Collective dynamics of phase oscillator populations with three-body interactions

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Many-body interactions between dynamical agents have caught particular attention in recent works that found wide applications in physics, neuroscience, and sociology. In this paper we investigate such higher order (nonadditive) interactions on collective dynamics in a system of globally coupled heterogeneous phase oscillators. We show that the three-body interactions encoded microscopically in nonlinear couplings give rise to added dynamic phenomena occurring beyond the pairwise interactions. The system in general displays an abrupt desynchronization transition characterized by irreversible explosive synchronization via an infinite hysteresis loop. More importantly, we give a mathematical argument that such an abrupt dynamic pattern is a universally expected effect. Furthermore, the origin of this abrupt transition is uncovered by performing a rigorous stability analysis of the equilibrium states, as well as by providing a detailed description of the spectrum structure of linearization around the steady states. Our work reveals a self-organized phenomenon that is responsible for the rapid switching to synchronization in diverse complex systems exhibiting critical transitions with nonpairwise interactions.

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I. INTRODUCTION

Collective synchronization in large systems consisting of many interacting agents is a universal emergent behavior that can be observed in a wide variety of realistic systems [1]. Noted examples range from fireflies flashing [2] and colonies of yeast cells [3] to applause formation in a large audience [4]. In addition, as one of the most well-known cooperative phenomena in nature, synchronization process plays a prominent role for the functionality in a wide range of applications including arrays of Josephson junction [5], power grids [6], and wireless communication components [7]. Unraveling the fundamental mechanism behind such organizational behaviors has been an object of study in the area of nonlinear dynamics and network science that extends our understanding of the macroscopic dynamics in a vast number of complex systems [8–10].

A system of coupled phase oscillators has been a particularly important paradigm used to shed light on collective synchrony in many rich parts of science. The remarkable example is the Kuramoto model introduced in 1975 [11], which consists of a system of globally coupled phase oscillators with the natural frequencies distributed across the population and interacting via the sinusoidal function of phase difference between each pair. In this simple framework, the onset of synchronization can be elucidated as a result of phase transition characterized by the order parameter occurring from the incoherence to coherence [12]. In terms of various applications in physics, biology, and social systems, the Kuramoto-like oscillators are the most popular and widely used models for exploring synchronization that afford analytical insights in uncovering the emergence of collective rhythm in diverse systems containing self-sustained oscillators [13–17].

Most existing models defined by the Kuramoto dynamics have the strong assumption that the interactions between phase oscillators are pairwise. Specifically, the net interaction of a phase oscillator can be expressed in terms of the linear superposition of two-body interactions that are quite general in classical mechanics and many other fields of physics. However, recent works in neuronal networks and social science have highlighted the potential importance of many-body interactions between dynamic units, i.e., three- or four-way interactions [18,19]. Such higher order (many-body) interactions take place between three or more dynamical units that are organized via a simplicial complex or the hypernetwork topology formed by different dimensions [20–22]. Using phase reduction, it has been shown that the higher order interactions arise in generic oscillatory systems [23,24], which have significant influences on shaping collective dynamics that one can find beyond the typical pairwise interactions [25,26]. Investigating the dynamical effects induced by the simplicial complex has been a topic of high interest across a number of disciplines including brain dynamics, social interactions, data analysis, and so on [27–33].

Very recently, research has revealed that there is an interconnection between the many-body interactions and the irreversible explosive desynchronization transitions characterized by an abrupt switch between the coherent and incoherent state [34–37]. This type of explosive synchronization can be achieved in oscillator networks by incorporating higher order structures in addition to establishing correlations between the oscillator's dynamics and network topology [38,39]. Despite these advances, a basic problem in this direction still lies in

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understanding the mathematical foundation of the observed abrupt synchronization transition generated by the many-body interactions and the associated structure properties of the eigenspectrum of the equilibrium states in the thermodynamic limit.

The aim of this paper is to study the generic collective dynamics induced by many-body interactions. Specifically, we investigate synchronization dynamics in a large ensemble of globally coupled heterogeneous phase oscillators by incorporating three-body interactions with arbitrary combination coefficients. We uncover that the abrupt desynchronization characterized by an irreversible explosive synchronization via an infinite hysteresis loop is an intrinsic dynamic phenomenon that is not present in the absence of higher order interactions. Importantly, we present a general framework for analytically capturing the collective dynamics emerging from three-body interactions, and for unveiling the dynamical nature for such abrupt transition to happen. In particular, we establish an explicit criterion for the occurrence of abrupt desynchronization, and the dynamical stability of equilibrium states involved in the coupled system is addressed by analyzing the corresponding spectrum structure of linearization. Our work demonstrates that the many-body interactions provide a natural mechanism for the emergence of abrupt transitions in the coupled oscillatory systems.

The remainder of this paper is organized as follows. In Sec. II we present the dynamic model by considering three-body interactions with arbitrary combination coefficients. In Sec. III we formulate the self-consistent equations describing the generalized order parameters characterizing the long term macroscopical dynamics of the system. In Sec. IV we explore the stability of equilibrium states described by the self-consistent equations providing a complete description of the associated spectrum structure of linearization by invoking in the celebrated Ott-Antonsen manifold. A general criterion for the emergence of abrupt desynchronization is obtained. Section V illustrates with an example to get a sense of dynamics with many-body interactions. Finally, we conclude with a discussion of our results in Sec. VI.

II. DYNAMIC MODEL

We propose an extension of the classical Kuramoto model by considering three-body interactions, whose motion of equations are formulated as

$$\dot{\theta}_i = \boldsymbol{\omega}_i + \frac{K}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\sin(\alpha\theta_j + \beta\theta_k + \gamma\theta_i). \tag{1}$$

Here θ_i is the phase of oscillator i, with $i = 1, \dots, N$, $\boldsymbol{\omega}_i$ is its natural frequency chosen randomly from a prescribed distribution $g(\omega)$ [40]. $K > 0$ accounts for the global attracting coupling strength of the system. The combination coefficients involved in the coupling function α, β, γ are integers.

Unlike the conventional Kuramoto model and its variants, in which only the phase difference between each pair of individual oscillators is considered (pairwise interactions). Equation (1) involves a triplet $(\theta_j, \theta_k, \theta_i)$ with arbitrary combination coefficients. We proceed with the analysis by assuming that $\alpha + \beta + \gamma = 0$. Physically, this constraint implies that the system given by Eq. (1) remains invariant under the global phase shift, i.e., the coupling term vanishes so long as the indexes j, k, i are identical.

Equation (1) represents a typical model of three-way interactions, which comes from the phase reduction of limit cycle oscillators [23,41]. It defines the 2-simplex interactions corresponding to a fully connected hypergraph. In particular, if α (or β) $= 0$ and $\gamma \neq 0$, Eq. (1) reduces to the generalized Kuramoto model, where the interactions are defined between each pair of nodes (links). In contrast, for $\alpha, \beta, \gamma \neq 0$, the interactions are defined on the triangle formed by the nodes θ_j, θ_k, and θ_i.

