Orbital diamagnetism of two-dimensional quantum systems in a dissipative environment: Non-Markovian effect and application to graphene

E. Kh. Alpomishev,[1,2] G. G. Adamian,[1] and N. V. Antonenko[1,3]
[1]*Joint Institute for Nuclear Research, 141980 Dubna, Russia*
[2]*Institute of Nuclear Physics, 100214 Tashkent, Uzbekistan*
[3]*Tomsk Polytechnic University, 634050 Tomsk, Russia*

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The non-Markovian dynamics of a charged particle confined in the harmonic oscillator and linearly coupled to a neutral bosonic heat bath is investigated in the external uniform magnetic field. The analytical expressions are derived for the time-dependent and asymptotic orbital angular momenta. The transition from non-Markovian dynamics to Markovian dynamics and the transition from a confined charge particle to a free charge particle are considered. The orbital diamagnetism of graphene in a dissipative environment and an external uniform magnetic field is studied and compared with existing experimental data. The results are presented for the electric conductivity and resonance behavior of the mass magnetization in graphene.

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I. INTRODUCTION

The electronic band structure, vibrational, mechanical, and magnetotransport properties of graphene have been intensively studied both experimentally and theoretically [1–8]. Graphene can be considered as the basis for all graphite-like materials, including graphite, nanotube, and fullerene. As known, there is almost no band gap in graphene, and it has semimetallic and semiconducting properties, depending on external factors (e.g., temperature, defects, etc.) Graphene possesses high mechanical strength and belongs to the optical transparent two-dimensional (2D) materials known to date [8,9]. Several important properties of graphene allow it to be used in many fields, for example, in nanotechnology, telecommunications, biotechnology, electronics, as an active medium for amplifying light, and others. Therefore, a comprehensive study of its electromagnetic, mechanical, and optical properties is of great interest.

In graphene [10,11], several experiments have been conducted showing the dependencies of orbital diamagnetism on temperature in the range of about from 10 to 300 K and on magnetic fields ranged from 0 to 8 T. In these experiments the graphene was not pure but derived from the thermal decomposition of SiC crystals. Experimental possibilities are limited, first, by the inaccessibility of graphene in its ideal form; second, by the influence of external factors (substrates, defects, and impurities) on graphene. A nonlinear dependence of diamagnetic magnetization M_z on the external magnetic field B at low temperatures T has been experimentally observed in Ref. [10]. The magnetic susceptibility of graphene is about two to three times stronger than that of normal materials and graphite [10]. A grand canonical thermodynamic potential and zero chemical potential for the undoped graphene have been used [10] to explain the magnetization of graphene. In Ref. [12], the orbital magnetism of graphene has been also studied within the effective mass approximation.

The purpose of the present paper is to consider the dependencies of graphene magnetization on temperature and homogeneous magnetic field within the quantum Langevin approach and to find the temperature dependencies of the mobility and density of charge carriers in graphene. For the orbital diamagnetism, the role of non-Markovian effect in two-dimensional quantum systems (a free charge particle and a charged particle enclosed in an oscillator potential) is presented and analyzed. The basic idea of our model is the following. We consider the charge carrier as a quantum particle coupled to a neutral bosonic environment (heat bath) through the particle-phonon interactions. For example, the underlayer could be the heat bath for graphene. Note that the quantum Langevin approach has been widely applied to find the effects of fluctuations and dissipation in macroscopic systems [13–33]. The Markovian and non-Markovian dynamical calculations of the orbital diamagnetism of quantum system in a dissipative environment and an uniform magnetic field have been performed within classical and quantum Langevin approaches [21,27,32].

The paper is organized as follows. In Sec. II, we introduce the Hamiltonian of the total system. Solving the second order Heisenberg equations for the heat bath degrees of freedom, the generalized non-Markovian Langevin equations are explicitly obtained for a quantum particle. Using the solutions of these equations, we derive the z component of time-dependent and asymptotic angular moments L_z (or magnetizations M_z) for the two–dimensional charged oscillator and free charged particle. The model developed is used in Sec. III to study the influence of non-Markovian and dissipative effects on the magnetization for free charged particle and charged particle confined in the oscillator potential. We also describe the experimental data on the graphene magnetization and predict the temperature dependence of mobility and density of charge carriers in graphene. Some properties on the electrical conductivity in graphene are predicted. A summary is given in Sec. IV. In

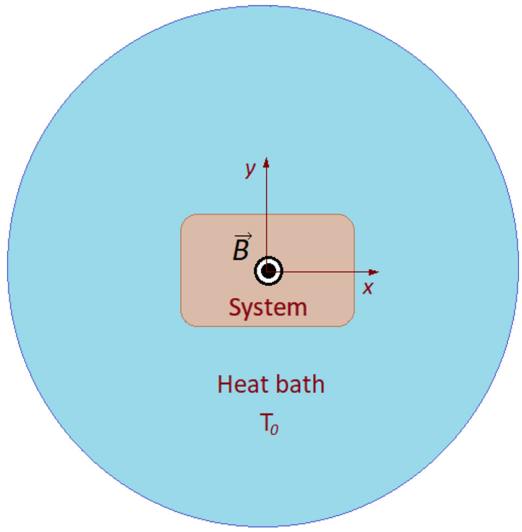

FIG. 1. Schematic of the model. The 2D subsystem (on the xy plane) is in a neutral heat bath with the temperature T_0. The constant magnetic field **B** is directed along the z axis.

addition, the solution of the system of Langevin equations and the derivation of the expressions for orbital magnetism of the free charged particle are presented in two Appendices.

II. NON-MARKOVIAN LANGEVIN EQUATIONS WITH AN EXTERNAL UNIFORM MAGNETIC FIELD

In order to investigate the influence of external fields on the dynamics of open quantum systems, we consider the motion of charged particle with effective mass m and negative charge e is considered in a 2D parabolic potential (on the xy plane) surrounded by the neutral bosonic heat bath in the presence of a perpendicular axisymmetric (along the z axis) magnetic field (see Fig. 1). In the case of linear coupling in coordinates between this particle and the heat bath, the total Hamiltonian of the collective subsystem (the charged particle confined in the 2D oscillator potential) plus the heat bath is as follows:

$$\hat{H} = \frac{1}{2m}[\hat{\mathbf{p}} + |e|\hat{\mathbf{A}}(\hat{x},\hat{y})]^2 + \frac{m}{2}(\omega_x^2\hat{x}^2 + \omega_y^2\hat{y}^2) + \sum_\nu \hbar\omega_\nu \hat{b}_\nu^+\hat{b}_\nu + \sum_\nu (\alpha_\nu\hat{x} + g_\nu\hat{y})(\hat{b}_\nu^+ + \hat{b}_\nu) + \sum_\nu \frac{1}{\hbar\omega_\nu}(\alpha_\nu\hat{x} + g_\nu\hat{y})^2, \tag{1}$$

where $\hat{\mathbf{A}} = (-\frac{1}{2}\hat{y}B, \frac{1}{2}\hat{x}B, 0)$ is the vector potential of the constant magnetic field with the strength $B = |\mathbf{B}|$, $\hat{\mathbf{p}}$ is the canonically conjugated momentum, ω_x and ω_y are the frequencies of the 2D oscillator potential, $\hat{b}_\nu^+$ and $\hat{b}_\nu$ are the phonon creation and annihilation operators of the heat bath, and α_ν and g_ν are the coupling parameters. The bosonic heat bath is modeled by an ensemble of noninteracting harmonic oscillators with frequencies ω_ν. The coupling between the heat bath and the collective subsystem is linear in coordinates. The last term in Eq. (1) compensates the renormalizations of the stiffness coefficients that arise because the collective and heat-bath subsystems are coupled.

For convenience, we introduce the new definitions for momenta,

$$\hat{\pi}_x = \hat{p}_x - \tfrac{1}{2}m\omega_c\hat{y}, \quad \hat{\pi}_y = \hat{p}_y + \tfrac{1}{2}m\omega_c\hat{x},$$

where $\omega_c = |e|B/m$ is the cyclotron frequency and $[\hat{\pi}_x, \hat{\pi}_y] = -[\hat{\pi}_y, \hat{\pi}_x] = -i\hbar m\omega_c$. Therefore, the total Hamiltonian (1) is transformed into the form

$$\hat{H} = \frac{1}{2m}(\hat{\pi}_x^2 + \hat{\pi}_y^2) + \frac{m}{2}(\omega_x^2\hat{x}^2 + \omega_y^2\hat{y}^2) + \sum_\nu \hbar\omega_\nu\hat{b}_\nu^+\hat{b}_\nu + \sum_\nu(\alpha_\nu\hat{x} + g_\nu\hat{y})(\hat{b}_\nu^+ + \hat{b}_\nu) + \sum_\nu \frac{1}{\hbar\omega_\nu}(\alpha_\nu\hat{x} + g_\nu\hat{y})^2. \tag{2}$$

The system of the Heisenberg equations for the operators $\hat{x}$, $\hat{y}$, $\hat{\pi}_x$, $\hat{\pi}_y$, and the bath phonon operators $\hat{b}_\nu$, $\hat{b}_\nu^+$ is obtained by commuting them with $\hat{H}$,

$$\begin{aligned}
\dot{\hat{x}}(t) &= \frac{i}{\hbar}[\hat{H}, \hat{x}] = \frac{\hat{\pi}_x(t)}{m},\\
\dot{\hat{y}}(t) &= \frac{i}{\hbar}[\hat{H}, \hat{y}] = \frac{\hat{\pi}_y(t)}{m},\\
\dot{\hat{\pi}}_x(t) &= \frac{i}{\hbar}[\hat{H}, \hat{\pi}_x]\\
&= -\omega_c\hat{\pi}_y(t) - m\omega_x^2\hat{x}(t) - 2\sum_\nu \frac{\alpha_\nu}{\hbar\omega_\nu}[\hat{b}_\nu^+(t) + \hat{b}_\nu(t)],\\
\dot{\hat{\pi}}_y(t) &= \frac{i}{\hbar}[\hat{H}, \hat{\pi}_y]\\
&= \omega_c\hat{\pi}_x(t) - m\omega_y^2\hat{y}(t) - 2\sum_\nu \frac{g_\nu}{\hbar\omega_\nu}[\hat{b}_\nu^+(t) + \hat{b}_\nu(t)],
\end{aligned} \tag{3}$$

and

$$\dot{\hat{b}}_\nu(t) = \frac{i}{\hbar}[\hat{H}, \hat{b}_\nu] = -i\omega_\nu\hat{b}_\nu(t) - \frac{i}{\hbar}[\alpha_\nu\hat{x}(t) + g_\nu\hat{y}(t)]. \tag{4}$$

