

Group synchrony, parameter mismatches, and intragroup connectionsShirin Panahi  and Francesco Sorrentino *University of New Mexico, Albuquerque New Mexico 87106, USA* (Received 28 September 2021; accepted 5 November 2021; published 30 November 2021)

Group synchronization arises when two or more synchronization patterns coexist in a network formed of oscillators of different types, with the systems in each group synchronizing on the same time evolution, but systems in different groups synchronizing on distinct time evolutions. Group synchronization has been observed and characterized when the systems in each group are identical and the couplings between the systems satisfy specific conditions. By relaxing these constraints and allowing them to be satisfied in an approximate rather than exact way, we observe that stable group synchronization may still occur in the presence of small deviations of the parameters of the individual systems and of the couplings from their nominal values. We analyze this case and provide necessary and sufficient conditions for stability through a master stability function approach, which also allows us to quantify the synchronization error. We also investigate the stability of group synchronization in the presence of intragroup connections and for this case extend some of the existing results in the literature. Our analysis points out a broader class of matrices describing intragroup connections for which the stability problem can be reduced in a low-dimensional form.

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A large literature has considered synchronization in networks of coupled oscillators, see for example the review in Ref. [1]. In the case of identical oscillators, stability of the synchronous solution for arbitrary network topologies can be investigated through the master stability function approach, introduced in Ref. [2]. More complex patterns of synchronous dynamics may arise in networks formed of different types of oscillators. Group synchronization in networks of coupled oscillators was first analyzed in Ref. [3], followed by further analysis in Ref. [4] and an experimental realization of group synchronization [5]. Moreover, group synchronization in networks of fractional-order chaotic oscillators was investigated in Ref. [6].

According to the definition in Ref. [3], a *group* is a set of oscillators characterized by the same (uncoupled) dynamics, with oscillators of different types belonging to different groups. Group synchronization is achieved when the oscillators in each group synchronize on the same time evolution, with these synchronized dynamics being different from group to group. Cluster synchronization can be considered as a particular case of group synchronization but for which the oscillators in different clusters are of the same type. Most recently, a canonical transformation for simultaneous block diagonalization of matrices has been proposed to decouple the cluster synchronization stability problem into subproblems of minimal dimensionality [7]. The more general case of cluster synchronization in multilayer networks formed of oscillators of different types has been studied in Ref. [8].

An experimental realization of group synchronization was performed in Ref. [5], where stable group synchronization was observed under different configurations for a

small network of four coupled optoelectronic oscillators. This observation suggests that group synchronization is robust with respect to small parametric mismatches that are inevitable in experiments. The analysis of stability of the group-synchronous solution in the presence of such small parametric mismatches (affecting both the individual nodes being coupled and the strengths of the couplings between them) is studied in this paper. The effects of parameter mismatches in a network of *identical* oscillators achieving complete synchronization have been studied in Refs. [9–11] and achieving cluster synchronization in Refs. [12,13]. A second-order expansion to study complete synchronization in networks of identical nodes with parameter mismatches has been studied in Refs. [14,15]. However, to the best of our knowledge, no paper has investigated the role of parameter mismatches in networks formed of systems of different types, which is the main focus of this paper. The other subject of this paper is the study of the conditions under which group synchronization may arise in the presence of intragroup connections, which was first considered in Ref. [4].

II. GROUP SYNCHRONIZATION IN THE PRESENCE OF PARAMETRIC MISMATCHES

Though our results can be generalized to the case of an arbitrary number of groups (for the case in which the stability of the group-synchronous solution can be reduced in a low-dimensional form [4]), for simplicity in what follows we will focus on the case of two groups. For the time being, we also maintain the assumption introduced in Ref. [3] that the network topology is bipartite, i.e., connections exist only from group X to group Y and vice versa (this assumption will be removed later in Sec. III).

We first present the ideal case that parameter mismatches are absent. For this very special case, the dynamical equations are as follows:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i, \mathbf{u}_i^x) = \mathbf{F}\left(\mathbf{x}_i, \sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} \mathbf{H}(\mathbf{y}_j)\right), \quad i = 1, \dots, N_x. \quad (1a)$$

$$\dot{\mathbf{y}}_j = \mathbf{G}(\mathbf{y}_j, \mathbf{u}_j^y) = \mathbf{G}\left(\mathbf{y}_j, \sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} \mathbf{L}(\mathbf{x}_i)\right), \quad j = 1, \dots, N_y. \quad (1b)$$

where \mathbf{x}_i (\mathbf{y}_j) is an n_x -dimensional (n_y -dimensional) state vector of systems in group X and in group Y , respectively, $\mathbf{u}_i^x = \sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} \mathbf{H}(\mathbf{y}_j)$ [$\mathbf{u}_j^y = \sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} \mathbf{L}(\mathbf{x}_i)$] is the input received by node i in group X (j in group Y). The functions $\mathbf{F} : \mathbf{R}^{n_x} \times \mathbf{R}^{n_x} \rightarrow \mathbf{R}^{n_x}$ and $\mathbf{G} : \mathbf{R}^{n_y} \times \mathbf{R}^{n_y} \rightarrow \mathbf{R}^{n_y}$ define the time evolution of the systems in group X and group Y , respectively, and the interaction functions $\mathbf{H} : \mathbf{R}^{n_y} \rightarrow \mathbf{R}^{n_x}$ and $\mathbf{L} : \mathbf{R}^{n_x} \rightarrow \mathbf{R}^{n_y}$ define the output of systems in group Y and group X , respectively. N_x is the number of systems in group X and N_y is the number of systems in group Y . In what follows we assume without loss of generality that $N_x \geq N_y$. The input \mathbf{u}_i^x received by node i in group X (\mathbf{u}_j^y received by node j in group Y) corresponds to a superposition of the outputs from the nodes in group Y (group X) through the coefficients A_{ij}^{NOM} (B_{ji}^{NOM}). A_{ij}^{NOM} is an $N_x \times N_y$ coupling matrix, whose entries A_{ij}^{NOM} represent the nominal strength of the direct interaction from system j in the Y group to system i in the X group. Analogously, B_{ji}^{NOM} is an $N_y \times N_x$ coupling matrix, whose entries B_{ji}^{NOM} represent the nominal strength of the direct interaction from system i in the X group to j in the Y group. Note that Eqs. (1) are a generalization of those considered in Refs. [3,4].

In what follows, we consider the following two types of small parameter mismatches from nominal conditions: (i) deviations from identity of the uncoupled dynamics of the individual systems in each group and (ii) deviations from nominal conditions of the coupling strengths between systems.

If the following conditions hold:

$$\sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} = a \neq 0, \quad i = 1, \dots, N_x, \quad (2a)$$

$$\sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} = b \neq 0, \quad j = 1, \dots, N_y, \quad (2b)$$

then the synchronous solution exists,

$$\dot{\mathbf{x}}_s = \mathbf{F}(\mathbf{x}_s, a\mathbf{H}(\mathbf{y}_s)), \quad (3a)$$

$$\dot{\mathbf{y}}_s = \mathbf{G}(\mathbf{y}_s, b\mathbf{L}(\mathbf{x}_s)). \quad (3b)$$

We note here that by replacing $a\mathbf{H} \rightarrow \mathbf{H}$ and $A^{\text{NOM}}/a \rightarrow A^{\text{NOM}}$ ($b\mathbf{L} \rightarrow \mathbf{L}$ and $B^{\text{NOM}}/b \rightarrow B^{\text{NOM}}$), it is always possible to set $a = b = 1$. Hence, without loss of generality, in what follows we set $a = b = 1$. This has the significant advantage that the synchronous solution, which obeys Eqs. (3) with $a =$

$b = 1$, is independent of the particular choice of the bipartite network.

Stability of this synchronized solution can be reduced in a master stability form [3]. Unfortunately, exact satisfaction of condition (2) is difficult to implement in experiments and in real-world situations. The other requirement whose exact satisfaction is hardly accomplished in experiments is that the evolution of all the systems in the X group (in the Y group) is exactly characterized by the same function \mathbf{F} (\mathbf{G}). What can be realistically achieved is that $\mathbf{F}(\mathbf{x}_i, \mathbf{u}_i^x) \rightarrow \mathbf{F}(\mathbf{x}_i, \mathbf{u}_i^x, \mu_i)$, $i = 1, \dots, N_x$ in Eq. (1a) and $\mathbf{G}(\mathbf{y}_j, \mathbf{u}_j^y) \rightarrow \mathbf{G}(\mathbf{y}_j, \mathbf{u}_j^y, \nu_j)$, $j = 1, \dots, N_y$ in Eq. (1b) where μ_i and ν_j are scalar parameters that slightly vary from oscillator to oscillator.¹ Analogously, it is realistic to assume that the couplings A_{ij} and B_{ji} are affected by small mismatches with respect to their nominal values,

$$A_{ij} = A_{ij}^{\text{NOM}} + \delta A_{ij}, \quad (4a)$$

$$B_{ji} = B_{ji}^{\text{NOM}} + \delta B_{ji}, \quad (4b)$$

where δA_{ij} and δB_{ji} are small deviations. Under these assumptions, Eqs. (1) become

$$\dot{\mathbf{x}}_i = \mathbf{F}\left(\mathbf{x}_i, \sum_{j=1}^{N_y} A_{ij} \mathbf{H}(\mathbf{y}_j), \mu_i\right), \quad i = 1, \dots, N_x, \quad (5a)$$

$$\dot{\mathbf{y}}_j = \mathbf{G}\left(\mathbf{y}_j, \sum_{i=1}^{N_x} B_{ji} \mathbf{L}(\mathbf{x}_i), \nu_j\right), \quad j = 1, \dots, N_y, \quad (5b)$$

which describe the dynamics of the two groups under realistic circumstances, such as those that would be observed in any experimental setting.

