Space-time fractional porous media equation: Application on modeling of S&P500 price return

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We present the fractional extensions of the porous media equation (PME) with an emphasis on the applications in stock markets. Three kinds of "fractionalization" are considered: *local*, where the fractional derivatives for both space and time are local; *nonlocal*, where both space and time fractional derivatives are nonlocal; and *mixed*, where one derivative is local, and another is nonlocal. Our study shows that these fractional equations admit solutions in terms of generalized q-Gaussian functions. Each solution of these fractional formulations contains a certain number of free parameters that can be fitted with experimental data. Our focus is to analyze stock market data and determine the model that better describes the time evolution of the probability distribution of the price return. We proposed a generalized PME motivated by recent observations showing that q-Gaussian distributions can model the evolution of the probability distribution. Various phases (weak, strong super diffusion, and normal diffusion) were observed on the time evolution of the probability distribution of the price return separated by different fitting parameters [Phys. Rev. E **99**, 062313 (2019)]. After testing the obtained solutions for the S&P500 price return, we found that the local and nonlocal schemes fit the data better than the classic porous media equation.

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I. INTRODUCTION

Local differential operators have been widely applied to formulate the physical laws in terms of partial differential equations. This includes various stochastic processes such as classical diffusion, which are described in terms of local stochastic differential equations as well as the Fokker-Planck equation. The best-known example of the local differential equations in the stock market with an application of Brownian motion is the classical Black-Scholes (BSE) equation (see [1]). However, the situation is different for many physical, financial, and economic processes that cannot be described in terms of local differential operators. Such processes are of a global nature, for which the growth or change of a field depends on the field configuration over the *whole space-time*. The anomalous diffusion is an example of the scaling between space and time described by a power-law relation with an anomalous exponent. This constitutes a wide-ranging class of physical processes to which many systems are mapped, such as turbulence [2–6], light scattering of clouds [7], and stock markets [8].

Despite the widespread use of BSE to describe the price evolution in markets with inherent risk in their expected returns, the BSE fails in many option pricing applications [9]. The BSE limitations open up an opportunity to model the stock market price by nonlocal operators by considering three aspects: (1) The experimental measurements of stock market price return X exhibits mean-square displacements that scale as a power law with time as $\langle X^2\rangle \sim t^\alpha$. In the S&P500 stock market index, α is larger than the value of 1 given by classical diffusion, so that is an anomalous diffusion exponent [8]. (2) The observed probability distribution function (PDF) of price return is non-Gaussian with heavy tails that rules out normal diffusion. (3) The time series of price return exhibits short-time correlations.

A scenario for explaining these observations is to assume that any stock market price change causes an impact across a time interval, leading to nonlocal effects. As a consequence, a mathematical description of the stock market price change should include nonlocal operators [10–12], one way of which is fractionalizing the differential operators in the governing equations. The fractional calculus emerges from considerations that the spatiotemporal correlations of a stochastic process can be translated into time or space fractional differentials [13]. Nonlocal operators are employed in this field to describe the nonlocal features of the system, be it spatial, capturing the nonlocal interactions in space or temporal capturing the temporal correlations in the system.

Fractional diffusion equations for anomalous diffusion govern random walks where a random waiting time separates random particle jumps. A power-law probability distribution for particle jumps (*Levy flights*) leads to fractional derivatives in space. The Levy flights are defined as Markovian random walks that converge to stable densities because of power-law distributed jumps [14–16]. Power-law waiting times correspond to time fractional derivatives. Particle traces are random fractals, whose dimension relates to the orders of the fractional derivatives [17]. Other examples in financial economics

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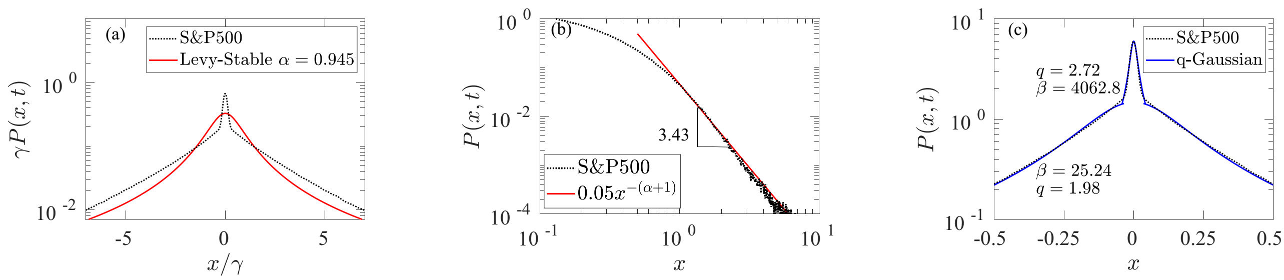

FIG. 1. Comparison of the probability distribution of the price return of S&P500 for $\Delta t = 1$ min with two different fittings. (a) For the Levy-Stable distribution, the fitting values after applying a least-squares fitting method are $\alpha = 0.95$ and $\gamma = 0.11$. (b) A second calculation was made to obtain the characteristic exponent from the power law of the tails of the PDF. The value of $\alpha = 2.43$ lies outside the Levy regime, ruling out the Levy-stable distribution function. (c) The q-Gaussian distribution function (blue). Two q-Gaussians capture better the anatomy probability distribution of the price return of S&P500.

are the pricing contracts, where the fractional partial differential equations have proved to be a useful tool.

Fractionalization of the classical diffusion equation using the Riesz fractional derivative gives the α-stable Levy distribution as its solution, which has been widely used in the literature of the stock markets. For fractional calculus and applications in finance, see [9,18–21].

Although being promising in some aspects in stock markets, the Levy flights have shown some restrictions to model stock market prices due to its infinite variance and uncorrelated time series. The presence of short time-correlations of the price return rules out the main hypothesis of the Levy regime of independent identically distributed random variables [22]. The Levy-stable distribution provides only an estimation of the stock market fluctuations at low frequencies where the correlations can be neglected. However, correlations during the first minutes on the price fluctuations were observed at high frequencies, making the Levy regime no longer applicable. Additionally, the characteristic exponents applying to model the power-law tails of the price return's PDF lie outside the Levy regime. These occur since most of the equations modeling physical phenomena of this kind are nonlinear. Based on the results shown in Fig. 1, it is clear that the Levy-stable distribution does not capture the PDF of the stock market's price return adequately.

An alternative to describe the Levy diffusion process is through the porous media equation (PME). The PME is a generalization (nonlinear version) of normal diffusion that shows connections with the fluctuations in stock markets [8,23]. Initially the PME was employed to describe the correlated anomalous diffusion processes with finite variance [24]. Many analytical [25–33] and numerical [34–37] methods have been developed to study the properties of PME, which admits modeling anomalous processes in nonextensive statistical mechanics with the q-Gaussian function as its solution. Indeed, the PME is capable of being applied to self-similar and scale-invariant systems such as financial markets where q-Gaussian distributions for the price return are observed [8,38–40]. Gluing the two ideas mentioned above (self-similarity and nonlinearity), one ends up with the fractional PME (FPME), which has been studied in many papers [24,40–43], aiming to study anomalous diffusion in porous media and other problems related to PME, each of which with a particular (local or nonlocal) "fractionalization" scheme. The fractional derivative in FPME is responsible for the power-law behavior of the mean-square displacement and the heavy-tailed waiting times in the corresponding continuous-time random walk scheme.

The aim of the present paper is twofold: (1) derive an alternative diffusion equation for the stock market by fractionalization of the time derivative of the PME (accounting for the strong short-time correlation observed in the stock market), and (2) generalize the PME using both time and space derivatives and study the possible solutions. In physical systems, the space fractionalization accounts for nonlocal interaction (see [44]). In stock markets, the space variable represents the price return of the stock index so that the space fractionalization accounts for the nonlocal interaction of the price fluctuations. We then discuss the properties of a class of solutions of local and nonlocal FPME. We show that these solutions are normalizable and smooth for any given order of nonlinearity. Then we apply the findings to the S&P500 stock market index and argue that the solutions of the nonlocal FPME are more suitable and reliable for this application. We attribute this to the nonlocal nature of S&P500 markets.

The paper is organized as follows: In the next section we briefly describe the solutions of PME and its fractionalization. This section gives a feeling how the q-Gaussian PDF resulted from a PME. We then describe the local-local, local-nonlocal, and mixed variants of the space and time fractional PME in Secs. III–V, respectively. We present the results of the application of the model to stock markets in Sec. VI. The relationship between the notion of nonlocality with the empirical study done in this work is discussed in the last section. We close the paper with a conclusion. Since the paper contains quite a few mathematical parts, we collected the definitions and some properties of the model in Tables I– IV.

II. OVERVIEW: q-GAUSSIAN PDF AND FRACTIONAL PME

The PME is one of the simplest examples of a nonlinear diffusion widely used to describe processes that involve fluid flows and heat transfer [53]. Also, it can be used to investigate any diffusion process where the diffusion coefficient depends on the state variable. The classical PME is

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P^{2-q}(x,t)}{\partial x^2}. \quad (1)$$

TABLE I. Definitions of the most common local (1) and nonlocal (2–4) fractional derivatives.

N	Fractional derivative	Definition	Ref.
1	Katugampola	$\mathcal{D}^{\alpha} f(x) = \lim_{\epsilon\to 0} \frac{f^{(n)}(xe^{\epsilon x^{n-\alpha}}) - f^{(n)}(x)}{\epsilon} \quad (n < \alpha \leqslant n+1)$	[45]
2	Riemann-Liouville	$^{RL}_{a}\mathcal{D}^{\alpha,x} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}\frac{f(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau \quad (n-1 < \alpha \leqslant n)$ $^{RL}\mathcal{D}^{\alpha,x}_{b} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{b}\frac{f(\tau)}{(\tau-x)^{\alpha-n+1}}d\tau \quad (n-1 < \alpha \leqslant n)$	[46–48]
3	Caputo	$^{C}_{a}\mathcal{D}^{\alpha,x} f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau \quad (n-1 < \alpha \leqslant n)$ $^{C}\mathcal{D}^{\alpha,x}_{b} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}\frac{f^{n}(\tau)}{(\tau-x)^{\alpha-n+1}}d\tau \quad (n-1 < \alpha \leqslant n)$	[47–49]
4	Rietz	$\left(\frac{d}{d\lvert x\rvert}\right)^{\alpha} f = \frac{(^{RL}_{-\infty}\mathcal{D}^{\alpha,x} + ^{RL}\mathcal{D}^{\alpha,x}_{\infty}) f(x)}{2\cos(\pi\alpha/2)}$	[47–52]

A solution of this partial differential equation (PDE) is the Barenblatt function for $q > 1$ and $t > 0$,

$$P(x,t) = \frac{1}{(Dt)^{\frac{1}{3-q}}}\left(C - \frac{1-q}{2(2-q)(3-q)}\frac{x^2}{(Dt)^{\frac{2}{3-q}}}\right)^{\frac{1}{1-q}}, \quad (2)$$

where C is an integration constant. Equation (1) has been generalized to analyze several physical situations that present anomalous diffusion [54].

Using fractional calculus, it is possible to obtain useful dynamical models, where fractional differential operators in the time and space variables describe the strong short-time correlation and nonlocal spatial properties of the complex processes and media [13]. Intuitively, a nonlocal operator is defined as the operator that needs the information in a finite interval upon its operation on a function, contrary to local operators that need only the information at one point in its close vicinity (see [55], and Appendix A). In general, many usual properties of the local (first-order) derivative $\mathcal{D}_x$ are not satisfied for fractional derivative operators $\mathcal{D}_x^{\alpha}$. For example, a product rule, chain rule, semigroup property have strongly complicated analogs for the operators $\mathcal{D}_x^{\alpha}$. A natural generalization of the local derivatives to fractional order is the *local fractional derivatives*. The concept of local fractional derivatives keeps some of the properties of ordinary derivatives. Nevertheless, they lose the memory condition of fractional order derivatives [56]. The Katugampola operator is an important example of the local fractional derivative operators [57]. Recently, fractional local operators have been used to model phenomena of turbulence [58], and anomalous diffusion [59].