The solution of Eq. (4) is

$$\begin{aligned}
\hat{b}_\nu(t) = \hat{f}_\nu(t) - \frac{\alpha_\nu\hat{x}(t) + g_\nu\hat{y}(t)}{\hbar\omega_\nu} &+ \frac{\alpha_\nu}{\hbar\omega_\nu}\int_0^t d\tau\,\dot{\hat{x}}(\tau)e^{-i\omega_\nu(t-\tau)}\\
&+ \frac{g_\nu}{\hbar\omega_\nu}\int_0^t d\tau\,\dot{\hat{y}}(\tau)e^{-i\omega_\nu(t-\tau)},
\end{aligned} \tag{5}$$

where

$$\hat{f}_\nu(t) = \left[\hat{b}_\nu(0) + \frac{i}{\hbar\omega_\nu}\hat{\Gamma}_\nu(0)\right]e^{-i\omega_\nu t},$$

$$\hat{\Gamma}_\nu(0) = \alpha_\nu\hat{x}(0) + g_\nu\hat{y}(0).$$

Therefore,

$$\begin{aligned}
\hat{b}_\nu^+(t) + \hat{b}_\nu(t) = \hat{f}_\nu^+(t) + \hat{f}_\nu(t) - 2\frac{\alpha_\nu\hat{x}(t) + g_\nu\hat{y}(t)}{\hbar\omega_\nu} &+ \frac{2\alpha_\nu}{\hbar\omega_\nu}\int_0^t d\tau\,\dot{\hat{x}}(\tau)\cos[\omega_\nu(t-\tau)]\\
&+ \frac{2g_\nu}{\hbar\omega_\nu}\int_0^t d\tau\,\dot{\hat{y}}(\tau)\cos[\omega_\nu(t-\tau)].
\end{aligned} \tag{6}$$

Substituting (6) into (3), we eliminate the bath variables from the equations of motion of the collective subsystem and obtain the nonlinear integrodifferential stochastic dissipative equations,

$$\begin{aligned}
\dot{\hat{x}}(t) &= \frac{\hat{\pi}_x(t)}{m}, \quad \dot{\hat{y}}(t) = \frac{\hat{\pi}_y(t)}{m},\\
\dot{\hat{\pi}}_x(t) &= -\omega_c\hat{\pi}_y(t) - m\omega_x^2\hat{x}(t)\\
&\quad - \frac{1}{m}\int_0^t d\tau\, K_\alpha(t-\tau)\hat{\pi}_x(\tau) - \hat{F}_\alpha(t),\\
\dot{\hat{\pi}}_y(t) &= \omega_c\hat{\pi}_x(t) - m\omega_y^2\hat{y}(t)\\
&\quad - \frac{1}{m}\int_0^t d\tau\, K_g(t-\tau)\hat{\pi}_y(\tau) - \hat{F}_g(t).
\end{aligned} \tag{7}$$

The presence of the integral parts in these equations indicates the non-Markovian dynamics. The operators,

$$\begin{aligned}
\hat{F}_\alpha(t) &= \sum_\nu \hat{F}_\alpha^\nu(t) = \sum_\nu \alpha_\nu[\hat{f}_\nu^+(t) + \hat{f}_\nu(t)],\\
\hat{F}_g(t) &= \sum_\nu \hat{F}_g^\nu(t) = \sum_\nu g_\nu[\hat{f}_\nu^+(t) + \hat{f}_\nu(t)]
\end{aligned} \tag{8}$$

play a role of random forces in the coordinates, and Eqs. (7) are the generalized nonlinear quantum Langevin equations. Following the usual procedure of statistical mechanics, we identify these operators as fluctuations because of the uncertainty in the initial conditions for the bath operators. To specify the statistical properties of the fluctuations, we consider an ensemble of initial states in which the fluctuations have the Gaussian distribution with zero average value,

$$\langle\langle \hat{F}_\alpha^\nu(t)\rangle\rangle = \langle\langle \hat{F}_g^\nu(t)\rangle\rangle = 0. \tag{9}$$

Here, the symbol $\langle\langle\cdots\rangle\rangle$ denotes the average over the bath. We assume that there are no correlations between $\hat{F}_\alpha^\nu(t)$ and $\hat{F}_g^\nu(t)$ so that

$$\sum_\nu \frac{\alpha_\nu g_\nu}{\hbar\omega_\nu} \equiv 0. \tag{10}$$

The dissipative kernels in Eqs. (7) are

$$\begin{aligned}
K_\alpha(t-\tau) &= 2\sum_\nu \frac{\alpha_\nu^2}{\hbar\omega_\nu}\cos[\omega_\nu(t-\tau)],\\
K_g(t-\tau) &= 2\sum_\nu \frac{g_\nu^2}{\hbar\omega_\nu}\cos[\omega_\nu(t-\tau)].
\end{aligned} \tag{11}$$

Because these kernels do not contain the phonon occupation numbers, they are independent of temperature T_0 (in the energy units) of the heat bath. The temperature enters in the analysis through the distribution of initial conditions. We use the Bose-Einstein statistics for the heat bath,

$$\begin{aligned}
\langle\langle \hat{f}_\nu^+(t)\hat{f}_{\nu'}^+(t')\rangle\rangle &= \langle\langle \hat{f}_\nu(t)\hat{f}_{\nu'}(t')\rangle\rangle = 0,\\
\langle\langle \hat{f}_\nu^+(t)\hat{f}_{\nu'}(t')\rangle\rangle &= \delta_{\nu,\nu'} n_\nu e^{i\omega_\nu(t-t')},\\
\langle\langle \hat{f}_\nu(t)\hat{f}_{\nu'}^+(t')\rangle\rangle &= \delta_{\nu,\nu'}(n_\nu+1)e^{-i\omega_\nu(t-t')},
\end{aligned} \tag{12}$$

with occupation numbers for phonons $n_\nu = [\exp(\hbar\omega_\nu/T_0) - 1]^{-1}$ depending on T_0. Using the properties of random forces, we obtain the quantum fluctuation-dissipation relations,

$$\sum_\nu \varphi_\alpha^\nu(t,t')\frac{\tanh\left[\frac{\hbar\omega_\nu}{2T_0}\right]}{\hbar\omega_\nu} = K_\alpha(t-t'),$$

$$\sum_\nu \varphi_g^\nu(t,t')\frac{\tanh\left[\frac{\hbar\omega_\nu}{2T_0}\right]}{\hbar\omega_\nu} = K_g(t-t'),$$

where

$$\begin{aligned}
\varphi_\alpha^\nu(t,t') &= 2\alpha_\nu^2(2n_\nu+1)\cos[\omega_\nu(t-t')],\\
\varphi_g^\nu(t,t') &= 2g_\nu^2(2n_\nu+1)\cos[\omega_\nu(t-t')]
\end{aligned}$$

are the symmetrized correlation functions $\varphi_k^\nu(t,t') = \langle\langle \hat{F}_k^\nu(t)\hat{F}_k^\nu(t') + \hat{F}_k^\nu(t')\hat{F}_k^\nu(t)\rangle\rangle$, $k=\alpha, g$. The quantum fluctuation-dissipation relations differ from the classical ones and are reduced to them in the limit of high temperature T_0 (or $\hbar\to 0$): $\sum_\nu \varphi_\alpha^\nu(t,t') = 2T_0K_\alpha(t-t')$, $\sum_\nu \varphi_g^\nu(t,t') = 2T_0K_g(t-t')$.

It is convenient to introduce the spectral density D_ω of the heat bath excitations which allows us to replace the sum over different oscillators ν by the integral over frequency: $\sum_\nu\cdots \to \int_0^\infty d\omega\, D_\omega\cdots$. This is accompanied by the following replacements: $\alpha_\nu\to\alpha_\omega$, $g_\nu\to g_\omega$, $\omega_\nu\to\omega$, and $n_\nu\to n_\omega$. Let us consider the following spectral functions:

$$D_\omega\frac{\alpha_\omega^2}{\omega} = \frac{\lambda_x^2}{\pi}\frac{\gamma^2}{\gamma^2+\omega^2}, \quad D_\omega\frac{g_\omega^2}{\omega} = \frac{\lambda_y^2}{\pi}\frac{\gamma^2}{\gamma^2+\omega^2},$$

where the memory time γ^{-1} of dissipation is inverse to the phonon bandwidth of the heat bath excitations and the coefficients,

$$\lambda_x = \frac{1}{m}\int_0^\infty d\tau\, K_\alpha(t-\tau), \quad \lambda_y = \frac{1}{m}\int_0^\infty d\tau\, K_g(t-\tau)$$

are the friction coefficients in the Markovian limit [15]. This Ohmic dissipation with the Lorenzian cutoff (Drude dissipation) results in the dissipative kernels,

$$K_\alpha(t) = m\lambda_x\gamma e^{-\gamma|t|}, \quad K_g(t) = m\lambda_y\gamma e^{-\gamma|t|}.$$

The relaxation time of the heat bath should be much less than the period of the collective oscillator, i.e., $\gamma \gg \omega_{x,y}$.

The system of Eqs. (7) is solved by applying the Laplace transformations. After the tedious algebra we obtain the solution of this system of equations,

$$\begin{aligned}
\hat{x}(t) &= A_1(t)\hat{x}(0) + A_2(t)\hat{y}(0) + A_3(t)\hat{\pi}_x(0) + A_4(t)\hat{\pi}_y(0) - \hat{I}_x(t) - \hat{I}_x'(t),\\
\hat{y}(t) &= B_1(t)\hat{x}(0) + B_2(t)\hat{y}(0) + B_3(t)\hat{\pi}_x(0) + B_4(t)\hat{\pi}_y(0) - \hat{I}_y(t) - \hat{I}_y'(t),\\
\hat{\pi}_x(t) &= C_1(t)\hat{x}(0) + C_2(t)\hat{y}(0) + C_3(t)\hat{\pi}_x(0) + C_4(t)\hat{\pi}_y(0) - \hat{I}_{\pi_x}(t) - \hat{I}_{\pi_x}'(t),\\
\hat{\pi}_y(t) &= D_1(t)\hat{x}(0) + D_2(t)\hat{y}(0) + D_3(t)\hat{\pi}_x(0) + D_4(t)\hat{\pi}_y(0) - \hat{I}_{\pi_y}(t) - \hat{I}_{\pi_y}'(t),
\end{aligned} \tag{13}$$

where all time-dependent coefficients are given in Appendix A.