We write $\mu_i = \bar{\mu} + \delta\mu_i$ where $\bar{\mu} = (N_x)^{-1} \sum_{i=1}^{N_x} \mu_i$ and $\delta\mu_i$ is a small deviation. Similarly, we write $\nu_j = \bar{\nu} + \delta\nu_j$ where $\bar{\nu} = (N_y)^{-1} \sum_{j=1}^{N_y} \nu_j$ and $\delta\nu_j$ is a small deviation. Note that by construction $\sum_{i=1}^{N_x} \delta\mu_i = 0$ and $\sum_{j=1}^{N_y} \delta\nu_j = 0$.

We can also write

$$\sum_{j=1}^{N_y} A_{ij} = a + \sum_{j=1}^{N_y} \delta A_{ij} = a + \delta\bar{a} + \delta a_i, \quad (6a)$$

$$\sum_{i=1}^{N_x} B_{ji} = b + \sum_{i=1}^{N_x} \delta B_{ji} = b + \delta\bar{b} + \delta b_j, \quad (6b)$$

where

$$\delta\bar{a} = N_x^{-1} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \delta A_{ij}, \quad (7a)$$

$$\delta\bar{b} = N_y^{-1} \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} \delta B_{ji}, \quad (7b)$$

¹Even though we do not consider the case of two or more parameters that vary from system to system, the extension of our results to this case is straightforward.

are the average sums of the rows of the matrices A and B and

$$\delta a_i = \left(\sum_{j=1}^{N_y} \delta A_{ij} \right) - \delta \bar{a} = \left(\sum_{j=1}^{N_y} \delta A_{ij} \right) - N_x^{-1} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \delta A_{ij}, \quad (8a)$$

$$\delta b_j = \left(\sum_{i=1}^{N_x} \delta B_{ji} \right) - \delta \bar{b} = \left(\sum_{i=1}^{N_x} \delta B_{ji} \right) - N_y^{-1} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \delta B_{ji}, \quad (8b)$$

are small deviations. The deviations δa_i and δb_j are calculated with respect to the average row-sums $\delta \bar{a}$ and $\delta \bar{b}$ (hence they sum to zero), that is, $\sum_{i=1}^{N_x} \delta a_i = 0$ and $\sum_{j=1}^{N_y} \delta b_j = 0$.

Unfortunately, for the case of Eqs. (5), different from the case of Eqs. (1), an exact synchronous solution does not exist. It is possible, however, that the individual trajectories stabilize in a nearly synchronous state, where the trajectories of the nodes in the X group (and in the Y group) remain close to each other, i.e., for which both

$$\delta \mathbf{x}_i(t) = [\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)], \quad (9a)$$

$$\delta \mathbf{y}_j(t) = [\mathbf{y}_j(t) - \bar{\mathbf{y}}(t)], \quad (9b)$$

remain small in time, where $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{y}}(t)$ are the average solutions,

$$\bar{\mathbf{x}}(t) = (N_x^{-1}) \sum_{i=1}^{N_x} \mathbf{x}_i(t), \quad (10a)$$

$$\bar{\mathbf{y}}(t) = (N_y^{-1}) \sum_{j=1}^{N_y} \mathbf{y}_j(t), \quad (10b)$$

obeying

$$\dot{\bar{\mathbf{x}}} = (N_x)^{-1} \sum_{i=1}^{N_x} \mathbf{F} \left(\mathbf{x}_i, \sum_{j=1}^{N_y} A_{ij} \mathbf{H}(\mathbf{y}_j), \mu_i \right), \quad (11a)$$

$$\dot{\bar{\mathbf{y}}} = (N_y)^{-1} \sum_{j=1}^{N_y} \mathbf{G} \left(\mathbf{y}_j, \sum_{i=1}^{N_x} B_{ji} \mathbf{L}(\mathbf{x}_i), \nu_j \right). \quad (11b)$$

By differentiating (9) with respect to time, we obtain

$$\delta \dot{\mathbf{x}}_i = \left[\mathbf{F} \left(\mathbf{x}_i, \sum_{j=1}^{N_y} A_{ij} \mathbf{H}(\mathbf{y}_j), \mu_i \right) - (N_x)^{-1} \sum_{i=1}^{N_x} \mathbf{F} \left(\mathbf{x}_i, \sum_{j=1}^{N_y} A_{ij} \mathbf{H}(\mathbf{y}_j), \mu_i \right) \right], \quad i = 1, \dots, N_x, \quad (12a)$$

$$\delta \dot{\mathbf{y}}_j = \left[\mathbf{G} \left(\mathbf{y}_j, \sum_{i=1}^{N_x} B_{ji} \mathbf{L}(\mathbf{x}_i), \nu_j \right) - (N_y)^{-1} \sum_{j=1}^{N_y} \mathbf{G} \left(\mathbf{y}_j, \sum_{i=1}^{N_x} B_{ji} \mathbf{L}(\mathbf{x}_i), \nu_j \right) \right], \quad j = 1, \dots, N_y. \quad (12b)$$

By expanding to first order the function \mathbf{F} in (12) about the point $(\bar{\mathbf{x}}, \mathbf{H}(\bar{\mathbf{y}}), \bar{\mu})$ and the function \mathbf{G} in (12b) about the point $(\bar{\mathbf{y}}, \mathbf{L}(\bar{\mathbf{x}}), \bar{\nu})$, we obtain

$$\begin{aligned} \mathbf{F} \left(\mathbf{x}_i, \sum_{j=1}^{N_y} A_{ij} \mathbf{H}(\mathbf{y}_j), \mu_i \right) &\simeq \mathbf{F} \left(\bar{\mathbf{x}}, \sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} \mathbf{H}(\bar{\mathbf{y}}), \bar{\mu} \right) + D\mathbf{F}_x \left(\bar{\mathbf{x}}, \sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} \mathbf{H}(\bar{\mathbf{y}}), \bar{\mu} \right) \delta \mathbf{x}_i \\ &\quad + D\mathbf{F}_u \left(\bar{\mathbf{x}}, \sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} \mathbf{H}(\bar{\mathbf{y}}), \bar{\mu} \right) \left[\sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} D\mathbf{H}(\bar{\mathbf{y}}) \delta \mathbf{y}_j + \sum_{j=1}^{N_y} \mathbf{H}(\bar{\mathbf{y}}) \delta A_{ij} \right] \\ &\quad + D\mathbf{F}_\mu \left(\bar{\mathbf{x}}, \sum_{j=1}^{N_y} A_{ij}^{\text{NOM}} \mathbf{H}(\bar{\mathbf{y}}), \bar{\mu} \right) \delta \mu_i \\ \mathbf{G} \left(\mathbf{y}_j, \sum_{i=1}^{N_x} B_{ji} \mathbf{L}(\mathbf{x}_i), \nu_j \right) &\simeq \mathbf{G} \left(\bar{\mathbf{y}}, \sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} \mathbf{L}(\bar{\mathbf{x}}), \bar{\nu} \right) + D\mathbf{G}_y \left(\bar{\mathbf{y}}, \sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} \mathbf{L}(\bar{\mathbf{x}}), \bar{\nu} \right) \delta \mathbf{y}_j \\ &\quad + D\mathbf{G}_u \left(\bar{\mathbf{y}}, \sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} \mathbf{L}(\bar{\mathbf{x}}), \bar{\nu} \right) \left[\sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} D\mathbf{L}(\bar{\mathbf{x}}) \delta \mathbf{x}_i + \sum_{i=1}^{N_x} \mathbf{L}(\bar{\mathbf{x}}) \delta B_{ji} \right] \\ &\quad + D\mathbf{G}_\nu \left(\bar{\mathbf{y}}, \sum_{i=1}^{N_x} B_{ji}^{\text{NOM}} \mathbf{L}(\bar{\mathbf{x}}), \bar{\nu} \right) \delta \nu_j. \end{aligned}$$