For instance, if $\gamma = -2$, $\alpha = \beta = 1$, previous studies demonstrated that Eq. (1) stands for a sort of higher order nonlinear coupling displaying a number of novel collective behaviors, such as clustering, multibranch entrainment, and finite size induced phase transitions [42,43]. More importantly, the interesting dynamical features induced by the high order interactions play a center role in brain dynamics. The formation of multiclusters, as well as the extensive multistability accompanied by a continuum of abrupt desynchronization transitions, naturally enables the system to store information and memory [39,43,44]. The dynamic mechanisms of such nontrivial collective behaviors were further addressed from both macroscopic and microscopic perspectives in recent works [36,37].

To get a complete understanding of the dynamics described by three-body interactions in large coupled phase oscillator systems, here we restrict our consideration to the case $|\gamma| = 1$. Without loss of generality, we set $\gamma = -1$ (since the case $\gamma = 1$ can be discussed in a similar way). Accordingly, combination coefficients should satisfy the condition

$$\alpha + \beta = 1. \tag{2}$$

Our goal is to uncover the generic dynamical properties induced by the three-body interactions by providing a nontrivial and analytically tractable model described by the Kuramoto dynamics.

To facilitate the analysis, we make the following two simplifying assumptions. First, the size of the system N is large enough ($N \to \infty$), so that the state of the ensemble can be described by a phase distribution. Second, $g(\omega)$ is unimodal and symmetric, i.e., $g(\omega) = g(-\omega)$, and $g(\omega)$ is nonincreasing for $\omega > 0$.

III. MEAN-FIELD THEORY

Having introduced the model, we turn our focus to finding the equilibrium states given by the system of Eq. (1). To do so, it is useful to introduce the generalized complex order parameters, which are

$$Z_\alpha(t) = |Z_\alpha(t)|e^{i\Theta_\alpha(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\alpha\theta_j(t)}, \tag{3}$$

with $\alpha = 0, \pm 1, \pm 2, \dots$. $|Z_\alpha|$ and Θ_α are the amplitude and argument of Z_α, respectively. In particular, $Z_0 = 1$, Z_1 corresponds to the classical Kuramoto order parameter, and Z_α with $|\alpha| > 1$ are the so called higher order parameters [45,46]. Additionally, it follows that $Z_{-\alpha}(t) = \bar{Z}_\alpha(t)$, here and in the following, "−" denotes the complex conjugate. We thus have $|Z_\alpha| = |Z_{-\alpha}|$ and $\Theta_{-\alpha} = -\Theta_\alpha$. As we shall see below, the generalized order parameters play the same role as the classic Kuramoto order parameter. In other words, $|Z_\alpha|$ characterizes the degree of the coherence of the system, while Θ_α denotes the phase of the centroid of the configuration $\{e^{i\alpha\theta_j}\}$. Broadly speaking, in the case $|Z_\alpha| = 0$ with $\alpha \neq 0$, the oscillators are spread roughly around the unit circle, while for the case $|Z_\alpha| \approx 1$, a number of oscillators are tightly concentrated around a common phase. We thus interpret $|Z_\alpha| = 0$ as the incoherent state and $|Z_\alpha| > 0$ as the partially synchronized states ($\alpha \neq 0$).

The definition of Eq. (3) allows us to rewrite Eq. (1) in a more simple form as

$$\dot{\theta}_i = \boldsymbol{\omega}_i + K|Z_\alpha||Z_\beta|\sin(\Theta_\alpha + \Theta_\beta - \theta_i). \tag{4}$$

In this setting we may interpret each oscillator as interacting not necessarily with the others individually, but collectively through a simple mean-field $Z_\alpha Z_\beta$.

We next briefly outline the self-consistent approach proposed by Kuramoto. First, passing to the limit $\mathrm{N} \to \infty$, where the overall state of the oscillators may be well characterized by a distribution function $\rho(\boldsymbol{\theta}, \boldsymbol{\omega}, t)$, $\rho(\boldsymbol{\theta}, \boldsymbol{\omega}, t)d\boldsymbol{\theta}d\boldsymbol{\omega}$ represents the fraction of the oscillators with the natural frequencies between ω and $\omega + d\boldsymbol{\omega}$ and phases between θ and $\theta + d\boldsymbol{\theta}$ for a fixed time. Furthermore, $\rho(\boldsymbol{\theta}, \boldsymbol{\omega}, t)$ should satisfy the normalization condition $\int_0^{2\pi} \rho(\boldsymbol{\theta}, \boldsymbol{\omega}, t)d\boldsymbol{\theta} = 1$. Second, the conservation of oscillator number implies a continuity equation for the phase distribution of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial \theta} = 0, \tag{5}$$

with the velocity $v(\boldsymbol{\theta}, \boldsymbol{\omega}, t)$ given by

$$v(\boldsymbol{\theta}, \boldsymbol{\omega}, t) = \omega + K\mathrm{Im}(Z_\alpha Z_\beta e^{-i\theta}), \tag{6}$$

Eq. (5) defines a nonlinear partial differential equation, whose steady solutions comprise two parts, i.e., the phase-locked case and the drifting ones. In both cases, the macroscopic order parameters Z_α and Z_β are time independent. Note that the assumed symmetry of $g(\omega)$ and constraint about the combination coefficients Eq. (2) imply that one may set $\Theta_\alpha(t) = \Theta_\beta(t) = 0$ by shifting initial conditions.

To ease notation, let $Q = K|Z_\alpha||Z_\beta|$. In the first scenario, $|\omega| < Q$, $v = 0$, the oscillators are entrained by the mean field, and the stationary distribution formed by the trapped oscillators is expressed as

$$\rho_l(\theta, \omega) = \delta(\theta - \theta_\omega), \tag{7}$$

with

$$\sin\theta_\omega = \frac{\omega}{Q} \tag{8}$$

and

$$\cos\theta_\omega = \sqrt{1 - \left(\frac{\omega}{Q}\right)^2}. \tag{9}$$

However, in the second scenario, $|\omega| > Q$, where $v \neq 0$. The corresponding stationary distribution formed by the drifting oscillators (untrapped) is given by

$$\rho_d(\theta, \omega) = \frac{C_\omega}{\omega - Q\sin\theta}, \tag{10}$$

with C_ω being the normalization constant obtained as

$$C_\omega = \frac{\mathrm{sgn}(\omega)\sqrt{\omega^2 - Q^2}}{2\pi}. \tag{11}$$

Here $\mathrm{sgn}(\omega)$ stands for the sign function with respect to ω. For the sake of mathematical simplicity, it is convenient to introduce two intergration symbols

$$\hat{g}\psi(\omega) = \int_{-\infty}^{+\infty} g(\omega)\psi(\omega)d\omega \tag{12}$$

and

$$[\phi(\theta), \varphi(\theta)] = \int_0^{2\pi} d\theta\phi(\theta)\varphi(\theta). \tag{13}$$