A. Asymptotic angular momenta for a charged particle confined in an oscillator potential and free charged particle

1. Non-Markovian case

Using Eqs. (13), (A1), and the correlations of the random forces at different times, we find the time dependence of the z component of angular momentum (or magnetic moment per unit volume $M(t) = -\frac{n_0|e|L_z(t)}{2m}$, where n_0 is the concentration of charge carriers),

$$\begin{aligned} L_z(t) &= \langle \hat{x}(t)\hat{\pi}_y(t) - \hat{y}(t)\hat{\pi}_x(t)\rangle \\ &= \frac{m\hbar\gamma^2}{\pi}\int_0^\infty\int_0^t\int_0^t \frac{d\omega\, d\tau\, d\tilde{\tau}\ \omega\ \coth[\hbar\omega/(2T_0)]}{\omega^2+\gamma^2}\cos(\omega[\tau-\tilde{\tau}]) \\ &\quad\times\{\lambda_x[A_3(\tau)D_3(\tilde{\tau}) - B_3(\tau)C_3(\tilde{\tau})] + \lambda_y[A_4(\tau)D_4(\tilde{\tau}) - B_4(\tau)C_4(\tilde{\tau})]\}. \end{aligned} \tag{14}$$

Here, the symbol $\langle\cdots\rangle$ denotes the average over the whole system (the collective plus the heat bath subsystems) and $L_z(t=0) = 0$ is assumed for the simplicity. Based on the above Eq. (14), we find the asymptotic value,

$$L_z(\infty) = \frac{2\hbar\omega_c\gamma^2}{\pi}\int_0^\infty d\omega\,\omega^3\coth\left[\frac{\hbar\omega}{2T_0}\right]\frac{(\omega^2+\gamma^2)\left[\lambda_x\left(\omega^2-\omega_y^2\right)+\lambda_y\left(\omega^2-\omega_x^2\right)\right]-2\omega^2\gamma\lambda_x\lambda_y}{\left|s_1^2+\omega^2\right|^2\left|s_2^2+\omega^2\right|^2\left|s_3^2+\omega^2\right|^2}, \tag{15}$$

where the values of s_{1-3} are defined in Appendix A.

Employing the residue theorem, closing the contour in the upper half-plane, and using the cotangent function poles at $\hbar\omega/(2T_0) = i\pi n$ with an integer n, we calculate analytically the integral over ω in Eq. (15) and obtain

$$L_z(\infty) = \hbar\omega_c\gamma^2(I + I^* - I_s), \tag{16}$$

where

$$\begin{aligned} I = &\frac{s_1^2\left\{\left(\gamma^2-s_1^2\right)\left[\lambda_x\left(s_1^2+\omega_y^2\right)+\lambda_y\left(s_1^2+\omega_x^2\right)\right]-2s_1^2\gamma\lambda_x\lambda_y\right\}}{\left(s_1^2-s_1^{*2}\right)\left(s_1^2-s_2^2\right)\left(s_1^2-s_2^{*2}\right)\left(s_1^2-s_3^2\right)\left(s_1^2-s_3^{*2}\right)}\cot\left[\frac{\hbar s_1}{2T_0}\right] \\ &+\frac{s_2^2\left\{\left(\gamma^2-s_2^2\right)\left[\lambda_x\left(s_2^2+\omega_y^2\right)+\lambda_y\left(s_2^2+\omega_x^2\right)\right]-2s_2^2\gamma\lambda_x\lambda_y\right\}}{\left(s_2^2-s_1^2\right)\left(s_2^2-s_1^{*2}\right)\left(s_2^2-s_2^{*2}\right)\left(s_2^2-s_3^2\right)\left(s_2^2-s_3^{*2}\right)}\cot\left[\frac{\hbar s_2}{2T_0}\right], \\ &+\frac{s_3^2\left\{\left(\gamma^2-s_3^2\right)\left[\lambda_x\left(s_3^2+\omega_y^2\right)+\lambda_y\left(s_3^2+\omega_x^2\right)\right]-2s_3^2\gamma\lambda_x\lambda_y\right\}}{\left(s_3^2-s_1^2\right)\left(s_3^2-s_1^{*2}\right)\left(s_3^2-s_2^2\right)\left(s_3^2-s_2^{*2}\right)\left(s_3^2-s_3^{*2}\right)}\cot\left[\frac{\hbar s_3}{2T_0}\right], \end{aligned} \tag{17}$$

and

$$I_s = 32\pi^3\frac{T_0^4}{\hbar^4}\sum_{n=1}^\infty\frac{\left\{(\gamma^2-n^2\pi^2)\left[\lambda_x\left(x_n^2+\omega_y^2\right)+\lambda_y\left(x_n^2+\omega_x^2\right)\right]-2x_n^2\gamma\lambda_x\lambda_y\right\}n^3}{\left|x_n^2-s_1^2\right|^2\left|x_n^2-s_2^2\right|^2\left|x_n^2-s_3^2\right|^2}. \tag{18}$$

Here, $x_n = 2\pi nT_0/\hbar$, $\mathrm{Re}(s_1) < 0$, $\mathrm{Re}(s_2) < 0$, and $\mathrm{Re}(s_3) < 0$. At high temperatures, the contribution from the sum in Eq. (16) becomes negligible.

2. Markovian case

For the Markovian dynamics ($\gamma\to\infty$), we obtain from Eq. (15) the asymptotics,

$$L_z(\infty) = \frac{2\hbar\omega_c}{\pi}\int_0^\infty d\omega\,\omega^3\coth\left[\frac{\hbar\omega}{2T_0}\right]\frac{\lambda_x\left(\omega^2-\omega_y^2\right)+\lambda_y\left(\omega^2-\omega_x^2\right)}{\left|s_1^2+\omega^2\right|^2\left|s_2^2+\omega^2\right|^2}, \tag{19}$$

where s_i and s_i^* $(i = 1, 2)$ are the roots of the equation,

$$D(s) = \left(\omega_x^2+s^2+s\lambda_x\right)\left(\omega_y^2+s^2+s\lambda_y\right)+s^2\omega_c^2 = 0. \tag{20}$$

Note that one can obtain Eq. (20) from Eq. (A2) in the limit $\gamma\to\infty$.

Calculating analytically the integral over ω in Eq. (19), we derive

$$L_z(\infty) = \hbar\omega_c(J + J^* - J_s), \tag{21}$$

where

$$J = \frac{s_1^2\left[\lambda_x\left(s_1^2+\omega_y^2\right)+\lambda_y\left(s_1^2+\omega_x^2\right)\right]}{\left(s_1^2-s_1^{*2}\right)\left(s_1^2-s_2^2\right)\left(s_1^2-s_2^{*2}\right)}\cot\left[\frac{\hbar s_1}{2T_0}\right]+\frac{s_2^2\left[\lambda_x(s_2^2+\omega_y^2)+\lambda_y\left(s_2^2+\omega_x^2\right)\right]}{\left(s_2^2-s_1^2\right)\left(s_2^2-s_1^{*2}\right)\left(s_2^2-s_2^{*2}\right)}\cot\left[\frac{\hbar s_2}{2T_0}\right], \tag{22}$$

and

$$J_s = 32\pi^3 \frac{T_0^4}{\hbar^4} \sum_{n=1}^{\infty} \frac{\left[\lambda_x\left(x_n^2+\omega_y^2\right)+\lambda_y\left(x_n^2+\omega_x^2\right)\right]n^3}{\left|x_n^2-s_1^2\right|^2\left|x_n^2-s_2^2\right|^2}. \tag{23}$$

Here, $\mathrm{Re}(s_1) < 0$ and $\mathrm{Re}(s_2) < 0$. At high temperatures, Eq. (21) is simplified to

$$L_z(\infty) = \hbar\omega_c(J_0 + J_0^*), \tag{24}$$

where

$$J_0 = \frac{\hbar s_1^3\left[\lambda_x\left(s_1^2+\omega_y^2\right)+\lambda_y\left(s_1^2+\omega_x^2\right)\right]}{6T_0\left(s_1^{*2}-s_1^2\right)\left(s_1^2-s_2^2\right)\left(s_1^2-s_2^{*2}\right)} + \frac{\hbar s_2^3\left[\lambda_x\left(s_2^2+\omega_y^2\right)+\lambda_y\left(s_2^2+\omega_x^2\right)\right]}{6T_0\left(s_2^{*2}-s_2^2\right)\left(s_2^2-s_1^2\right)\left(s_2^2-s_1^{*2}\right)}. \tag{25}$$

In the case of $\omega_c \gg \omega_{x,y}$, the roots of Eq. (20) are well approximated as

$$s_1 = i\omega_c - \frac{1}{2}(\lambda_x+\lambda_y), \quad s_2 = \frac{\omega_x\omega_y}{\omega_c^2+\frac{1}{4}(\lambda_x+\lambda_y)^2}\left[i\omega_c - \frac{1}{2}(\lambda_x+\lambda_y)\right]. \tag{26}$$