The present paper proposes a generalized form of the PME that admits a broader range of results:

$$_{a}\mathcal{D}^{\xi,t^{n}} P(x,t) = D\left(^{C}\mathcal{D}^{\gamma,x^{\alpha}}_{b} P^{\nu}(x,t)\right), \quad (3)$$

where $^{C}\mathcal{D}^{\gamma,x^{\alpha}}_{b}$ denotes the Caputo fractional derivative of the PDF with respect to x^{α} (see Appendix B). The fractional derivative $\mathcal{D}$ of orders ξ and γ is a function of three variables: the limits (a and b), the arguments (t and x), and the degree order of the arguments (n and α). This last type of variable allows us to have a derivative with respect to a function when $n, \alpha \neq 1$.

A particular case of Eq. (3), when $n = \alpha = 1$ and $a = b = 0$, is

$$\frac{\partial^{\xi}}{\partial t^{\xi}} P(x,t) = D\frac{\partial^{\gamma}}{\partial x^{\gamma}} P^{\nu}(x,t). \quad (4)$$

No general solution of Eq. (4) is known. In the present paper we aim to show that a particular solution to this equation is the Green function. This function is obtained from the boundary condition $P(x,t) = 0$, when $x \to \pm\infty$, and the initial condition $P(x,0) = \delta(x)$, where $\delta(x)$ is the Dirac δ function. The Green functions can be expressed in terms of well-known distributions. Some cases are the Gaussian, the Levy-stable [54], and the q-Gaussian distributions. We show that Eq. (4) admits exact solutions that vary depending on the definition of the fractional derivatives applied. The definitions of the most commonly used fractional derivatives are contained in Table I. Equation (4) allows space and time to scale differently, and as a consequence, different solutions can be obtained.

In searching for the solutions of Eq. (4), we exploit the fact that they follow the self-similarity law,

$$P(x,t) = \frac{1}{\phi(t)} F\left[\frac{x}{\phi(t)}\right]. \quad (5)$$

Equation (5) has often been used to model the price return in stock markets. This price return obeys $\phi(t) \sim t^{H}$, H being the characteristic exponent of the PDF, and it is known as the Hurst exponent.

A summary of these self-similar solutions is contained in Table II. The first four equations of Table II have been solved previously by applying integer or fractional derivatives [40,70,73]. The last four equations have been solved in this manuscript after applying local and nonlocal derivatives. The local derivatives were used to solve equations N.5 and N.6 in Table II. The solution of equation N.6 is the *first generalized* q-Gaussian function. The fractional derivatives used to solve equations N.7 and N.8 were of nonlocal character. The equation N.7 in Table II is proposed as an improvement of equation

TABLE II. Summary of the FPMEs and their Green functions. Equations marked with (*) are used to fit the PDF of the S&P500 index, while (⋆) represents the best model.

N	Equation	Definition	Green function	Fractional derivatives	Authors
1	Diffusion (*)	$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}$	$P(x,t) = \frac{1}{\sqrt{2Dt}} g(\frac{x}{\sqrt{2Dt}}), \quad g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$	Integer derivatives	Bachelier [60], A. Einstein [61,62]
2	Anomalous super-diffusion	$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^\gamma P(x,t)}{\partial x^\gamma}$, $0 < \gamma < 2$	$P(x,t) = \frac{1}{(Dt)^{1/\gamma}} L_\gamma\Big(\frac{x}{(Dt)^{1/\gamma}}\Big)$, $L_\gamma(x) = \frac{1}{\pi}\int_0^\infty e^{-\lvert k\rvert^\gamma}\cos(kx)dk$	Riesz	P. Levy [63,64], W. Feller [64–67]
3	Classical PME (*)	$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)^{2-q}}{\partial x^2}$, $5/3 < q < 3$	$P(x,t) = \frac{1}{\sqrt{3-q}(Dt)^{\frac{1}{3-q}}} g_q(\frac{x}{\sqrt{3-q}(Dt)^{\frac{1}{3-q}}})$, $g_q(x) = \frac{1}{C_q} e_q(-x^2), \quad e_q(x) = [1+(1-q)x]^{\frac{1}{1-q}}$, $Cq = \sqrt{\frac{\pi}{q-1}}\frac{\Gamma[(3-q)/(2(q-1))]}{\Gamma[1/(q-1)]}$	Integer derivatives	Barenblatt [68,69] C.Tsallis [41,43,70,71]
4	Space-FPME	$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^\gamma P(x,t)^\nu}{\partial x^\gamma}$, $\nu = \frac{2-\gamma}{1+\gamma}$	$P(x,t) = \frac{1}{C_q (k_1 t)^{(\gamma+1)/(\gamma^2-\gamma+1)}} (\frac{z^{\gamma(\gamma+1)}}{(1+bz)^{1-\gamma^2}})^{\frac{1}{1-2\gamma}}$ $z = \frac{x}{(\lvert k_1\rvert t)^{(\gamma+1)/(\gamma^2-\gamma+1)}} \quad b = cte$, $\frac{1}{C_q} = [k\frac{\Gamma(\beta)}{\Gamma(\alpha+1)}]^{\frac{1+\gamma}{1-2\gamma}}, \quad \alpha = \frac{(2-\gamma)\gamma}{(1-2\gamma)}, \quad \beta = -\frac{\gamma^2-3\gamma+2}{1-2\gamma}$	Riemann-Liouville (RL)	C.Tsallis [40,72]
5	Time-FPME (*)	$\frac{\partial^\xi P(x,t)}{\partial t^\xi} = D\frac{\partial^2 P(x,t)^{2-q}}{\partial x^2}$	$P(x,t) = \frac{1}{(\frac{Dt^\xi}{\xi})^{\frac{1}{3-q}}} g_q(\frac{x}{(\frac{Dt^\xi}{\xi})^{\frac{1}{3-q}}})$	Katugampola	Eq. (30)
6	Time-Space-FPME (*)	$\frac{\partial^\xi}{\partial t^\xi}P(x,t) = D\frac{\partial^\gamma}{\partial x^\gamma}P^{2-q}(x,t)$, $0 < \xi \leqslant 1,\ 1 < \gamma \leqslant 2,\ 1 < q < 3$	$P(x,t) = \frac{1}{(Bt)^{\frac{\xi}{1-q+\gamma}}} g_q^\gamma(\frac{x}{(Bt)^{\frac{\xi}{1-q+\gamma}}})$, $g_q^\gamma(x) = \frac{1}{C_q^\gamma} e_q(-x^\gamma), \quad e_q(x) = [1+(1-q)x]^{\frac{1}{1-q}}$, $C_q^\gamma = \frac{2\Gamma(\frac{1}{q-1}-\frac{1}{\gamma})\Gamma(1+\frac{1}{\gamma})}{(q-1)^{1/\gamma}\Gamma(\frac{1}{q-1})}$, $B = -[\frac{\xi}{D\gamma(\gamma-1)(2-q)(1-q+\gamma)}]^{\frac{\gamma}{1-q+\gamma}}$	Katugampola	Eq. (12)
7	Time-Space-FPME	$\frac{\partial^\xi}{\partial t^\xi}P(x,t) = D\frac{\partial^\gamma}{\partial x^\gamma}P^\nu(x,t)$, $0 < \xi \leqslant 1,\ 0 < \gamma < \frac{1}{2},\ \nu > -1$	$P(x,t) = \frac{A}{t^{1/\sigma}}(\frac{x}{t^{1/\sigma}})^{\alpha\gamma}(c_1 + c_2\frac{x}{t^{1/\sigma}})^{-\alpha(1-\gamma)}$	Katugampola (time) and RL (space)	Eq. (21)
8	Particular case of the Generalized PME (⋆)	${}_0\mathcal{D}_t^{\xi,\frac{1}{t}} P(x,t) = D({}_x^C\mathcal{D}_1^{\gamma,x^\alpha} P^\nu(x,t)) \qquad \xi,\gamma > 0$, $\xi = 1+\lambda+1/\alpha$, $\nu = -1-\frac{2}{\lambda+1/\alpha}$, $\gamma = (1/\alpha-\lambda)(1+\frac{1}{\lambda+1/\alpha})$	$P(x,t) = \frac{1}{(Bt)^{\frac{1}{\alpha}}} g_q^{\alpha,\lambda}(\frac{x}{(Bt)^{\frac{1}{\alpha}}})$, $g_q^{\alpha,\lambda}(x) = \frac{1}{C_q^{\alpha,\lambda}} e_q^{\alpha,\lambda}(-x^\alpha), \quad e_q^\lambda(x) = [1+(1-q)x]^\lambda$, $C_q^{\alpha,\lambda} = 2\lambda^{1/\alpha}\frac{\Gamma(-\lambda-\frac{1}{\alpha})\Gamma(1+\frac{1}{\alpha})}{\Gamma(-\lambda)}$, $B = -\frac{1}{\eta_q^{NL}(\alpha,\lambda,D)}$ Eq. (28)	RL (time) and Caputo (space)	Eq. (27)

N.4. The equation N.7 presents fractional derivatives on both variables, time and space. After applying local derivatives on time and nonlocal derivatives on the space, the solution of equation N.7 is not a q-Gaussian and does not satisfy the self-similar law $P(x,t) \approx t^{-H}F(xt^{-H})$. Then, a particular form of Eq. (3) is proposed as equation N.8 in Table II. The equation N.8 has been solved by applying nonlocal fractional definitions on time and space. The solution of Eq. (3) is called a *second generalized* q-Gaussian function, and it presents a self-similar form, $P(x,t) \sim t^{-H}F(xt^{-H})$. By replacing $\lambda = \frac{1}{1-q}$ in the *second generalized* q-Gaussian, the *first generalized* q-Gaussian can be recovered.

Here, we aim to get show that local and nonlocal FPME admit *generalized* q-Gaussian solutions, so that for local fractional derivation, the Katugampola definition is applied; for nonlocal fractional derivation, the Riemann-Liouville and Caputo derivatives are applied. Especially, for the nonlocal fractional derivation, we consider a fractional derivative with respect to another function, in the sense of Caputo derivative. We consider both local and nonlocal FPMEs focusing on three distinct cases: (LL) referring to the case where both time and space derivatives are local, (LN) or (NL) where one of them is local and the other is nonlocal, and the (NN) referring to the case where both derivatives of time and space are nonlocal.

The main finding of the present paper is that the *local* and *nonlocal* cases admit generalized q-Gaussian functions as their Green function solutions. The difference between them is the number and the form of the fitting parameters. As an application, the local and nonlocal generalized q-Gaussian distributions are used to describe the regimes observed during the time evolution of the PDF of the S&P500 index. After addressing the q-Gaussian distribution function as a self-similar solution of the PME in the next section, we present the solution of the FPME with the local fractional derivative (LL) in Sec. III. Sections IV and V contain the analysis of the FPME with (LN) and (NN) fractionalization. In Sec. VI we present an application of the generalized q-Gaussian distribution to describe the price return of S&P 500 from the past 24 years, and we compare the results with previous solutions.

