

**Asymptotic expansion of the solution of the master equation and its application to the speed limit**Satoshi Nakajima \* and Yasuhiro Utsumi *Department of Physics Engineering, Faculty of Engineering, Mie University, Tsu, Mie 514-8507, Japan* (Received 20 July 2021; accepted 5 November 2021; published 29 November 2021)

We investigate an asymptotic expansion of the solution of the master equation under the modulation of control parameters. In this case, the nondecaying part of the solution becomes the dynamical steady state expressed as an infinite series using the pseudoinverse of the Liouvillian, whose convergence is not granted in general. We demonstrate that for the relaxation time approximation model, the Borel summation of the infinite series is compatible with the exact solution. By exploiting the series expansion, we obtain the analytic expression of the heat and the activity. In the two-level system coupled to a single bath, under the linear modulation of the energy as a function of time, we demonstrate that the infinite series expression is the asymptotic expansion of the exact solution. The equality of a trade-off relation between the speed of the state transformation and the entropy production [Shiraishi, Funo, and Saito, *Phys. Rev. Lett.* **121**, 070601 (2018)] holds in the lowest order of the frequency of the energy modulation in the two-level system. To obtain this result, the heat emission and absorption at edges (the initial and end times) or the differences of the Shannon entropy between the instantaneous steady state and the dynamical steady state at edges are essential: If we ignore these effects, the trade-off relation can be violated.

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Time-dependent open systems have been studied actively in recent years. These studies relate to quantum pump [1–8], excess entropy production [9–13], efficiency and power of heat engine [14–18], shortcuts to adiabaticity [19–21], and speed limits [22–24]. In the studies of quantum pumping and excess entropy production for systems governed by the master equation, the time dependence is described using the pseudoinverse of the Liouvillian [1–3,5–7,12,13,16,25]. However, the solution using the pseudoinverse of the Liouvillian is an asymptotic expansion and does not converge in general: The solution is obtained by iterative applications of the pseudoinverse and the time derivative, which can make the series expansion divergent similarly to the case for the adiabatic iteration [26]. In the two-level system (like spinless one-level quantum dot) coupled to a single bath, the solution expressed by the pseudoinverse is an asymptotic expansion of the exact solution and provides a good approximation in the first few terms.

One of the trade-off relations between the speed of state transformation and the entropy production for the classical stochastic process is given in Ref. [23]: An inequality

$$r := \frac{2\sigma A}{L^2} \tau \geq 1 \quad (1)$$

holds if the Liouvillian satisfies the local detailed balance condition. Here,  $L$  is the total variation distance between the states at the initial time  $t = 0$  and the final time  $t = \tau$ ,  $A$  is the average activity, and  $\sigma$  is the total entropy production [for the definitions, see Sec. IV; we derive (1) in Appendix A]. The

authors of Ref. [23] considered a two-level system and provided a specific protocol, which realizes  $r = (5/2) \ln(3/2) = 1.01366\dots$  in the limit of slow driving. In the present paper, based on the asymptotic expansion of the two-level system, we will demonstrate protocols that achieve the equality of (1).

The structure of the paper is as follows. First, we give an infinite series expression of the dynamical steady state which is the solution of the master equation under the modulation of control parameters (Sec. II). Next, in Sec. III A, we introduce the relaxation time approximation model (which is analytically tractable) and show that the Borel summation of the infinite series expression becomes the exact solution. In Sec. III B, for the two-level system, we consider linear modulation of the energy as a function of time and show that the infinite series expression is the asymptotic expansion of the exact solution. In Sec. IV, we show that the equality of (1) holds in the lowest order of the frequency of the energy modulation in the two-level system. In Sec. V, we summarize this paper. In Appendix A, we derive the trade-off relation of Shiraishi-Funo-Saito. In Appendix B, we derive the exact solution of the master equation. In Appendix C, we explain a prescription getting the exact solution from the asymptotic expansion in the two-level system. In Appendix D, we discuss the higher derivative and oscillation. In Appendix E, we give an instance of the convergent infinite series expression of the dynamical steady state. In Appendix F, we calculate the heat current and heat using the asymptotic expansion of the dynamical steady state for the relaxation time approximation model.

**II. GENERAL THEORY**

In this section, we give an infinite series expression of the dynamical steady state using the pseudoinverse of the Liouvillian.

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We consider a master equation

$$\frac{d}{dt}p_i(t) = \sum_{j=0}^{N-1} K_{ij}(\alpha_t)p_j(t). \quad (2)$$

Here,  $p_i$  is the probability of state  $i$  ( $= 0, 1, \dots, N-1$ ) at time  $t$  and  $\alpha_t$  is the value of the set of the control parameters at time  $t$ . We assume that the master equation has a unique instantaneous steady state  $p_i^{\text{ss}}(\alpha)$  which satisfies  $\sum_j K_{ij}(\alpha)p_j^{\text{ss}}(\alpha) = 0$  and  $\sum_i p_i^{\text{ss}}(\alpha) = 1$ . The pseudoinverse of the Liouvillian  $K_{ij}(\alpha)$  is defined by

$$\sum_k R_{ik}(\alpha)K_{kj}(\alpha) = \delta_{ij} - p_i^{\text{ss}}(\alpha). \quad (3)$$

Applying the pseudoinverse  $R(\alpha_t)$  to the master equation (2), we obtain

$$\left[1 - R(\alpha_t)\frac{d}{dt}\right]\delta p(t) = R(\alpha_t)\frac{dp^{\text{ss}}(\alpha_t)}{dt}, \quad (4)$$

where  $\delta p(t) := p(t) - p^{\text{ss}}(\alpha_t)$ . Here,  $R$  denotes the matrix ( $R_{ij}$ ) and  $p$  is the vector ( $p_0, p_1, \dots, p_{N-1}$ )<sup>T</sup>. The formal solution of (4) is given by an infinite series [1,5]

$$\delta p^{\text{dss}}(t) := \sum_{n=1}^{\infty} \left[ R(\alpha_t)\frac{d}{dt} \right]^n p^{\text{ss}}(\alpha_t) =: \sum_{n=1}^{\infty} p^{(n)}(t). \quad (5)$$

The general solution of (4) is

$$\delta p(t) = \delta p^{\text{dss}}(t) + \tilde{\delta}p(t). \quad (6)$$

Here,  $\tilde{\delta}p(t)$  is the solution of

$$\left[1 - R(\alpha_t)\frac{d}{dt}\right]\tilde{\delta}p(t) = 0 \quad (7)$$

under  $\tilde{\delta}p(0) = \delta p(0) - \delta p^{\text{dss}}(0)$ .  $\tilde{\delta}p(t)$  also satisfies  $\frac{d}{dt}\tilde{\delta}p(t) = K(\alpha_t)\tilde{\delta}p(t)$  and damps exponentially as a function of time [13]. Then, the general solution of the master equation is given by [5,13]

$$p(t) = p^{\text{dss}}(t) + \tilde{\delta}p(t), \quad (8)$$

where

$$\begin{aligned} p^{\text{dss}}(t) &:= p^{\text{ss}}(\alpha_t) + \delta p^{\text{dss}}(t) \\ &= p^{\text{ss}}(\alpha_t) + \sum_{n=1}^{\infty} p^{(n)}(t). \end{aligned} \quad (9)$$

We call  $p^{\text{dss}}(t)$  the *dynamical steady state*. The general solution consists of the exponentially damping term  $\tilde{\delta}p(t)$  and the dynamical steady state, which is composed of the instantaneous steady state and the correction  $\delta p^{\text{dss}}(t)$ .