Substituting (26) into Eq. (21) and assuming $\omega_x = \omega_y = \omega_0$ and $\lambda_x = \lambda_y = \lambda$, we obtain the following expression:

$$\begin{aligned} L_z(\infty) = &-\frac{\hbar\left(\omega_c^2+\lambda^2\right)^2}{2\left[\left(\omega_c^2+\lambda^2\right)^2-\omega_0^4\right]}\left\{\frac{\omega_0^2\coth\left[\frac{\hbar\omega_0^2(\omega_c+i\lambda)}{2T_0(\omega_c^2+\lambda^2)}\right]}{\omega_0^2+(\omega_c-i\lambda)^2} + \frac{\omega_0^2\coth\left[\frac{\hbar\omega_0^2(\omega_c-i\lambda)}{2T_0(\omega_c^2+\lambda^2)}\right]}{\omega_0^2+(\omega_c+i\lambda)^2}\right. \\ &\left.+\frac{2\left[\omega_0^2\left(\omega_c^2-\lambda^2\right)+\left(\omega_c^2+\lambda^2\right)^2\right]\sinh\left[\frac{\hbar\omega_c}{T_0}\right]+4\omega_0^2\omega_c\lambda\sin\left[\frac{\hbar\lambda}{T_0}\right]}{\left[\omega_0^2+(\omega_c-i\lambda)^2\right]\left[\omega_0^2+(\omega_c+i\lambda)^2\right]\left(\cos\left[\frac{\hbar\lambda}{T_0}\right]-\cosh\left[\frac{\hbar\omega_c}{T_0}\right]\right)}\right\} \\ &-\sum_{n=1}^{\infty}\frac{64n^3\pi^3T_0^4\hbar^3\omega_c\lambda\left(\omega_c^2+\lambda^2\right)^2}{\left[16n^4\pi^4T_0^4+8n^2\pi^2\hbar^2T_0^2\left(\omega_c^2-\lambda^2\right)+\hbar^4\left(\omega_c^2+\lambda^2\right)^2\right]} \\ &\times\frac{\left(4\pi^2n^2T_0^2+\hbar^2\omega_0^2\right)}{\left[16\pi^4n^4T_0^4\left(\omega_c^2+\lambda^2\right)^2+8\pi^2n^2T_0^2\hbar^2\omega_0^4\left(\omega_c^2-\lambda^2\right)+\hbar^4\omega_0^8\right]}. \end{aligned} \tag{27}$$

At high temperatures, it is transformed into the simple formula,

$$L_z(\infty) = \frac{\hbar^2\omega_c\left(\omega_c^2+\lambda^2\right)}{6T_0\left(\omega_0^2+\omega_c^2+\lambda^2\right)}. \tag{28}$$

Note that Eq. (28) does not depend on the friction at $\omega_0 \to 0$ and

$$L_z(\infty) = \frac{\hbar^2\omega_c}{6T_0}. \tag{29}$$

Thus, the Bohr–Van Leeuwen theorem (there is no diamagnetism in the classical system) is only restored in the limit of high temperature.

In the limiting case of zero friction coefficients, $\lambda_x = \lambda_y = \lambda = 0$, and $\omega_x = \omega_y = \omega_0$, Eq. (27) is simplified to

$$\begin{aligned} L_z(\infty) = &-\frac{\hbar}{2\sqrt{4\omega_0^2+\omega_c^2}}\left\{\left(\sqrt{4\omega_0^2+\omega_c^2}-\omega_c\right)\coth\left[\frac{\hbar\left(\sqrt{4\omega_0^2+\omega_c^2}-\omega_c\right)}{4T_0}\right]\right. \\ &\left.-\left(\sqrt{4\omega_0^2+\omega_c^2}+\omega_c\right)\coth\left[\frac{\hbar\left(\sqrt{4\omega_0^2+\omega_c^2}+\omega_c\right)}{4T_0}\right]\right\}. \end{aligned} \tag{30}$$

In the case of $\lambda_x \neq \lambda_y$ and $\omega_0 \to 0$, we obtain

$$\begin{aligned} L_z(\infty) = &-\frac{8T_0\omega_c}{4\omega_c^2+(\lambda_x+\lambda_y)^2} + \frac{\hbar\,\sinh[\hbar\omega_c/T_0]}{2\{\sinh^2[\hbar\omega_c/(2T_0)]+\sin^2[\hbar(\lambda_x+\lambda_y)/(4T_0)]\}} \\ &-\sum_{n=1}^{\infty}\frac{128n\pi T_0^2\hbar^3\omega_c(\lambda_x+\lambda_y)}{|4n\pi T_0+\hbar(2i\omega_c-\lambda_x-\lambda_y)|^2|4n\pi T_0-\hbar(2i\omega_c-\lambda_x-\lambda_y)|^2}. \end{aligned} \tag{31}$$

If the friction is isotropic, $\lambda_x = \lambda_y = \lambda$, then the expression,

$$L_z(\infty) = -\frac{2T_0\omega_c}{\omega_c^2+\lambda^2} - \frac{\hbar\,\sinh[\hbar\omega_c/T_0]}{\cos[\hbar\lambda/T_0]-\cosh[\hbar\omega_c/T_0]} - \sum_{n=1}^{\infty}\frac{16n\pi T_0^2\hbar^3\lambda\omega_c}{16n^4\pi^4T_0^4+8\hbar^2n^2\pi^2T_0^2\left(\omega_c^2-\lambda^2\right)+\hbar^4\left(\omega_c^2+\lambda^2\right)^2} \tag{32}$$

follows from Eq. (31). As noted above, the contribution of the sum to $L_z(\infty)$ [i.e., to Eqs. (31) or (32)] is negligible at high temperatures.

In the limit of zero friction ($\lambda \to 0$), Eq. (32) results in the Landau formula or the Langevin function,

$$L_z(\infty) = -\hbar\left(\frac{2T_0}{\hbar\omega_c} - \coth\left[\frac{\hbar\omega_c}{2T_0}\right]\right). \tag{33}$$

So, Eqs. (31) and (30) naturally generalize the Landau formula (33) in the case of dissipative system. In Appendix B, we show the impossibility of obtaining Eq. (33) by starting directly from the total Hamiltonian for free charged particle. So, the important effect of confined boundaries [34] should be considered starting directly from the total Hamiltonian (1). Equation (33) can be also derived from Eq. (30) in the limit $\omega_0 \to 0$. Note that Eqs. (30), (32), and (33) are also derived in Ref. [21]. Here, we derive them from the non-Markovian expressions.

At $\hbar\omega_c \ll 2T_0$, we again obtain Eq. (29) but from Eq. (33). Because Eq. (33) is transformed into $L_z(\infty) = \hbar$ at $\hbar\omega_c \gg 2T_0$, we find the quantization conditions $[T_0/(\hbar\omega_c) \to 0,\ \gamma \to \infty,\ \lambda \to 0]$ for the orbital angular momentum in a dissipative environment. As seen, for large values of the cyclotron frequency, the asymptotic magnetization is equal to one (negative) Bohr magneton if $m = m_e$ (m_e is the mass of the electron). The reason for this is the localization of the charged particles with increasing magnetic field, when the variance $|\sigma^2_{x\pi_y}(\infty)| = L_z(\infty)/2$ reaches the minimal value of $\hbar/2$ [27].

III. CALCULATED RESULTS

A. Influence of non-Markovian and dissipative effects on magnetization

In the Markovian ($\gamma \to \infty$) and non-Markovian $[\hbar\gamma/(2T_0) = 12]$ cases, the dependencies of z component of the asymptotic angular momentum $L_z(\infty)$ on $\hbar\lambda/(2T_0)$ are shown in Fig. 2. As seen, the value of $L_z(\infty)$ decreases monotonically and approaches zero with increasing friction coefficients ($\lambda_x = \lambda_y = \lambda$). The rate of this approach decreases with increasing cyclotron frequency ω_c. So, this is diamagnetism of the system even in the presence of a dissipative environment. The absolute value of $L_z(\infty)$ increases with decreasing frequency of oscillator ($\omega_x = \omega_y = \omega_0$) and reaches its maximal value for the free charged particle ($\omega_0 = 0$). As seen in Fig. 2 at $\hbar\omega_c/(2T_0) = 0.5$ and $\lambda = 0$, the average value of the angular momentum of a free charged particle is about 1.5 times larger than that of charged particle in the harmonic oscillator with $\hbar\omega_0/(2T_0) = 2$. This difference between angular momenta decreases with growing $\hbar\lambda/(2T_0)$. It should be noted that without taking the contribution of the "boundary" charge carriers [34] into account, the angular momentum of the free charged particle is much larger [31] than that of the charged particle in the harmonic oscillator.

The influence of non-Markovian effect on $L_z(\infty)$ is rather weak (Figs. 2 and 3). For the charged particle confined in the oscillator potential, the non-Markovian $L_z(\infty)$ is slightly larger than the Markovian one. For a free charged particle, there is almost no difference between the Markovian and the non-Markovian cases. The values of $L_z(\infty)$ obtained in the Markovian and non-Markovian cases are almost the same (Fig. 3). The difference between both cases weakly changes with increasing frequency of the oscillator. As seen in Fig. 3 for the oscillator with $\omega_0/\lambda = 2$, the value of $L_z(\infty)$ is almost independent of T_0 at very low temperatures. This means that the heat bath phonons cannot significantly affect the oscillator because their energies are small.

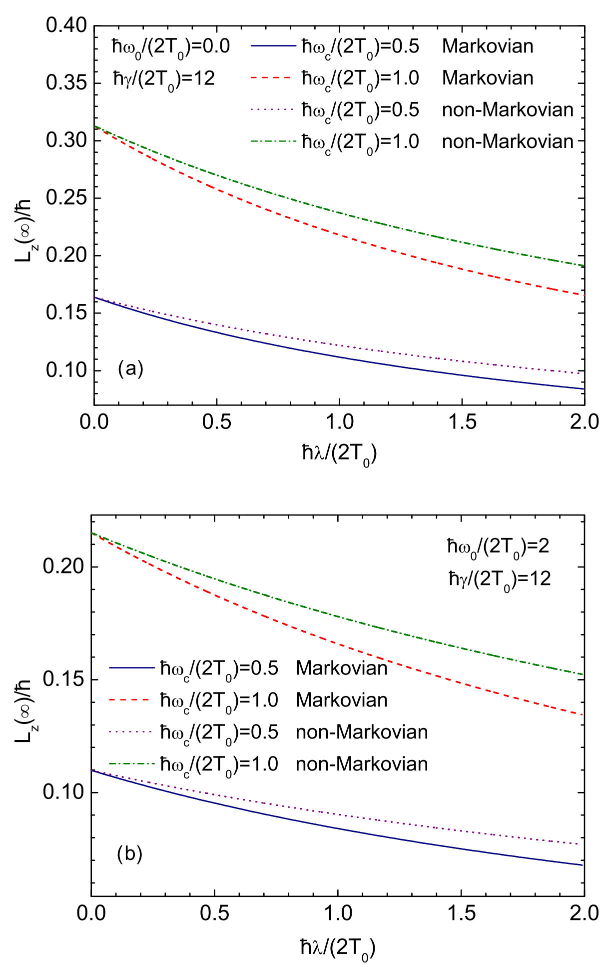

FIG. 2. The calculated dimensionless asymptotic z component of angular momentum $L_z(\infty)/\hbar$ as a function of $\hbar\lambda/(2T_0)$ at (a) $\hbar\omega_0/(2T_0) = 0$ and (b) 2. Here, $\lambda_x = \lambda_y = \lambda$ and $\omega_x = \omega_y = \omega_0$. The values of $\hbar\omega_c/(2T_0)$ used are indicated. In the Markovian case, Eqs. (27) and (32) are used. In the non-Markovian case, Eq. (16) with $\omega_0 \neq 0$ and Eq. (16) in the limit $\omega_0 \to 0$ are used.