III. PME WITH LL FRACTIONAL OPERATORS

The generalized forms of PME are obtained by replacing the first time derivative or second space derivative by fractional orders derivatives in the classical PME. These generalized PMEs may model more efficiently certain real-world phenomena, especially when the dynamics are affected by constraints inherent to the system. Typically, fractional derivatives are defined with an integral representation. Consequently, they are nonlocal in character. There exist several definitions for fractional derivatives and fractional integrals like the Riemann-Liouville, Caputo, Hadamard, Riesz, Grünwald-Letnikov. However, some usual properties of these fractional derivatives are different from ordinary derivatives, such as the Leibniz rule, the chain rule, and the semigroup property. Consequently, these fractional derivatives cannot be applied for local scaling or differentiability properties. For further details, we refer the reader to [74,75] and Appendix C. Recently, the Katugampola [57] local fractional operator was used as a limit-based fractional derivative that allows zero as a possible order of the derivative. The Katugampola operator maintains many of the familiar properties of standard derivatives such as the product, quotient, and chain rules.

A. A local fractional nonlinear time-space diffusion equation

In general, finding solutions for nonlinear anomalous diffusion equations is a challenge since, besides its difficulty to get exact analytical solutions, the principle of superposition is not applicable as in the linear case, so that the Fourier analysis cannot be done. Despite this huge interest and theoretical studies on the problem, very limited information is available concerning the possible solutions of these equations and their properties, especially the dependence of the solutions on the fractionalization parameters.

In this section we solved a time-space FPME (TS-FPME) with (LL) fractionalization. Throughout this section we consider the Katugampola derivative and integral (Katugampola operators) to solve the generalized PME. By applying this local derivative, our solution will be a generalized q-Gaussian distribution. Information about Katugampola's definition and its properties can be found in Table I, Appendix A, and Table IV. The Katugampola fractional definition was applied for the time and space fractional operators. Such FPME can be written as

$$\frac{\partial^{\xi}}{\partial t^{\xi}}P(x,t) = D\frac{\partial^{\gamma}}{\partial x^{\gamma}}P^{\nu}(x,t), \tag{6}$$

where $0 < \xi \leqslant 1 < \gamma \leqslant 2, |\nu| < 1$ are a set of three free parameters, and D is the diffusion coefficient. To solve Eq. (6), we express the function $P(x,t)$ in its self-similar form:

$$P(x,t) = \frac{1}{\phi(t)}F\left(\frac{x}{\phi(t)}\right), \tag{7}$$

where $\phi(t)$ is a function to be identified. Equation (7) is consistent with a symmetric probability distribution.

By considering, $z = \frac{x}{\phi(t)}$ and inserting Eq. (7) into Eq. (6) we have the following two equations:

$$\partial_x^{\gamma}P^{\nu}(x,t) = \frac{1}{\phi^{\nu+\gamma}}\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu},$$
$$\frac{\partial^{\xi}P(x,t)}{\partial t^{\xi}} = \frac{-1}{\phi^2(t)}\frac{\partial^{\xi}\phi}{\partial t^{\xi}}\left[F + z\frac{d}{dz}F\right],$$

so that,

$$\frac{-1}{\phi^2(t)}\frac{\partial^{\xi}\phi}{\partial t^{\xi}}\frac{d}{dz}[zF] = \frac{D}{\phi^{\nu+\gamma}}\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu}.$$

In the above equation, the properties of the Katugampola derivative were used (see Appendix A and Table IV). Then we arrange everything in such a way that all quantities in one side are only a function of z and in the other side are a sole function of t. This procedure leads us to obtain the two following independent equations:

$$\phi^{\nu+\gamma-2}\frac{\partial^{\xi}\phi}{\partial t^{\xi}} = \frac{\xi}{\nu+\gamma-1},$$
$$-\frac{\xi}{\nu+\gamma-1}\frac{d}{dz}[zF] = D\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu}.$$

TABLE III. Summary of particular forms of the generalized PME Eq. (3) to model the time evolution of the PDFs of price return. The C-PME, T-FPME, TS-FPME, and G-PME are obtained after setting a specific value of the parameters in the general form of the PME. The fittings were performed by setting the parameters as shown in the table.

	Classical (C-PME)	Time fractional (T-FPME)	Time-space fractional (TS-FPME)	Particular case of the generalized PME (G-PME)
Forms of PME	$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial P^{2-q}(x,t)}{\partial x^2}$ $1 < q < 3$	$\frac{\partial^\xi P(x,t)}{\partial t^\xi} = D\frac{\partial^2 P^{2-q}(x,t)}{\partial x^2}$ $0 < \xi \leqslant 1,$ (30) $1 < q < 3$	$\frac{\partial^\xi P(x,t)}{\partial t^\xi} = D\frac{\partial^\gamma P^{2-q}(x,t)}{\partial x^\gamma}$ $0 < \xi \leqslant 1,$ $1 < \gamma \leqslant 2,$ $1 < q < 3$	${}_0\mathcal{D}^{\xi,\frac{1}{t}} P(x,t) = D({}^C\mathcal{D}_1^{\gamma,x^\alpha} P^\nu(x,t))$ $\xi, \gamma > 0$
Parameters after comparing with general PME	$q = 2-\nu,$ $\gamma = 2,$ $\xi = 1$	$q = 2-\nu,$ $\gamma = 2,$ $\xi = \frac{3-q}{\alpha}$	$q = 2-\nu,$ $\gamma > 1,$ $\xi = \frac{1-q+\gamma}{\alpha}$	$\xi = 1+\lambda+1/\alpha,$ $\nu = -1-\frac{2}{\lambda+1/\alpha},$ $\gamma = (1/\alpha-\lambda)(1+\frac{1}{\lambda+1/\alpha})$

The solution of the first equation is $\phi \propto t^{\frac{\xi}{\nu+\gamma-1}}$, and for the second one we have

$$\frac{d}{dz}[zF] = F + z\frac{d}{dz}F = k\frac{d^\gamma}{dz^\gamma}F^\nu,$$

with $k = \frac{-D(\nu+\gamma-1)}{\xi}$, which can be rewritten as

$$\frac{d}{dz}[zF] = k\frac{d^{\gamma-1}}{dz^{\gamma-1}}\frac{d}{dz}[F^\nu]. \tag{8}$$

By applying the property $\mathcal{D}^\mu(f) = \mathcal{D}^{\mu-1}\mathcal{D}^1(f)$ for $1 < \mu \leqslant 2$, and taking local fractional integral with respect to z in Eq. (8), the following expression is obtained:

$$\int^t \frac{d}{dz}[zF]\frac{dz}{z^{2-\gamma}} = k\int^t \frac{z^{2-\gamma}}{z^{2-\gamma}}\frac{d}{dz}\left[\frac{d}{dz}F^\nu\right]dz. \tag{9}$$

In the right-hand side, the integration by parts is used, $\frac{z^{\gamma-1}F}{\gamma-1} + c$, choosing $c = 0$. By considering $F = (c_1 + c_2 z^\gamma)^{\frac{1}{\nu-1}}$ to ob-

TABLE IV. Comparison of properties between Katugampola, Riemman-Liouville and Caputo fractional derivatives.

Property	Katugampola [45,100]	Riemann-Liouville [77,102–104]	Caputo [102–104]
Key property	$\mathcal{D}^\alpha f(t) = \lim_{\epsilon\to 0}\frac{f(te^{\epsilon t^{-\alpha}})-f(t)}{\epsilon}$ $\mathcal{D}^\alpha f(t) = t^{1-\alpha}\frac{df(t)}{dt}$	${}^{RL}\mathcal{D}^\alpha f(t) = \mathcal{D}^n\mathcal{I}^{n-\alpha}f(t)$ $n-1 < \alpha < n, \quad \mathcal{D}^n = \frac{d^n}{dt^n},$	${}^{C}\mathcal{D}^\alpha f(t) = \mathcal{I}^{n-\alpha}\mathcal{D}^n f(t)$ $\mathcal{I}^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)(t-\tau)^{\alpha-1}d\tau$
Cte. function	$\mathcal{D}^\alpha c = 0$	${}^{RL}\mathcal{D}^\alpha c = \frac{c}{\Gamma(1-\alpha)}t^{-\alpha}$	${}^{C}\mathcal{D}^\alpha c = 0$
Linearity		$\mathcal{D}^\alpha(af(t)+g(t)) = a\mathcal{D}^\alpha f(t) + \mathcal{D}^\alpha g(t)$	
Product (Leibniz)	$\mathcal{D}^\alpha(f(t)g(t)) = f(t)\mathcal{D}^\alpha g(t) + g(t)\mathcal{D}^\alpha f(t)$	${}^{RL}\mathcal{D}^\alpha(f(t)g(t)) = \sum_{k=0}^{\infty}\binom{\alpha}{k}({}^{RL}\mathcal{D}^{\alpha-k}f(t))g^k(t)$	${}^{C}\mathcal{D}^\alpha(f(t)g(t)) = {}^{RL}\mathcal{D}^\alpha(f(t)g(t))\ldots. -\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}[(f(t)g(t))^k(0)]$
Quotient rule	$\mathcal{D}^\alpha(\frac{f(t)}{g(t)}) = \frac{g(t)\mathcal{D}^\alpha f(t)-f(t)\mathcal{D}^\alpha g(t)}{g(t)^2}$		
Chain rule	$\mathcal{D}^\alpha(fog) = \frac{df}{dg}\mathcal{D}^\alpha g(t)$	${}^{RL}_{a}\mathcal{D}^{\alpha,x}(fog) = \frac{1}{\Gamma(1-\alpha)}(\frac{1}{g'(x)}\frac{d}{dx})^n\int_a^x\frac{g'(\tau)f(\tau)d\tau}{[g(x)-g(\tau)]^{1+\alpha-n}}$ ${}^{RL}\mathcal{D}_b^{\alpha,x}(fog) = \frac{(-1)^n}{\Gamma(1-\alpha)}(\frac{1}{g'(x)}\frac{d}{dx})^n\int_x^b\frac{g'(\tau)f(\tau)d\tau}{[g(\tau)-g(x)]^{1+\alpha-n}}$	${}^{C}_{a}\mathcal{D}^{\alpha,x}(fog) = \frac{1}{\Gamma(1-\alpha)}\int_a^x\frac{g'(\tau)f_g^{(n)}(\tau)d\tau}{[g(x)-g(\tau)]^{1+\alpha-n}},$ ${}^{C}\mathcal{D}_b^{\alpha,x}(fog) = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_x^b\frac{g'(\tau)f_g^{(n)}(\tau)d\tau}{[g(\tau)-g(x)]^{1+\alpha-n}},$ $f_g^n(\tau) = (\frac{1}{g'(x)}\frac{d}{dx})^n f(\tau)$
Power function	$\mathcal{D}^\alpha(t^p) = pt^{p-\alpha}$	${}^{RL}\mathcal{D}^\alpha(t^p) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha},$ $p > -1, p \in \mathbb{R}$	${}^{C}\mathcal{D}^\alpha(t^p) = {}^{RL}\mathcal{D}^\alpha(t^p),$ $p > n-1, p \in \mathbb{R}.$ ${}^{C}\mathcal{D}^\alpha(t^p) = 0,$ $p \leqslant n-1, p \in \mathbb{N}$

tain a special solution (where c_1 and c_2 are constants), the following expression is obtained:

$$\frac{d}{dz}[F^{\nu}] = \frac{\gamma \nu c_2}{\nu - 1} z^{\gamma-1} F.$$

After incorporating the previous expression into Eq. (9), we obtain

$$c_2 = \frac{-(\nu-1)\xi}{D\gamma(\gamma-1)\nu(\nu+\gamma-1)}.$$

Therefore the general solution is

$$P(x,t) \propto \frac{1}{t^{\frac{\xi}{\nu+\gamma-1}}}\left(c_1 + \frac{(\nu-1)\xi}{D\gamma(\gamma-1)\nu(\nu+\gamma-1)}\frac{x^{\gamma}}{t^{\frac{\gamma\xi}{\nu+\gamma-1}}}\right)^{\frac{1}{\nu-1}}, \tag{10}$$

where c_1 is removed after applying the normalization condition in Eq. (10). Then, by defining $\nu = 2 - q$, considering $\alpha = \frac{1-q+\gamma}{\xi}$, and

$$\eta_q^L(\xi,\gamma,q,D) = \frac{\xi}{D\gamma(\gamma-1)\nu(\nu+\gamma-1)} = \frac{\xi}{D\gamma(\gamma-1)(2-q)(1-q+\gamma)}, \tag{11}$$

the following equation is reached:

$$P(x,t) = \frac{A_q^L}{t^{\frac{1}{\alpha}}}\left[1 + (1-q)\eta_q^L(\beta,\gamma,q,D)\frac{x^{\gamma}}{t^{\frac{\gamma}{\alpha}}}\right]^{\frac{1}{1-q}}, \tag{12}$$

where A_q^L is a normalization factor. In most of physical systems cases, the $P(x,t)$ is symmetric with respect to x. This point leads us to make $x \to |x|$ (i.e., its absolute value), or we can consider some values of γ that satisfy this property.