In general, the infinite series  $p^{\text{dss}}(t)$  dose not converge. In the next section, we compare  $p^{\text{dss}}(t)$  and an exact solution of the master equation.

### III. EXACT SOLUTION AND BOREL SUMMATION

In this section, first, we study the relaxation time approximation model for an  $N$ -level system and show that the Borel summation of the infinite series  $p^{\text{dss}}(t)$  is the exact solution of the master equation. Next, we study the two-level system and demonstrate that  $p^{\text{dss}}(t)$  is identical with the asymptotic expansion of the exact solution.

#### A. $N$ -level system

We introduce a Liouvillian

$$K_{ij}(\alpha) = \gamma[p_i^{\text{ss}}(\alpha) - \delta_{ij}] \quad (10)$$

with  $\gamma > 0$  as an analytically tractable model. This Liouvillian satisfies the detailed balance condition  $K_{ij}(\alpha)p_j^{\text{ss}}(\alpha) = K_{ji}(\alpha)p_i^{\text{ss}}(\alpha)$  [27]. This model is not just a toy model because it induces the following Liouvillian:

$$K(\alpha) = \gamma \begin{pmatrix} -f & 1-f \\ f & -(1-f) \end{pmatrix}, \quad (11)$$

which is the same form with a spinless one-level quantum dot coupled to leads. Here,  $f = \sum_b \frac{\gamma_b}{\gamma} \frac{1}{e^{\beta_b \Delta} + 1}$  and  $\sum_b \gamma_b = \gamma$ .  $\gamma_b$  is the coupling strength and  $\beta_b$  is the inverse temperature of bath  $b$ .  $\Delta$  is the energy level difference between the two levels. The master equation of (10) is given by

$$\frac{d}{dt}p_i(t) = -\gamma[p_i(t) - p_i^{\text{ss}}(\alpha_t)]. \quad (12)$$

This is the relaxation time approximation of which the relaxation time is  $1/\gamma$ . A pseudoinverse of (10) is given by

$$R_{ij}(\alpha) = \frac{1}{\gamma}[p_i^{\text{ss}}(\alpha) - \delta_{ij}]. \quad (13)$$

To simplify, we assume  $\gamma$  is a constant in the following of this paper. Substituting (5) and (13) into (9), we obtain

$$\tilde{F}_i(t) := p_i^{\text{dss}}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma^n} \frac{d^n}{dt^n} p_i^{\text{ss}}(\alpha_t). \quad (14)$$

In Sec. III B, we show that  $\tilde{F}_i(t)$  diverges in the two-level system.

We calculate the Borel summation of  $\tilde{F}_i(t)$ . For a series  $\tilde{S}(t) := \sum_{n=0}^{\infty} a_n(t)$ , if (i) the Borel function  $\mathcal{B}[\tilde{S}(t)](s) := \sum_{n=0}^{\infty} \frac{a_n(t)}{n!} s^n$  converges to an analytic function for  $0 \leq s < \rho$  ( $\rho > 0$ ) and can be analytically continued along the positive real axis and (ii)  $S(t) := \int_0^{\infty} ds e^{-s} \mathcal{B}[\tilde{S}(t)](s)$  is well defined,  $S(t)$  is called the Borel summation of  $\tilde{S}(t)$  [28]. The Borel function of  $\tilde{F}_i(t)$  is given by

$$\mathcal{B}[\tilde{F}_i(t)](s) = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \frac{(-1)^n}{\gamma^n} \frac{d^n}{dt^n} p_i^{\text{ss}}(t), \quad (15)$$

where  $p_i^{\text{ss}}(t)$  denotes  $p_i^{\text{ss}}(\alpha_t)$ . This series converges to an analytic function  $p_i^{\text{ss}}(t - \frac{s}{\gamma})$  for small  $s$ . Then, the Borel summation of  $\tilde{F}_i(t)$  is given by

$$\begin{aligned} \int_0^{\infty} ds e^{-s} \mathcal{B}[\tilde{F}_i(t)](s) &= \int_0^{\infty} ds e^{-s} p_i^{\text{ss}}\left(t - \frac{s}{\gamma}\right) \\ &=: F_i(t). \end{aligned} \quad (16)$$

Next, we show that the exact solution of the master equation is identical with the Borel summation of  $\tilde{F}_i(t)$ . Using the general theory of a first-order linear ordinary differential equation, the exact solution under the initial condition  $p_i(t=0) = p_i(0)$  is given by

$$p_i(t) = e^{-\gamma t} \left[ p_i(0) + \gamma \int_0^t du p_i^{\text{ss}}(u) e^{\gamma u} \right]. \quad (17)$$

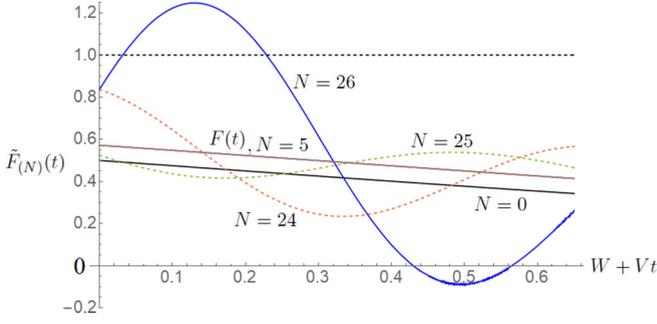


FIG. 1.  $F(t)$  [ $:= F_1(t)$ ] and  $\tilde{F}_{(N)}(t)$  for  $N = 0, 5, 24, 25$ , and  $26$  at  $V/\gamma = 0.3$ .  $F(t)$  and  $\tilde{F}_{(5)}(t)$  are overlapping. The asymptotic expansion above 24th order deviates significantly from the exact solution.

This equation can be rewritten as (Appendix B)

$$p_i(t) = e^{-\gamma t} [p_i(0) - F_i(0)] + F_i(t), \quad (18)$$

where  $F_i(t)$  was introduced in (16). The first term of (18) corresponds to  $\bar{\delta}p(t)$  because this term damps exponentially. The second term of (18) corresponds to the dynamical steady state. We note that the exact solution of the master equation is identical with the Borel summation of  $p_i^{\text{dss}}(t)$  even if  $\gamma$  is time dependent.

In the following of this paper, we assume that the instantaneous steady state  $p^{\text{ss}}(\alpha)$  is the Gibbs distribution  $p^{\text{ss}}(\alpha) = e^{-\beta E_i}/Z$  where  $E_i$  is the energy of the state  $i$ ,  $\beta$  is the inverse temperature, and  $Z := \sum_i e^{-\beta E_i}$  is the partition function. We focus on the energy modulation

$$E_i = \alpha(t) E_i^{(0)} \quad (19)$$

with fixed  $E_i^{(0)}$  and  $\beta$  and study the linear modulation  $\alpha(t) = h_i + \omega t$  in particular. In this case, the control parameter is  $\alpha(t)$ .