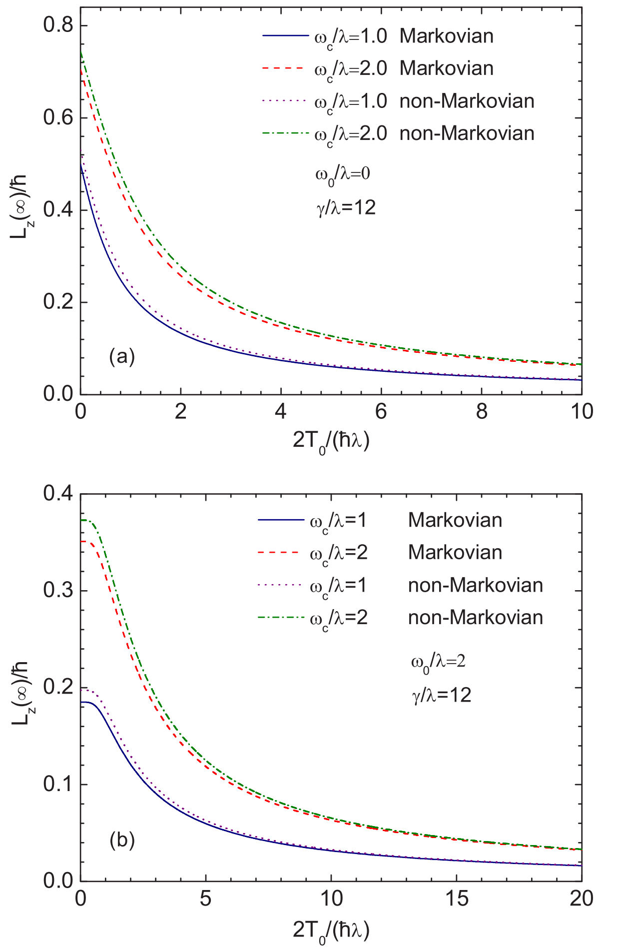

FIG. 3. The calculated dimensionless asymptotic z component of angular momentum $L_z(\infty)/\hbar$ as a function of $2T_0/(\hbar\lambda)$ at (a) $\hbar\omega_0/(2T_0) = 0$ and (b) 2. Here, $\lambda_x = \lambda_y = \lambda$ and $\omega_x = \omega_y = \omega_0$. The values of ω_c/λ used are indicated. In the Markovian case, Eqs. (27) and (32) are used. In the non-Markovian case, Eq. (16) with $\omega_0 \neq 0$ and Eq. (16) in the limit $\omega_0 \to 0$ are used.

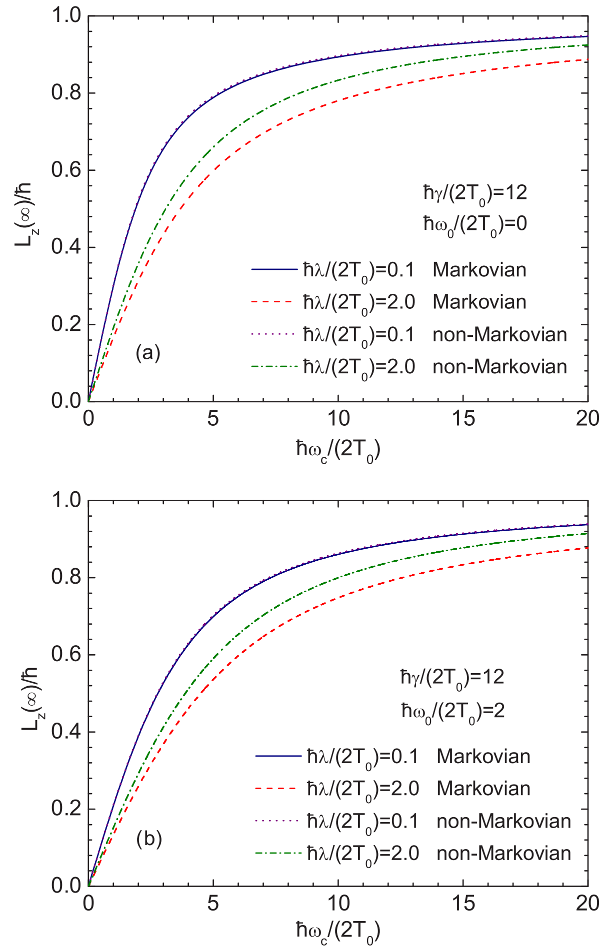

FIG. 4. The calculated dimensionless asymptotic z component of angular momentum $L_z(\infty)/\hbar$ as a function of $\hbar\omega_c/(2T_0)$ at (a) $\hbar\omega_0/(2T_0) = 0$ and (b). Here, $\lambda_x = \lambda_y = \lambda$ and $\omega_x = \omega_y = \omega_0$. The values of $\hbar\lambda/(2T_0)$ used are indicated. In the Markovian case, Eqs. (27) and (32) are used. In the non-Markovian case, Eq. (16) with $\omega_0 \neq 0$ and Eq. (16) in the limit $\omega_0 \to 0$ are used.

We calculate the z component of angular momentum for the charged oscillator settled in increasing the external magnetic field at different $\hbar\lambda/(2T_0)$'s (Fig. 4). The results indicate the values of $L_z(\infty)$ in the Markovian and non-Markovian cases are close to each other even at large B. At large B, the value of $L_z(\infty)$ approaches $\hbar$, which means it tends to the usual quantization of the z component of angular momentum even in the dissipative system. It should be noted that the dependencies of $L_z(\infty)$ on $\hbar\omega_c/(2T_0)$ at $\hbar\omega_0/(2T_0) = 0$ and 2 are almost the same.

B. Magnetization and electrical conductivity of graphene

Because the values of $L_z(\infty)$ are almost the same in the Markovian and non-Markovian cases and there is almost no band gap in graphene, we employ Eq. (32) for the free charged particle with the Markovian dynamics for the description of the magnetization of graphene. Note that for carbon materials the main contribution to the diamagnetism is made by the free electrons (Landau's diamagnetism) [35]. In the calculations we set $\lambda_x = \lambda_y = \lambda$ and $\omega_x = \omega_y = \omega_0$. In order to turn to the observable values, all parameters λ and ω_c in the expressions are multiplied by the ratio $\frac{m}{|e|}$,

$$\lambda \to \frac{m}{|e|}\lambda = \tilde{\lambda} = \mu^{-1}, \quad \omega_c \to \frac{m}{|e|}\omega_c = B, \tag{34}$$

and the temperature T_0 is replaced by $k_B T_0$, where k_B is the Boltzmann constant. Using Eq. (32), we determine the mass magnetization,

$$M_z(\infty) = -\frac{n_0|e|L_z(\infty)}{2m\rho}$$
$$= C_0\left[\frac{2T_0B}{A(B^2+\tilde{\lambda}^2)} + \frac{\sinh[AB/T_0]}{\cos[A\tilde{\lambda}/T_0]-\cosh[AB/T_0]} + \sum_{n=1}^{\infty}\frac{16n\pi T_0^2A^2B\tilde{\lambda}}{16n^4\pi^4T_0^4+8n^2\pi^2T_0^2A^2(B^2-\tilde{\lambda}^2)+A^4(B^2+\tilde{\lambda}^2)^2}\right], \quad (35)$$

where ρ is the density of the material, $m = 0.04m_e$ is the effective mass of the charge carriers [3,8], $A = \hbar|e|/(k_B m)$, $\tilde{\lambda}^{-1} = \mu = \mu(T_0)$ is the temperature-dependent charge carriers mobility and $C_0 = C_0(T_0) = n_0|e|\hbar/(2m\rho)$ is the effective charge carriers density. The values of $\mu(T_0)$ and $C_0(T_0)$ are the free parameters of our model, and they are extracted from the experimental data.

The dependencies of the magnetization M_z of graphene on the magnetic field and temperature are presented in Figs. 5 and 6. They are described separately for low (approximately $T_0 \leqslant 45$ K) and high (approximately $T_0 \geqslant 50$ K) temperature regimes since the properties of graphene change drastically at around $T_0 \approx 45$–50 K. This is reflected in the fact that we use two sets of $\mu(T_0)$ and $C_0(T_0)$ to describe the graphene magnetization,

$$\mu = (2.198\,057 - 0.001\,9185T_0 - 0.117\,6485\ \ln\ T_0)^{-1},$$
$$C_0 = 1.48 \quad (36)$$

for the low temperature regime, and

$$\mu = (1.5497 + 0.0404T_0)^{-1},$$
$$C_0 = 1.824\,8912 + 0.011\,9269T_0 - 0.000\,0188T_0^2 \quad (37)$$

for the high temperature regime. Here, the charge carriers mobility is in units of $\mathrm{m^2\,V^{-1}\,s^{-1}}$ and effective charge carriers density is in units of emu/g. Note that the temperature dependence of the mobility means that for the description of the properties of graphene we need the temperature-dependent coupling between the charge carrier and the heat bath. In other words, the temperature affects this coupling.

The convergence of the series contained in Eq. (35) depends on temperature T_0 and magnetic field B. For example, if we choose the numerical error to be less than 1% and use the temperature-dependent mobility μ from Eq. (36), then at $B = 0.5$ T ($B = 3$ T) the maximum numbers of terms of the series are 3.5×10^5 (1.15×10^6), 300 (1000), and 7 (25) at $T_0 = 1$ mK, 1, and 40 K, respectively.