Then the normalization factor was identified as follows:

$$A_q^L = \frac{1}{2}[\eta(\xi,\gamma,q,D)]^{\frac{1}{\gamma}} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} - \frac{1}{\gamma}\right)\Gamma\left(1+\frac{1}{\gamma}\right)}, \tag{13}$$

where

$$\eta(\xi,\gamma,q,D) = \frac{\eta_q^L(\beta,\gamma,q,D)}{(\nu-1)^{-1}}.$$

For $\gamma = 2$, these parameters become $\alpha = \frac{3-q}{\xi}$, $A_q^L = \sqrt{\pi^{-1}\eta(\xi,q,D)}\frac{\Gamma(\frac{1}{q-1})}{\Gamma(\frac{3-q}{2(q-1)})}$, and $\eta_q^L(\xi,q,D) = \frac{\xi}{2D(2-q)(3-q)}$.

We call Eq. (12) the local q-Gaussian (Lq-Gaussian) distribution, which is the Green function of Eq. (6) obtained from a TS-FPME with the Katugampola fractional derivative (local fractional definition). The Lq-Gaussian has been defined as $g_q^{\gamma}(x)$, equation N.6 in Table II.

In Fig. 2 we show the Lq-Gaussians for different q values as indicated in the plot for $t = 1, \xi = 1, \gamma = 2$. In Fig. 2(a), by increasing q, the peak of curves increase and the distribution becomes narrower (the tails become heavier). In the case of Fig. 2(b) something similar occurs, where we denote the PDFs of the Lq-Gaussian for different ξ values as indicated in the plot for $t = 1, q = 1.5$, and $\gamma = 2$. The reverse occurs for Fig. 2(c): by increasing γ (considering $t = 1, q = 1.5, \xi = 1$), the peak of the PDFs decreases. Also, the time evolution of the Green function of Eq. (6) is shown in Fig. 2(d).

B. A connection between the (q,α)-stable distributions and Lq-Gaussians

In the previous section, we obtained Lq-Gaussian distributions by solving the TS-FPME. In fact, these Lq-Gaussians are generalized q-Gaussians by considering $|x|^{\gamma/2}$, $q > 1$, i.e., a q exponential in the variable $|x|^{\gamma}$. From the definition of the q exponential, it follows that $f \sim C_f|x|^{-\gamma/(q-1)}$, $C_f > 0$, as $|x| \to \infty$. Analogously, for any q-Gaussian, $g \sim C_g|x|^{-2/(q-1)}$, $C_g > 0$, as $|x| \to \infty$. By comparing the order of the power law of the asymptotes, we verify that for a fixed $1 < \gamma < 2$ and for any $1 < q < 2$ there exists a proportionality from Lq-Gaussians to q-Gaussian. For further details, see [76].

Let us denote the class of random variables with (q,γ)-stable distributions by $\mathcal{L}_q[\gamma]$. A random variable $X \in \mathcal{L}_q[\gamma]$ has a symmetric density $f(x)$ with asymptotes $f \sim C|x|^{-(1+\gamma)/(1+\gamma(q-1))}$, $|x| \to \infty$, where $1 \leqslant q < 2$, $1 < \gamma < 2$, and C is a positive constant. On the other hand, any Lq-Gaussian behaves asymptotically when $C_1/|x|^{\gamma/q-1}$. Especially any Lq_γ-Gaussian behaves asymptotically when $C_2/|x|^{\gamma/(q_\gamma-1)}$. Hence we obtain the following relationship:

$$\frac{1+\gamma}{1+\gamma(q-1)} = \frac{\gamma}{q_\gamma - 1}. \tag{14}$$

Solving this equation with respect to q_γ, we have

$$q_\gamma = \frac{\gamma Q_\gamma + 1}{\gamma + 1}, \quad Q_\gamma = 2 + \gamma(q-1). \tag{15}$$

Three parameters were linked: γ, the parameter of the γ-*stable Levy distributions*, q, the parameters of correlations, and q_γ, the parameters of attractors in terms of Lq_γ-Gaussians. Then under Eq. (15) the density of $X \in \mathcal{L}_q[\gamma]$ is asymptotically equivalent to Lq_γ-Gaussian.

The Lq-Gaussians have an interesting property. Its successive derivatives and integrations with respect to $|x|^{\gamma}$ correspond to $q_{\gamma,n}$ exponentials in the same variable $|x|^{\gamma}$, where $q_{\gamma,n} = \frac{\gamma q - n(q-1)}{\gamma - n(q-1)}$ [76].

By considering $\mathcal{G}_{q_{\gamma,n}}[\gamma]$ as a set of functions $\{be_{q_{\gamma,n}}^{-\beta|\xi|^{\gamma}}, b > 0, \beta > 0\}$ and $\mathcal{F}_{q_{\gamma,n}}$ to be the $q_{\gamma,n}$ Fourier transform, the following expression is obtained:

$$\mathcal{F}_{q_{\gamma,n}} : \mathcal{G}_{q_{\gamma,n}}[\gamma] \to \mathcal{G}_{q_{\gamma,n+1}}[\gamma], \quad -\infty < n \leqslant [\gamma/(q-1)].$$

This is similar to the q exponential with the variable $|\xi|^{\gamma}$, i.e., $e_q^{-\beta|\xi|^{\gamma}}$, $\beta > 0$, which is the q-Fourier transform of (q,γ)-stable distributions [76].

C. The local fractional nonlinear time-space diffusion equation with the drift

The drift is often an inevitable part of stochastic systems that should be analyzed in detail for every case study to control its effects, although it is suggested to define the equations for the general drift term. For the case where it depends only on time (as the case for many physical systems of interest), the situation becomes easier. In this case, the governing equation

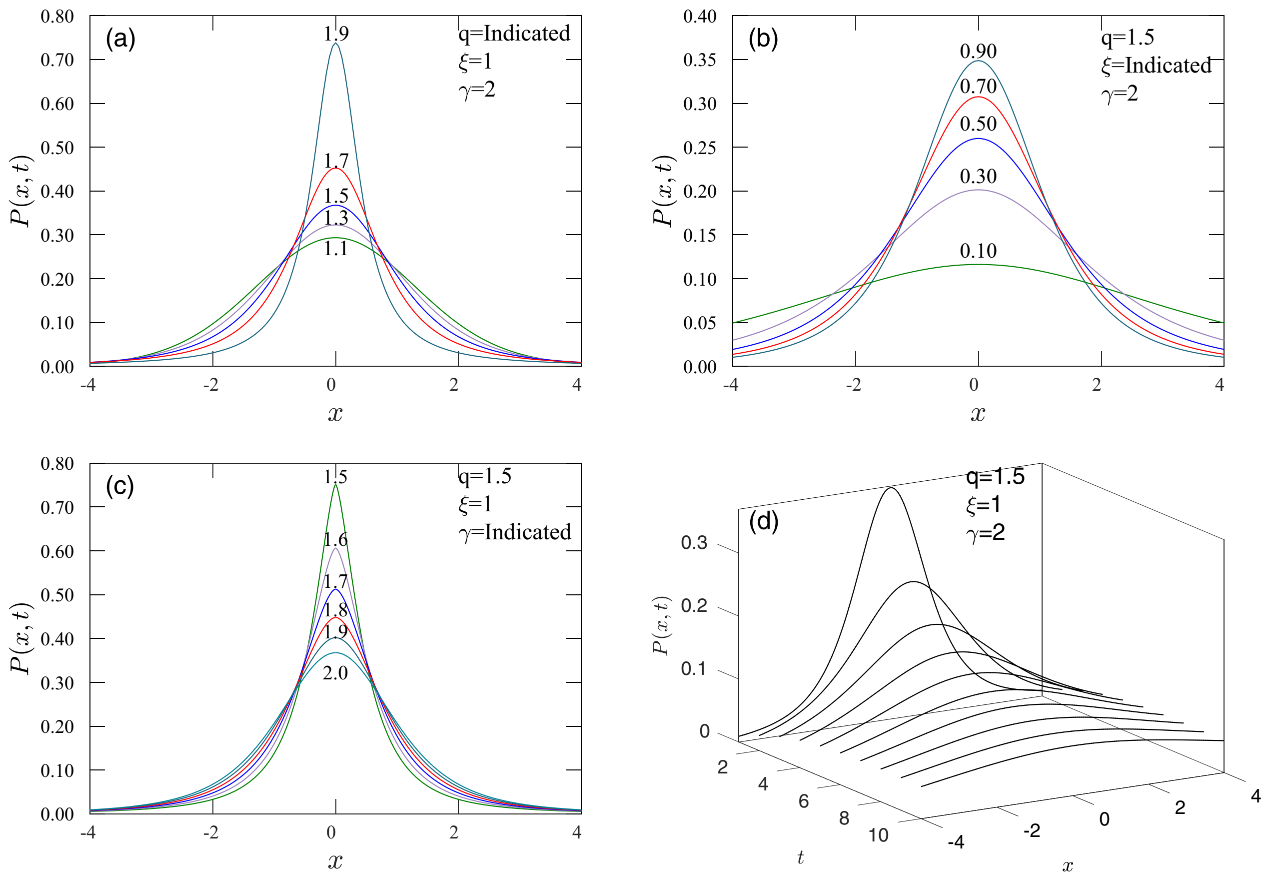

FIG. 2. (a) Lq-Gaussians for different q values as indicated in the plot for $t = 1, \xi = 1, \gamma = 2$. (b) L$q$-Gaussians for different ξ values as indicated in the plot for $t = 1, q = 1.5, \gamma = 2$. (c) L$q$-Gaussians for different γ values as indicated in the plot for $t = 1, q = 1.5, \xi = 1$. (d) Time evolution of the Green function of Eq. (6). Over time, the peaks of curves decrease and the PDFs lose the behavior of heavier tails; in contrast with (c), by increasing γ, the peaks of curves and behavior of heavier tails decreases. In (a), by increasing q, the peaks of curves increase and the distribution becomes narrower (the tails become heavier).

is

$$\frac{\partial^\xi}{\partial t^\xi}P(x,t) = -a(t)\frac{\partial P(x,t)}{\partial x} + D\frac{\partial^\gamma P^\nu(x,t)}{\partial x^\gamma},$$
$$0 < \xi \leqslant 1 < \gamma \leqslant 2, \quad |\nu| < 1. \tag{16}$$

By change of variable $\tau = t^\xi$ and the definition of the Katugampola derivative, we have

$$\partial_\tau P(x,t) = -a'(\tau)\partial_x P(x,t) + D\partial_x^\gamma P^\nu(x,t),$$

where $a'(\tau) = \frac{1}{\xi}a(t(\tau))$. By using the change of variable $(s, y) \equiv (\tau, x - x_0 - f(\tau))$, where $f(\tau) = \int_0^\tau a'(\tau')d\tau'$, and using the fact that $\frac{\partial y}{\partial \tau} = -a'(\tau)$ and $\partial_\tau + a'(\tau)\partial_x = \partial_s$, one finds that the governing equation $P(y,\tau)$ is

$$\partial_\tau P(y,\tau) = D\partial_y^\gamma P^\nu(y,\tau),$$

for which the solution is ($x_0 \equiv 0$)

$$P(y,\tau) = \frac{A}{\tau^{\frac{1}{1+\gamma-q}}}\left(c_1 - \frac{1-q}{2D\gamma(\gamma-1)(2-q)(1+\gamma-q)} \times \frac{y^\gamma}{\tau^{\frac{\gamma}{1+\gamma-q}}}\right)^{\frac{1}{1-q}}.$$

Let us equate the $P(y,\tau)$:

$$P(x,t) = \frac{\partial y}{\partial x}P(y,\tau(t)).$$

Then, we obtain that

$$P(x,t) = \frac{A}{t^{\frac{\xi}{1+\gamma-q}}}\left(c_1 - \frac{1-q}{2D\gamma(\gamma-1)(2-q)(1+\gamma-q)} \times \frac{(x-f'(t))^\gamma}{t^{\frac{\gamma\xi}{1+\gamma-q}}}\right)^{\frac{1}{1-q}}, \tag{17}$$

where $f'(t) = f(\tau(t))$, A is a normalization factor, and c_1 is a constant. Equation (17) is a Lq-Gaussian solution with a drift.