## B. Demonstration in two-level system

In this subsection, we consider the two-level system (11) and show that  $\tilde{F}_i(t)$  ( $i = 0, 1$ ) diverges and is identical with the asymptotic expansion of the exact solution  $F_i(t)$ .

We consider (11) with  $\Delta(t) = (h_i + \omega t)\Delta$  and fixed  $\gamma$  and  $\beta$  in the following of this section. In this case,  $\tilde{F}(t) := \tilde{F}_1(t)$  is given by

$$\tilde{F}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma^n} \frac{d^n}{dt^n} f(W + Vt) \quad (20)$$

with  $f(x) := 1/(e^x + 1)$ . Here,  $W := \beta \Delta h_i$ ,  $V := \beta \Delta \omega > 0$ . Figure 1 shows  $F(t)$  [ $:= F_1(t)$ ] and  $\tilde{F}_{(N)}(t) := \sum_{n=0}^N \frac{(-1)^n}{\gamma^n} \frac{d^n}{dt^n} f(W + Vt)$  for  $N = 0, 5, 24, 25$ , and  $26$  at  $V/\gamma = 0.3$ .  $\tilde{F}(t)$  does not converge. For  $N = 5$ , the asymptotic expansion  $\tilde{F}_{(5)}$  well agrees with the exact solution  $F(t)$ . However, for large  $N = 24, 25$ , and  $26$ , the deviations increase and the asymptotic expansions oscillate. Although probability should be between 0 and 1,  $\tilde{F}_{(26)}(t)$  is not.

The dynamical steady state is given by

$$\begin{aligned} F(t) &= \int_0^\infty ds e^{-s} f\left(W + V\left(t - \frac{s}{\gamma}\right)\right) \\ &= {}_2F_1(1, T; T + 1; -e^{W+Vt}) \\ &= f(W + Vt) {}_2F_1(1, 1; T + 1; 1 - f(W + Vt)) \end{aligned} \quad (21)$$

with  $T := \gamma/V$ .  ${}_2F_1(a, b; c; z)$  is the hypergeometric function. In the third equality of (21), we used the Kummer relation  ${}_2F_1(a, b; c; z) = \frac{1}{(1-z)^a} {}_2F_1(a, c-b; c; \frac{z}{z-1})$ .

We show that  $\tilde{F}(t)$  is an asymptotic expansion of  $F(t)$ . A divergent series  $\sum_{k=0}^{\infty} a_k/T^k$  is the asymptotic expansion of a function  $G(T)$  if  $R_n(T) := T^n[G(T) - \sum_{k=0}^n a_k/T^k]$  satisfies  $|R_n(T)| \rightarrow 0$  ( $T \rightarrow \infty$ ). In the third line of (21),  ${}_2F_1(1, 1; T + 1; z)$  can be expanded in  $z$  as

$$\begin{aligned} &{}_2F_1(1, 1; T + 1; z) \\ &= 1 + \sum_{k=1}^{\infty} \frac{k!}{(T+1)(T+2)\cdots(T+k)} z^k \\ &=: 1 + \sum_{k=1}^n \frac{k!}{(T+1)(T+2)\cdots(T+k)} z^k + \delta_n(T, z). \end{aligned} \quad (22)$$

The second term of the third line of (22) can be expanded in  $1/T$  as

$$\begin{aligned} &\sum_{k=1}^n \frac{k!}{(T+1)(T+2)\cdots(T+k)} z^k \\ &= \sum_{k=1}^n \frac{a_k(z)}{T^k} + r_n(T, z) \end{aligned} \quad (23)$$

with  $r_n(T, z) = O(1/T^{n+1})$ . Here,  $a_1(z) = z$ ,  $a_2(z) = -z + 2z^2$ ,  $a_3(z) = z - 6z^2 + 6z^3$ ,  $a_4(z) = -z + 14z^2 - 36z^3 + 24z^4$ , and  $a_5(z) = z - 30z^2 + 150z^3 - 240z^4 + 120z^5$ . Then,  ${}_2F_1(1, 1; T + 1; z)$  is given by

$${}_2F_1(1, 1; T + 1; z) = 1 + \sum_{k=1}^n \frac{a_k(z)}{T^k} + \frac{R_n(T, z)}{T^n} \quad (24)$$

with  $R_n(T, z) := T^n[r_n(T, z) + \delta_n(T, z)]$ . The first and second terms of the right-hand side provide  $\tilde{F}_{(n)}(t)$ . If  $T$  is large enough,  ${}_2F_1(1, 1; T + 1; z)$  ( $0 \leq z \leq 1$ ) is well approximated by first few terms. The truncation error  $\delta_n(T, z)$  satisfies

$$\begin{aligned} |\delta_n(T, z)| &\leq \delta_n(T, 1) \\ &= \frac{(n+1)!}{(T-1)(T+1)(T+2)\cdots(T+n)}. \end{aligned} \quad (25)$$

Here, we used the Gauss summation theorem  ${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a-b)\Gamma(c-b)}$  [ $\text{Re}(c) > 0$ ,  $\text{Re}(c-a-b) > 0$ ], where  $\Gamma(z)$  is the gamma function. The above inequality leads to  $|R_n(T, z)| \rightarrow 0$  ( $T \rightarrow \infty$ ). Then,  $\tilde{F}(t)$  is the asymptotic expansion of  $F(t)$ . We obtain  $\tilde{F}(t)$  from  $F(t)$  if we expand  ${}_2F_1(1, 1; T + 1; z)$  regarding  $T$  as larger than any natural number  $n$ . In addition, we explain a prescription getting  $F(t)$  from  $\tilde{F}(t)$  in Appendix C.

The oscillation of  $\tilde{F}_i(t)$  is not limited to a two level-system. In Appendix D, we discuss the higher derivative and oscillation for a wide class of functions.

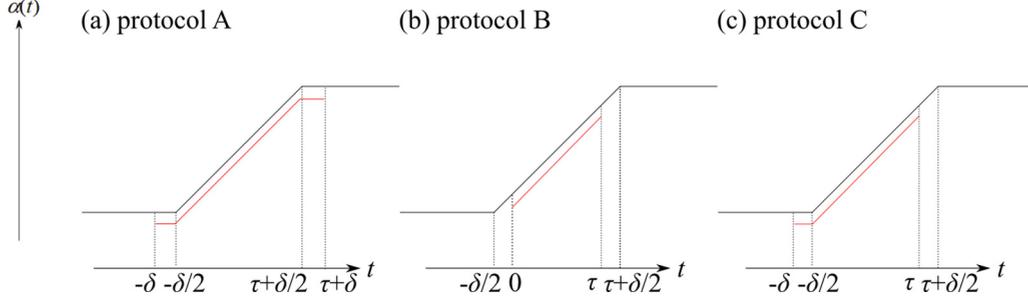


FIG. 2. Protocols A, B, and C.

We note that any infinite series expansion (9) is not necessarily divergent. For instance, the protocol discussed in Ref. [23] leads to a convergent series expansion. In Appendix E, we give an instance of the convergent infinite series expression of the dynamical steady state.

