. Bialas and J. Łuczka, Kinetic energy of a free quantum Brownian particle, Entropy **20**, 123 (2018).

[2] J. Spiechowicz, P. Bialas, and J. Łuczka, Quantum partition of energy for a free Brownian particle: Impact of dissipation, Phys. Rev. A **98**, 052107 (2018).

[3] P. Bialas, J. Spiechowicz, and J. Łuczka, Partition of energy for a dissipative quantum oscillator, Sci. Rep. **8**, 16080 (2018).

[4] P. Bialas, J. Spiechowicz, and J. Łuczka, Quantum analogue of energy equipartition theorem, J. Phys. A **52**, 15LT01 (2019).

[5] J. Łuczka, Quantum counterpart of classical equipartition of energy, J. Stat. Phys. **179**, 839 (2020).

[6] J. Spiechowicz and J. Łuczka, Energy of a free Brownian particle coupled to thermal vacuum, Sci. Rep. **11**, 4088 (2021).

[7] J. Waterston and J. W. Strutt, On the physics of media that are composed of free and perfectly elastic molecules in a state of motion, Philos. Trans. R. Soc. A **183**, 1 (1892).

[8] L. Boltzmann, Ìber die Natur der Gasmoleküle, Wien. Ber. **74**, 553 (1876).

[9] J. C. Maxwell, Illustrations of the dynamical theory of gases, in *The Scientific Papers of James Clerk Maxwell*, edited by W. D. Niven (Dover, New York, 1890), Vol. 1, pp. 377–409.

[10] P. Ullersma, An exactly solvable model for Brownian motion: IV. Susceptibility and Nyquist's theorem, Physica (Amsterdam) **32**, 27 (1966).

[11] G. W. Ford, M. Kac, and P. Mazur, Statistical mechanics of assemblies of coupled oscillators, J. Math. Phys. **6**, 504 (1965).

[12] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Quantum Oscillator in a Blackbody Radiation Field, Phys. Rev. Lett. **55**, 2273 (1985).

[13] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Quantum oscillator in a blackbody radiation field II. Direct calculation of the energy using the fluctuation-dissipation theorem, Ann. Phys. (NY) **185**, 270 (1988).

[14] G. W. Ford and M. Kac, On the quantum Langevin equation, J. Stat. Phys. **46**, 803 (1987).

[15] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Quantum Langevin equation, Phys. Rev. A **37**, 4419 (1988).

[16] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002).

[17] P. Massignan, A. Lampo, J. Wehr, and M. Lewenstein, Quantum Brownian motion with inhomogeneous damping and diffusion, Phys. Rev. A **91**, 033627 (2015).

[18] P. Hanggi and G.-L. Ingold, Quantum Brownian motion and the third law of thermodynamics, Acta Phys. Pol. B **37**, 1537 (2006).

[19] D. Boyanovsky and D. Jasnow, Heisenberg-Langevin versus quantum master equation, Phys. Rev. A **96**, 062108 (2017).

[20] H. Z. Shen, S. L. Su, Y. H. Zhou, and X. X. Yi, Non-Markovian quantum Brownian motion in one dimension in electric fields, Phys. Rev. A **97**, 042121 (2018).

[21] M. Bandyopadhyay, Quantum thermodynamics of a charged magneto-oscillator coupled to a heat bath, J. Stat. Mech. (2009) P05002.

[22] S. Gupta and M. Bandyopadhyay, Quantum Langevin equation of a charged oscillator in a magnetic field and coupled to a heat bath through momentum variables, Phys. Rev. E **84**, 041133 (2011).

[23] M. Bandyopadhyay, Dissipative cyclotron motion of a charged quantum-oscillator and third law, J. Stat. Phys. **140**, 603 (2010).

[24] L. Landau, Diamagnetismus der Metalle, Z. Phys. **64**, 629 (1930)

[25] J. H. van Vleck, *The Theory of Electric and Magnetic Susceptibilities* (Oxford University Press, London, 1932).

[26] S. Dattagupta, A. M. Jayannavar, and N. Kumar, Landau diamagnetism revisited, Curr. Sci. **80**, 861 (2001).

[27] M. Bandyopadhyay, Dissipative tunnelling in 2DEG: Effect of magnetic field, impurity and temperature, J. Stat. Mech. (2006) L03001.

[28] M. Bandyopadhyay, Orbital magnetism of two-dimensional electron gas in a crossed electromagnetic field: The effect of spin-orbit interaction, confined geometries and defects, J. Stat. Mech. (2006) P10010.

[29] R. B. Laughlin, Quantized Hall conductivity in two dimensions, Phys. Rev. B **23**, 5632 (1981).

[30] K. V. Klitzing, G. Dorda, and M. Pepper, New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance, Phys. Rev. Lett. **45**, 494 (1980).

[31] H. Friedrich and D. Wintgen, The hydrogen atom in a uniform magnetic field—An example of chaos, Phys. Rep. **183**, 37 (1989).

[32] L. Jacak, P. Hawrylak, and A. Wojs, *Quantum Dots* (Springer, Berlin, 1997).

[33] B. Tanatar and D. M. Ceperley, Ground state of the two-dimensional electron gas, Phys. Rev. B **39**, 5005 (1989).

[34] M. G. Vavilov and I. L. Aleiner, Magnetotransport in a two-dimensional electron gas at large filling factors, Phys. Rev. B **69**, 035303 (2004).

[35] J. W. Park, D. F. Bogorin, C. Cen, D. A. Felker, Y. Zhang, C. T. Nelson, C. W. Bark, C. M. Folkman, X. Q. Pan, M. S. Rzchowski, J. Levy, and C. B. Eom, Creation of a two-dimensional electron gas at an oxide interface on silicon, Nat. Commun. **1**, 94 (2010).

[36] X. L. Li, G. W. Ford, and R. F. O'Connell, Magnetic-field effects on the motion of a charged particle in a heat bath, Phys. Rev. A **41**, 5287 (1990).

[37] X. L. Li, G. W. Ford, and R. F. O'Connell, Charged oscillator in a heat bath in the presence of a magnetic field, Phys. Rev. A **42**, 4519 (1990).

[38] X. L. Li and R. F. O'Connell, Green's function and position correlation function for a charged oscillator in a heat bath and a magnetic field, Phys. A (Amsterdam, Neth.) **224**, 639 (1996).

[39] X. L. Li, G. W. Ford, and R. F. O'Connell, Dissipative effects on the localization of a charged oscillator in a magnetic field, Phys. Rev. E **53**, 3359 (1996).

[40] Y. Subası, C. H. Fleming, J. M. Taylor, and B. L. Hu, Equilibrium states of open quantum systems in the strong coupling regime, Phys. Rev. E **86**, 061132 (2012).

[41] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Butterworth-Heinemann, 1980), Part 1.

[42] *Tables of Integral Transforms, Based, in Part, on Notes Left by Harry Bateman and Compiled by the Staff of the Bateman Manuscript Project*, edited by A. Erdelyi, M. F. Oberhettinger, and F. G. Tricomi (McGraw-Hill, New York, 1954), Vol. 1.

[43] M. Bandyopadhyay and S. Dattagupta, Landau-Drude diamagnetism: Fluctuation, dissipation and decoherence, J. Phys.: Condens. Matter **18**, 10029 (2006).

[44] M. Bandyopadhyay and S. Dattagupta, Dissipative diamagnetism—A case study for equilibrium and nonequilibrium statistical mechanics, J. Stat. Phys. **123**, 1273 (2006).