IV. PME WITH LOCAL AND NONLOCAL FRACTIONAL OPERATORS

To be self-contained, we consider the case where one fractional derivative is local and the other is nonlocal. In this section we solve a TS-FPME with (LN) fractionalization (the equation N.7 from Table II),

$$\frac{\partial^{\xi}}{\partial t^{\xi}}P(x,t) = D\frac{\partial^{\gamma}}{\partial x^{\gamma}}P^{\nu}(x,t),$$
$$0<\xi\leqslant 1,\ 0<\gamma<\frac{1}{2},\ \nu>-1, \tag{18}$$

where $\frac{\partial^{\xi}}{\partial t^{\xi}}$ and $\frac{\partial^{\gamma}}{\partial x^{\gamma}}$ denote the Katugampola and Riemann-Liouville fractional derivatives, respectively (see Table II). We consider the decomposition of Eq. (7). Then, by using the property of $\frac{d^{\gamma}}{dx^{\gamma}}F(ax) = a^{\gamma}\frac{d^{\gamma}}{dz^{\gamma}}F(z)$, and some properties of the Katugampola derivative, we obtain

$$\partial_x^{\gamma}P^{\nu}(x,t) = \frac{1}{\phi^{\nu+\gamma}}\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu},$$
$$\frac{\partial^{\xi}P(x,t)}{\partial t^{\xi}} = \frac{-1}{\phi^2(t)}\frac{\partial^{\xi}\phi}{\partial t^{\xi}}\left[F+z\frac{d}{dz}F\right],$$

so that

$$\frac{-1}{\phi^2(t)}\frac{\partial^{\beta}\phi}{\partial t^{\xi}}\frac{d}{dz}[zF] = \frac{D}{\phi^{\nu+\gamma}}\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu}.$$

To continue, similar strategies applied in Sec. III A were used. We transform the previous equation into two independent equations:

$$\phi^{\nu+\gamma-2}\frac{\partial^{\xi}\phi}{\partial t^{\xi}} = \frac{\xi}{\nu+\gamma-1},$$
$$-\frac{\xi}{\nu+\gamma-1}\frac{d}{dz}[zF] = D\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu},$$

where the solution of the first equation is $\phi = t^{\frac{\xi}{\nu+\gamma-1}}$. For the second, the solution is

$$\frac{d}{dz}[zF] = F + z\frac{d}{dz}F = -D\left[\frac{\nu+\gamma-1}{\xi}\right]\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu},$$

or $\frac{d}{dz}[zF] = k\frac{d^{\gamma}}{dz^{\gamma}}F^{\nu}$ with $k = \frac{-D(\nu+\gamma-1)}{\xi}$. By integrating with respect to z, we obtain

$$zF = k\frac{d^{\gamma-1}}{dz^{\gamma-1}}[F^{\nu}] + c, \tag{19}$$

where c is a constant, and we have set $c = 0$. By considering $F(z) = z^{\mu}(c_1 + c_2 z)^{\lambda}$ and using the following property for the RL operators,

$$\mathcal{D}_x^{\delta}[x^{\alpha}(a+bx)^{\beta}] = a^{\delta}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\delta)}x^{\alpha-\delta}(a+bx)^{\beta-\delta}, \tag{20}$$

we find that

$$\frac{d^{\gamma-1}}{dz^{\gamma-1}}[F^{\nu}] = c_1^{\gamma-1}\frac{\Gamma(\nu\mu+1)}{\nu\mu+2-\gamma}z^{\nu\mu-\gamma+1}(c_1+c_2z)^{\nu\lambda-\gamma+1}.$$

We put this result into Eq. (19), and the following expressions are obtained:

$$\nu = \frac{2-\gamma}{1+\gamma},\quad \mu = \frac{\gamma(1+\gamma)}{1-2\gamma},\quad \lambda = \frac{1-\gamma^2}{2\gamma-1},$$
$$c_1^{\gamma-1} = \frac{-\beta(1+\gamma)\Gamma\{[2+\gamma(\gamma-3)](1-2\gamma)^{-1}\}}{D[1+\gamma(\gamma-1)]\Gamma[(1-\gamma^2)(1-2\gamma)^{-1}]}.$$

The recent expressions reveal that the master equation admits the following solution:

$$P(x,t) = \frac{A}{t^{\frac{1}{\sigma}}}\left(\frac{x}{t^{\frac{1}{\sigma}}}\right)^{\alpha\gamma}\left(c_1 + c_2\frac{x}{t^{\frac{1}{\sigma}}}\right)^{-\alpha(1-\gamma)}, \tag{21}$$

where $\alpha = \frac{1+\gamma}{1-2\gamma}$, $\sigma = \frac{1-\gamma(1-\gamma)}{\beta(1+\gamma)}$, $c_2 = 1$, and

$$A = \frac{\Gamma[\alpha(1-\gamma)]}{c_1^{\gamma}\Gamma(1+\alpha\gamma)\Gamma(\gamma)}.$$

This solution is named a local-nonlocal (LNL) q-Gaussian distribution. The solution is not a generalized q-Gaussian distribution. This occurs due to the factor $x^{\alpha\gamma}$, which is located in front of the solution, assuring that this class of PDFs tend to zero when $x \to 0$. This makes a great difference with the previous function, the Lq-Gaussian Eq. (12).

Figure 3 shows the LNLq-Gaussian distribution functions. All the graphs tend to zero as $x \to 0$ for all times, as stated above, and do not have symmetric form around their peaks in terms of x. They manifested an abrupt increase before the peak and long-range tails beyond it, with peaks depending on q and ξ. The peak values increase by increasing both γ and ξ.

V. PME WITH NONLOCAL FRACTIONAL OPERATORS

Nonlocal fractional operators in PMEs generate nonlocal effects in the dynamics. More precisely, a nonlocal time derivative implies that the time variation of the PDF depends on the dynamics of the price return in all times, which should be compared with the local derivatives for which the change of PDF depends only on the prices in its close neighboring times [10]. It is stated that nonlocal derivatives are "aware of" all points in the phase space, here all times. For a complete reference see [10,12]. In the stock markets it is not a surprising fact, since the local change of the prices and consequently the price PDF depends not only the prices in the close neighboring times, but also the history of the changes of the price return, and also the predictions. Therefore one may expect that the PME with nonlocal fractional derivatives is a good choice to describe the dynamics of price return [10–12].

In this section we solve one particular case of the G-PME displayed in Eq. (3), when $n = -1$, $a = 0$, and $b = 1$. The solution is obtained by considering a hybrid case, where the time derivative is the Riemann-Liouville (RL) operator, and the space derivative is the Caputo operator (Appendix B). The other cases (RL-RL, Caputo-Caputo, and Caputo-RL derivatives) are straightforward to be processed following the same lines as this study. The resulting FPME equation is

$${}_0\mathcal{D}^{\xi,\frac{1}{t}}P(x,t) = D\big({}^{C}\mathcal{D}_1^{\gamma,x^{\alpha}}P^{\nu}(x,t)\big),\quad \xi,\gamma>0, \tag{22}$$

where $\mathcal{D}$ is the RL operator for the time derivative, and ${}^{C}\mathcal{D}$ is the Caputo operator acting on "space" coordinate x. To construct our solution, we need to restrict ourselves to the case $\nu = \frac{1+\xi}{1-\xi}$, leaving two parameters free for fitting, γ and α.

We again search for the solutions of the form of Eq. (7), where the parameters were defined in Sec. III. In the following we show that the above equation admits the solution of the form $F(z) = (1+bz^{\alpha})^{\lambda}$, where $b \neq 0$ and $\alpha \neq 1$. When established, this solution serves as another variant of the generalized q-Gaussian solution. By inserting this form in the

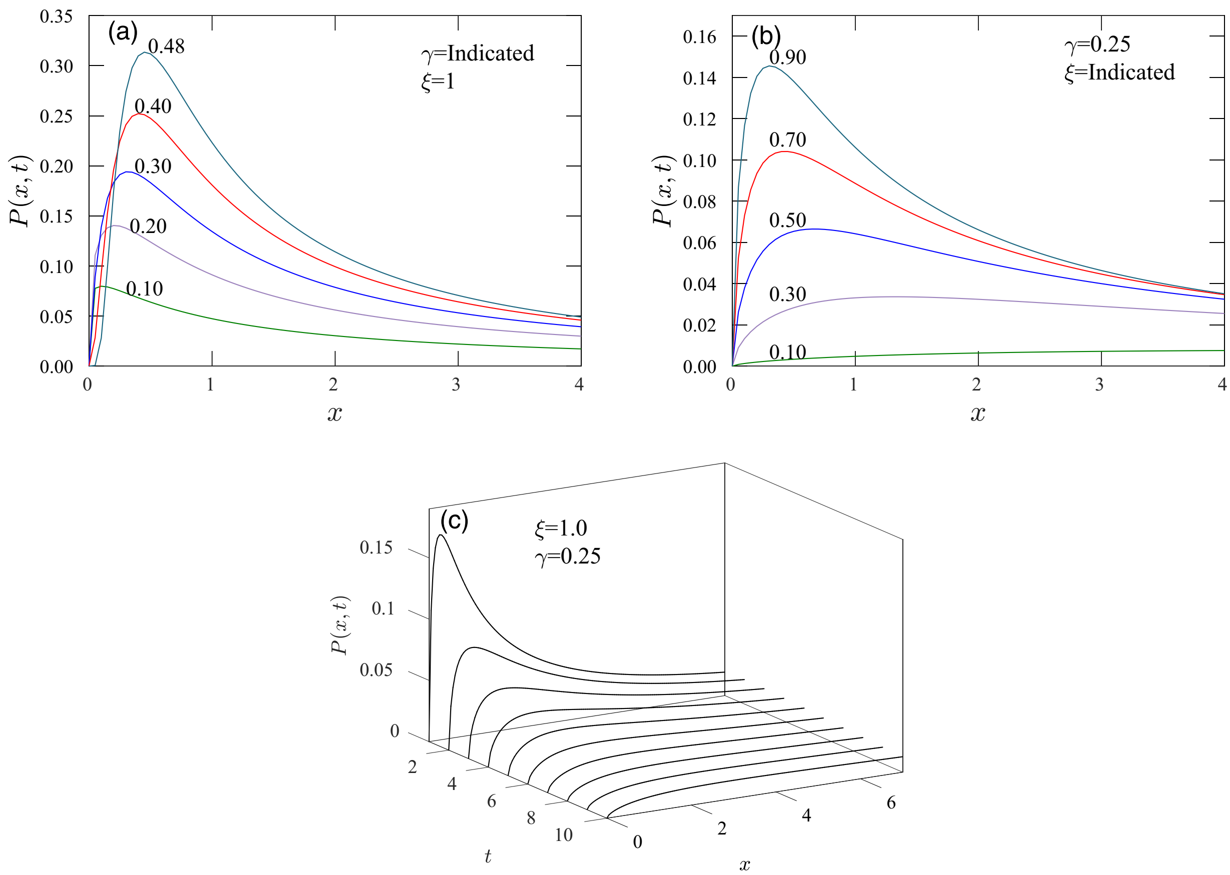

FIG. 3. (a) The PDFs of the LNLq-Gaussian distributions for different values of γ and $\xi = 1$. (b) The PDFs of the LNLq-Gaussian distributions for different values of ξ and $\gamma = 0.25$. (c) Time evolution of the Green function of Eq. (18).