#### IV. TRADE-OFF RELATION

In this section, we show that the equality of a trade-off relation of Shiraishi-Funo-Saito [23] holds in the lowest order of the frequency of the energy modulation in the two-level system.

If the Liouvillian satisfies the local detailed balance condition, the trade-off relation (1) holds [23]. There,  $L := \sum_i |p_i(\tau) - p_i(0)|$  is the  $l_1$  norm [29]. The average activity

$$A := \frac{1}{\tau} \int_0^\tau dt A(t) \quad (26)$$

is calculated from the activity

$$A(t) := \sum_{i \neq j} K_{ij}(t) p_j(t). \quad (27)$$

The total entropy production  $\sigma$  is given by

$$\sigma := \beta Q + S_{\text{Sh}}(p(\tau)) - S_{\text{Sh}}(p(0)), \quad (28)$$

where  $Q$  is the heat flowing into the bath from the system and  $S_{\text{Sh}}(p) := -\sum_i p_i \ln p_i$  is the Shannon entropy of the system.

We consider three protocols (Fig. 2). In the following, we suppose that  $\alpha(t)$  is constant for  $t \leq -\delta/2$  and  $t \geq \tau + \delta/2$ , and  $\alpha(t) = h_i + \omega t$  while  $-\delta/2 < t < \tau + \delta/2$ , where  $\delta$  is a small value:

$$\alpha(t) = \begin{cases} h_i - \omega \frac{\delta}{2} & (t \leq -\frac{\delta}{2}), \\ h_i + \omega t & (-\frac{\delta}{2} \leq t \leq \tau + \frac{\delta}{2}), \\ h_i + \omega(\tau + \frac{\delta}{2}) & (\tau + \frac{\delta}{2} \leq t). \end{cases} \quad (29)$$

In the first protocol (protocol A), we consider  $-\delta \leq t \leq \tau + \delta$ . At time  $t = -\delta$  and  $t = \tau + \delta$ , the state of the system is the instantaneous steady state  $p^{\text{ss}}$ . In the second protocol (protocol B), we consider  $0 \leq t \leq \tau$  and suppose that the state of the system at  $t = 0$  and  $t = \tau$  is the dynamical steady state  $p^{\text{dss}}$ . In the third protocol (protocol C), we consider  $-\delta \leq t \leq \tau$  and suppose that the state of the system at  $t = -\delta$  is the instantaneous steady state. Protocols B and C can be regarded as observations of the process of protocol A at different time intervals. The ratio  $r$  defined by (1) can be expanded as

$$r = r_0 + r_1 \omega + r_2 \omega^2 + \dots \quad (30)$$

In the following, we show that  $r_0 = 1$  for protocols A, B, and C in the two-level system (11). To calculate  $r_0$ , we estimate  $\sigma$  up to first order of  $\omega$  (we denote this by  $\sigma^{(1)}$ ) and the average activity  $A$  and  $L$  up to zeroth order. In the zeroth order of  $\omega$ ,  $A$  is given by  $A^{\text{ss}} := \frac{1}{\tau} \int_0^\tau dt A^{\text{ss}}(\alpha_t)$  with  $A^{\text{ss}}(\alpha_t) := \sum_{i \neq j} K_{ij}(\alpha_t) p_j^{\text{ss}}(\alpha_t)$ , and  $L$  is given by  $L^{\text{ss}} := \sum_i |p_i^{\text{ss}}(\alpha_\tau) - p_i^{\text{ss}}(\alpha_0)|$ .

We estimate  $A^{\text{ss}}$  in the relaxation time approximation model (10). In this case, we obtain

$$A^{\text{ss}}(\alpha_t) = \gamma \left[ 1 - \sum_i (p_i^{\text{ss}})^2 \right]. \quad (31)$$

In the two-level system (11) with  $\Delta(t) = \alpha(t)\Delta$ ,  $A^{\text{ss}}$  becomes

$$A^{\text{ss}} = \frac{2\gamma}{\Delta(\beta_f - \beta_i)} [f(\beta_i \Delta) - f(\beta_f \Delta)]. \quad (32)$$

Here,  $\beta_i := h_i \beta$ ,  $\beta_f := h_f \beta$ , and  $h_f := h_i + \omega \tau$ .

We evaluate the heat current

$$q(t) := - \sum_i E_i \frac{d}{dt} p_i \quad (33)$$

using the asymptotic expansion (9). We denote the heat current contributed from  $p^{\text{ss}}(\alpha_t)$  and  $p^{(n)}(t)$  by  $q_0(t)$  and  $q_n(t)$ , respectively. We obtain

$$q_0(t) = - \sum_i E_i \frac{d p_i^{\text{ss}}}{dt}, \quad (34)$$

$$q_1(t) = - \sum_{i,k} E_i \left[ R_{ik} \frac{d^2 p_k^{\text{ss}}}{dt^2} - \left( R \frac{dK}{dt} R \right)_{ik} \frac{d p_k^{\text{ss}}}{dt} \right]. \quad (35)$$

Substituting (19) into (34), we obtain

$$q_0(t) = \alpha'(t) \alpha(t) \beta [\langle (E^{(0)})^2 \rangle - \langle E^{(0)} \rangle^2], \quad (36)$$

where  $\langle (E^{(0)})^n \rangle := \sum_i (E_i^{(0)})^n p_i^{\text{ss}}(\alpha_t)$  is the  $n$ th moment of energy.  $q_0(t)$  is proportional to the variance of the energy and does not depend on the Liouvillian. It is also expressed by using the Fisher information [22,29,30]. For (29), the zeroth order of the heat  $Q_0 = \int_0^\tau dt q_0(t)$  is given by

$$Q_0 = \frac{1}{\beta} [\beta_i \langle E^{(0)} \rangle (\beta_i) + \ln Z(\beta_i) - \beta_f \langle E^{(0)} \rangle (\beta_f) - \ln Z(\beta_f)]. \quad (37)$$

The Shannon entropy is given by  $S_{\text{Sh}}(p) = S_{\text{Sh}}(\beta_t) + \delta S_{\text{Sh}}$  with  $S_{\text{Sh}}(\beta_t) := \beta_t \langle E^{(0)} \rangle (\beta_t) + \ln Z(\beta_t)$  and  $\beta_t := \alpha(t) \beta$ .