Then Eqs. (12) can be rewritten,

$$\delta \dot{\mathbf{x}}_i(t) = D\mathbf{F}_x^* \delta \mathbf{x}_i(t) + \sum_{j=1}^{N_y} \tilde{A}_{ij} D\mathbf{F}_u^* D\mathbf{H}_y(\bar{\mathbf{y}}) \delta \mathbf{y}_j + \mathbf{U}_i^x, \quad i = 1, \dots, N_x, \quad (13a)$$

$$\delta \dot{\mathbf{y}}_j(t) = DG_y^* \delta \mathbf{y}_j(t) + \sum_{i=1}^{N_x} \tilde{B}_{ji} DG_u^* DL_x(\bar{\mathbf{x}}) \delta \mathbf{x}_i + \mathbf{U}_j^y, \quad j = 1, \dots, N_y, \quad (13b)$$

where $\mathbf{U}_i^x = DF_u^* \mathbf{H}(\bar{\mathbf{y}}) \delta a_i + DF_\mu^* \delta \mu_i$ and $\mathbf{U}_j^y = DG_u^* \mathbf{L}(\bar{\mathbf{x}}) \delta b_j + DG_v^* \delta v_j$, and the superscript * denotes that the partial derivatives of the functions \mathbf{F} and \mathbf{G} are evaluated about $(\bar{\mathbf{x}}, H(\bar{\mathbf{y}}), \bar{\boldsymbol{\mu}})$ and $(\bar{\mathbf{y}}, L(\bar{\mathbf{x}}), \bar{\mathbf{v}})$, respectively. The matrices $\tilde{\mathbf{A}} = \{\tilde{A}_{ij}\}$ and $\tilde{\mathbf{B}} = \{\tilde{B}_{ij}\}$ in (13) are defined as follows:

$$\tilde{A}_{ij} = (A_{ij}^{\text{NOM}} - a_j), \quad i = 1, \dots, N_x, \quad j = 1, \dots, N_y, \quad (14a)$$

$$\tilde{B}_{ji} = (B_{ji}^{\text{NOM}} - b_i), \quad i = 1, \dots, N_x, \quad j = 1, \dots, N_y, \quad (14b)$$

where $a_j = (N_x)^{-1} \sum_{i=1}^{N_x} A_{ij}^{\text{NOM}}$ is the mean value over the elements in the j th column of the matrix A^{NOM} and $b_i = (N_y)^{-1} \sum_{j=1}^{N_y} B_{ji}^{\text{NOM}}$ is the mean value over the elements in the i th column of the matrix B^{NOM} . Note that in order to obtain (13) we have used Eqs. (8) and the following properties $\sum_{i=1}^{N_x} \delta \mu_i = 0$, $\sum_{j=1}^{N_y} \delta v_j = 0$, $\sum_{i=1}^{N_x} \delta a_i = 0$, $\sum_{j=1}^{N_y} \delta b_j = 0$, $\sum_{i=1}^{N_x} \delta \mathbf{x}_i = 0$, and $\sum_{j=1}^{N_y} \delta \mathbf{y}_j = 0$. Note also that the two matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ have sums over both their columns and rows equal to zero.

In what follows, we will study stability of the system (13). If this system is found to be stable, then that implies that the trajectories (1) will tend to remain close to the average solution (10) [see the definition of the variations in (9)].

In what follows, we assume that for each connection from node i in group X to node j in group Y , there is a connection from node j in group Y to node i in group X and vice versa. We note that we must still satisfy the assumption (which we imposed at the beginning of the paper without loss of generality) that the sums over the entries in the rows of both matrices A^{NOM} and B^{NOM} are equal to 1. With these considerations in mind, we take $B^{\text{NOM}} = eA^{\text{NOM}T}$, where $e = N_y/N_x$. It then follows that $\tilde{\mathbf{B}} = e\tilde{\mathbf{A}}^T$.

We then rewrite Eq. (13) in vectorial form,

$$\delta \dot{\mathbf{x}} = [I_{N_x} \otimes DF_x^*] \delta \mathbf{x} + [\tilde{\mathbf{A}} \otimes DF_u^* DH_y(\bar{\mathbf{y}})] \delta \mathbf{y} + \mathbf{U}_x, \quad (15a)$$

$$\delta \dot{\mathbf{y}} = [I_{N_y} \otimes DG_y^*] \delta \mathbf{y} + [e\tilde{\mathbf{A}}^T \otimes DG_u^* DL_x(\bar{\mathbf{x}})] \delta \mathbf{x} + \mathbf{U}_y. \quad (15b)$$

By the substitution $\delta \mathbf{y} \leftarrow \delta \mathbf{y} / \sqrt{e}$, Eq. (15) becomes

$$\delta \dot{\mathbf{x}} = [I_{N_x} \otimes DF_x^*] \delta \mathbf{x} + [\sqrt{e}\tilde{\mathbf{A}} \otimes DF_u^* DH_y(\bar{\mathbf{y}})] \delta \mathbf{y} + \mathbf{U}_x, \quad (16a)$$

$$\delta \dot{\mathbf{y}} = [I_{N_y} \otimes DG_y^*] \delta \mathbf{y} + [\sqrt{e}\tilde{\mathbf{A}}^T \otimes DG_u^* DL_x(\bar{\mathbf{x}})] \delta \mathbf{x} + \mathbf{U}_y / \sqrt{e}. \quad (16b)$$

By stacking all perturbation vectors together in one vector $\mathbf{z} = [\delta \mathbf{x}^T, \delta \mathbf{y}^T]^T$ and by introducing the vector

$\mathbf{U} = [\mathbf{U}_x^T, \mathbf{U}_y^T / \sqrt{e}]^T$, Eq. (15) can be rewritten as

$$\begin{aligned} \dot{\mathbf{z}} = & \begin{pmatrix} I_{N_x} \otimes DF_x^* & 0 \\ 0 & 0_{N_y} \otimes DG_y^* \end{pmatrix} \mathbf{z} \\ & + \begin{pmatrix} 0_{N_x} \otimes DF_x^* & 0 \\ 0 & I_{N_y} \otimes DG_y^* \end{pmatrix} \mathbf{z} \\ & + \sqrt{e} \begin{pmatrix} 0 & \tilde{\mathbf{A}} \otimes DF_u^* DH_y(\bar{\mathbf{y}}) \\ \tilde{\mathbf{A}}^T \otimes DG_u^* DL_x(\bar{\mathbf{x}}) & 0 \end{pmatrix} \mathbf{z} + \mathbf{U}. \end{aligned} \quad (17)$$

We define the matrices,

$$E^x = \begin{pmatrix} I_{N_x} & 0 \\ 0 & 0_{N_y} \end{pmatrix}, \quad E^y = \begin{pmatrix} 0_{N_x} & 0 \\ 0 & I_{N_y} \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} 0_{n_x} & \tilde{\mathbf{A}} \\ \tilde{\mathbf{A}}^T & 0_{n_y} \end{pmatrix} \quad (18)$$

and seek to compute the $N_x + N_y$ -dimensional transformation matrix T that simultaneously block diagonalizes the matrices in Eq. (18),

$$T = SBD(E^x, E^y, \tilde{M}), \quad (19)$$

where here by SBD we indicate that the transformed matrices TE^xT^T , TE^yT^T , and $T\tilde{M}T^T$ are in the same block-diagonal form with blocks of minimal dimension [16–20]. To compute the matrix T we first need to find a matrix P that commutes with the symmetric matrices E^x , E^y , and \tilde{M} . The matrix T has for columns the eigenvectors of the matrix P [17]. In order for the matrix P to commute with the diagonal matrices E^x and E^y , it needs to be in the following block-diagonal form:

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \quad (20)$$

where the square block P_1 has dimension N_x and the square block P_2 has dimension N_y . Finally, the matrix P in (20) needs also to commute with \tilde{M} . Then the following two relations must be satisfied:

$$\tilde{\mathbf{A}}P_2 = P_1\tilde{\mathbf{A}}, \quad (21a)$$

$$\tilde{\mathbf{A}}^T P_1 = P_2\tilde{\mathbf{A}}^T. \quad (21b)$$

A simple solution to the above set of equations is given by

$$P_1 = \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T, \quad (22a)$$

$$P_2 = \tilde{\mathbf{A}}^T\tilde{\mathbf{A}}. \quad (22b)$$

From Eq. (20) it follows that the matrix T has the following block-diagonal structure:

$$T = \begin{pmatrix} T^x & 0 \\ 0 & T^y \end{pmatrix}, \quad (23)$$

where the columns of the matrix T^x are the eigenvectors of the matrix $\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T$ and the columns of the matrix T^y are the

eigenvectors of the matrix $\tilde{A}^T \tilde{A}$. As both matrices $\tilde{A} \tilde{A}^T$ and $\tilde{A}^T \tilde{A}$ are symmetric, the matrices T^x and T^y are orthogonal. Hence, T is also an orthogonal matrix.