The temperature dependencies of $\mu(T_0)$ at low and high temperature regimes are shown in Fig. 7. The extracted $\mu(T_0)$ and $C_0(T_0)$ lead to the following conclusions. (1) At low temperatures, the charge carriers mobility in graphene increases with temperature. This behavior is similar to that of polycrystalline graphite [35]. (2) At low temperatures, the density of charge carriers in graphene is almost independent of temperature. In this case, the charge carriers cannot be removed from the valence band and, correspondingly, the number of charge carriers in the conduction band does not change. (3) At the high temperatures, the charge carriers mobility changes as T_0^{-1}, such as in the metals. (4) In the high-temperature regime, the charge carrier density monotonically increases with temperature. (5) The extracted charge carrier mobilities at low and high temperatures are well consistent with the experimental data [8].

As seen in Figs. 5 and 6, the calculated mass magnetization are in good agreement with the experimental data. At high temperatures and weak magnetic fields, $M_z \sim -B/T_0$ that is consistent with Eq. (29). At low temperatures, the mass magnetization weakly depends on T_0. The predicted dependencies M_z on T_0 at $B = 5$ and 8 T are similar to those at weaker magnetic fields (Fig. 6). We also predict the resonance in the dependence of the magnetization M_z on T_0 at very low temperatures.

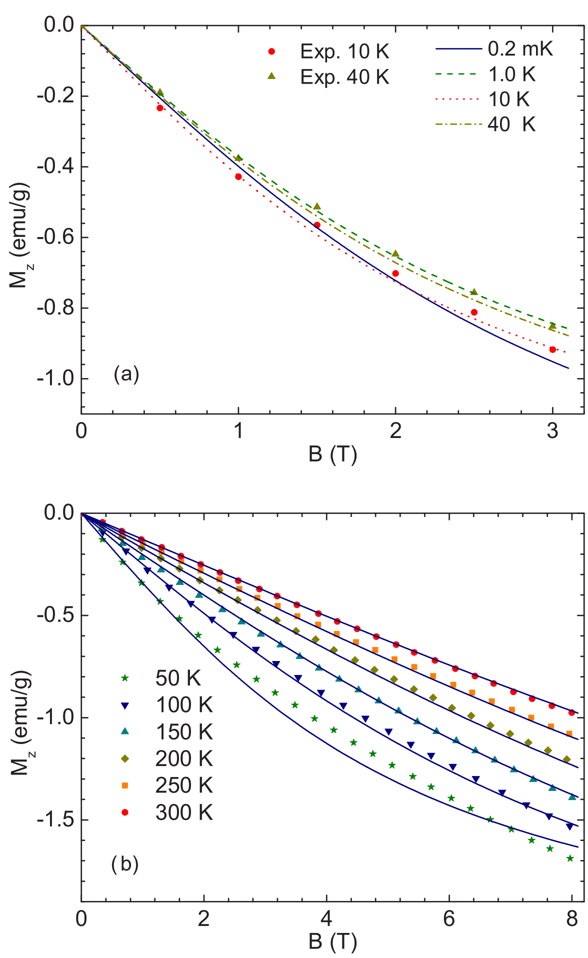

FIG. 5. The calculated (lines) dependence of the mass magnetization in graphene on the external magnetic field at indicated temperatures. In the calculations, Eqs. (35), (36), and (37) are used. The symbols denote experimental data [10].

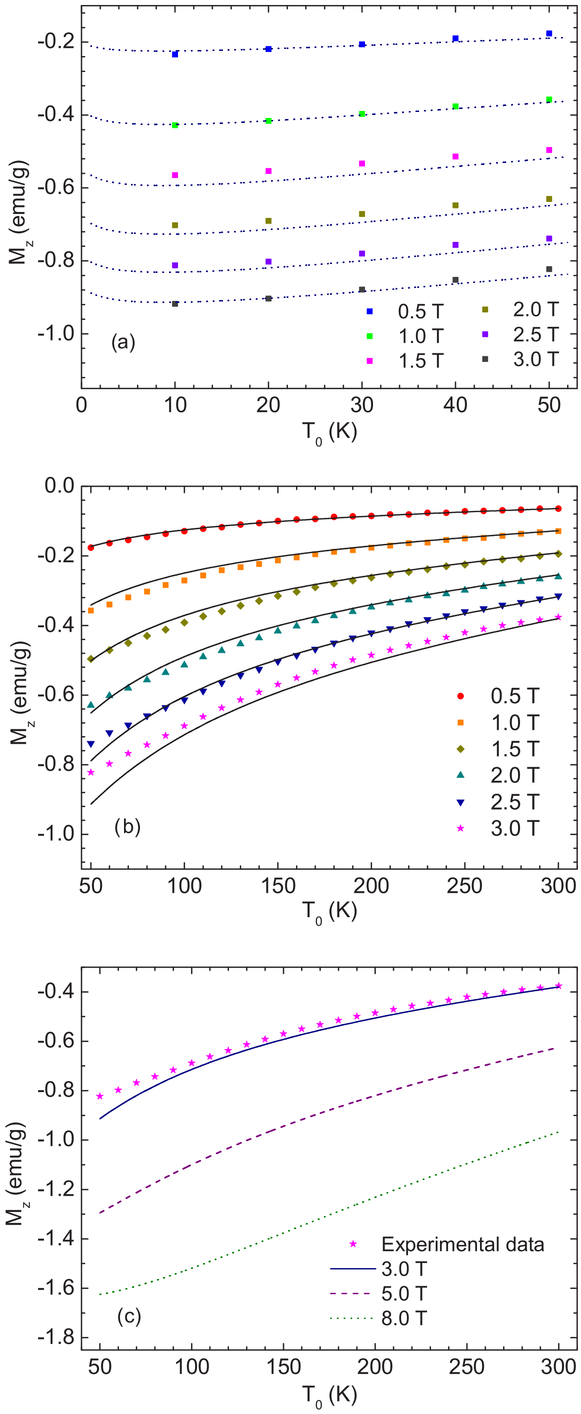

FIG. 6. The calculated (lines) dependencies of the mass magnetization of graphene on temperature at indicated external magnetic fields. In the calculations, Eqs. (35), (36), and (37) are used. The symbols denote experimental data [10].

For example, the resonance condition approximately corresponds to $T_0 = 1$ mK at $B = 0.5$ and $B = 3.0$ T (Fig. 8). The resonance temperature is almost independent of the magnetic field. The resonance becomes more pronounced at small T_0 (Fig. 8). Because of this resonance behavior of the magnetization, in Fig. 5,

$$|M_z(T_0 = 0.2 \text{ mK})| > |M_z(T_0 = 10 \text{ K})| > |M_z(T_0 = 1 \text{ K})|.$$

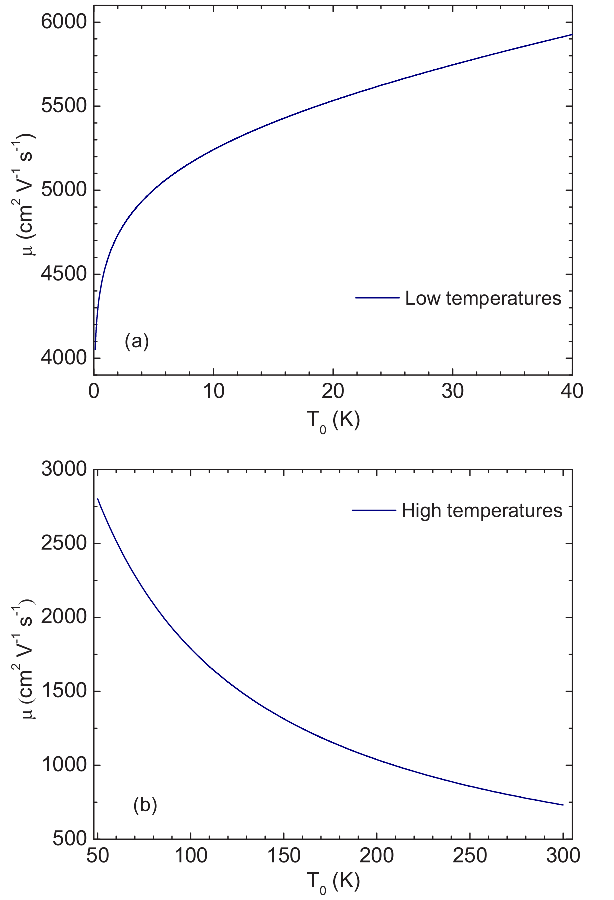

FIG. 7. The calculated temperature dependencies of charge carriers mobility in graphene at (a) low and (b) high temperature regimes. In the calculations, Eqs. (36) and (37) are used.

Note that the resonance type behavior of M_z was also reported in Ref. [11].

Using the extracted $\mu(T_0)$ and $C_0(T_0)$ from Eqs. (36) and (37) for low and high temperature regimes, the temperature dependencies of the diagonal σ_{xx} and nondiagonal σ_{xy} elements of the Drude electric conductivity [32] in graphene are predicted in Fig. 9. As seen, at very low temperatures and $B \leqslant 1$ T, the values of σ_{xx} and σ_{xy} strongly decrease with decreasing temperature. These calculated values of electric conductivity are quite close to the experimental data [3,8]. As seen from the description of the experimental data on the mass magnetization (Figs. 5 and 6), the graphene properties change dramatically around $T_0 \approx 45$–50 K and, as a consequence, the charge carrier mobility (Fig. 7) and, accordingly, electrical conductivity (Fig. 9) have a pronounced discontinuities between $T_0 = 45$ and 50 K.