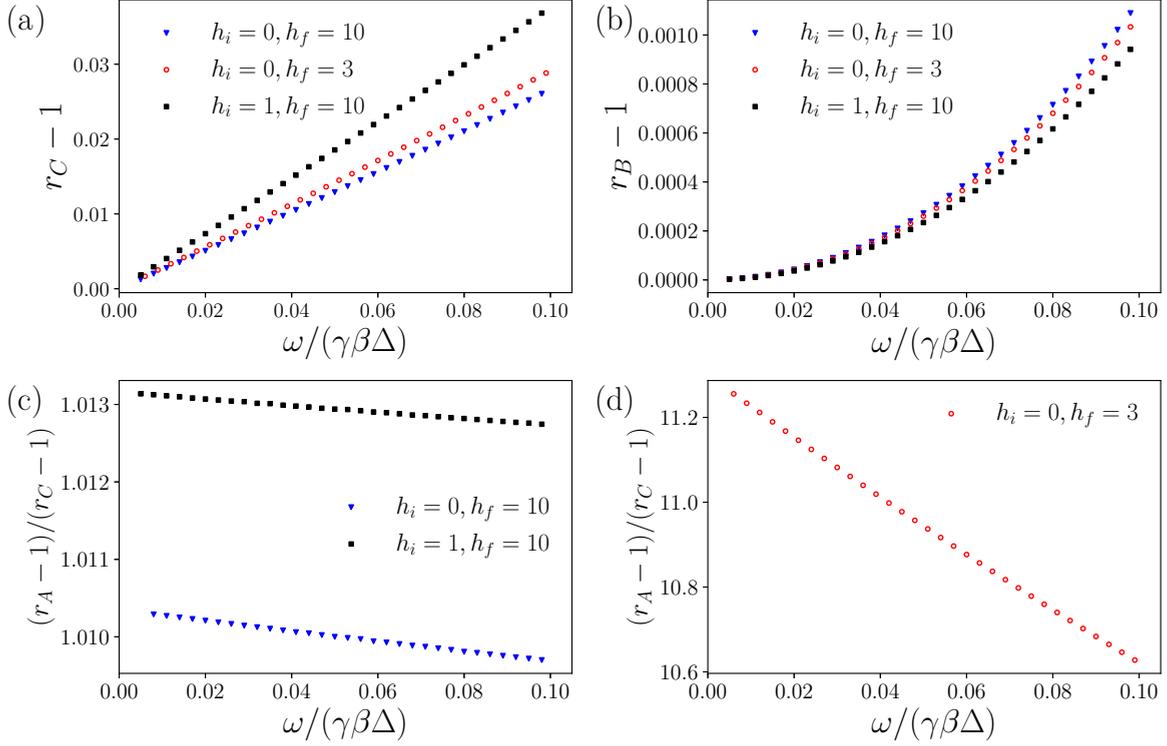


FIG. 3. (a)  $(r - 1)$  for protocol C, (b)  $(r - 1)$  for protocol B, (c)  $(r_A - 1)/(r_C - 1)$  and (d)  $(r_A - 1)/(r_C - 1)$  for  $(h_i, h_f) = (0, 10)$  (triangle),  $(h_i, h_f) = (1, 10)$  (square), and  $(h_i, h_f) = (0, 3)$  (circle). Here,  $r_A$  and  $r_C$  denote  $r$  for protocols A and C, respectively. In these figures, we set  $\beta\Delta = 1$ .

Therefore, the zeroth order of the entropy production vanishes:  $\beta Q_0 + S_{\text{Sh}}(\beta_f) - S_{\text{Sh}}(\beta_i) = 0$ . For the relaxation time approximation model,  $q_1(t)$  is given by

$$q_1(t) = \frac{\alpha(t)}{\gamma} \{ [\alpha'(t)]^2 \beta^2 C_3(\beta_t) - \alpha''(t) \beta C_2(\beta_t) \}. \quad (38)$$

Here,  $C_n(\beta_t)$  is the  $n$ th-order cumulant of  $E_i^{(0)}$ :

$$C_n(\beta_t) = (-1)^n \frac{\partial^n \ln Z(\beta_t)}{\partial \beta_t^n} \quad (39)$$

and  $Z(\beta_t) := \sum_i e^{-\beta_t E_i^{(0)}}$ . In Appendix F, we calculate  $q_n(t)$ .

We calculate  $\sigma^{(1)}$  for the relaxation time approximation model. For each protocol,  $\sigma^{(1)}$  is given by

$$\begin{aligned} \sigma^{(1)} &= \beta(Q_1 + \delta Q_1|_{t=\tau} - \delta Q_1|_{t=0}) \quad (\text{protocol A}), \\ \sigma^{(1)} &= \beta Q_1 + \delta S_{\text{Sh}}|_{t=\tau} - \delta S_{\text{Sh}}|_{t=0} \quad (\text{protocol B}), \\ \sigma^{(1)} &= \beta(Q_1 - \delta Q_1|_{t=0}) + \delta S_{\text{Sh}}|_{t=\tau} \quad (\text{protocol C}). \end{aligned} \quad (40)$$

Here,

$$\begin{aligned} Q_1 &:= \int_0^\tau dt q_1(t) \\ &= \frac{\omega}{\gamma} [\beta_i C_2(\beta_i) + C_1(\beta_i) - \beta_f C_2(\beta_f) - C_1(\beta_f)]. \end{aligned} \quad (41)$$

$\delta Q_1$  is the heat contributed from the term  $[\alpha''(t)]$  including the delta function in (38) and given by

$$\delta Q_1 = \frac{\omega}{\gamma} \beta_t C_2(\beta_t). \quad (42)$$

$\delta S_{\text{Sh}}$  is given by

$$\delta S_{\text{Sh}} = - \sum_i p_i^{(1)} \ln p_i^{\text{ss}} = \frac{\omega}{\gamma} \beta \beta_t C_2(\beta_t). \quad (43)$$

Then,  $\sigma^{(1)}$  becomes

$$\sigma^{(1)} = \frac{\omega}{\gamma} \beta [C_1(\beta_i) - C_1(\beta_f)] \quad (44)$$

for protocols A, B, and C. In the two-level system,

$$\sigma^{(1)} = \frac{\omega}{\gamma} \beta \Delta [f(\beta_i \Delta) - f(\beta_f \Delta)] \quad (45)$$

holds.

From (32), (45), and

$$L^{\text{ss}} = 2[f(\beta_i \Delta) - f(\beta_f \Delta)], \quad (46)$$

we obtain

$$r_0 = \frac{2\sigma^{(1)} A^{\text{ss}}}{(L^{\text{ss}})^2} \tau = 1 \quad (47)$$

for protocols A, B, and C. To obtain (47), the heat emission and absorption at edges ( $t = -\delta/2, \tau + \delta/2$ ) or the correction of the Shannon entropy  $\delta S_{\text{Sh}}$  at edges ( $t = 0, \tau$ ) are essential. If we ignore these corrections,  $r_0$  can be less than 1.

Figure 3 shows the numerical results of  $(r - 1)$  for the three protocols calculated from (26) and (33) by exploiting the exact solution for the two-level system (21). For all protocols, during the duration  $0 \leq t \leq \tau$ , the system is driven as  $\alpha(t) = h_i + \omega t$ . The initial states for the protocols A, B, and C

are the instantaneous steady state  $p_1(t=0) = f(W)$ , the dynamical steady state  $p_1(t=0) = F(0)$ , and the instantaneous steady state  $p_1(t=0) = f(W)$ , respectively. In addition, for the protocol A, after we turn off the driving at time  $\tau$ , we fix  $\alpha(t) = h_f = h_i + \omega\tau$  ( $\tau \leq t \leq \tau + 30/\gamma$ ) and wait until the system relaxes to the instantaneous steady state. In this duration the heat is emitted from the system.