For the order parameters we have that

$$Z_\alpha = \hat{g}(e^{i\alpha\theta}, \rho) = \hat{g}(e^{i\alpha\theta}, \rho_l) + \hat{g}(e^{i\alpha\theta}, \rho_d), \tag{14}$$

the first term on the right-hand side of Eq. (14) corresponds to the contribution to the order parameter from the phase-locked oscillators, while the second term is generated by the drifting ones. Substituting Eqs. (7)–(11) into Eq. (14), we obtain

$$\begin{aligned}
\hat{g}(e^{i\alpha\theta}, \rho_l) &= \hat{g}e^{i\alpha\theta_\omega} \\
&= \hat{g}(\cos\theta_\omega + i\sin\theta_\omega)^\alpha \\
&= \hat{g}\sum_{n=0}^{\alpha} \frac{\alpha!}{n!(\alpha-n)!}(\cos\theta_\omega)^{\alpha-n}(i\sin\theta_\omega)^n \\
&= \hat{g}\cos\alpha\theta_\omega + i\hat{g}\sin\alpha\theta_\omega = \hat{g}\cos\alpha\theta_\omega \\
&= \sum_{n\in even}^{\alpha} \frac{\alpha!}{n!(\alpha-n)!}\int_{-\infty}^{+\infty} d\omega g(\omega) \\
&\quad \times (\cos\theta_\omega)^{\alpha-n}(i\sin\theta_\omega)^n,
\end{aligned} \tag{15}$$

the result above uses the fact that $\sin\theta_\omega$ is an odd function with respect to ω, and so is $\sin\alpha\theta_\omega$, hence the integral $\hat{g}\sin\alpha\theta_\omega$ vanishes automatically.

Regarding the drifting populations, the calculations become more complex, since

$$\begin{aligned}
(e^{i\alpha\theta}, \rho_d) &= \int_0^{2\pi} e^{i\alpha\theta}\frac{C_\omega}{\omega - Q\sin\theta}d\theta \\
&= \oint_{|z|=1} \frac{z^\alpha C_\omega}{i\omega z - Q(z^2-1)/2}dz,
\end{aligned} \tag{16}$$

where we have used $z = e^{i\theta}$. Note that the integral above has two simple poles $z_d^\pm = \frac{i}{Q}(\omega \pm \sqrt{\omega^2 - Q^2})$ located, respectively, within and outside the unit circle on the complex plane. Therefore, Eq. (16) can be evaluated by using Cauchy residue theorem, which yields

$$\hat{g}(e^{i\alpha\theta}, \rho_d) = \hat{g}\left(i\frac{\omega - \mathrm{sgn}(\omega)\sqrt{\omega^2 - Q^2}}{Q}\right)^\alpha = \hat{g}[z_d(\omega)]^\alpha. \tag{17}$$

Clearly $z_d(\omega)$ is an odd function to ω, if $\alpha \in$ odd, $\hat{g}(z_d)^\alpha = 0$, implying that the untrapped oscillators have no contribution to the mean field. Otherwise, for $\alpha \in$ even, $\hat{g}(z_d)^\alpha \neq 0$, indicating that the contribution to the order parameters from the drifting populations may not be neglected. Hence, the order parameters Z_α reduce to a simple form

$$Z_\alpha = |Z_\alpha| = \hat{g}\cos\alpha\theta_\omega + \hat{g}[z_d(\omega)]^\alpha. \tag{18}$$

Likewise,

$$Z_\beta = |Z_\beta| = \hat{g}\cos\beta\theta_\omega + \hat{g}[z_d(\omega)]^\beta. \tag{19}$$

Considering the definition of Q, we lastly arrive at the self-consistent equation describing the equilibrium states yielding

$$\frac{1}{K} = F(Q), \tag{20}$$

where the characteristic function of the coupled system is defined by

$$F(Q) = Q^{-1}Z_\alpha Z_\beta = Q^{-1}\{\hat{g}\cos\alpha\theta_\omega + \hat{g}[z_d(\omega)]^\alpha\}\{\hat{g}\cos\beta\theta_\omega + \hat{g}[z_d(\omega)]^\beta\}. \tag{21}$$

The self-consistent equation (20) gives an implicit function dependence of the generalized order parameters on the coupling strength K. Specifically, for a fixed value of K, we may solve for Q from Eq. (20), which in turn determines Z_α, Z_β according to Eqs. (18) and (19), respectively.

In particular, $\alpha = 1, \beta = 0, F(Q) = Q^{-1}\hat{g}\cos\theta_\omega$. Then we have

$$\frac{1}{K} = \frac{1}{Q}\int_{-Q}^{Q} d\omega g(\omega)\sqrt{1-(\frac{\omega}{Q})^2}, \tag{22}$$

which reproduces the self-consistent equation of the classic Kuramoto model [12]. In contrast, for the case $\alpha \neq 0$ and 1, $F(Q)$ becomes more complex. For example, if $\alpha = 2$ and $\beta = -1$, we have that

$$\frac{1}{K} = \frac{1}{Q}\int_{-Q}^{Q} d\omega g(\omega)\sqrt{1-\left(\frac{\omega}{Q}\right)^2}\left[\int_{-Q}^{Q} d\omega g(\omega)\left(1-2\frac{\omega^2}{Q^2}\right) - \int_{|\omega|>Q} d\omega g(\omega)\left(\frac{\omega - \mathrm{sgn}(\omega)\sqrt{\omega^2-Q^2}}{Q}\right)^2\right]. \tag{23}$$

In addition to the incoherent state $Q = 0$ that exists for all K, we here summarize the properties of the self-consistent equation given by Eq. (20) with the aim of clarifying synchronization transitions with three-body interactions. In fact, Eq. (20) defines a map from $Q \in \mathbb{R}$ to $K \in \mathbb{R}$, and the associated inverse map gives the synchronization profile and bifurcation features of the coupled system. For instance, if $\alpha = 1$ and $\beta = 0$, the resulting expression of $F(Q)$ is considerably simplified by choosing a unimodal and symmetrical $g(\omega)$. More precisely, the function $F(Q)$ is strictly decreasing for $Q > 0$, the corresponding inverse map is one to one [the red solid line of Fig. 1(a)]. Only one branch of solution exists for a sufficiently large K. Accordingly, the phase transition from incoherence to coherence is of second order that takes place at a critical point $K_c = \lim_{Q\to 0^+} F(Q)^{-1}$ [Fig. 1(b)]. This picture coincides with the previous results about the classic Kuramoto model, which behaves qualitatively the same as long as the distribution $g(\omega)$ remains unimodal.