right side of Eq. (22), we obtain

$$ {}^{C}\mathcal{D}_1^{\gamma,x^\alpha} P^\nu(x,t) = \left(\frac{1}{t^{\frac{1}{\alpha}}}\right)^{\nu} {}^{C}\mathcal{D}_1^{\gamma,x^\alpha}(1+bz^\alpha)^{\lambda\nu} = b^\gamma \left(\frac{1}{t}\right)^{\frac{\nu}{\alpha}+\gamma}\frac{\Gamma(\lambda\nu+1)}{\Gamma(\lambda\nu+1-\gamma)}(1+bz^\alpha)^{\lambda\nu-\gamma}. \tag{23} $$

To obtain the above equation, we have used the following property of the fractional derivatives, which is valid for all the fractional differential operators considered here:

$$ \mathcal{D}_t^\mu[f(at)] = a^\mu \mathcal{D}_x^\mu[f(x)]\,|_{x=at} \,. $$

Additionally, the following property of the fractional derivative of a function *with respect to another function* [77] was applied:

$$ \mathcal{D}_b^{\alpha,\psi}(\psi(b)-\psi(x))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(b)-\psi(x))^{\beta-\alpha-1}, \quad \alpha > 0,\ n < \beta \in \mathbb{R}, \tag{24} $$

where $n = \alpha$, if $\alpha \in \mathbb{N}$ and $n = [\alpha] + 1$, if $\alpha \notin \mathbb{N}$.

By inserting Eq. (7) in the left side of Eq. (22) and then applying Eq. (20) for the RL operators with $\delta = \alpha + \beta + 1$, we get

$$ {}_0\mathcal{D}^{\xi,\frac{1}{t}} P(x,t) = {}_0\mathcal{D}^{\xi,\frac{1}{t}}\left\{\frac{1}{t^{\frac{1}{\alpha}}}(1+bz^\alpha)^\lambda\right\} = {}_0\mathcal{D}^{\xi,\frac{1}{t}}\left\{\frac{1}{t^{\frac{1}{\alpha}}}\left(1+b'\frac{1}{t}\right)^\lambda\right\} = \frac{\Gamma(\frac{1}{\alpha}+1)}{\Gamma(\frac{1}{\alpha}+1-\xi)}(1+bz^\alpha)^{\lambda-\xi}\left(\frac{1}{t}\right)^{\frac{1}{\alpha}-\xi}, \tag{25} $$

where $b' = bx^\alpha$. By equating Eqs. (23) and (25), one finds that they match each other, yielding to

$$ \xi = 1+\lambda+1/\alpha,\ \ \nu = -1-\frac{2}{\lambda+1/\alpha}, \quad \gamma = (1/\alpha-\lambda)\left(1+\frac{1}{\lambda+1/\alpha}\right). \tag{26} $$

Therefore we see that the solution is

$$ P(x,t) = A_q^{NL} t^{\frac{-1}{\alpha}}\left[1+(1-q)\eta_q^{NL}(\alpha,\lambda,D)\frac{x^\alpha}{t}\right]^\lambda, \tag{27} $$

where the prefactor A_q^{NL} is a normalization constant, and $\eta_q^{NL}(\alpha,\lambda,D)$ is a constant depending on A_q^{NL}. If we again suppose that the distribution is symmetric with respect to x, therefore, $x \to |x|$ (i.e., its absolute value), then the normal-

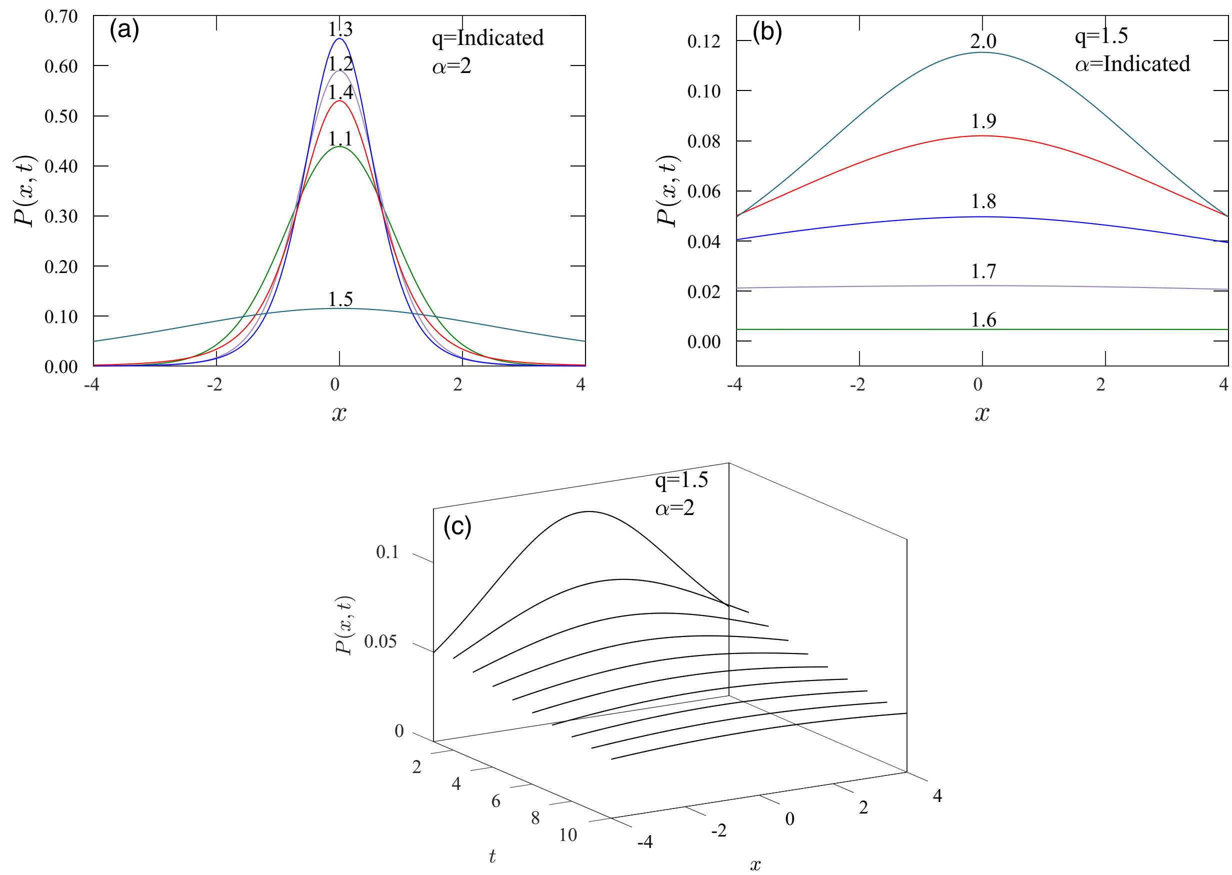

FIG. 4. (a) The PDFs of the NLq-Gaussian distributions for $\alpha = 2$ and indicated values of q. (b) The PDFs of the NLq-Gaussian distributions for $q = 1.5$ and indicated value of α. (c) The time evolution of PDFs of the NLq-Gaussian distributions for $q = 1.5$ and $\alpha = 2$. These are the Green functions of Eq. (22). We can see the NLq-Gaussian for a constant value α, and different q values show a multiple behavior; for $q = 1.1, 1.2, 1.3$ the peaks of curves and behavior of heavier tails increase, in contrast with $q = 1.4, 1.5$. Also, for a constant value q and different α values, by increasing α, the peaks of curves and behavior of heavier tails increase. In time evolution of Lq-Gaussian, over time, the peaks of curves decrease and the PDFs lost behavior of heavier tails.

ization constant is

$$A_q^{NL} = \frac{1}{2}\eta^{\frac{1}{\alpha}}(\alpha, \lambda, D)\frac{\Gamma(-\lambda)}{\Gamma\big(-\lambda - \frac{1}{\alpha}\big)\Gamma\big(1 + \frac{1}{\alpha}\big)},$$

where $\eta(\alpha, \lambda, D) = \frac{\eta_q^{NL}(\alpha,\lambda,D)}{\lambda}$. From this expression one can calculate the final expression of $\eta_q^{NL}(\alpha, \lambda, D)$,

$$\eta_q^{NL}(\alpha, \lambda, D) = \cdots \lambda\left[2^{1-\nu}\alpha^{\nu} D B^{\nu}\left(-\lambda - \frac{1}{\alpha}, \frac{1}{\alpha}\right)\right.$$
$$\left.\times\, \Gamma(\lambda\nu + 1)\Gamma\left(-\lambda - \frac{1}{\alpha}\right)\right]^{1/\xi}, \qquad (28)$$

where $B(.,.)$ is the Beta function. By defining $\lambda = \frac{1}{1-q}$ in Eq. (27) (where $q > 1$), we recover the *first generalized* q-Gaussian distribution with two free independent parameters. We named Eq. (27) as the nonlocal q-Gaussian (NLq-Gaussian) distribution.

The NLq-Gaussian distribution Eq. (27) is the Green function of Eq. (22), obtained using the fractional derivatives of Riemann-Liouville and Caputo (nonlocal fractional definitions), for time and space, respectively. The NLq-Gaussian has been defined as $g_q^{\lambda,\alpha}(x)$, equation N.8 in Table II. The plots for NLq-Gaussian solutions are shown in Fig. 4 for various α and q values. With respect to the local case, the behavior for the nonlocal case is more complicated. As shown in Fig. 4(a), the case where α is kept constant and q increases is evaluated. For $q < 1.3$, the peak rises, and for $q \geqslant 1.3$, it decreases. In Fig. 4(b), for a constant q, however, the peak increases when α increases. Figure 4(c) shows the time evolution of the Green function of Eq. (22), where the solution is the NLq-Gaussian distribution. The example was made for the values of $q = 1.5$ and $\alpha = 2$. We have shown that the distribution widens as time goes on.

VI. AN APPLICATION OF Lq-GAUSSIAN IN S&P500 STOCK MARKETS

The price return in stock markets exhibits remarkable characteristic features. Some characteristic features have been described mostly by Fokker-Planck [78,79] or PME [80] approaches. The most largely observed feature of price return in recent studies is the self- similarity law, where the PDF obeys

$$P(x, t) = \frac{1}{(Bt)^H} F\left(\frac{x}{(Bt)^H}\right), \qquad (29)$$

in which F is a normalized distribution that is usually fit to a q-Gaussian. In earlier work the function F was assumed as a Levy-stable distribution function, L_γ. The Levy-stable distribution was obtained from the analytical approach of a nonlinear PME to model price dynamics at large scale [81–84]. However, the Levy-stable distribution has the drawback that it presents infinite standard deviation, and it does not obey the empirical power-law tails [85,86]. For short-time returns, it has been proved that F is a q-Gaussian distribution function exhibiting short-time correlations, weak long time correlations, and power-law tails [8,87]. The q-Gaussian distribution was obtained from different generalized forms of the PME to model price dynamics [88,89]. For long time returns, F is a Gaussian distribution function, where the price return behaves like independent and identically distributed random variables but still following the self-similar principle. The Gaussian distribution can be obtained from the PME when $q = 1$ due to a slow convergence to the normal distribution function.