In Fig. 3, we plot  $(r_C - 1)$ ,  $(r_B - 1)$ , and  $(r_A - 1)/(r_C - 1)$  for various  $h_i$  and  $h_f$ :  $(h_i, h_f) = (0, 10)$  (triangle),  $(h_i, h_f) = (1, 10)$  (square), and  $(h_i, h_f) = (0, 3)$  (circle) at  $\beta\Delta = 1$ . Here,  $r_A$  and  $r_C$  denote  $r$  for protocols A and C, respectively. Figure 3 shows that  $r_0 = 1$ ,  $r_1 > 0$  for protocols A and C and  $r_0 = 1$ ,  $r_1 = 0$ ,  $r_2 > 0$  for protocol B. Namely, for small  $\omega$ , the correction  $(r - 1)$  is a linear function for protocols A and C and a quadratic function for protocol B.

In systems described by the master equation having the Liouvillian (10) or more general Liouvillians, in general,  $r_0 > 1$  if  $N \geq 3$ . For instance, in the relaxation time approximation model,  $r_0$  of protocol A of an  $N$ -level ( $N \geq 2$ ) system of which energies are  $E_0^{(0)} = 0$  and  $E_i^{(0)} = \Delta$  ( $i = 1, 2, \dots, N-1$ ) is given by

$$r_0 = \frac{N}{2(N-1)} + \frac{N-2}{2(N-1)^2} \frac{\ln \frac{1+(N-1)e^{-\beta_i\Delta}}{1+(N-1)e^{-\beta_f\Delta}}}{\frac{1}{N-1+e^{\beta_i\Delta}} - \frac{1}{N-1+e^{\beta_f\Delta}}}. \quad (48)$$

## V. SUMMARY

In summary, for the relaxation time approximation model, we calculated the infinite series expansion of the solution of the driven master equation using the pseudoinverse of the Liouvillian. We demonstrated that the Borel summation of the series becomes the exact solution. For the two-level system coupled to a single bath, we considered the linear modulation of the energy and showed that the series expression is the asymptotic expansion of the exact solution. Based on the asymptotic expansion for the relaxation time approximation model, we demonstrated that the equality of the trade-off relation of Shiraishi-Funo-Saito [23] holds in the lowest order of the frequency of the energy modulation in the two-level system. To our knowledge, this is the first instance where the equality sign holds in the trade-off relation of Ref. [23]. If we ignore the heat emission and absorption at edges (the initial and end times) or the differences of the Shannon entropy between the instantaneous steady state and the dynamical steady state at edges, the trade-off relation can be broken.

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## APPENDIX A: DERIVATION OF THE TRADE-OFF RELATION

To be self-contained, we repeat the derivations of (1) for the single heat bath case (Ref. [23] studied the multiple heat baths case). Suppose the local detailed balance condition

$$K_{ij}(t)e^{-\beta(t)E_j(t)} = K_{ji}(t)e^{-\beta(t)E_i(t)}. \quad (A1)$$

Here,  $\beta(t)$  is the inverse temperature of the bath. The entropy production rate  $\dot{\sigma}(t)$  is defined by

$$\dot{\sigma}(t) := \frac{d}{dt} S_{\text{Sh}}(p(t)) + \beta(t)q(t). \quad (A2)$$

The heat current  $q(t)$  defined by (33) can be rewritten as

$$q(t) = \sum_{i \neq j} K_{ji}(t)p_i(t)[E_i(t) - E_j(t)]. \quad (A3)$$

Here, we used the master equation

$$\frac{dp_i}{dt} = \sum_{j(\neq i)} [K_{ij}(t)p_j(t) - K_{ji}(t)p_i(t)]. \quad (A4)$$

Using (A3), (A4), and the local detailed balance condition, (A2) becomes

$$\begin{aligned} \dot{\sigma}(t) &= \frac{1}{2} \sum_{i \neq j} (K_{ji}p_i - K_{ij}p_j) \ln \frac{K_{ji}p_i}{K_{ij}p_j} \\ &\geq \sum_{i \neq j} \frac{(K_{ji}p_i - K_{ij}p_j)^2}{K_{ji}p_i + K_{ij}p_j}. \end{aligned} \quad (A5)$$

Here, we used the inequality  $(a-b) \ln \frac{a}{b} \geq 2 \frac{(a-b)^2}{a+b}$ , which is valid for non-negative  $a$  and  $b$ . The distance  $L = \sum_i |p_i(\tau) - p_i(0)|$  is evaluated as

$$L \leq \int_0^\tau dt \sum_i \left| \frac{dp_i}{dt} \right|. \quad (A6)$$

Here,

$$\begin{aligned} \sum_i \left| \frac{dp_i}{dt} \right| &= \sum_i \left| \sum_{j(\neq i)} (K_{ij}p_j - K_{ji}p_i) \right| \\ &\leq \sum_i \sum_{j(\neq i)} |K_{ij}p_j - K_{ji}p_i| \\ &\leq \sum_i \sqrt{\left( \sum_{j(\neq i)} \frac{(K_{ij}p_j - K_{ji}p_i)^2}{(K_{ij}p_j + K_{ji}p_i)} \right) \sum_{j(\neq i)} (K_{ij}p_j + K_{ji}p_i)} \\ &\leq \sqrt{\left( \sum_{j \neq i} \frac{(K_{ij}p_j - K_{ji}p_i)^2}{(K_{ij}p_j + K_{ji}p_i)} \right) \sum_{j \neq i} (K_{ij}p_j + K_{ji}p_i)} \\ &\leq \sqrt{2\dot{\sigma}(t)A(t)} \end{aligned} \quad (A7)$$

holds. Here, we used the Schwarz inequality in the third and fourth line. Then, we obtain

$$L \leq \int_0^\tau dt \sqrt{2\dot{\sigma}(t)A(t)} \leq \sqrt{2\sigma\tau A}. \quad (A8)$$

Here, we used  $\sigma := \int_0^\tau dt \dot{\sigma}(t)$ , (26) and the Schwarz inequality. The above equation leads to (1). If  $\beta(t)$  is time independent,  $\sigma$  is given by (28).

If

$$\begin{aligned} &|K_{ij}(t)p_j(t) - K_{ji}(t)p_i(t)| \\ &= \sqrt{\frac{\sigma}{2\tau A}} [K_{ij}(t)p_j(t) + K_{ji}(t)p_i(t)] \end{aligned} \quad (A9)$$

holds for all  $i \neq j$ , the inequalities for the third and fourth lines in (A7) hold exactly and the second inequality in (A8) is satisfied with a relative error  $O(\omega^2)$  where  $\omega$  is the modulation frequency. For the two-level system in Sec. IV, the above equation is satisfied up to the first order of  $\omega$  while  $0 \leq t \leq \tau$ . The equality of the last line in (A7) holds with a relative error  $O(\omega^2)$ .

#### APPENDIX B: DERIVATION OF (18)

By variable transformation  $s = \gamma(t - u)$ , (17) becomes

$$p_i(t) = e^{-\gamma t} p_i(0) + \int_0^\infty ds p_i^{\text{ss}}\left(t - \frac{s}{\gamma}\right) e^{-s} - \int_{\gamma t}^\infty ds p_i^{\text{ss}}\left(t - \frac{s}{\gamma}\right) e^{-s}. \quad (\text{B1})$$

Using variable transformation  $-\frac{s'}{\gamma} = t - \frac{s}{\gamma}$  in the last term, we obtain (18).