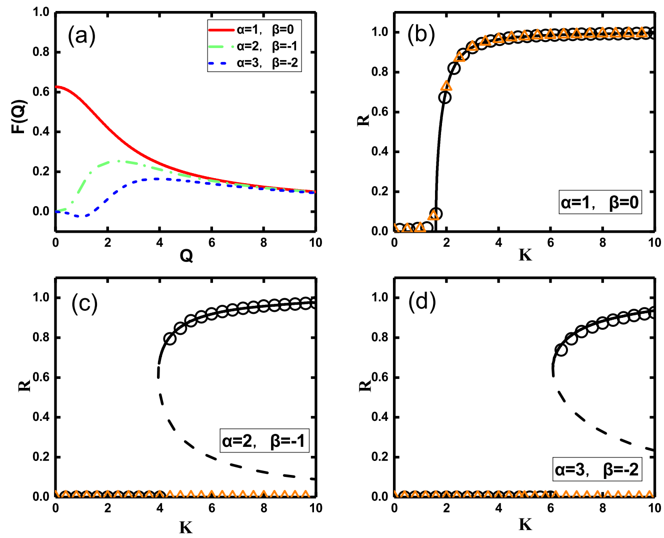

FIG. 1. (a) The characteristic functions $F(Q)$ with Gaussian distribution $g(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}}$. $\alpha = 1, \beta = 0$ (red solid line), $\alpha = 2, \beta = -1$ (green dash-dotted line), $\alpha = 3, \beta = -2$ (blue short-dashed line). (b)–(d) Corresponding phase diagrams of the Kuramoto order parameter $R = |Z_1|$ versus the global coupling strength K, respectively. The orange triangles and the black circles represent forward and backward transitions of the numerical simulations with $N = 100\,000$, respectively. In the forward continuations, the initial conditions $\{\theta_i(0)\}$ for each K are uniformly distributed in $[0, 2\pi]$. In the backward continuations, the initial phases for each K are identical, i.e., $\{\theta_i(0)\} = 0, \forall i = 1, \dots, N$. The black solid lines are the stable branch of solutions predicted by the mean-field theory, the black dashed lines represent the unstable solutions of the self-consistent equation.

Apparently there are important consequences when the three-body interactions are considered. On the one hand, it is knowing that the incoherent state $Q = 0$ (Z_α and $Z_\beta = 0$) remains stable for all $K > 0$ due to the higher order interactions. This nonlinearity implies that a transition from the unsynchronized state to the synchronized states can never occur spontaneously. On the other hand, the function $F(Q)$ exhibits a complicated structure approaching the maximum at some $Q_c \in (0, +\infty)$, and $\lim_{Q\to+\infty} F(Q) = 0$ [Figs. 1(a) and 2(a)]. All equivalently, the inverse of Eq. (20) does not exist at all for $K > K_c = F(Q_c)^{-1}$, beyond which there are two branches of solution and the abrupt desynchronization transition is manifested [Figs. 1(c) and 1(d) and Figs. 2(c) and 2(d)]. In the next sections we will prove that Q_c determines a critical point marking the bifurcations and phase transitions of the system, and the solutions along the branch $Q > Q_c$ are attracting, while as the solutions along the branch $Q < Q_c$ are unstable. Particularly, Q_c determines the critical case, the corresponding solutions are metastable, and the phase transition is hybrid [the red solid line of Figs. 2(a) and 2(b)] [12].

IV. STABILITY ANALYSIS

A. Ott-Antonsen reduction

The self-consistency approach gives a basic understanding of macroscopic behaviors as well as phase transitions towards synchrony. Moreover, the characteristic function $F(Q)$

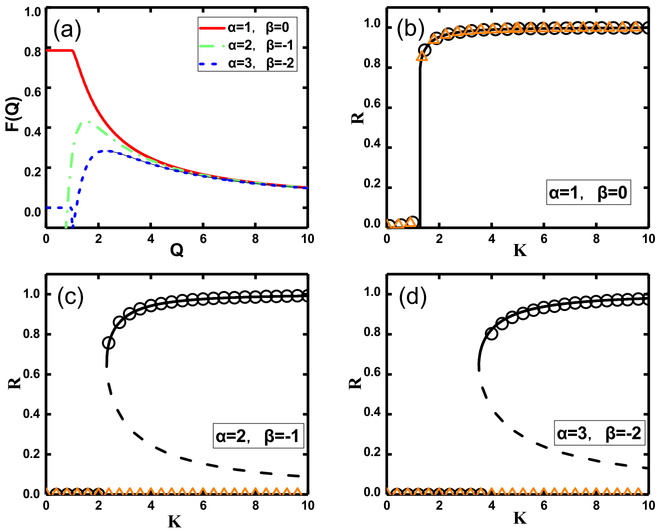

FIG. 2. (a) The characteristic functions $F(Q)$ with uniform distribution $g(\omega)=\frac{1}{2}, \omega\in[-1,1]$. $\alpha=1, \beta=0$ (red solid line), $\alpha=2, \beta=-1$ (green dash-dotted line), $\alpha=3, \beta=-2$ (blue short-dashed line). (b)–(d) Corresponding phase diagrams of the Kuramoto order parameter $R=|Z_1|$ versus the global coupling strength K, respectively. The orange triangles and the black circles represent forward and backward transitions of the numerical simulations with $N=100\,000$, respectively. In the forward continuations, the initial conditions $\{\theta_i(0)\}$ for each K are uniformly distributed in $[0,2\pi]$. In the backward continuations, the initial phases for each K are identical, i.e., $\{\theta_i(0)\}=0, \forall i=1,\dots,N$. The black solid lines are the stable branch of solutions predicted by the mean-field theory, the black dashed lines represent the unstable solutions of the self-consistent equation. In all cases, $Q_c\geqslant 1$ implying that the partially synchronized states with $Q<1$ are unstable.

features the bifurcation that leads to transitions between different collective states as the system's parameters are varied. It can be seen that the three-body interactions are crucial ingredients for the formation of abrupt desynchronization transitions.

In order to better understand the generic properties of this phase transition, we now turn our attention to explore the asymptotical stability of the equilibrium states described by Eqs. (18)–(20). In doing so, we give the main sketch of the Ott-Antonsen method pioneered in [47,48], which consists of an invariant manifold of solutions of Eq. (5). Note that $\rho(\theta,\omega,t)$ is a 2π-periodic function with respect to θ allowing for the Fourier expansion as

$$\rho(\theta,\omega,t)=\frac{1}{2\pi}\left[1+\sum_{n=1}^{+\infty}\bar{z}_n(\omega,t)e^{in\theta}+\sum_{n=1}^{+\infty}z_n(\omega,t)e^{-in\theta}\right], \tag{24}$$

where the complex quantity $z_n(\omega,t)$ represents the nth Fourier coefficient, and $\bar{z}_n(\omega,t)$ denotes its complex conjugate. In this form, determining the distribution function $\rho(\theta,\omega,t)$ is equivalent to determining the set of $\{z_n(\omega,t)\}$. Furthermore, the Ott-Antonsen ansatz pointed out that all Fourier coefficients collapse onto a condition, i.e., $z_n(\omega,t)=z^n(\omega,t)$. Inserting the ansatz Eq. (24) into Eq. (5), the Fourier coefficient $z(\omega,t)$ satisfies a single differential equation

$$\frac{dz}{dt}=i\omega z+\frac{K}{2}[X(t)-\bar{X}(t)z^2], \tag{25}$$

with $X(t)=Z_\alpha(t)Z_\beta(t)$.

Notice that Eq. (25) must be closed by using the fact that in the thermodynamic limit $N\to\infty$, the generalized order parameters take the form

$$Z_{\alpha,\beta}=\int_{-\infty}^{+\infty}g(\omega)d\omega\int_0^{2\pi}e^{i(\alpha,\beta)\theta}\rho(\theta,\omega,t)d\theta=\hat{g}z_{\alpha,\beta}(\omega,t). \tag{26}$$

Equation (25) together with Eq. (26) forms a closed description of the macroscopic dynamics of Eq. (1). Specifically, Eq. (25) governs the evolution of $z(\omega,t)$ in terms of $Z_\alpha Z_\beta$, while Eq. (26) determines $Z_{\alpha,\beta}(t)$ as a function of $z(\omega,t)$.