Particular cases of the generalized PME are presented in Table III. The solutions of each of these partial differential equations obey the self-similar law given in Eq. (29) and are related to the q-Gaussian distribution function.

In this part we analyze the S&P500 stock market data during the 24-year period from January 1996 to August 2020 with a frequency of 1 min. This analysis is made with the aim of comparing the efficiency of previous fractional PME with two particular cases proposed in this manuscript, the TS-FPME, and the G-PME. These several generalizations are presented in Table III. The detrended price return is defined as

$$x(t) = I^*(t_o + t) - I^*(t_o), \tag{31}$$

where $I^*(t_o)$ is the detrended stock market index at time t_o, and $I^*(t)$ is the detrended stock market index for any time $t > t_o$. The PDF of the detrended price return has been fitted using the q-Gaussian distribution [8], where different zones had been captured from the strong- to superdiffusion regime previously [8,90]. Figure 5 shows the evolution of the PDF and its regimes in the (x, t) space. Initially, the PDF has a pronounced bump in the center that fully disappears close to 80 min. During the first 35 min there is a power-law relation of time against the end points of the bumps at the PDFs (see Fig. 1(e) in Ref. [8]). This zone is defined as the strong superdiffusion regime (zone A). The remaining area during that time and the following next area close to 10 days corresponds to the weak superdiffusion regime (zone B). Finally, the last regime corresponds to a normal diffusion process (zone C) and is reached after approximately 30 days.

We have reconstructed the time evolution of the PDFs of the detrended price return, and we collapsed them after applying the corresponding rescaling factor. Four equations of Table II have been used to model this behavior: the classical PME (C-PME), the time FPME (T-FPME), the time-space FPME (TS-FPME), and a particular case of the generalized PME (G-PME). These solutions obey Eq. (29) and are presented in Table III with more detail.

The TS-FPME and the particular solution of the G-PME proposed in this manuscript are options to model the time evolution of the detrended price return. The Lq-Gaussian and NLq-Gaussian, which are the solutions of the TS-FPME and the particular case of the G-PME, respectively, fit the collapse of the PDFs of the detrended price return well. Figure 6 shows the result of these fittings.

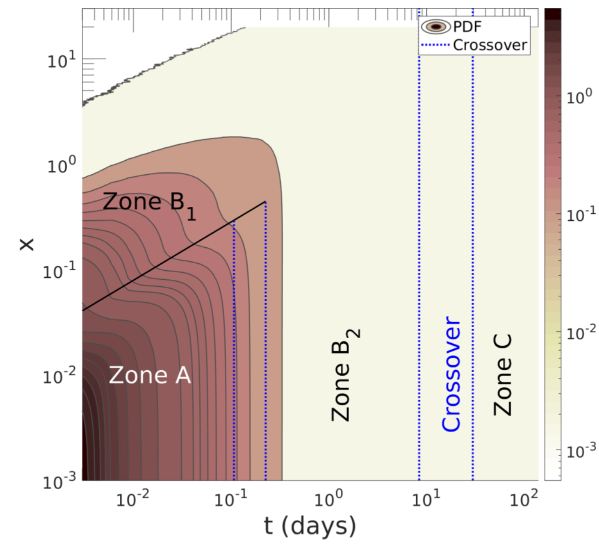

FIG. 5. Time evolution of the PDF of the S&P500 price return. Three different zones were determined based on an abrupt slope change of the fitting parameters α and q. The contour plot represents the PDF of the detrended price return. The black circles represent the end points of the strong superdiffusion regime (zone A) from $t = 0$ to $t = 35$ min. These points are the ends of the bump obtained from the two points at the PDF with an abrupt change of slope (Fig. 1(c) in Ref. [8]). The remaining area during the first 35 min corresponds to a weak superdiffusion regime (zone B_1). From 80 min to 10 days, the zone corresponds to a weak superdiffusion regime (zone B_2). A normal diffusion process is reached after 30 days. The gray dashed lines represent the transitions between each zone.

The first best option to fit the price return was obtained by replacing F as the Lq-Gaussian (g_q^γ) in Eq. (29); this equation can be written as

$$P(x,t) = \frac{1}{(Bt)^{\frac{\xi}{1-q+\gamma}}} \frac{1}{C_q^\gamma} \left(1 - (1-q) \frac{x^\gamma}{(Bt)^{\frac{\gamma\xi}{1-q+\gamma}}} \right)^{\frac{1}{1-q}}, \tag{32}$$

where C_q^γ is the normalization constant. B is related with the diffusion term, and both of them are detailed in Table II.

The second best option was obtained by replacing F as the NLq-Gaussian ($g_q^{\alpha,\lambda}$) in Eq. (29), this second equation can be written as

$$P(x,t) = \frac{1}{(Bt)^{\frac{1}{\alpha}}} \frac{1}{C_q^{\lambda,\alpha}} \left(1 - (1-q) \frac{x^\alpha}{(Bt)} \right)^{\lambda}. \tag{33}$$

$C_q^{\lambda,\alpha}$ is the normalization constant, and B is related with the diffusion term. The definitions of these parameters are expressed in Table II. By considering $\lambda = \frac{1}{1-q}$, Eq. (32) is recovered.

The results of fitting the collapse of the PDF of the detrended price return are shown in Fig. 6. Figure 6 presents the collapses of the PDFs of price return for the specific zones presented in Fig. 5. Each collapse has been fitted by the four solutions of the equations presented in Table III. The best fitting for the four cases is the NLq-Gaussian. However, the

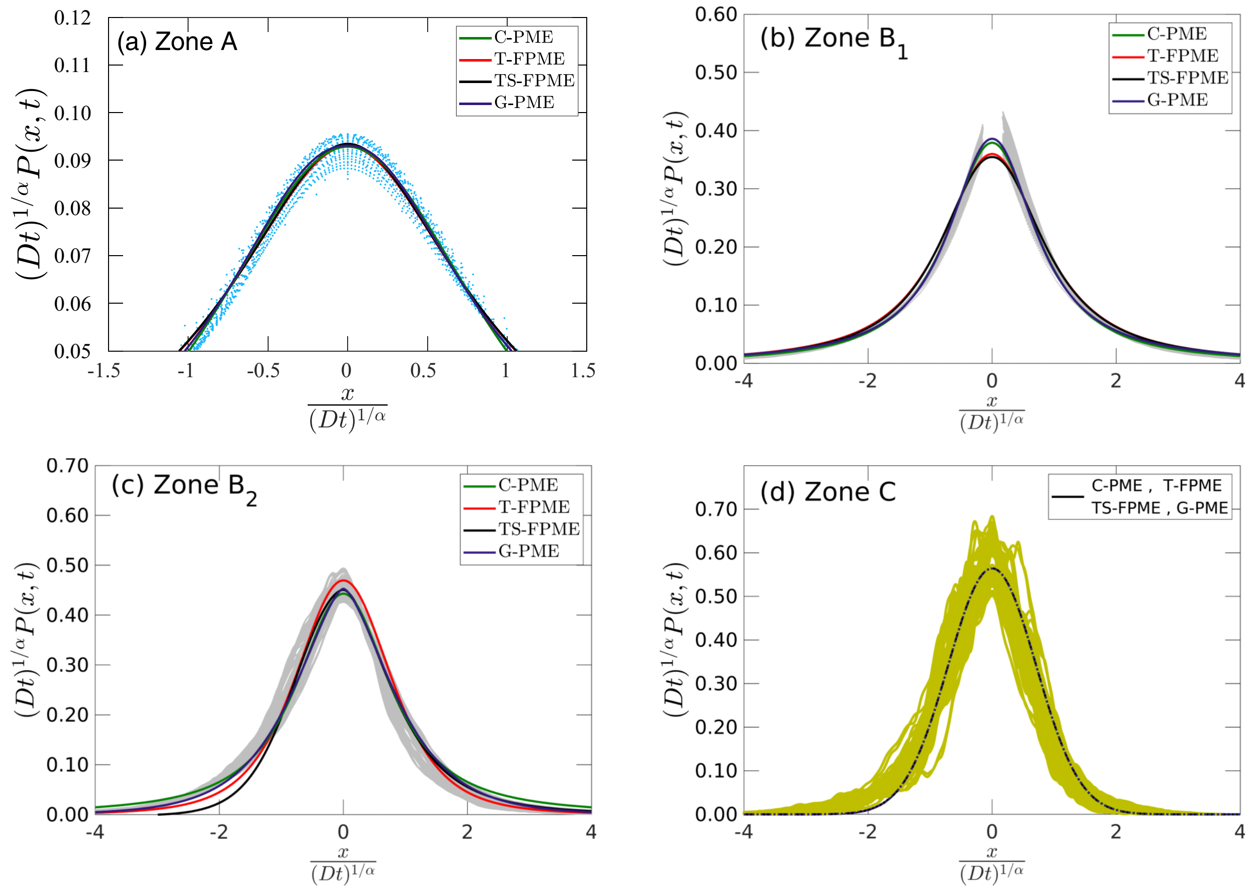

FIG. 6. This figure shows the collapse of the time evolution of the PDFs for each zone displayed previously in Fig. 5. (a) The collapse of the PDFs of the strong superdiffusion regime is represented in zone A. The collapse in zone A has been fitted with the solution of the C-PME ($\gamma = 2, q = 1.94$), T-FPME ($\gamma = 2, q = 2.73, \xi = 0.21$), TS-FPME ($\gamma = 1.97, q = 2.70, \xi = 0.21$), and G-PME ($\alpha = 2.18, q = 3.14, \lambda = -0.52$). (b) The collapse of the PDFs of the weak superdiffusion regime for the first 35 min while the bump remains in the PDFs occurs in zone B_1. The collapse in zone B_1 has been fitted with the solution of the C-PME ($\gamma = 2, q = 1.85$), T-FPME ($\gamma = 2, q = 1.72, \xi = 0.82$), TS-FPME ($\gamma = 2.09, q = 1.91, \xi = 0.75$), and G-PME ($\alpha = 2.05, q = 2.44, \lambda = -0.99$). (c) The collapse of the PDFs during the weak superdiffusion regime after 100 min, when the bump disappears completely, occurs in zone B_2. The collapse in zone B_2 has been fitted with the solution of the C-PME ($\gamma = 2, q = 1.81$), T-FPME ($\gamma = 2, q = 1.42, \xi = 0.92$), TS-FPME ($\gamma = 1.73, q = 1.41, \xi = 0.77$), and G-PME ($\alpha = 1.56, q = 1.19, \lambda = -4.62$). (d) The collapse of the PDFs for the normal diffusion regime occurs in zone C. The collapsed data is fitted by a Gaussian distribution function, which is a concurrent solution for the four previous PDEs when $\gamma = 2, q = 1.00, \xi = 1.00$.

four solutions constitutes an acceptable solution for the correspondent collapses of the PDFs. A convergence to a Gaussian normal distribution is observed for long time returns.