IV. SUMMARY

The influence of the external uniform magnetic field on the open 2D quantum system linearly coupled in coordinates to the neutral bosonic heat bath was studied beyond the Markov approximation. In order to average the influence of bosonic heat bath on the charged particle, we applied the spectral function of heat-bath excitations which describes the Drude

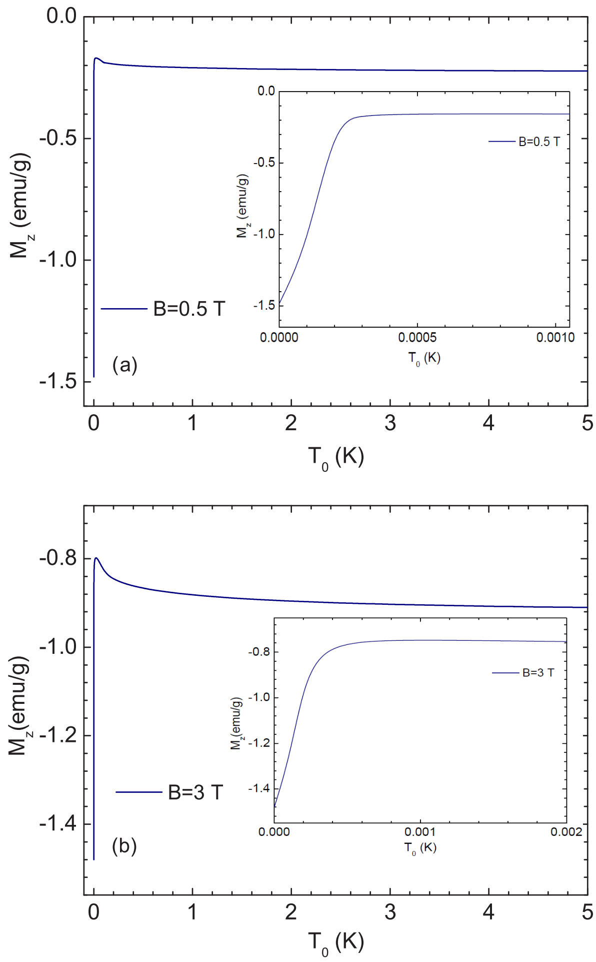

FIG. 8. The calculated dependence of the mass magnetization in graphene on temperature at indicated external magnetic fields. In the calculations, Eqs. (35) and (36) are used.

dissipation with Lorentzian cutoffs. Our formalism is valid at arbitrary coupling strengths, and, hence, at arbitrary low temperature. At the initial time interval, the magnetic field acts on the quantum particle through its contribution to the Lorentz force. The dissipation and external magnetic field do affect each other due to the non-Markovian dynamics of the quantum system. The combined action of the constant magnetic field and random forces leads to the emergence of angular momentum. The explicit expressions for the asymptotic angular momenta were obtained for the two-dimensional charged quantum harmonic oscillator and for the free charged particle in the Markovian and non-Markovian cases. We found the weak influence of the non-Markovian effect on the orbital diamagnetism of the open 2D quantum systems.

Using the analytical Markovian expression (35) for the asymptotic angular momentum of the free charged particle and the temperature-dependent coupling between the charge carriers and the environment, we have described well the dependencies of diamagnetic magnetization in graphene on the magnetic field and temperature. For graphene, the temperature dependencies of the mobility and density of charge carriers were extracted [Eqs. (36) and (37)]. A pronounced discontinuity of the mobility between $T_0 = 45$ and 50 K was

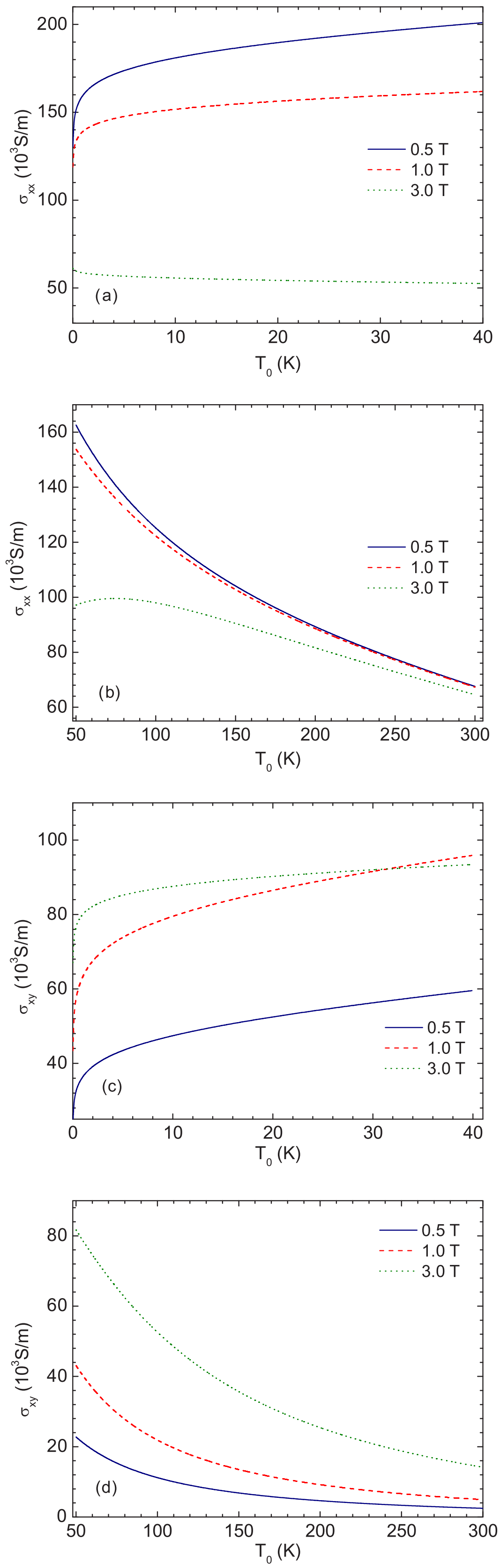

FIG. 9. The calculated dependencies of diagonal (a) and (b) σ_{xx} and (c) and (d) nondiagonal σ_{xy} components of the Drude electric conductivity in graphene on temperature at indicated external magnetic fields.

found. The dependencies of the mass magnetization on the temperature (magnetic field) at $B = 5$ and 8 T ($T_0 = 1$ and 0.2 mK) were predicted. The resonance behavior of the diamagnetic magnetization of graphene was predicted at very low temperatures ($T_0 \approx 1$ mK). Using the extracted mobility and density of charge carriers in graphene, we also found the temperature dependence of the electric conductivity on temperature at different magnetic fields. At very low temperatures and weak magnetic fields, a strong decrease in the electric conductivity with decreasing temperature was demonstrated.

ACKNOWLEDGMENTS

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APPENDIX A: SOLUTION OF EQS. (7)

The coefficients in Eq. (13) are

$$\hat{I}_x(t) = \int_0^t A_3(\tau)\hat{F}_\alpha(t-\tau)d\tau, \quad \hat{I}'_x(t) = \int_0^t A_4(\tau)\hat{F}_g(t-\tau)d\tau,$$
$$\hat{I}_y(t) = \int_0^t B_3(\tau)\hat{F}_\alpha(t-\tau)d\tau, \quad \hat{I}'_y(t) = \int_0^t B_4(\tau)\hat{F}_g(t-\tau)d\tau,$$
$$\hat{I}_{\pi_x}(t) = \int_0^t C_3(\tau)\hat{F}_\alpha(t-\tau)d\tau, \quad \hat{I}'_{\pi_x}(t) = \int_0^t C_4(\tau)\hat{F}_g(t-\tau)d\tau,$$
$$\hat{I}_{\pi_y}(t) = \int_0^t D_3(\tau)\hat{F}_\alpha(t-\tau)d\tau, \quad \hat{I}'_{\pi_y}(t) = \int_0^t D_4(\tau)\hat{F}_g(t-\tau)d\tau,$$

where

$$A_1(t) = \sum_{i=1}^{6}\beta_i[(s_i+\gamma)\big[s_i(s_i+\gamma)\big(s_i^2+\omega_y^2+\omega_c^2\big)+\lambda_x\gamma\big(s_i^2+\omega_y^2\big)\big]+s_i\lambda_y\gamma(s_i(s_i+\gamma)+\lambda_y\gamma)]e^{s_it},$$
$$A_2(t) = \omega_c\omega_y^2\sum_{i=1}^{6}\beta_i(s_i+\gamma)^2e^{s_it}, \quad A_3(t) = \frac{1}{m}\sum_{i=1}^{6}\beta_i(s_i+\gamma)\big[(s_i+\gamma)\big(s_i^2+\omega_y^2\big)+s_i\lambda_y\gamma\big]e^{s_it},$$
$$A_4(t) = -\frac{\omega_c}{m}\sum_{i=1}^{6}\beta_is_i(s_i+\gamma)^2e^{s_it},$$
$$B_1(t) = -A_2(t)|_{x\leftrightarrow y}, \quad B_2(t) = A_1(t)|_{x\leftrightarrow y}, \quad B_3(t) = -A_4(t)|_{x\leftrightarrow y}, \quad B_4(t) = A_3(t)|_{x\leftrightarrow y},$$
$$C_1(t) = -m^2\omega_x^2A_3(t), \quad C_2(t) = m\dot{A}_2(t), \quad C_3(t) = m\dot{A}_3(t), \quad C_4(t) = m\dot{A}_4(t),$$
$$D_1(t) = m\dot{B}_1(t), \quad D_2(t) = -m^2\omega_y^2B_4(t), \quad D_3(t) = m\dot{B}_3(t), \quad D_4(t) = m\dot{B}_4(t). \tag{A1}$$

Here, s_i are the roots of the equation,

$$D(s) = (s_i+\gamma)\big\{\big[s_i^4+\omega_x^2\omega_y^2+s_i^2\big(\omega_c^2+\omega_x^2+\omega_y^2\big)\big](s_i+\gamma)+s_i\gamma\lambda_x\big(s_i^2+\omega_y^2\big)\big\}+s_i\gamma\lambda_y\big[\big(s_i^2+\omega_x^2\big)(s_i+\gamma)+s_i\gamma\lambda_x\big]=0, \tag{A2}$$

and $\beta_i = [\prod_{j\neq i}(s_i-s_j)]^{-1}$ with $i, j = 1$–6. These roots satisfy the conditions $s_4 = s_1^*,\ s_5 = s_2^*$, and $s_6 = s_3^*$. Equation (A2) is transformed into Eq. (20) in the limit $\gamma \to \infty$.