We can then define the $N_x n_x + N_y n_y$ -dimensional orthogonal matrix

$$\hat{T} = (T^x \otimes I_{n_x}) \oplus (T^y \otimes I_{n_y}) \quad (24)$$

and use it to block-diagonalize Eq. (17). We introduce $\mathbf{w} = \hat{T} \mathbf{z}$, then we have:

$$\begin{aligned} \dot{\mathbf{w}} = & \begin{pmatrix} I_{N_x} \otimes D\mathbf{F}_x^* & 0 \\ 0 & I_{N_y} \otimes D\mathbf{G}_y^* \end{pmatrix} \mathbf{w} \\ & + \sqrt{e} \begin{pmatrix} 0 & S_1 \otimes D\mathbf{F}_u^* D\mathbf{H}_y(\bar{\mathbf{y}}) \\ S_1^T \otimes D\mathbf{G}_u^* D\mathbf{L}_x(\bar{\mathbf{x}}) & 0 \end{pmatrix} \mathbf{w} \\ & + T\mathbf{U}, \end{aligned} \quad (25)$$

where

$$S_1 = T^x \tilde{A} T^{yT}. \quad (26)$$

As mentioned, both T^x and T^y are orthogonal matrices, and hence $T^{xT} T^x = I_{N_x}$ and $T^{yT} T^y = I_{N_y}$. From (26), we see that S_1 is the N_x by N_y matrix whose entries $S_{1kk} = s_k$ are equal to the singular values of the matrix \tilde{A} (all the other entries are zero).

There are many possible numerical methods to compute the singular values of a matrix [21]. We note here that the eigenvalues of the matrix \tilde{M} in (18) are equal to $\{\pm s_1, \pm s_2, \dots, \pm s_r\} \cup \underbrace{\{0, 0, \dots, 0\}}_{N_x - N_y \text{ times}}$ [3]. From a computa-

tional point of view the method of computing the singular values from the eigenvalues of the matrix \tilde{M} is more stable than direct calculation of the matrix S_1 in (26).

We note that through the transformation $\mathbf{w} = \hat{T} \mathbf{z}$, each row of the matrix T^x , say, $T^{x\kappa}$, is associated with either a singular value s_κ or a zero value; analogously, each row of the matrix T^y , say, $T^{y\kappa}$, is associated with a singular value s_κ .

We then introduce the matrix:

$$Q = \begin{pmatrix} 0 & S_1 \\ S_1^T & 0 \end{pmatrix} = \left(\begin{array}{cccccc|cccc} & & & & & & s_1 & 0 & \cdots & 0 \\ & & & & & & 0 & s_2 & \cdots & 0 \\ & & & & & & 0 & 0 & \ddots & 0 \\ & & & & & & 0 & 0 & \cdots & s_r \\ & & & & & & 0 & 0 & \cdots & 0 \\ & & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & & 0 & 0 & \cdots & 0 \\ & & & & & & \hline s_1 & 0 & 0 & 0 & 0 & \cdots & 0 & & & \\ 0 & s_2 & 0 & 0 & 0 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & 0 & s_r & 0 & \cdots & 0 & & & 0 \end{array} \right). \quad (27)$$

Based on our assumption that $N_x \geq N_y$, we have that $r = N_y$. We call $d = (N_x - N_y)$, $d + r = N_x$.

Now by permuting rows and columns of matrix Q , we obtain the block-diagonal matrix,

$$\tilde{Q} = \left(\begin{array}{cccccc|cccc} \tilde{Q}_1 & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \tilde{Q}_2 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{Q}_r & & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \end{array} \right), \quad (28)$$

where each \tilde{Q}_i $i = 1, 2, \dots, r$ is a 2 by 2 block,

$$\tilde{Q}_\kappa = \begin{pmatrix} 0 & s_\kappa \\ s_\kappa & 0 \end{pmatrix} \quad \kappa = 1, 2, \dots, r. \quad (29)$$

One of the blocks $\kappa = 1$ corresponds to motion parallel to synchronization manifold and the other $r - 1$ blocks, $i = 2, \dots, r$ correspond to motion transverse to the synchronization manifold [3]. By applying the same permutation to the first term on the right-hand side of Eq. (25), for each $\kappa = 1, \dots, r$ we can write:

$$\dot{\mathbf{p}}_\kappa = \left[\begin{pmatrix} D\mathbf{F}_x^* & 0 \\ 0 & D\mathbf{G}_y^* \end{pmatrix} + \begin{pmatrix} 0 & \lambda_\kappa D\mathbf{F}_u^* D\mathbf{H}_y(\bar{\mathbf{y}}) \\ \lambda_\kappa D\mathbf{G}_u^* D\mathbf{L}_x(\bar{\mathbf{x}}) & 0 \end{pmatrix} \right] \mathbf{p}_\kappa + \left[\begin{pmatrix} \left(\sum_i^{N_x} T_i^{x\kappa} \delta a_i \right) D\mathbf{F}_u^* \mathbf{H}(\bar{\mathbf{y}}) + \left(\sum_i^{N_x} T_i^{x\kappa} \delta \mu_i \right) D\mathbf{F}_\mu^* \\ \left(\sum_i^{N_y} T_i^{y\kappa} \delta b_i \right) D\mathbf{G}_u^* \mathbf{L}(\bar{\mathbf{x}}) + \left(\sum_i^{N_y} T_i^{y\kappa} \delta v_i \right) D\mathbf{G}_v^* \end{pmatrix} \right], \quad (30)$$

where $\lambda_\kappa = \sqrt{e}s_\kappa$, $T_i^{x\kappa}$ ($T_i^{y\kappa}$) is entry i of row $T^{x\kappa}$ ($T^{y\kappa}$). Analogously, to the right-lower block of the matrix \tilde{Q} [Eq. (28)], we can associate d equations,

$$\dot{\mathbf{q}}_\kappa = D\mathbf{F}_x^* \mathbf{q}_\kappa + \left(\sum_i T_i^{x\kappa} \delta a_i \right) D\mathbf{F}_u^* \mathbf{H}(\bar{\mathbf{y}}) + \left(\sum_i T_i^{y\kappa} \delta \mu_i \right) D\mathbf{F}_\mu^*, \quad \kappa = 1, 2, \dots, d. \quad (31)$$

The homogeneous part of Eqs. (30) and (31) coincides with the reduction obtained in Ref. [3].

The vector \mathbf{w} can be written as $[\mathbf{p}_1^T, \mathbf{p}_2^T, \dots, \mathbf{p}_r^T, \mathbf{q}_1^T, \mathbf{q}_2^T, \dots, \mathbf{q}_d^T]^T$, where \mathbf{p}_1^T is a parallel perturbation and all the remaining perturbations are transverse.

We now define the following master stability functions associated with Eqs. (30) and (31):

$$\mathcal{M}_\kappa(s_\kappa, \alpha_\kappa, \beta_\kappa, \gamma_\kappa, \delta_\kappa) = \lim_{\tau \rightarrow \infty} \sqrt{\tau^{-1} \int_0^\tau \|\mathbf{p}_\kappa(t)\|^2 dt}, \quad (32a)$$

$$\mathcal{M}_0(0, \alpha_\kappa, \beta_\kappa) = \lim_{\tau \rightarrow \infty} \sqrt{\tau^{-1} \int_0^\tau \|\mathbf{q}_\kappa(t)\|^2 dt}, \quad (32b)$$

where $\alpha_\kappa = (\sum_i T_i^{x\kappa} \delta a_i)$, $\beta_\kappa = (\sum_i T_i^{y\kappa} \delta \mu_i)$, $\gamma_\kappa = (\sum_i T_i^{y\kappa} \delta b_i)/\sqrt{e}$, and $\delta_\kappa = (\sum_i T_i^{y\kappa} \delta v_i)/\sqrt{e}$.

Moreover, as the system of Eqs. (30) and (31) is linear, in case of stability, the master stability functions (32) can be approximated as follows:

$$\mathcal{M}_\kappa = c_{\alpha 1}(s)|\alpha_\kappa| + c_{\beta 1}(s)|\beta_\kappa| + c_{\gamma 1}(s)|\gamma_\kappa| + c_{\delta 1}(s)|\delta_\kappa|, \quad (33a)$$

$$\mathcal{M}_0 = c_{\alpha 2}|\alpha_\kappa| + c_{\beta 2}|\beta_\kappa|, \quad (33b)$$

where $c_{\alpha 1}$, $c_{\beta 1}$, $c_{\gamma 1}$, $c_{\delta 1}$, $c_{\alpha 2}$, and $c_{\beta 2}$ are appropriate coefficients (see also Ref. [11]). Note that Eqs. (33) relate the master stability functions with the parameter mismatches through the coefficients α_κ , β_κ , γ_κ , and δ_κ defined above.