Before closing this section, it is worth discussing the relation between the governing equation (fractional Fokker-Planck equation or PME) and the stochastic differential equation (Langevin equation) of their associated time series. More specifically, we want to discuss the relation between the autocorrelation of the time series and the fractionalization scheme in the governing equation. The key point is the scaling properties of the governing equation describing a self-similar time series. In fact, denoting a self-similar time series by $\{Y(t)\}_{t\in\mathbb{Z}}$ with the property

$$\sqrt{\langle Y^2\rangle} \propto t^H, \tag{34}$$

where $\langle . \rangle$ denotes time average, the corresponding governing equation should have the same symmetry, i.e., it should be invariant under the transformation $Y \to cY$ and $t \to c^{1/H}t$. (For any positive real number c, note also that the invariance of the governing equation means the invariant of the PDF up to a scaling factor, i.e., $P(Y,t) = t^{-H} f(Y/t^H)$, where f is a function to be fixed by the governing equation.) One can show by inspection that a governing diffusion equation comprised of ordinary time and space differential operators is not invariant under this transformation. (Note that the symmetry transformation of the normal diffusion equation is $Y \to cY$ and $t \to c^2 t$.) Therefore the governing equation for a self-similar time series should include fractional operators, or one should use space- or time-dependent diffusion coefficient with power-law dependence. The former strategy is more convenient and is often taken, since the latter results in nonanalytical solutions. The symmetry of the system can easily be found in the corresponding PDF of the time series, and also by calculating the second moment of Y, i.e., the fractionalization exponent is manifest in the PDF. Note also that this transformation argument only fixes the order (exponent) of fractionalization, not the exact form of the function f which should be obtained by solving the governing equation. In fact, the scheme of fractionalization (local or nonlocal, and also the definition of the fractional derivatives) is determined by the physics of the system under study. As a toy model for self-similar correlated time series, consider the fractional Brownian motions (fBms), which includes the notion of nonlocality; see, for example, Ref. [91], which is a particular case of the classical fractional stochastic volatility model of [92]. fBm is a non-Markovian self-similar Gaussian stochastic process with stationary power-law correlated increments. Its mean-square displacement fulfills the relation (34) [93,94]. fBms are described by an exponential PDF $P(Y,t) \propto t^{-H}\exp[-\frac{1}{2}(\frac{Y}{t^H})^2]$ with nontrivial symmetry transformation. Let us consider a general H-self-similar time series $\{Y(t)\}_{t\in\mathbb{Z}}$ defined by the relation $\{Y(ct)\}_{t\in\mathbb{Z}} \equiv \{c^H Y(t)\}_{t\in\mathbb{Z}}$, where c is any positive number, and H is the Hurst exponent. If this process has stationary increments $X_n = Y(n) - Y(n-1)$, $n \in \mathbb{Z}$, then the autocorrelation $\gamma_X(k) \equiv \langle X_k X_0\rangle - \langle X_k\rangle\langle X_0\rangle$ behaves like [95] k^{2d-1} as $k\to\infty$, where $d = H - \frac{1}{2}$, and $0 < d < 1/2$, ensuring that $\sum_{k=-\infty}^{\infty}\gamma_X(k) = \infty$. From a spectral domain perspective, the spectral density of $\{X_n\}$ behaves as ω^{-2d} as the frequency $\omega\to 0$. This relates a self-similar time series with nonlocality, which is applied to the price return time series which becomes stationary by normalizing the detrended data set [see Eq. (31)] [96]. Following these facts, we suggest a relation between the Hurst exponent $H = 1/\alpha = \xi - \lambda - 1$ given in Eqs. (22) and (26), with a self-similar function Eq. (27) for the nonlocal-nonlocal case. A similar argument holds for the local-local case.

Although we did not propose any Langevin equation corresponding to our FPME (for which the Itô stochastic calculus does not apply [97]), in some limits this equation is already known. As an example, for the fractional-linear Fokker-Planck equation, a fractional stochastic equation is used [98]. Another example is the stochastic differential equation [99], which is found for the nonlinear normal derivative Fokker-Planck equation.

VII. CONCLUSIONS

Some limitations of the previous approaches based on Levy processes opens up an opportunity to model the stock market price by considering (1) correlations during the first minutes on the price fluctuations were observed at high frequencies, making the Levy regime no longer applicable, and (2) the characteristic exponents applying to model the power-law tails of the price return's PDFs lies outside the Levy regime. To solve these problems, this paper deals with two issues: first the solutions of the FPME were obtained and discussed. In the second part we applied the solutions obtained in the first to the S&P500 stock markets. For the first part we provided different solutions for the generalized form of the FPME by considering generalized q-Gaussian trial functions and inserting them into the anomalous PME. The solutions were built by considering the local and nonlocal fractional derivatives assuming a Dirac's δ function as the initial condition. More precisely, the fractional derivatives were classified as local and nonlocal, where the Katugampola's is the local fractional derivative and the Riemann-Liouville, Caputo, and Riesz are nonlocal fractional derivatives. Our analysis proves that the generalized PME admits generalized q-Gaussian PDF as a class of solutions which obey a self-similar law. For the local derivatives the resulting solution was proved to be the Lq-Gaussian, which is called the *first generalized* Lq-Gaussian function. This solution fits the PDF of the detrended price return well (Sec. VI). The second analyzed class of G-PME is the one in which the time and space derivatives are given by nonlocal fractional generalizations, Riemann-Liouville and Caputo, both of which are based on the Laplace transform and proved to admit so-called *second generalized* NLq-Gaussian solution, which is symmetric about its mean (peak). The NLq-Gaussians hold a different self-similar law. The main difference is that for the Lq-Gaussians we have three free parameters (exponents) ξ and γ and q, whereas for the NLq-Gaussians there are two free self-similarity parameters α, γ. For the mixed (hybrid) equation where the time fractional derivative is considered to be local and the spatial one is nonlocal, the solution is again proportional to the generalized q-Gaussian (LNLq-Gaussian), but they obey a power in x that causes the PDF to vanish in the limit $x\to 0$ and are not symmetric in x.

The Lq-Gaussian and NLq-Gaussian have been used to model the detrended price return of S&P500. Although both distribution functions describe well the fitting of the detrended price return, we found that the NLq-Gaussian is the best model to fit the probability of the detrended price return. The solutions presented here assume Dirac delta initial conditions. For arbitrary initial conditions, the generalized form of the PME could be solved by applying the q-Fourier analysis. The ordinary Fourier analysis only applies for linear operators. The generalized PME contains nonlinear operators, preventing us from using the ordinary Fourier analysis.

APPENDIX A: PROPERTIES OF KATUGAMPOLA DERIVATIVE

Here we give a brief summary of the definition of the Katugampola fractional operator and some of its properties. This local fractional operator is used to construct the TS-FPME in Sec. III. If $0 \leqslant \alpha < 1$, the Katugampola operator generalizes the classical calculus properties of polynomials [100]. Furthermore, if $\alpha = 1$, the definition is equivalent to the classical definition of the first-order derivative of the function f. The Katugampola derivative is defined as

$$\mathcal{D}^\alpha f(t) = \lim_{\epsilon\to 0}\frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon} \tag{A1}$$

for $t > 0$ and $\alpha \in (0, 1]$. When $\alpha \in (n, n+1]$ (for some $n \in \mathbb{N}$, and f is an n-differentiable at $t > 0$), the above definition generalizes to

$$\mathcal{D}^\alpha f(x) = \lim_{\epsilon\to 0}\frac{f^{(n)}(xe^{\epsilon x^{n-\alpha}}) - f^{(n)}(x)}{\epsilon},$$

and if f is $(n+1)$-differentiable at $t > 0$, then

$$\mathcal{D}^\alpha f(t) = t^{n+1-\alpha}f^{(n+1)}(t).$$

In the rest, we review some properties of the Katugampola derivative in Table IV. If f is α-differentiable in some $(0, a)$, $a > 0$, and $f^{(\alpha)}(0) = \lim_{t\to 0^+}\mathcal{D}^\alpha f(t)$ exists, the following properties hold for the Katugampola derivative. For

f, g, to be α-differentiable at a point $t > 0$,

$$\begin{aligned}\mathcal{D}^\alpha f(at) &= f'(at)\mathcal{D}^\alpha(at)\\ &= af'(at)\mathcal{D}^\alpha t\\ &= af'(at)t^{1-\alpha}\\ &= (at)^{1-\alpha}a^\alpha f'(at).\end{aligned}$$

One can define the inverse of the $\mathcal{D}^\alpha$ operator as a fractional integral,

$$(\mathcal{D}^\alpha)^{-1} \equiv \mathcal{D}^{-\alpha} \equiv I^\alpha = \int^t dx \frac{(.)}{x^{1-\alpha}},$$

where the (.) symbol is serving as place holder for the function to be operated upon. One verifies that

$$\mathcal{I}^\alpha[\mathcal{D}^\alpha(f)] = \int^t dx \frac{x^{1-\alpha} f'}{x^{1-\alpha}} = f,$$

where f vanishes at the lower limit. Then

$$\begin{aligned}\mathcal{D}^\alpha[\mathcal{I}^\alpha(f)] &= \mathcal{D}^\alpha\left[\int^t dx \frac{f}{x^{1-\alpha}}\right] = t^{1-\alpha}\left(\int^t dt \frac{f}{x^{1-\alpha}}\right)'\\ &= t^{1-\alpha}\frac{f}{t^{1-\alpha}} = f.\end{aligned}$$

APPENDIX B: CAPUTO FRACTIONAL DERIVATIVE OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

This section contains definitions of nonlocal fractional operators that are used in this paper to construct the TS-FPMEs. The Riemann-Liouville fractional derivative is a fractional operator that is used in Secs. IV and V as a nonlocal fractional operator to construct the TS-FPMEs with (LN) and (NN) fractionalizations. The integral representation for this operator is

$${}^{RL}_{a}\mathcal{D}^{\alpha,x} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{dt\, f(t)}{(x-t)^{\alpha+1}},$$

where $n-1 < \alpha \leqslant n$ [74]. A recent variation of the RL operator is the Caputo derivative [101], defined as

$${}^{C}\mathcal{D}^{\alpha,x}_{b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{dt\, f^{(n)}(t)}{(x-t)^{\alpha+1-n}}, \quad n-1 < \alpha \leqslant n,$$

where C stands for Caputo and $f^{(n)}$ is the nth derivative of f. The main advantage of the Caputo derivative is that the derivate of a constant is zero, which is not the case of the RL operator. Substantially, this kind of fractional derivative is a formal generalization of the integer derivative under Laplace transform [40].

A generalized fractional operator that we used to construct the TS-FPME with (NN) fractionalization in Sec. V is the Caputo fractional derivative of a function with respect to another function [77], and is defined as

$$\begin{aligned}{}^{C}\mathcal{D}^{\alpha,\psi(x)}_{b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\cdots\\ \times \int_x^b \psi(t)'(\psi(t)-\psi(x))^{n-\alpha-1}\\ \times\left(\frac{1}{\psi(t)'}\frac{d}{dt}\right)^n f(t)dt.\end{aligned}$$

Note that the recent integral representation in the special case $\psi(x) = x$ is reduced to the integral representation of the Caputo derivative.

We solve a particular case of the generalized PME, Eq. (3), described by a fractional derivative of a function with respect to another function. This innovative approach will be useful to solve other physical problems that present a self-similar pattern and can be modeled by a q-Gaussian.

APPENDIX C: FRACTIONAL DERIVATIVES—DEFINITION AND PROPERTIES

In this section we give a short review of the properties of the fractional derivatives: Katugampola, Riemann-Liouville, and Caputo. The Katugampola is one definition for the local fractional derivative. The Riemann-Liouville and Caputo are definitions of nonlocal derivatives. A comparison between each property of these fractional derivatives is presented in Table IV.

Katugampola's corresponds to the ordinary derivative when $\alpha = 0$ and $\alpha = 1$. The Riemann-Liouville and Caputo derivatives are an analytical continuation of the ordinary derivatives. The main difference between them is that the Caputo derivative of a constant is zero, a property that does not hold for the Riemann-Liouville derivative. This desirable property is satisfied by the Katugampola local derivative, too.

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