We next seek solutions of Eq. (25) in terms of the equilibriums states and imposing small perturbations away from them. By equilibrium states, it means that $\dot{z}=0$ and $X(t)$ is time dependent, which yields

$$z_0(\omega)=\begin{cases}\frac{\sqrt{Q^2-\omega^2}+i\omega}{Q}=e^{i\theta_\omega}, & |\omega|<Q,\\ i\frac{\omega-sgn(\omega)\sqrt{\omega^2-Q^2}}{Q}, & |\omega|>Q.\end{cases} \tag{27}$$

It is noteworthy that $|z(\omega,t)|\leqslant 1$ and $Q>0$, therefore, other branches of solution of $z_0(\omega)$ have been ruled out. Remarkably, the first branch of Eq. (27) is precisely Eq. (7) corresponding to the trapped oscillators, while the second one is exactly Eq. (17) corresponding to the drifting populations.

B. Stability of the incoherent state

The first task is to study the stability of the incoherent state, in which $z_0(\omega)=0$ and $Z_\alpha=Z_\beta=Q=0$. Let

$$z(\omega,t)=0+\varepsilon\eta(\omega,t), \tag{28}$$

with $0<\varepsilon\ll 1$ being the magnitude of perturbation function. For clarity of representation, we consider only the case $\alpha\geqslant 1$, and the result can then be generalized to the case $\beta\geqslant 1$. According to Eqs. (3) and (26) with $z_\alpha=z^\alpha$ and $z_\beta=z^\beta$, we have that $Z_\beta=\hat{g}z^\beta=\bar{Z}_{-\beta}=\hat{g}(\bar{z})^{-\beta}=\hat{g}(\bar{z})^{\alpha-1}$.

The corresponding perturbed mean-field $X(t)$ is $X(t)=\hat{g}z^\alpha\hat{g}z^\beta=\hat{g}z^\alpha\hat{g}(\bar{z})^{\alpha-1}=\varepsilon^{2\alpha-1}\hat{g}\eta^\alpha\hat{g}(\bar{\eta})^{\alpha-1}$. Substituting these terms into Eq. (25) up to the leading order of ε, we then have

$$\frac{d\eta}{dt}=i\omega\eta+\frac{K}{2}\hat{g}\eta, \quad \alpha=1, \tag{29}$$

and

$$\frac{d\eta}{dt}=i\omega\eta, \quad \alpha>1. \tag{30}$$

Let $\lambda\in\mathbb{C}$ be the eigenvalues of linear evolution equation [Eqs. (29) and (30)], i.e., $d\eta/dt=\lambda\eta$ for $\alpha=1$, multiplying both sides of Eq. (29) by the operator $\hat{g}$, the self-consistent equation for λ reads

$$\frac{1}{K}=\frac{1}{2}\int_{-\infty}^{+\infty}d\omega\frac{g(\omega)}{\lambda-i\omega}, \quad \lambda\neq i\omega. \tag{31}$$

The critical point K_c characterizing the instability of the incoherent state is obtained by imposing $\lambda\to 0^+$, then

$(\lambda - i\omega)^{-1} \rightarrow \pi\delta(0)$ and $K_c = 2/\pi g(0)$ corresponding to the result of the classic Kuramoto model, where the eigenvalues are empty for $K < K_c$, and the incoherent state remains stable [49]. However, the eigenvalues with positive real parts appear as long as $K > K_c$ implying the instability of the incoherent state. In contrast, for $\alpha > 1$, Eq. (30) shows that $\lambda = i\omega$ (continuous spectrum) indicating that the incoherent state is stable for all K. This is because the high order interactions involved in Eq. (25) do no contribute to linear equation of the perturbation making no effects on the emergence of discrete spectrum of the linearization. This result establishes that the asynchronous state is always stable as long as the many-body interactions are considered.

C. Stability of the coherent states

We now carry out the stability analysis of the coherent states $z_0(\omega)$ given by Eq. (27). To this end, we write $z(\omega, t)$ as a sum of a steady part and time-dependent small perturbation, i.e.,

$$z(\omega, t) = z_0(\omega) + \varepsilon\eta(\omega, t). \tag{32}$$

Recall that the mean-field $X(t)$ under perturbation becomes $X(t) = X_0 + \varepsilon X_1(t)$, where $X_0 = \hat{g}z_0^\alpha \hat{g}z_0^\beta$ and $X_1(t) = a\hat{g}(\eta z_0^{\alpha-1}) + b\hat{g}(\bar{\eta}\bar{z}_0^{\alpha-2})$, with $a = \alpha\hat{g}z_0^\beta$ and $b = -\beta\hat{g}z_0^\alpha$.

Substituting this definitions into Eq. (25) and neglecting the high order terms of ε, we obtain the linear evolution equation of $\eta(\omega, t)$, which is

$$\frac{d\eta}{dt} = c_\omega \eta + \frac{K}{2}\big(X_1 - \bar{X}_1 z_0^2\big), \tag{33}$$

where the coefficient c_ω is given by

$$c_\omega = i\omega - K\bar{X}_0 z_0 = \begin{cases} -\sqrt{Q^2 - \omega^2}, & |\omega| < Q, \\ i\mathrm{sgn}(\omega)\sqrt{\omega^2 - Q^2}, & |\omega| > Q, \end{cases} \tag{34}$$

which corresponds to the continuous spectrum.

For completeness, we need to consider the linear evolution of $\bar{\eta}(\omega, t)$ at the same time. Defining a vector $\mathbf{V} = (\eta, \bar{\eta})^T$, then Eq. (33) together with its complex conjugate can be written in a compact form

$$\frac{d\mathbf{V}}{dt} = \mathbf{MV} + \frac{K}{2}\mathbf{S}\hat{\mathbf{G}}\mathbf{V}, \tag{35}$$

where the multiplication matrix $\mathbf{M}$ is $\mathbf{M} = \begin{pmatrix} c_\omega & 0 \\ 0 & \bar{c}_\omega \end{pmatrix}$, and the 2×2 matrix $\mathbf{S}$ is given by $\mathbf{S} = \begin{pmatrix} a & -z_0^2\bar{b} \\ -\bar{z}_0^2 a & \bar{b} \end{pmatrix}$, and the operator matrix $\hat{\mathbf{G}} = \begin{pmatrix} \hat{g}z_0^{\alpha-1} & \frac{b}{a}\hat{g}\bar{z}_0^{\alpha-2} \\ \hat{g}z_0^{\alpha-2} & \frac{\bar{a}}{\bar{b}}\hat{g}\bar{z}_0^{\alpha-1} \end{pmatrix}$.