Next, following Refs. [10,11], we will show how the low-dimensional approach [and the master stability functions defined in (32)] can be used to quantify the synchronization error observed in the high-dimensional system. The main motivation for performing this analysis is that typically dealing with the low-dimensional systems (30) and (31) is computationally more convenient than with the high-dimensional system (13) (see also Refs. [10,11]).

We define the synchronization error,

$$E(t) = \sqrt{\sum_{i=1}^{N_x} \delta \mathbf{x}_i^T(t) \delta \mathbf{x}_i(t) + \sum_{i=1}^{N_y} \delta \mathbf{y}_i^T(t) \delta \mathbf{y}_i(t)} \\ = \sqrt{\sum_{\kappa=2}^r \mathbf{p}_\kappa^T(t) \mathbf{p}_\kappa(t) + \sum_{\kappa=1}^d \mathbf{q}_\kappa^T(t) \mathbf{q}_\kappa(t)}. \quad (34)$$

Assuming stability, the time average of the synchronization error can be computed in terms of the master stability functions,

$$\langle E(t) \rangle_t = \sqrt{\sum_{\kappa=2}^r \mathcal{M}_\kappa^2 + \sum_{\kappa=1}^d \mathcal{M}_0^2}, \quad (35)$$

where with the symbol $\langle \dots \rangle_t$, we indicate a time average. Note that $\mathcal{M}_0(0, \alpha_\kappa, \beta_\kappa)$ still depends on κ through both the arguments α_κ and β_κ , see Eq. (32b).

A. Spectrum of the matrix \tilde{M}

We have already stated that the singular values of the matrix \tilde{A} can be computed from the eigenvalues of the matrix \tilde{M} . We want to show that there is a direct relation between the eigenvalues of the matrix \tilde{M} and those of the nominal matrix

$$M^{\text{NOM}} = \begin{pmatrix} 0 & A^{\text{NOM}} \\ A^{\text{NOM}T} & 0 \end{pmatrix}, \quad (36)$$

where here we retain the assumption (with no loss of generality) that the sum of the entries in the rows of the matrix A^{NOM} is equal to 1. This also implies that the sum of the entries in the columns of the matrix A^{NOM} is equal to $1/e$. We know from Ref. [3] that the spectrum of the matrix M^{NOM} is characterized by the following properties: (i) the spectrum is symmetric with respect to the real and imaginary axes, (ii) at least $d = (N_x - N_y)$ eigenvalues are equal to zero, (iii) two eigenvalues are equal to $+\sqrt{e^{-1}}$ and $-\sqrt{e^{-1}}$, and (iv) the eigenvectors of the matrix M^{NOM} , $\mathbf{v}^\kappa = [\mathbf{v}^{x\kappa} \ \mathbf{v}^{y\kappa}]^T$ can be of either one of two different types:

(1) Type I eigenvectors associated with eigenvalues $\pm\sqrt{e^{-1}}$, $\mathbf{v}^1 = [1, 1, \dots, 1, 0, 0, \dots, 0]$ and $\mathbf{v}^2 = [0, 0, \dots, 0, 1, 1, \dots, 1]$. These correspond to modes parallel to the synchronization manifold.

(2) Type II eigenvectors associated with the remaining $(N_x + N_y - 2)$ eigenvalues, for which $\sum_{l=1}^{N_x} v_l^{x\kappa} = 0$ and $\sum_{l=1}^{N_y} v_l^{y\kappa} = 0$, $\kappa = 3, \dots, N_x + N_y$. These correspond to modes orthogonal to the synchronization manifold.

In the first part of this section we have seen that, in case of parametric mismatches, the master stability function (32) and the extended master stability function (33) depend on the eigenvalues of the matrix \tilde{M} rather than those of the nominal matrix M^{NOM} . We want to show that the spectrum of the matrix \tilde{M} is closely related to that of the matrix M^{NOM} , namely,

$$M^{\text{NOM}} \mathbf{v}^\kappa = \zeta^\kappa \mathbf{v}^\kappa \Leftrightarrow \tilde{M} \mathbf{v}^\kappa = \vartheta^\kappa \mathbf{v}^\kappa, \quad (37)$$

where (a) $\vartheta^1 = \vartheta^2 = 0$ and (b) $\vartheta^\kappa = \zeta^\kappa$, $\kappa = 3, \dots, N_x + N_y$.

To prove property (a) we just need to observe that from the structure of the matrix \tilde{M} in (18) and the property that the sums over the entries in the rows and in the columns of the matrix \tilde{A} is equal to zero, it follows that $\tilde{M} \mathbf{v}^\kappa = \mathbf{0}$ for $\kappa = 1, 2$. Hence, \mathbf{v}^1 and \mathbf{v}^2 are still eigenvectors for the matrix \tilde{M} but with associated eigenvalue 0.

To prove property (b), we see that the left eigenvalue equation for the matrix \tilde{M} corresponds to the following

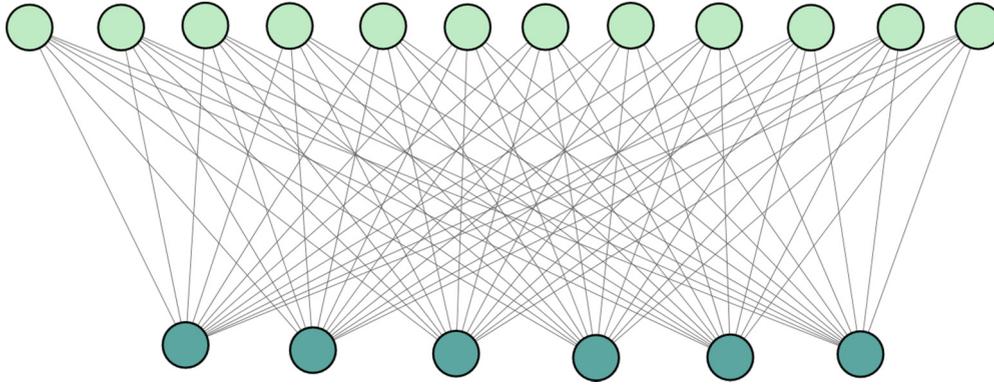


FIG. 1. A bipartite network with $N_x = 12$ nodes in the X group, shown in light green, and $N_y = 6$ nodes in the Y group, shown in dark green. The network is bipartite so it contains just inter group connections.

equations: $\mathbf{v}^{x\kappa^T} \tilde{A} = \zeta^\kappa \mathbf{v}^{y\kappa^T}$ and $\mathbf{v}^{y\kappa^T} \tilde{A}^T = \zeta^\kappa \mathbf{v}^{x\kappa^T}$. We then note that from Eq. (14) we can write that the matrix $\tilde{A} = (A^{\text{NOM}} - \Delta)$, where the entries over column j of the matrix Δ are all the same and equal to a_j , $j = 1, \dots, N_y$. Then for $\kappa = 3, \dots, N_x + N_y$, $\mathbf{v}^{x\kappa^T} \tilde{A} = \mathbf{v}^{x\kappa^T} (A^{\text{NOM}} - \Delta) = \zeta^\kappa \mathbf{v}^{y\kappa^T}$ since $\mathbf{v}^{x\kappa^T} \Delta = \mathbf{0}$ from the property that the sum over the entries of the vector $\mathbf{v}^{x\kappa}$ is equal to zero, for $\kappa = 3, \dots, N_x + N_y$. Hence, for $\kappa = 3, \dots, N_x + N_y$, \mathbf{v}^κ is still an eigenvector for the matrix \tilde{M} with associated eigenvalue ζ^κ .

$$A^{\text{NOM}} = 0.2 \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad (38)$$

B. Example

Consider the example of the bipartite network with two groups, $N_x = 12, N_y = 6$ shown in Fig. 1.