#### APPENDIX C: A PRESCRIPTION GETTING $F(t)$ FROM $\tilde{F}(t)$

We can obtain  $F(t)$  from  $\tilde{F}(t)$  by the following prescription. Because of the analytic continuation, it is enough to consider the  $|e^{W+Vt}| < 1$  case. Then,  $F(t)$  is given by

$$F(t) = {}_2F_1(1, T; T + 1; -e^{W+Vt}) = \sum_{n=0}^{\infty} \frac{1}{1 + n/T} (-e^{W+Vt})^n. \quad (\text{C1})$$

On the other hand,  $\tilde{F}(t)$  is expressed as

$$\begin{aligned} \tilde{F}(t) &= \sum_{m=0}^{\infty} \left(-\frac{1}{\gamma}\right)^m \frac{d^m}{dt^m} \sum_{n=0}^{\infty} (-e^{W+Vt})^n \\ &\stackrel{\text{w}}{=} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(-\frac{1}{\gamma}\right)^m (nV)^m (-e^{W+Vt})^n \\ &= \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \left(-\frac{n}{T}\right)^m \right] (-e^{W+Vt})^n \\ &\stackrel{\text{w}}{=} \sum_{n=0}^{\infty} \frac{1}{1 + n/T} (-e^{W+Vt})^n. \end{aligned} \quad (\text{C2})$$

In the second line, we changed the order of the sum, and in the fourth line, we suppose that  $T > n$  for all  $n$ . Then,  $\tilde{F}(t)$  becomes  $F(t)$ .

#### APPENDIX D: HIGHER DERIVATIVE

$f(W + Vt)$  can be written as

$$f(W + Vt) = \frac{1}{2} - \frac{1}{V} \sum_{k=0}^{\infty} \left[ \frac{1}{t - [-W/V + i(\pi/V)(2k + 1)]} + \frac{1}{t - [-W/V - i(\pi/V)(2k + 1)]} \right]. \quad (\text{D1})$$

The dominant contribution of  $\frac{d^n}{dt^n} f(W + Vt)$  (for large  $n$ ) comes from two poles ( $k = 0$ ) most nearly above and below

$t$ . In general, the  $n$ th derivative of  $g(t) = \sum_k A_k / (t - a_k)$  ( $a_k$  are complex numbers) is given by

$$g^{(n)}(t) = (-1)^n n! \sum_k \frac{A_k}{(t - a_k)^{n+1}}. \quad (\text{D2})$$

The factorial increase with  $n$  of the coefficients underlies the divergence commonly encountered in asymptotic series, and, together with the increasingly fast oscillations, reflects the instability of differentiation [31]. In the studies of the full-counting statistics, the oscillations coming from the higher derivative by the counting field have been observed [2,32–34].

#### APPENDIX E: INSTANCE OF THE CONVERGENT INFINITE SERIES EXPRESSION OF THE DYNAMICAL STEADY STATE

In Ref. [23], a two-level system of which the Liouvillian is  $K_{10} = 1$  and  $K_{01}(\alpha) = \frac{4\tau+1}{2\tau-\alpha}$  with  $\alpha_t = t$  has been studied. In this case, the instantaneous steady state is given by  $p^{\text{ss}}(\alpha) = \left(\frac{2\tau+1+\alpha}{4\tau+1}, \frac{2\tau-\alpha}{4\tau+1}\right)^t$ . The pseudoinverse is given by

$$R(\alpha) = \begin{pmatrix} 0 & \frac{2\tau-\alpha}{4\tau+1} \\ \frac{2\tau-\alpha}{4\tau+1} & 0 \end{pmatrix}. \quad (\text{E1})$$

Then,  $p^{(n+1)}(t)$  ( $n = 0, 1, \dots$ ) becomes

$$p^{(n+1)}(t) = R(\alpha_t) \left[ \frac{dR(\alpha_t)}{dt} \right]^n \frac{dp^{\text{ss}}(\alpha_t)}{dt}. \quad (\text{E2})$$

The infinite series expression of the dynamical steady state (9) converges to  $p_1^{\text{dss}}(t) = \frac{1}{2} - \frac{t}{4\tau}$  and is identical with the exact solution.

Reference [23] set the energies as  $E_0 = 0$  and  $K_{01}(\alpha_t) = e^{\beta E_1(t)}$ . For  $\alpha_t = t$  for  $0 \leq t \leq \tau$  under the initial distribution  $p_0(0) = 1/2$ ,  $r$  defined by (1) converges to  $(5/2) \ln(3/2) = 1.01366\dots$  for large  $\tau$ .

#### APPENDIX F: HEAT CURRENT AND HEAT

For our energy modulation (19), we can also interpret that the inverse temperature is time dependent  $\beta_t := \alpha(t)\beta$ . Then, from (35), we obtain

$$q_1(t) = \beta \alpha(t) (-\{\alpha''(t) + 2\langle E^{(0)} \rangle \beta [\alpha'(t)]^2\} \mu_1(\beta_t) + \beta [\alpha'(t)]^2 \mu_2(\beta_t)), \quad (\text{F1})$$

where

$$\mu_1(\beta_t) := - \sum_{i,k} E_i^{(0)} R_{ik}^{(0)} E_k^{(0)} p_k^{\text{ss}}, \quad (\text{F2})$$

$$\begin{aligned} \mu_2(\beta_t) &:= - \sum_{i,k} E_i^{(0)} R_{ik}^{(0)} (E_k^{(0)})^2 p_k^{\text{ss}} \\ &\quad - \sum_{i,k} E_i^{(0)} \left( R^{(0)} \frac{dK}{d\beta_t} R^{(0)} \right)_{ik} E_k^{(0)} p_k^{\text{ss}}. \end{aligned} \quad (\text{F3})$$

Here,  $R_{ik}^{(0)}$  is defined by (3) and  $\sum_k R_{ik}^{(0)}(\alpha) p_k^{\text{ss}}(\alpha) = 0$  is called the Drazin inverse [35].

We consider the Liouvillian (10) with fixed  $\gamma$  and  $\beta$ . In this case, using the Faà di Bruno formula [36], we obtain

$$q_{n-1}(t) = \frac{(-1)^n}{\gamma^{n-1}} \alpha(t) \sum \frac{n!}{m_1! m_2! \cdots m_n!} \times (-1)^k \beta^k C_{k+1}(\beta_t) \prod_j \left( \frac{\alpha^{(j)}}{j!} \right)^{m_j}, \quad (\text{F4})$$

where  $k := \sum m_j$  and the sum is over all  $n$ -tuples of non-negative integers  $(m_1, \dots, m_n)$  satisfying the constraint  $\sum_{j=1}^n j m_j = n$ . If  $\alpha(t) = h_i + \omega t$  while  $0 < t < \tau$ , (F4) be-

comes

$$q_{n-1}(t) = \frac{1}{\gamma^{n-1}} \alpha(t) \omega^n \beta^n C_{n+1}(\beta_t). \quad (\text{F5})$$

The heat is given by

$$Q_n := \int_0^\tau dt q_n(t) = \left( \frac{\omega \beta}{\gamma} \right)^n \frac{1}{\beta} [\beta_i C_{n+1}(\beta_i) + C_n(\beta_i) - \beta_f C_{n+1}(\beta_f) - C_n(\beta_f)]. \quad (\text{F6})$$