Following the standard procedure for treating linear operator, let $d\mathbf{V}/dt = \lambda\mathbf{V}$ with $\lambda \in \mathbb{C}$ be the eigenvalues of the linearization. We get

$$\frac{K}{2}\mathbf{S}\hat{\mathbf{G}}\mathbf{V} = (\lambda\mathbf{I} - \mathbf{M})\mathbf{V}. \tag{36}$$

Multiplying both sides with the inverse operator $(\lambda\mathbf{I} - \mathbf{M})^{-1}$ and applying the operator $\hat{\mathbf{G}}$ to both sides of Eq. (36), we then have

$$\frac{K}{2}\mathbf{J}\hat{\mathbf{G}}\mathbf{V} = \hat{\mathbf{G}}\mathbf{V}, \tag{37}$$

with the matrix $\mathbf{J}$ being $\mathbf{J} = \hat{\mathbf{G}}[(\lambda\mathbf{I} - \mathbf{M})^{-1}\mathbf{S}]$, whose entries are respectively calculated as $J_{11}(\lambda) = a\hat{g}(\frac{z_0^{\alpha-1}}{\lambda - c_\omega}) - b\hat{g}(\frac{\bar{z}_0^{\alpha}}{\lambda - \bar{c}_\omega})$, $J_{12}(\lambda) = -\bar{b}\hat{g}(\frac{z_0^{\alpha+1}}{\lambda - c_\omega}) + \frac{|b|^2}{a}\hat{g}(\frac{\bar{z}_0^{\alpha-2}}{\lambda - \bar{c}_\omega})$, $J_{21}(\lambda) = a\hat{g}(\frac{z_0^{\alpha-2}}{\lambda - c_\omega}) - \frac{|a|^2}{\bar{b}}\hat{g}(\frac{\bar{z}_0^{\alpha+1}}{\lambda - \bar{c}_\omega})$, and $J_{22}(\lambda) = -\bar{b}\hat{g}(\frac{z_0^{\alpha}}{\lambda - c_\omega}) + \bar{a}\hat{g}(\frac{\bar{z}_0^{\alpha-1}}{\lambda - \bar{c}_\omega})$.

Clearly a nonzero solution of the vector $\hat{\mathbf{G}}\mathbf{V}$ requires λ to be a root of the determinant

$$\det\Big(\mathbf{I} - \frac{K}{2}\mathbf{J}\Big) = 0, \tag{38}$$

or equivalently,

$$\Big(1 - \frac{K}{2}\mathbf{J}_{11}\Big)\Big(1 - \frac{K}{2}\mathbf{J}_{22}\Big) - \frac{K^2}{4}\mathbf{J}_{12}\mathbf{J}_{21} = 0. \tag{39}$$

For a real eigenvalue $\lambda \in \mathbb{R}$, we show that $J_{11}(\lambda) = J_{22}(\lambda)$ and $J_{12}J_{21} = (ax - by)^2$ with $x = \hat{g}[z_0^{\alpha+1}/(\lambda - c_\omega)]$ and $y = \hat{g}[\bar{z}_0^{\alpha-2}/(\lambda - \bar{c}_\omega)]$. After some tedious calculations, we arrive at a pair of spectrum equations determining the eigenvalues of the linearization about the synchronized states, That are

$$\frac{1}{K} = H_c(\lambda) = \frac{1}{2}\int_{-\infty}^{+\infty} g(\omega)\frac{\big(az_0^{\alpha-1} + bz_0^{\alpha-2}\big)\big(1 - z_0^2\big)}{\lambda - c_\omega}d\omega, \tag{40}$$

$$\frac{1}{K} = H_s(\lambda) = \frac{1}{2}\int_{-\infty}^{+\infty} g(\omega)\frac{\big(az_0^{\alpha-1} - bz_0^{\alpha-2}\big)\big(1 + z_0^2\big)}{\lambda - c_\omega}d\omega. \tag{41}$$

The stability of the coherent states is entirely controlled by the sign of λ. They are stable for $\lambda < 0$, otherwise they are unstable. Particularly, $\lambda = 0$ indicates the bifurcation between different steady states.

D. Discussion about the eigenvalue equations

We introduce several parameters to simplify notation. Let $s = \omega/Q$, $q = \mathrm{sgn}(s)\sqrt{s^2 - 1}$, $h = s - q$. We also define two integral operators to separately consider the locked and drifting populations labeled as $\hat{g}_1 \rightarrow \hat{g}_{|\omega|<Q}$ and $\hat{g}_2 \rightarrow \hat{g}_{|\omega|>Q}$.

As the first step we are able to prove that $K^{-1} = H_s(0)$ holds for all values of $K > K_c$. This is because

$$H_s(0) = (\hat{g}_1 + \hat{g}_2)\frac{\big(az_0^{\alpha-1} - bz_0^{\alpha-2}\big)\big(1 + z_0^2\big)/2}{-c_\omega}. \tag{42}$$

Notice that $z_0 = e^{i\theta\omega}$ for $|s| < 1$, and $z_0 = ih$ for $|s| > 1$. Correspondingly, $c_\omega = -Q\cos\theta_\omega$ for $|s| < 1$, and $c_\omega = iQq$ for $|s| > 1$. Using these relations we have

$$H_s(0) = \hat{g}_1\frac{\big(az_0^{\alpha-1} - bz_0^{\alpha-2}\big)z_0\cos\theta_\omega}{Q\cos\theta_\omega} + \hat{g}_2\frac{\big(az_0^{\alpha-1} - bz_0^{\alpha-2}\big)(1 + isz_0)}{-iQq}, \tag{43}$$

where we have used the identity $(1 + z_0^2)/2 = z_0(z_0^{-1} + z_0)/2 = z_0\cos\theta_\omega$ for $|s| < 1$, and the relation $z_0^2 = 1 + 2isz_0$ [see Eq. (27)]. From now on we should always keep these in mind. Therefore, the first term on the right-hand side of Eq. (43) reduces to $\frac{1}{Q}\hat{g}_1(az_0^\alpha - bz_0^{\alpha-1})$.

Regarding the second term above, the calculations now become more involved. Here and in the following, assuming that

$\alpha \geqslant 1$ is an odd number, then $\beta \leqslant 0$, which is even. Recall that the functions $z_0(s)$, $h(s)$, $q(s)$, and s itself are odd functions with the argument s on the symmetrical interval $|s| > 1$, respectively. So the quantity $\hat{g}_2[az_0^{\alpha-1}(1 + isz_0)/(-iQq)]$ vanishes automatically leading the second term above to be

$$\frac{1}{Q}\hat{g}_2\frac{bz_0^{\alpha-2}z_0(1 + isz_0)}{iqz_0} = \frac{1}{Q}\hat{g}_2\frac{bz_0^{\alpha-1}(1 - sh)}{-qh} = -\frac{1}{Q}\hat{g}_2 bz_0^{\alpha-1}, \tag{44}$$

in which we have used the identity $(1 - sh)/(qh) = 1$, using this result we get

$$\begin{aligned} H_s(0) &= \frac{1}{Q}\left[\hat{g}_1\left(az_0^{\alpha} - bz_0^{\alpha-1}\right) - \hat{g}_2\left(bz_0^{\alpha-1}\right)\right] \\ &= \frac{1}{Q}\left(a\hat{g}_1 z_0^{\alpha} - b\hat{g}z_0^{-\beta}\right) = \frac{1}{Q}\left(\alpha\hat{g}z_0^{\beta}\hat{g}z_0^{\alpha} + \beta\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}\right) \\ &= \frac{1}{Q}\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}, \end{aligned} \tag{45}$$

where we have considered the relations $\hat{g}z_0^{-\beta} = \hat{g}z_0^{\beta}$ (since Z_β is real), $\hat{g}_1 z_0^{\alpha} = \hat{g}z_0^{\alpha}$ (because $\hat{g}_2 z_0^{\alpha} = 0$), and $\alpha + \beta = 1$. Remarkably, Eq. (45) is precisely the self-consistent equation obtained in Eq. (20), i.e., $H_s(0) = K^{-1}$. This identity shows that $\lambda = 0$ is always a trivial solution of Eq. (41) originating from the rotational invariance of the phase dynamics Eq. (1).