The matrix A^{NOM} for this network is

$r = d = 6$. By calculating the transformation matrix T and following Eq. (25), the matrix \tilde{Q} for this network is

$$\tilde{Q} = \frac{0.2}{\sqrt{1/2}} \begin{pmatrix} \boxed{0} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (39)$$

where each one of the r two-dimensional Q_i blocks is shown in a box. The first box (Q_1) corresponds to the parallel motion, with associated singular value $s_1 = 0$. The other blocks correspond to the transverse motions with associated singular values $s_2 = s_3 = \dots = s_6 = 0.2/\sqrt{1/2}$. By applying the same permutation applied to the matrix Q to the first term on the right-hand

side of Eq. (25), we have:

$$\begin{pmatrix} DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DG_y^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & DF_x^* \end{pmatrix}. \tag{40}$$

We further note that in case there are no mismatches, that is, $\delta a_i = 0$, $\delta b_j = 0$, $\delta \mu_i = 0$, and $\delta v_j = 0$, the low-dimensional system (30) reduces to the unforced system

$$\dot{\tilde{x}}(t) = DF_x^* \tilde{x}(t) + \lambda_\kappa DF_u^* DH_y^* \tilde{y}(t), \tag{41a}$$

$$\dot{\tilde{y}}(t) = DG_y^* \tilde{y}(t) + \lambda_\kappa DG_u^* DL_x^* \tilde{x}(t), \tag{41b}$$

which is the low-dimensional solution found in Ref. [3]. Note that Eqs. (25) is a system of linear time-varying nonhomogeneous differential equations and is characterized by the same stability range as the associated homogeneous system (41) [22]. The condition for stability is that the maximum Lyapunov exponents of (41) are negative for $\kappa = 2, \dots, r$, excluding $\lambda_1 = 0$, which is associated with perturbations tangent to the synchronization manifold [3].

It is important to emphasize that Eqs. (41) depend on $\bar{x}(t)$ and $\bar{y}(t)$, which are averaged trajectories over all the systems in the network. In a large network, calculating $\bar{x}(t)$ and $\bar{y}(t)$ may be computationally expensive, as it requires full integration of $(N_x + N_y)$ individual systems, see Eq. (11). However, for practical purposes, $\bar{x}(t)$ and $\bar{y}(t)$ in (41) can be replaced by $x_s(t)$ and $y_s(t)$ obeying Eq. (3) (see also Ref. [11]).

We conclude that in the presence of small parametric mismatches, we can still use (41) to determine whether the synchronous solution is stable. That also explains the excellent agreement found between the stability region predicted by the theory and experimentally observed in Ref. [5]. While this is not a surprising result, as it is expected that linear stability of a smooth dynamical system is not affected by small parametric variations, by using this approach we will be able to quantify the synchronization error as a function of the parameter mismatches, based on the low-dimensional reduction (30) and (31).

We now consider the following set of equations:

$$\dot{x}_{i1} = -\sigma_i x_{i1} + \sigma_i \sum_{j=1}^{N_y} A_{ij} y_j, \tag{42a}$$

$$\dot{x}_{i2} = x_{i1} \sum_{j=1}^{N_y} A_{ij} y_j - \xi x_{i2}, \tag{42b}$$

$$i = 1, \dots, N_x,$$

$$\dot{y}_j = -y_j + \sum_{i=1}^{N_x} B_{ji} x_{i1} \left(\rho - \sum_{i=1}^{N_x} B_{ji} x_{i2} \right), \tag{43}$$

$$j = 1, \dots, N_y.$$

We set the nominal parameters as follows: $\sigma_i = \sigma = 10$, $i = 1, \dots, N_x$, $\rho = 28$, and $\xi = 2$. If $\sum_j A_{ij} = 1$ and $\sum_i B_{ji} = 1$, then the synchronous dynamics corresponds to that of the chaotic Lorenz system [23]. Given the set of equations (42) and (43), Eqs. (41) become

$$\begin{aligned} \dot{\tilde{x}}_1 &= -\sigma \tilde{x}_1 + \sigma \lambda_\kappa \tilde{y}, \\ \dot{\tilde{x}}_2 &= \tilde{y} \tilde{x}_1 + \tilde{x}_1 \lambda_\kappa \tilde{y} - \xi \tilde{x}_2, \\ \dot{\tilde{y}} &= -\tilde{y} + (\rho - \tilde{x}_2) \lambda_\kappa \tilde{x}_1 - \tilde{x}_1 \lambda_\kappa \tilde{x}_2. \end{aligned}$$

The maximum Lyapunov exponent of this system is plotted versus λ_κ in Fig. 2, from which we see that the synchronous solution is stable for λ approximately less than 0.435.

Figure 3 shows c_{α_1} , c_{β_1} , c_γ , and c_δ as functions of λ_κ . The two constant values $c_{\alpha_2} = 51.75$ and $c_{\beta_2} = 0.8002$. Using this information and Eqs. (33), we can compute the master stability function as a function of λ_κ . This in turn allows us to approximate the synchronization error, using Eq. (35) from knowledge of α_κ , β_κ , γ_κ , and δ_κ , as long as all the λ_κ 's are in the region of stability, $0 \leq \lambda_\kappa \leq 0.435$. We wish to emphasize that this result is general and applies to any bipartite network with bidirectional connections, whose time evolution obeys Eqs. (42) and (43).

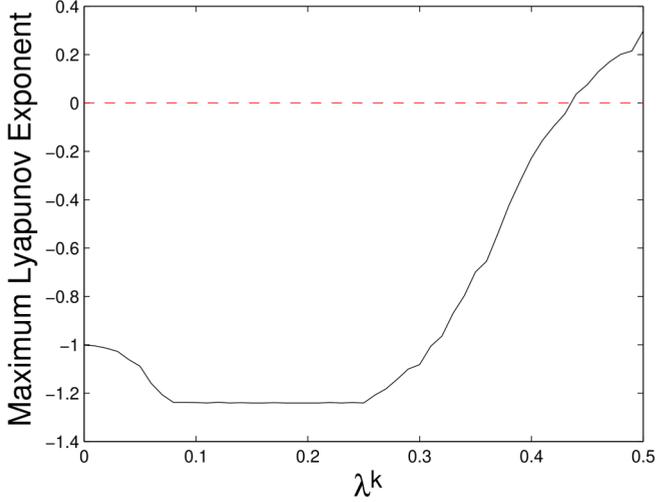


FIG. 2. Maximum Lyapunov exponent for the system (41) as a function of the parameter λ_κ . The dashed line is the 0-ordinate line.

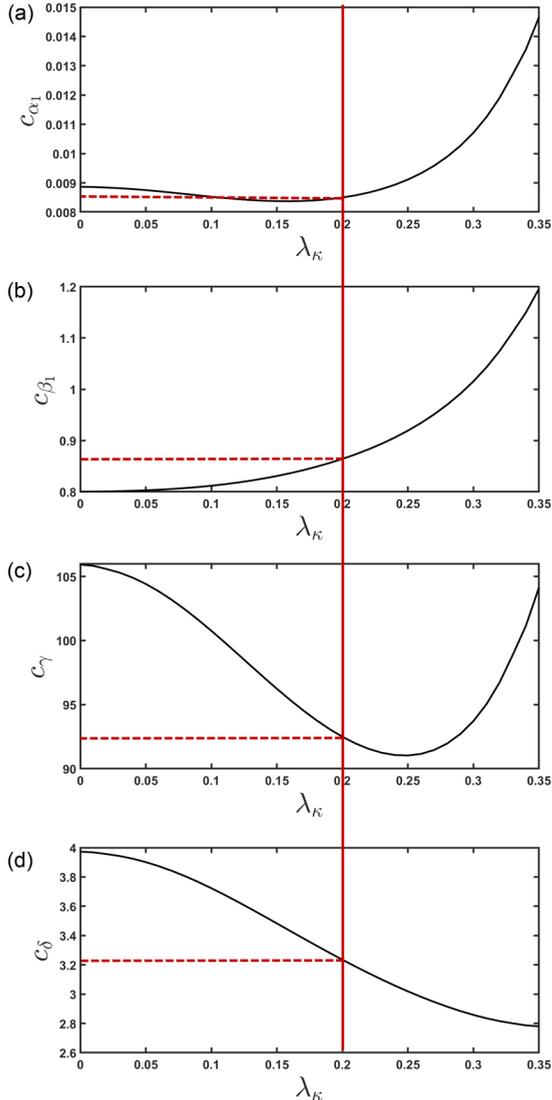


FIG. 3. c_{α_1} , c_{β_1} , c_γ , and c_δ versus λ_κ . The vertical lines highlight the values of the c_{α_1} , c_{β_1} , c_γ , and c_δ for $\lambda_\kappa = 0.2$.

TABLE I. The synchronization error observed in the high-dimensional system and the estimated synchronization error using the master stability function defined in Eq. (32) for two different cases.

Case	$\langle E(t) \rangle_t$	$\sqrt{\sum_{\kappa=2}^r \mathcal{M}_\kappa^2 + \sum_{\kappa=1}^d \mathcal{M}_0^2}$
(I)	1.529×10^{-3}	1.498×10^{-3}
(II)	1.609×10^{-3}	1.614×10^{-3}

In order to validate the theory, we consider the set of Eqs. (42) and (43) for the network topology in Fig. 1 [A^{NOM} in Eq. (38)] and study the following two cases:

(I) Small mismatches affecting the couplings, namely $A_{ij} = A_{ij}^{\text{NOM}}(1 + 5 \times 10^{-5}\phi_{ij})$, where ϕ_{ij} is a random number from a standard normal distribution.