APPENDIX B: FREE CHARGED PARTICLE IN CONSTANT MAGNETIC FIELD: NON-MARKOVIAN AND MARKOVIAN CASES

Let us consider the dynamics of the free charged particle in the constant magnetic field and neutral bosonic heat bath (see Sec. II). In this case, the Hamiltonian of total system is

$$\hat{H} = \frac{1}{2m}\big(\hat{\pi}_x^2+\hat{\pi}_y^2\big)+\sum_\nu\hbar\omega_\nu\hat{b}_\nu^+\hat{b}_\nu+\sum_\nu(\alpha_\nu\hat{x}+g_\nu\hat{y})(\hat{b}_\nu^++\hat{b}_\nu)+\sum_\nu\frac{1}{\hbar\omega_\nu}(\alpha_\nu\hat{x}+g_\nu\hat{y})^2. \tag{B1}$$

Using Eq. (B1), we obtain the nonlinear integrodifferential stochastic dissipative equations,

$$\dot{\hat{x}}(t) = \frac{\hat{\pi}_x(t)}{m}, \quad \dot{\hat{y}}(t) = \frac{\hat{\pi}_y(t)}{m}, \quad \dot{\hat{\pi}}_x(t) = -\omega_c\hat{\pi}_y(t)-\frac{1}{m}\int_0^t d\tau\, K_\alpha(t-\tau)\hat{\pi}_x(\tau)-\hat{F}_\alpha(t),$$
$$\dot{\hat{\pi}}_y(t) = \omega_c\hat{\pi}_x(t)-\frac{1}{m}\int_0^t d\tau\, K_g(t-\tau)\hat{\pi}_y(\tau)-\hat{F}_g(t), \tag{B2}$$

where $K_\alpha(t)$ and $K_g(t)$ are the dissipative kernels. Employing solutions of Eq. (B2), we derive the analytical expression for the z component of asymptotic angular momentum,

$$\begin{aligned}L_z(\infty) &= \frac{4\gamma^2\lambda\hbar\omega_c}{\pi}\int_0^\infty d\omega\,\omega\,\coth\left[\frac{\hbar\omega}{2T_0}\right]\frac{\omega^2+\gamma(\gamma-\lambda)}{\left|\omega^2+s_1^2\right|^2\left|\omega^2+s_2^2\right|^2}\\ &= \frac{4\gamma^2\lambda\hbar\omega_c}{\pi}\int_0^\infty d\omega\,\omega\,\coth\left[\frac{\hbar\omega}{2T_0}\right][\omega^2+\gamma(\gamma-\lambda)]/\{\left(\omega^2-\omega_c^2\right)^2(\omega^2+\gamma^2)^2\\ &\quad -4\omega^2\gamma\lambda\left(\omega^2-\omega_c^2\right)(\omega^2+\gamma^2)+2\gamma^2\lambda^2\left[3\omega^4-\omega^2\omega_c^2+\gamma^2\left(\omega^2+\omega_c^2\right)\right]-4\omega^2\gamma^3\lambda^3+\gamma^4\lambda^4\}.\end{aligned} \tag{B3}$$

Here $\lambda_x=\lambda_y=\lambda$ and $s_1, s_2, s_3=s_1^*,\ s_4=s_2^*$ are the roots of the equation,

$$\left(s^2+\omega_c^2\right)(s+\gamma)^2+2\gamma\lambda s(s+\gamma)+\gamma^2\lambda^2=0. \tag{B4}$$

Using the theory of residues, one can calculate analytically the integral in the first line of Eq. (B3) and derive the following expression:

$$L_z(\infty)=\hbar\omega_c\gamma^2(I_f+I_f^*-I_{fs}), \tag{B5}$$

where

$$I_f=\frac{2\lambda\left[\gamma(\gamma-\lambda)-s_1^2\right]}{\left(s_1^2-s_1^{*2}\right)\left(s_1^2-s_2^2\right)\left(s_1^2-s_2^{*2}\right)}\cot\left[\frac{\hbar s_1}{2T_0}\right]+\frac{2\lambda\left[\gamma(\gamma-\lambda)-s_2^2\right]}{\left(s_2^2-s_1^2\right)\left(s_2^2-s_1^{*2}\right)\left(s_2^2-s_2^{*2}\right)}\cot\left[\frac{\hbar s_2}{2T_0}\right], \tag{B6}$$

and

$$I_{fs}=16\pi\lambda\frac{T_0^2}{\hbar^2}\sum_{n=1}^\infty\frac{\left[\gamma(\gamma-\lambda)-x_n^2\right]n}{\left|x_n^2-s_1^2\right|^2\left|x_n^2-s_2^2\right|^2}. \tag{B7}$$

In the Markovian limit ($\gamma\to\infty$), we obtain from Eq. (B3),

$$L_z(\infty)=\frac{4\lambda\hbar\omega_c}{\pi}\int_0^\infty d\omega\frac{\omega\,\coth[\hbar\omega/(2T_0)]}{\omega^4-2\omega^2\left(\omega_c^2-\lambda^2\right)+\left(\omega_c^2+\lambda^2\right)^2}. \tag{B8}$$

The similar expression was derived in Refs. [21,31]. If the friction coefficient is zero, then

$$L_z(\infty)=\hbar\,\coth\left[\frac{\hbar\omega_c}{2T_0}\right]. \tag{B9}$$

As seen, Eq. (B9) contains only the second term of Eq. (33) and, correspondingly, it does not take into consideration the important contribution of the boundary charge carriers [34].

[1] K. S. Novoselov, A. K. Giem, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, Science **306**, 666 (2004).
[2] K. S. Novoselov, E. McCann, S. V. Morozov, V. I. Falako, M. I. Katsnelson, U. Zeitler, D. Jiang, F. Schedin, and A. K. Geim, Nat. Phys. **2**, 177 (2006).
[3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009).
[4] E. H. Hwang and S. Das Sarma, Phys. Rev. B **77**, 115449 (2008).
[5] J. H. Chen, L. Li, W. G. Cullen, E. D. Williams, and M. S. Fuhrer, Nat. Phys. **7**, 535 (2011).
[6] G. Qin, Z. Qin, H. Wanga, and M. Hu, Comput. Mater. Sci. **151**, 153 (2018).
[7] M. Tian, Y. Huang, W. Wang, R. Li, P. Liu, C. Liu, and Y. Zhang, J. Mater. Res. **29**, 1288 (2014).
[8] D. R. Cooper, B. D'Anjou, N. Ghattamaneni, B. Harack, M. Hilke, A. Horth, N. Majlis, M. Massicote, L. Vandsburger, E. Whiteway, and V. Yu, Int. Sch. Res. Notices **2012**, 501686 (2012).
[9] D. E. Sheehy and J. Schmalian, Phys. Rev. B **80**, 193411 (2009).
[10] Z. Li, L. Chen, S. Meng, L. Guo, J. Huang, Y. Liu, W. Wang, and X. Chen, Phys. Rev. B **91**, 094429 (2015).
[11] L. Chen, L. Guo, Z. Li, H. Zhang, J. Huang, S. Jin, and X. Chen, Sci. Rep. **3**, 2599 (2013).
[12] M. Koshino, Y. Arimura, and T. Ando, in *International Conference of Computational Methods in Sciences and Engineering 2009: (ICCMSE 2009)*, edited by T. E. and G. Maroulis, AIP Conf. Proc. No. 1504 (AIP, New York, 2012), p. 291.
[13] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[14] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981); Ann. Phys. (NY) **149**, 374 (1983).
[15] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed* (VCH, New York, 1990); Phys. Rev. A **30**, 568 (1984).
[16] V. V. Dodonov and O. V. Man'ko, Sov. J. Theor. Math. Phys. **65**, 1033 (1985).
[17] V. V. Dodonov and V. I. Man'ko, in *Density Matrices and Wigner Functions of Quasiclassical Quantum Systems*, edited by A. A. Komar, Proceedings of the Lebedev Physics Institute of Sciences (Nova Science, Commack, NY, 1987), Vol. 167.

[18] G. Y. Hu and R. F. O'Connell, Physica A **151**, 33 (1988); Phys. Rev. B **36**, 5798 (1987); X. L. Li, G. W. Ford, and R. F. O'Connell, Phys. Rev. A **41**, 5287 (1990); **42**, 4519 (1990); Physica A **193**, 575 (1993); Phys. Rev. E **53**, 3359 (1996).

[19] Y. Marathe, Phys. Rev. A **39**, 5927 (1989).

[20] A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu, and W. Scheid, Int. J. Mod. Phys. E **3**, 635 (1994).

[21] S. Dattagupta and J. Singh, Phys. Rev. Lett. **79**, 961 (1997).

[22] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1999).

[23] T. M. Nieuwenhuizen and A. E. Allahverdyan, Phys. Rev. E **66**, 036102 (2002).

[24] B. Bellomo, A. De Pasquale, G. Gualdi, and U. Marzolino, Phys. Rev. A **82**, 062104 (2010); J. Phys. A: Math. Theor. **43**, 395303 (2010).

[25] M. Carlesso and A. Bassi, Phys. Rev. A **95**, 052119 (2017).

[26] Z. Kanokov, Y. V. Palchikov, G. G. Adamian, N. V. Antonenko, and W. Scheid, Phys. Rev. E **71**, 016121 (2005).

[27] S. A. Kalandarov, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Phys. Rev. E **75**, 031115 (2007).

[28] V. V. Sargsyan, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Phys. Part. Nuclei **41**, 175 (2010).

[29] K. Wen, F. Sakata, Z.-X. Li, X.-Z. Wu, Y.-X. Zhang, and S.-G. Zhou, Phys. Rev. Lett. **111**, 012501 (2013).

[30] D. Lacroix, V. V. Sargsyan, G. G. Adamian, and N. V. Antonenko, Eur. Phys. J. B **88**, 89 (2015).

[31] S. A. Kalandarov, I. B. Abdurakhmanov, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Phys. Rev. A **99**, 062109 (2019).

[32] I. B. Abdurakhmanov, G. G. Adamian, N. V. Antonenko, and Z. Kanokov, Eur. Phys. J. B **91**, 223 (2018); Physica A **508**, 613 (2018).

[33] I. B. Abdurakhmanov, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Physica A **514**, 957 (2019).

[34] C. G. Darwin, Proc. Cambridge Philos. Soc. **27**, 86 (1930).

[35] S. Mrozowski, Phys. Rev. **85**, 609 (1952).