Turning to the nonzero eigenvalue $\lambda \neq 0$, we introduce the difference

$$\Delta(\lambda) = H_c(\lambda) - K^{-1}. \tag{46}$$

Next, we will prove the identity

$$\Delta(0) = QF'(Q). \tag{47}$$

To proceed, let $Y = [(az_0^{\alpha-1} + bz_0^{\alpha-2})(1 - z_0^2)/2]/(-c_\omega)$, we have

$$\Delta(0) = \hat{g}_1 Y + \hat{g}_2 Y - K^{-1}. \tag{48}$$

On the other hand, observe that

$$\begin{aligned} QF'(Q) &= Q\left(\frac{\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}}{Q}\right)' \\ &= Q\left[\frac{\left(\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}\right)'Q - \hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}}{Q^2}\right] = \left(\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}\right)' - K^{-1}. \end{aligned} \tag{49}$$

Therefore, proving the identity Eq. (47) is equivalent to proving the following identity:

$$\hat{g}_1 Y + \hat{g}_2 Y = \left(\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}\right)'. \tag{50}$$

The right-hand side of Eq. (50) can be directly obtained as

$$\begin{aligned} \left(\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}\right)' &= \hat{g}\left(z_0^{\alpha}\right)'\hat{g}z_0^{\beta} + \hat{g}z_0^{\alpha}\hat{g}\left(z_0^{\beta}\right)' \\ &= \hat{g}\left(z_0^{\alpha}\right)'\frac{a}{\alpha} - \frac{b}{\beta}\hat{g}\left(z_0^{\beta}\right)' = \hat{g}(z_0^{\alpha})'\frac{a}{\alpha} - \frac{b}{\beta}\hat{g}\left(\bar{z}_0^{-\beta}\right)' \\ &= \hat{g}\left(z_0^{\alpha-1}z_0'a + b\bar{z}_0^{-\beta-1}\bar{z}_0'\right) = \hat{g}\left(az_0^{\alpha-1}z_0' + b\bar{z}_0^{\alpha-2}\bar{z}_0'\right) \\ &= \hat{g}\left(az_0^{\alpha-1}z_0'\right) + \hat{g}\left(b\bar{z}_0^{\alpha-2}\bar{z}_0'\right) \\ &= \hat{g}\left(az_0^{\alpha-1}z_0'\right) + \frac{b}{\alpha-1}\left(\hat{g}\bar{z}_0^{\alpha-1}\right)' \\ &= \hat{g}\left(az_0^{\alpha-1}z_0'\right) + \frac{b}{\alpha-1}\left(\hat{g}z_0^{\alpha-1}\right)' \\ &= \hat{g}\left(az_0^{\alpha-1}z_0'\right) + \hat{g}\left(bz_0^{\alpha-2}z_0'\right), \end{aligned} \tag{51}$$

where we have used the fact $\hat{g}(z_0^{\alpha-1}) = Z_{\alpha-1}$ which is real. According to Eq. (27), the derivative of z_0 with respect to Q reads

$$z_0' = \begin{cases} \frac{-i\sin\theta_\omega z_0}{Q\cos\theta_\omega}, & |s| < 1, \\ \frac{sz_0}{Qq}, & |s| > 1, \end{cases} \tag{52}$$

substituting Eq. (52) into Eq. (51) we have that

$$\begin{aligned} \left(\hat{g}z_0^{\alpha}\hat{g}z_0^{\beta}\right)' &= \hat{g}_1\left[\left(az_0^{\alpha-1} + bz_0^{\alpha-2}\right)\frac{-i\sin\theta_\omega z_0}{Q\cos\theta_\omega}\right] \\ &\quad + \hat{g}_2\left[\left(az_0^{\alpha-1} + bz_0^{\alpha-2}\right)\frac{sz_0}{Qq}\right] \\ &= \hat{g}_1\left[\frac{\left(az_0^{\alpha-1} + bz_0^{\alpha-2}\right)\left(1 - z_0^2\right)/2}{Q\cos\theta_\omega}\right] \\ &\quad + \hat{g}_2\left[\frac{\left(az_0^{\alpha-1} + bz_0^{\alpha-2}\right)\left(1 - z_0^2\right)/2}{-iQq}\right] \\ &= \hat{g}_1 Y + \hat{g}_2 Y, \end{aligned} \tag{53}$$

in which we have used $\sin\theta_\omega = \frac{1}{2i}(e^{i\theta_\omega} - e^{-i\theta_\omega}) = \frac{1}{2i}(z_0 - z_0^{-1})$, for $|s| < 1$ and $1 - z_0^2 = -2isz_0$, for $|s| > 1$. This completes the proof.

The identity Eq. (47) is the most important result that can be used to justify the stability properties of the synchronized states. For instance, if the frequency distribution $g(\omega)$ is supported on a finite interval, e.g., $\omega \in [-1, 1]$, a fully locked state is achieved as long as $Q > 1$. In this scenario, the continuous spectrum $c_\omega = [-Q, -\sqrt{Q^2 - 1}]$ is confined to the negative real axis, and the integral $\hat{g}_2$ vanishes. The stability of the fully locked states become evident. It can be shown that both functions $H_c(\lambda)$ and $H_s(\lambda)$ are negative for $\lambda < -Q$ and are singular for $\lambda \in [-Q, -\sqrt{Q^2 - 1}]$. Hence, the location of an eigenvalue that is a solution to Eqs. (40) and (41) must satisfy $\lambda > -\sqrt{Q^2 - 1}$.

On the other hand, it can be seen that, both functions $H_c(\lambda)$ and $H_s(\lambda)$ are strictly decreasing for $\lambda \in (-\sqrt{Q^2 - 1}, +\infty)$. Moreover, $\lim_{\lambda \xrightarrow{0^-} -\sqrt{Q^2-1}} H_c(\lambda) = \lim_{\lambda \xrightarrow{0^-} -\sqrt{Q^2-1}} H_s(\lambda) = +\infty$, and $\lim_{\lambda \to +\infty} H_c(\lambda) = \lim_{\lambda \to +\infty} H_s(\lambda) = 0$. Therefore, only one root with $\lambda > -\sqrt{Q^2 - 1}$ to Eqs. (40) and (41) exists. We have proven that $\lambda = 0$ is a trivial root to Eq. (41) due to the rotational symmetry. Thus a nontrivial eigenvalue with $\lambda \neq 0$ is completely determined by Eq. (40).