(II) Small mismatches affecting both the couplings and the individual system parameters, namely $A_{ij} = A_{ij}^{\text{NOM}}(1 + 5 \times 10^{-5}\phi_{ij})$ and σ_i in Eq. (42) is given by $\sigma_i = 10(1 + 10^{-6}\psi_i)$, where both ϕ_{ij} and ψ_i are random numbers from a standard normal distribution.

All the transverse singular values of the matrix A^{NOM} in Eq. (38) are equal to $s_\kappa = 0.2/\sqrt{1/2}$. To these correspond only one value of $\lambda_\kappa = 0.2$. Table I compares the synchronization error obtained from integration of the full high-dimensional system (42) and (43) [left-hand side of Eq. (32)] and the estimated synchronization error using the master stability function [right-hand side of Eq. (32)] for both cases described above.

III. COUPLINGS AMONG MEMBERS OF THE SAME GROUP

So far we have only considered coupling from the X group to the Y group and vice versa. While in general group synchronization is possible in the presence of intragroup connections, stability of the group synchronous solution when also intragroup connections are present has not been fully elucidated. This problem was investigated in Ref. [4], where conditions were presented for the dimensionality reduction of the stability problem. Here we will extend some of the results in Ref. [4] and we will show that the stability of group synchronization can be reduced in a low-dimensional form for a broader class of networks.

Consider two groups of coupled oscillators, described by the following equations:

$$\dot{\mathbf{x}}_i = F(\mathbf{x}_i) + \sum_{j=1}^{N_y} A_{ij} H(\mathbf{y}_j) + \sum_{\ell=1}^{N_x} C_{i\ell} R(\mathbf{x}_\ell), \quad i = 1, \dots, N_x, \quad (44a)$$

$$\dot{\mathbf{y}}_j = G(\mathbf{y}_j) + \sum_{i=1}^{N_x} B_{ji} L(\mathbf{x}_i) + \sum_{\ell=1}^{N_y} D_{j\ell} S(\mathbf{y}_\ell), \quad j = 1, \dots, N_y, \quad (44b)$$

where the output functions $R : \mathbf{R}^{N_x} \rightarrow \mathbf{R}^{N_x}$ and $S : \mathbf{R}^{N_y} \rightarrow \mathbf{R}^{N_y}$ and the N_x -square matrix C and the N_y -square matrix D define

the intragroup connections of group X and group Y , respectively.

If $\sum_{j=1}^{N_y} A_{ij} = a \neq 0$ and $\sum_{\ell=1}^{N_x} C_{i\ell} = c \neq 0$ independent of i and $\sum_i^{N_x} B_{ji} = b \neq 0$ and $\sum_{\ell=1}^{N_y} D_{j\ell} = d \neq 0$ independent of j , then a synchronous solution exists, $x_1 = x_2 = \dots = x_{N_x} = x_s$, $y_1 = y_2 = \dots = y_{N_y} = y_s$, obeying

$$\dot{x}_s = F(x_s) + aH(y_s) + cR(x_s), \quad (45a)$$

$$\dot{y}_s = G(y_s) + bL(x_s) + dS(y_s). \quad (45b)$$

Note that by appropriately rescaling the functions H, L, R, S it is always possible to set $a = b = c = d = 1$. Therefore, without loss of generality, in what follows we will proceed under this assumption.

In order to analyze stability, we linearize (44) about the synchronous solution,

$$\delta\dot{x}_i = DF(x_s)\delta x_i + \sum_{j=1}^{N_y} A_{ij}DH(y_s)\delta y_j + \sum_{\ell=1}^{N_x} C_{i\ell}DR(x_s)\delta x_\ell, \quad (46a)$$

$$i = 1, \dots, N_x$$

$$\delta\dot{y}_j = DG(y_s)\delta y_j + \sum_{i=1}^{N_x} B_{ji}DL(x_s)\delta x_i + \sum_{\ell=1}^{N_y} D_{j\ell}DS(y_s)\delta y_\ell, \quad (46b)$$

$$j = 1, \dots, N_y.$$

In Ref. [4] it was shown that Eq. (46) can be reduced in a low-dimensional form, provided that the matrix $N = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ has the same set of eigenvectors as the matrix $M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. It was thus concluded in Ref. [4] that if this condition is satisfied, in a similar way to the case of bipartite topologies, then stability of the synchronous solution can be reduced in the following low-dimensional form,

$$\dot{\bar{x}}^\kappa = DF(x_s)\bar{x}^\kappa + \lambda^\kappa DH(y_s)\bar{y}^\kappa + \eta^\kappa DR(x_s)\bar{x}^\kappa, \quad (47a)$$

$$\dot{\bar{y}}^\kappa = DG(y_s)\bar{y}^\kappa + \lambda^\kappa DL(x_s)\bar{x}^\kappa + \eta^\kappa DS(y_s)\bar{y}^\kappa, \quad (47b)$$

$\kappa = 1, \dots, N_x + N_y$, where λ^κ and η^κ are the eigenvalues associated with the same eigenvector of the matrices M and N , respectively. By construction, there are always two κ eigenvectors which are associated with perturbations tangent to the synchronization manifold, and the corresponding pairs of eigenvalues ($\lambda^\kappa, \eta^\kappa$) do not need to be considered in order to assess stability. Without loss of generality, we label $\kappa = 1$ and $\kappa = 2$ the two tangent eigenmodes and $\kappa = 3, \dots, N_x + N_y$ the remaining transverse eigenmodes.

In what follows we extend the results in Ref. [4] and show that a reduction in a low-dimensional form is possible for any pair of matrices,

$$C = C' + J^x, \quad (48a)$$

$$D = D' + J^y. \quad (48b)$$

where (i) the matrix $\begin{bmatrix} C' & 0 \\ 0 & D' \end{bmatrix}$ has the same set of eigenvectors as the matrix M and (ii) J^x and J^y are any two N_x -square and

N_y -square matrices whose columns are composed of entries that are all the same. This form of the matrices J^x and J^y corresponds to a particular coupling configuration for which for any pair of nodes (i, j) in either group X or group Y , the coupling strength from node i to node j is a function of i but not of j .

Using the ansatz $\delta x_i(t) = p_i^\kappa \hat{x}^\kappa(t)$ and $\delta y_j(t) = q_j^\kappa \hat{y}^\kappa(t)$ in (46), we obtain

$$p_i^\kappa \dot{\hat{x}}^\kappa = DF(x_s)p_i^\kappa \hat{x}^\kappa + \sum_{j=1}^{N_y} A_{ij}q_j^\kappa DH(y_s)\hat{y}^\kappa + \sum_{\ell=1}^{N_x} C_{i\ell}DR(x_s)p_\ell^\kappa \hat{x}^\kappa, \quad (49a)$$

$$q_j^\kappa \dot{\hat{y}}^\kappa = DG(y_s)q_j^\kappa \hat{y}^\kappa + \sum_{i=1}^{N_x} B_{ji}p_i^\kappa DL(x_s)\hat{x}^\kappa + \sum_{\ell=1}^{N_y} D_{j\ell}DS(y_s)q_\ell^\kappa \hat{y}^\kappa. \quad (49b)$$

We now consider the left eigenvalue equation,

$$\omega^{\kappa T} M = \lambda^\kappa \omega^{\kappa T}, \quad (50)$$

where the left eigenvectors $\omega^{\kappa T} = (p_1^{\kappa l}, p_2^{\kappa l}, \dots, p_{N_x}^{\kappa l}, q_1^{\kappa l}, q_2^{\kappa l}, \dots, q_{N_y}^{\kappa l})$. By multiplying (49a) by $p_i^{\kappa l}$, summing over i , and dividing by $(\sum_i p_i^{\kappa l} p_i^\kappa)$ [by multiplying (49b) by $q_j^{\kappa l}$, summing over j , and dividing by $(\sum_j q_j^{\kappa l} q_j^\kappa)$], we obtain,

$$\dot{\hat{x}}^\kappa = DF(x_s)\hat{x}^\kappa + \lambda^\kappa DH(y_s)\hat{y}^\kappa + \eta^\kappa DR(x_s)\hat{x}^\kappa + \frac{(\sum_\ell \sum_i p_i^{\kappa l} J_{i\ell}^x p_\ell^\kappa)}{(\sum_i p_i^{\kappa l} p_i^\kappa)} DR(x_s)\hat{x}^\kappa, \quad (51a)$$

$$\dot{\hat{y}}^\kappa = DG(y_s)\hat{y}^\kappa + \lambda^\kappa DL(x_s)\hat{x}^\kappa + \eta^\kappa DS(y_s)\hat{y}^\kappa + \frac{(\sum_\ell \sum_j q_j^{\kappa l} J_{j\ell}^y q_\ell^\kappa)}{(\sum_j q_j^{\kappa l} q_j^\kappa)} DS(y_s)\hat{y}^\kappa, \quad (51b)$$

$\kappa = 1, \dots, N_x + N_y$.