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- [1] F. Cavaliere, M. Governale, and J. König, Nonadiabatic Pumping through Interacting Quantum Dots, *Phys. Rev. Lett.* **103**, 136801 (2009).
- [2] C. Flindt, T. Novotný, A. Braggio, and A.-P. Jauho, Counting statistics of transport through Coulomb blockade nanostructures: High-order cumulants and non-Markovian effects, *Phys. Rev. B* **82**, 155407 (2010).
- [3] H. L. Calvo, L. Classen, J. Splettstoesser, and M. R. Wegewijs, Interaction-induced charge and spin pumping through a quantum dot at finite bias, *Phys. Rev. B* **86**, 245308 (2012).
- [4] T. Yuge, T. Sagawa, A. Sugita, and H. Hayakawa, Geometrical pumping in quantum transport: Quantum master equation approach, *Phys. Rev. B* **86**, 235308 (2012).
- [5] S. Nakajima, M. Taguchi, T. Kubo, and Y. Tokura, Interaction effect on adiabatic pump of charge and spin in quantum dot, *Phys. Rev. B* **92**, 195420 (2015).
- [6] T. Pluecker, M. R. Wegewijs, and J. Splettstoesser, Gauge freedom in observables and Landsberg's nonadiabatic geometric phase: Pumping spectroscopy of interacting open quantum systems, *Phys. Rev. B* **95**, 155431 (2017).
- [7] E. Potanina, K. Brandner, and C. Flindt, Optimization of quantized charge pumping using full counting statistics, *Phys. Rev. B* **99**, 035437 (2019).
- [8] K. Takahashi, Y. Hino, K. Fujii, and H. Hayakawa, Full counting statistics and fluctuation-dissipation relation for periodically driven two-state systems, *J. Stat. Phys.* **181**, 2206 (2020).
- [9] K. Saito and H. Tasaki, Extended Clausius relation and entropy for nonequilibrium steady states in heat conducting quantum systems, *J. Stat. Phys.* **145**, 1275 (2011).
- [10] T. Sagawa and H. Hayakawa, Geometrical expression of excess entropy production, *Phys. Rev. E* **84**, 051110 (2011).
- [11] T. S. Komatsu, N. Nakagawa, S. Sasa, and H. Tasaki, Exact equalities and thermodynamic relations for nonequilibrium steady states *J. Stat. Phys.* **159**, 1237 (2015).
- [12] S. Nakajima and Y. Tokura, Excess entropy production in quantum system: Quantum master equation approach, *J. Stat. Phys.* **169**, 902 (2017).
- [13] S. Nakajima, Theoretical studies on quantum pump and excess entropy production: Quantum master equation approach, [arXiv:1710.05646](https://arxiv.org/abs/1710.05646).
- [14] N. Shiraishi, K. Saito, and H. Tasaki, Universal Trade-Off Relation between Power and Efficiency for Heat Engines, *Phys. Rev. Lett.* **117**, 190601 (2016).
- [15] K. Brandner and K. Saito, Thermodynamic Geometry of Microscopic Heat Engines, *Phys. Rev. Lett.* **124**, 040602 (2020).
- [16] Y. Hino and H. Hayakawa, Geometrical formulation of adiabatic pumping as a heat engine, *Phys. Rev. Research* **3**, 013187 (2021).
- [17] H. Tajima and K. Funo, Superconducting-like Heat Current: Effective Cancellation of Current-Dissipation Trade-Off by Quantum Coherence, *Phys. Rev. Lett.* **127**, 190604 (2021).
- [18] S. Kamimura, H. Hakoshima, Y. Matsuzaki, K. Yoshida, and Y. Tokura, Quantum enhanced heat engine by superabsorption, [arXiv:2106.10813](https://arxiv.org/abs/2106.10813).
- [19] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga, Shortcuts to adiabaticity: Concepts, methods, and applications, *Rev. Mod. Phys.* **91**, 045001 (2019).
- [20] K. Takahashi, Shortcuts to adiabaticity applied to nonequilibrium entropy production: An information geometry viewpoint, *New J. Phys.* **19**, 115007 (2017).
- [21] K. Funo, N. Lambert, F. Nori, and C. Flindt, Shortcuts to Adiabatic Pumping in Classical Stochastic Systems, *Phys. Rev. Lett.* **124**, 150603 (2020).
- [22] S. Ito, Stochastic Thermodynamic Interpretation of Information Geometry, *Phys. Rev. Lett.* **121**, 030605 (2018).
- [23] N. Shiraishi, K. Funo, and K. Saito, Speed Limit for Classical Stochastic Processes, *Phys. Rev. Lett.* **121**, 070601 (2018).
- [24] K. Funo, N. Shiraishi, and K. Saito, Speed limit for open quantum systems, *New J. Phys.* **21**, 013006 (2019).
- [25] D. V. Averin and J. P. Pekola, Reversing the Landauer's erasure: Single-electron Maxwell's demon operating at the limit of thermodynamic efficiency, *Phys. Status Solidi B* **254**, 1600677 (2017).
- [26] M. V. Berry, Quantum phase corrections from adiabatic iteration, *Proc. R. Soc. London A* **414**, 31 (1987).
- [27] Equation (10) does not need to satisfy the local detailed balance condition.
- [28] G. H. Hardy, *Divergent Series* (Clarendon Press, Oxford, 1949).
- [29] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. (Wiley-Interscience, New Jersey, 2006).
- [30] The Fisher information is given by  $J_{mm} := \sum_j p_i^{ss}(\alpha) [\partial_n \ln p_i^{ss}(\alpha)] [\partial_m \ln p_i^{ss}(\alpha)]$ , where  $\partial_n = \partial/\alpha^n$  and  $\alpha^n$  is the  $n$ th component of the parameters. For  $\alpha^n = \beta_t = \alpha(t)\beta$ ,  $J_{mm} = \langle (E^{(0)})^2 \rangle - \langle E^{(0)} \rangle^2$ .
- [31] M. V. Berry, Universal oscillations of high derivatives, *Proc. R. Soc. A* **461**, 1735 (2005).

- [32] C. Flindt, C. Fricke, F. Hohls, T. Novotný, K. Netočný, T. Brandes, and R. J. Haug, Universal oscillations in counting statistics, *Proc. Natl. Acad. Sci. USA* **106**, 10116 (2009).
- [33] D. S. Golubev, M. Marthaler, Y. Utsumi, and G. Schön, Statistics of voltage fluctuations in resistively shunted Josephson junctions, *Phys. Rev. B* **81**, 184516 (2010).
- [34] Y. Utsumi, O. Entin-Wohlman, A. Ueda, and A. Aharony, Full-counting statistics for molecular junctions: Fluctuation theorem and singularities, *Phys. Rev. B* **87**, 115407 (2013).
- [35] M. P. Drazin, Pseudo-inverses in associative rings and semi-groups, *Am. Math. Mon.* **65**, 506 (1958).
- [36] W. P. Johnson, The curious history of Faà di Bruno's formula, *Am. Math. Mon.* **109**, 217 (2002).