In the case of $\Delta(0) > 0$, which is equivalent to $F'(Q) > 0$, one must have a solution $\lambda > 0$ implying an unstable fully locked state. However, for the case $\Delta(0) < 0$, which is $F'(Q) < 0$, we have $\lambda < 0$ implying a stable fully locked state. Consequently, we conclude that the stability of the fully locked states are completely determined by the shape of the characteristic function $F(Q)$. Specifically, $F(Q)$ is increasing or $F'(Q) > 0$, the fully locked states are unstable. While $F(Q)$ is decreasing or $F'(Q) < 0$, the fully locked states are attractive along this branch. Particularly, $F'(Q_c) = 0$ locates a critical point Q_c manifesting a bifurcation wherein an abrupt desynchronization transition takes place (Fig. 2).

For the partially synchronized states with the coexistence of locked oscillators ($|s| < 1$) and the drifting ones ($|s| > 1$), in this situation ω ranges over $[-Q, +Q]$ and the second integral $\hat{g}_2$ must be considered. Accordingly, the continuous spectrum $c_\omega = [-Q, 0] \bigcup \{i\mathrm{sgn}(\omega)\sqrt{\omega^2 - Q^2}, |\omega| \geqslant Q\}$ exhibits a T shape on the complex plane [50]. Likewise, both functions $H_c(\lambda)$ and $H_s(\lambda)$ are negative for $\lambda < -Q$ and are singular for $\lambda \in [-Q, 0)$. So, the solutions to Eqs. (40) and (41), if they exist, must satisfy $\lambda \geqslant 0$. We remark that $H_c(\lambda)$ and $H_s(\lambda)$ are not necessarily decreasing functions with respect to λ for $\lambda \in [0, +\infty)$. The discussion about the stability of the partially synchronized states becomes more complicated. Nevertheless, in the case of $\Delta(0) > 0$, we must have $\lambda > 0$ since $\lim_{\lambda \to +\infty} H_c(\lambda) = 0$, which implies that the partially synchronized states are unstable provided that $F'(Q) > 0$. On the contrary, something different happens if one encounters $\Delta(0) < 0$ or $F'(Q) < 0$. In fact, the eigenvalues of Eqs. (40) and (41) with $\lambda \neq 0$ do not exist at all [51]. In this sense, the partially synchronized states are asymptotically stable, rather than linearly stable [52].

V. AN EXAMPLE

In this section we give an explicit example to get analytical insights of the dynamics of coupled phase oscillators with three-body interactions. For this purpose we make the special choice of a Lorentz distribution for $g(\omega)$, i.e., $g(\omega) = \frac{\gamma}{\pi}\frac{1}{\omega^2+\gamma^2}$ with γ being the half-width of the distribution. The order parameters can be evaluated using the Cauchy residue theorem by closing the contour to a semicircle with an infinitely large radius in the upper complex plane, which reads

$$\begin{aligned} Z_\alpha(t) &= \hat{g}z^\alpha(\omega, t) \\ &= \int_{-\infty}^{+\infty} \left(\frac{\gamma}{\pi}\frac{1}{\omega^2 + \gamma^2}\right)[z(\omega, t)]^\alpha d\omega \\ &= \frac{1}{2\pi i}\oint_C \left(\frac{1}{\omega - i\gamma} - \frac{1}{\omega + i\gamma}\right)[z(\omega, t)]^\alpha d\omega \\ &= [z(i\gamma, t)]^\alpha = Z^\alpha(t), \quad \alpha \geqslant 1. \end{aligned} \tag{54}$$

Similarly, $Z_\beta = \bar{Z}_{-\beta} = \hat{g}\bar{z}^{-\beta} = \bar{Z}^{-\beta}$ (since $\beta < 0$).

Using Eq. (25) by replacing ω with $i\gamma$, the evolution of the Kuramoto order parameter $Z(t)$ is obtained as

$$\frac{dZ}{dt} = -\gamma Z + \frac{K}{2}(Z^\alpha \bar{Z}^{-\beta} - \bar{Z}^\alpha Z^{-\beta} Z^2). \tag{55}$$

It is convenient to introduce polar coordinates. Let $Z = Re^{i\Theta}$. Then Eq. (55) reduces to

$$\frac{dR}{dt} = -\gamma R + \frac{K}{2}R^{\alpha-\beta}(1 - R^2), \tag{56}$$

$$\frac{d\Theta}{dt} = 0. \tag{57}$$

It becomes apparent that the term $\sim R^{\alpha-\beta}$ appears as a result of the many-body interactions. As a consequence, when $\alpha = 1$, the incoherent state $R = 0$ loses its stability at $K = K_c = 2\gamma$. Otherwise, for $\alpha > 1$, the factor $R^{\alpha-\beta}$ turns out to be a high order term making no effects on the stability of the incoherent state, which is always stable corresponding to an eigenvalue $\lambda = -\gamma$.

For the coherent states with $R \neq 0$, steady solution of Eq. (56) requires that $2\gamma K^{-1} = R^{\alpha-\beta-1}(1 - R^2)$. Obviously when K is sufficiently small, no solutions exist. However, when K is increased beyond a certain threshold K_c, two solutions appear located on either side of the critical point R_c, respectively. One is stable with $R > R_c$, and the other with $R < R_c$ is unstable. Straightforward calculations show that the associated critical points are

$$K_c = \frac{\gamma(\alpha - \beta - 1)^{\frac{1+\beta-\alpha}{2}}}{(\alpha - \beta + 1)^{\frac{\beta-\alpha-1}{2}}} \tag{58}$$

and

$$R_c = \sqrt{\frac{\alpha - \beta - 1}{\alpha - \beta + 1}}. \tag{59}$$

In particular, $\alpha = 1$ and $\beta = 0$, $K_c = 2\gamma$, and $R_c = 0$ characterizing a second order (continuous) phase transition towards synchronization, which is consistent with the classic Kuramoto model. When $\alpha > 1$ and $\beta < 0$, $R_c > 0$ giving rise to an abrupt desynchronization transition observed in Sec. III.

VI. CONCLUSIONS

In conclusion, this paper uncovers the effects of three-body interactions on collective synchronization of a population of globally coupled phase oscillators with distributed natural frequencies. We find that the nonlinearity in the coupling encoded by the simplicial interactions has significant dynamical consequences giving rise to abrupt desynchronization transitions that allow the system to switch between the incoherence and synchronized states via an explosive way. We provide a generic framework for analytically capturing the collective dynamics induced by the higher order interactions, and for systematically exploring the dynamical stability of the steady states. A universal condition for the onset of abrupt desynchronization was established in terms of characteristic functions. The mathematical mechanism of such an abrupt dynamical pattern was further addressed by providing a detailed description of spectrum structure of the equilibrium states in the Ott-Antonsen manifold. The goal is to clarify the dynamical consequences of many-body interactions by investigating a particularly simple and tractable extended Kuramoto model. Our hope is that such an investigation may bring us a step closer toward exploring the dynamics of many-body interactions with simplicial complexes in a variety of complex systems.

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