We shall now show that $\sum_i p_i^{\kappa l} J_{i\ell}^x = 0$, for $\kappa = 3, \dots, (N_x + N_y)$ in Eq. (51a) and $\sum_j q_j^{\kappa l} J_{j\ell}^y = 0$, for $\kappa = 3, \dots, (N_x + N_y)$ in Eq. (51b). Since by definition the matrices J^x and J^y have columns whose entries are all the same, it suffices to show that $\sum_i p_i^{\kappa l} = 0$, for $\kappa = 3, \dots, (N_x + N_y)$ and $\sum_j q_j^{\kappa l} = 0$, for $\kappa = 3, \dots, (N_x + N_y)$. We will prove that this latter property is satisfied by the $(N_x + N_y - 2)$ eigenvectors that are associated with the transverse eigenmodes of the matrix M .

Note that the left eigenvalue equation $\omega^{\kappa T} M = \lambda^\kappa \omega^{\kappa T}$ implies the following two equations: $q^{\kappa T} B = \lambda^\kappa p^{\kappa T}$ and $p^{\kappa T} A = \lambda^\kappa q^{\kappa T}$ where the vectors $p^{\kappa T} = (p_1^{\kappa l}, p_2^{\kappa l}, \dots, p_{N_x}^{\kappa l})$ and

$\mathbf{q}^{\kappa T} = (q_1^{\kappa l}, q_2^{\kappa l}, \dots, q_{N_y}^{\kappa l})$, or, equivalently,

$$\sum_{j=1}^{N_y} q_j^{\kappa l} B_{ji} = \lambda^\kappa p_i^{\kappa l}, \quad (52a)$$

$$\sum_{i=1}^{N_x} p_i^{\kappa l} A_{ij} = \lambda^\kappa q_j^{\kappa l}. \quad (52b)$$

By summing Eq. (52a) over $i = 1, \dots, N_x$ and Eq. (52b) over $j = 1, \dots, N_y$, we obtain

$$\sum_{j=1}^{N_y} q_j^{\kappa l} = \lambda^\kappa \sum_{i=1}^{N_x} p_i^{\kappa l}, \quad (53a)$$

$$\sum_{i=1}^{N_x} p_i^{\kappa l} = \lambda^\kappa \sum_{j=1}^{N_y} q_j^{\kappa l}. \quad (53b)$$

By plugging (53b) into (53a) we obtain $\sum_{j=1}^{N_y} q_j^{\kappa l} = (\lambda^\kappa)^2 \sum_{j=1}^{N_y} q_j^{\kappa l}$, which can be satisfied in two possible ways: either $\lambda^\kappa = \pm 1$, which applies to the parallel eigenmodes $\kappa = 1, 2$ of the matrix M , or $\sum_{j=1}^{N_y} q_j^{\kappa l} = 0$, which applies to the transverse eigenmodes $\kappa = 3, \dots, (N_x + N_y)$ of the matrix M . Analogously, by plugging (53a) into (53b), we can show that $\sum_{i=1}^{N_x} p_i^{\kappa l} = 0$ for $\kappa = 3, \dots, (N_x + N_y)$.

We can thus conclude that Eq. (51) reduces to (47) for $\kappa = 3, \dots, (N_x + N_y)$ (while in general it does not reduce to (47) for $\kappa = 1, 2$). This implies that when intragroup connections are in the general form of Eq. (48), transverse stability is still determined by the low-dimensional system (47).

Finally, we note that the analysis of parameter mismatches presented in Sec. II can be extended to the case that small mismatches affect the intragroup connections. In particular, if the intragroup connections are affected by small mismatches but are in a form that approximately satisfies Eq. (48), then the low-dimensional system (47) can still be used to predict stability of the synchronous state.

A. Numerical example

As a numerical example, we considered a small network for which $N_x = 2$, $N_y = 3$, and

$$A = \begin{pmatrix} w & 1-w & 0 \\ 0 & 1-w & w \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}. \quad (54)$$

In Ref. [3] we showed that for this choice of the coupling matrices A and B , the eigenvalues λ^κ were equal to $0, \pm\sqrt{w}, \pm 1$. Then, by setting $w = 0.64$ stability is only affected by the pair of eigenvalues $\lambda^\kappa, 0$, and $\sqrt{w} = 0.8$ [3]. In our numerical experiments, we retained the set of Eq. (42) and replaced (43) by the following:

$$\dot{y}_j = -y_j + \sum_{i=1}^{N_x} B_{ji} x_{i1} \left(\rho - \sum_{i=1}^{N_x} B_{ji} x_{i2} \right) + \sum_{\ell=1}^{N_y} D_{j\ell} S(y_\ell), \quad (55)$$

$j = 1, \dots, N_y$, where the last term on the right-hand side of (55) represents coupling between systems of the same group

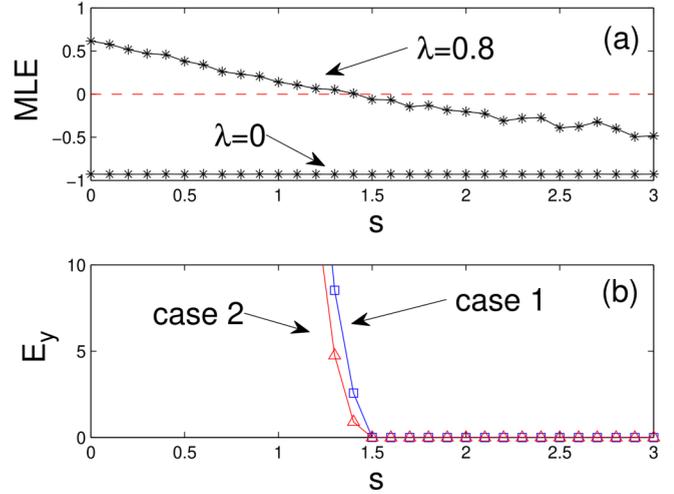


FIG. 4. (a) Maximum Lyapunov exponent (MLE) for the system (47) vs. s for $\lambda^\kappa = 0$ and $\lambda^\kappa = 0.8$. The dashed line corresponds to the 0-ordinate line. (b) Synchronization error E_y vs. s for the two choices of the matrix D in Eq. (56).

\mathbf{Y} and we set the coupling function $S(y_\ell) = sy_\ell$, with s being a variable scalar quantity.

We considered the following two cases of intragroup coupling matrices D :

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (56)$$

which we label case 1 and case 2, respectively. For both cases 1 and 2, the corresponding matrices M and N do not commute, so neither case can be studied within the framework presented in Ref. [4]. Note that both choices of D above are matrices whose columns are composed of entries that are all the same. Therefore, in both cases we expect stability to be described by the low-dimensional system (47) with $\eta^\kappa = 0$. Figure 4(a) shows the maximum Lyapunov exponent (MLE) for the system (47) as a function of the parameter s for $\lambda^\kappa = 0$ and $\lambda^\kappa = 0.8$, from which we see that stability is expected for $s \gtrsim 1.5$.

In Fig. 4(b) we run full numerical simulations of the network dynamics [Eqs. (42) and (55)] as we vary the parameter s . For each run, the network systems are evolved from initial conditions that are close to the synchronization manifold. The figure shows the final synchronization error,

$$E_y = N_y^{-1}(\tau)^{-1} \int_{t_1}^{t_1+\tau} \sum_{i=1}^{N_y} [y_i(t) - \bar{y}(t)]^2 dt, \quad (57)$$

with t_1 a large-enough time past the initial transient dynamics and τ a large-enough averaging window, versus s . As can be seen, perfect agreement is attained with the low-dimensional predictions of Fig. 4(a).

IV. CONCLUSIONS

The main contribution of this paper is to extend the analysis of the effects of parameter mismatches on the stability of the network synchronous solution [9–15] to the case of networks

formed of systems of different types. An important reference for our work is Ref. [10], which introduced a master stability function to characterize the synchronization error in networks with parameter mismatches.

Group synchronization was first studied in Ref. [3] and has been the subject of both theoretical [4] and experimental [5] investigations. Group synchronization has been observed and characterized when the systems in each group are identical and the couplings between the systems satisfy specific conditions. In this paper, we have defined a master stability function that describes stability of the group synchronization solution in the presence of mismatches on the individual parameters of the network oscillators and on the couplings between the oscillators in the groups. This is relevant to experimental realizations of group synchrony [5], for which parameter

mismatches are practically unavoidable. Our analysis applies to the case that for each connection from node i in group X to node j in group Y , there is a connection from node j in group Y to node i in group X . We have also extended the analysis presented in Ref. [4] to study stability of the group synchronization solution in the presence of intragroup couplings. Our analysis has pointed out a broader class of matrices describing intragroup connectivity for which the stability problem can be reduced in a low-dimensional form.

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