Prime numbers and random walks in a square grid

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In recent years, computer simulations have played a fundamental role in unveiling some of the most intriguing features of prime numbers. In this paper, we define an algorithm for a deterministic walk through a two-dimensional grid, which we refer to as a prime walk. The walk is constructed from a sequence of steps dictated by and dependent on the sequence of the last digits of the primes. Despite the apparent randomness of this generating sequence, the resulting structure—in both two and three dimensions—created by the algorithm presents remarkable properties and regularities in its pattern, which we proceed to analyze in detail.

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I. INTRODUCTION

One can argue that prime numbers present perplexing features, in a hybrid of local unpredictability and global regular behavior. It is this interplay between randomness and regularity that motivated searches for both local and global patterns that could potentially become signatures for certain underlying fundamental mathematical properties. Patterns such as the connections that are known to exist between the prime number sequence and the nontrivial zeros of the Riemann zeta function [1] constitute one of the most important open problems in mathematics [2].

Since the formulation of the Riemann hypothesis, much has been done, and yet much remains in the dark. It is often acknowledged that not a small number of mathematical discoveries have been accomplished after having assumed many conjectures or hypotheses to be valid *a priori*. For this reason, instead of attempting an analysis of the underlying behavior of the prime numbers, which has been an aspiration of mathematicians for centuries, we choose to perform ultralarge-scale computer calculations at a fundamental level.

In this paper, presented as an unbiased computational experiment, we observe, rather than prescribe, exactly how the motion of a deterministic walk defined over the prime number sequence conspires to produce radically different results when compared with a simple random walk.

The algorithm defined below that creates our prime walk (PW) is simple; yet again, the PW itself appears to be complex and unpredictable. It can thus be placed as an example of emergent complexity. In fact, within the subfield of prime number theory, many examples [3–7] can be found in which a simple algorithm defined over the prime number sequence gives rise to complex structures spontaneously, a complexity within which regularities can be found, thus opening a new avenue for research into the distribution of prime numbers.

II. METHODOLOGY

Here, we propose a way of number arrangement yielding an appealing visual structure in the form of a fractal plot. Inspired by Ulam's spiral [3], we assign positions to positive integers in a two-dimensional (2D) plane following these rules:

(i) The starting point is $(0, 0)$, assigned to $N = 1$.

(ii) Given the point (x, y) assigned to number N, if $N + 1$ is not a prime, the same point is assigned to it.

(iii) If $N + 1$ is a prime and its last digit is 1, we move up in the plane: $(x, y) \rightarrow (x, y + 1)$.

(iv) If $N + 1$ is a prime and its last digit is 3, we move down in the plane: $(x, y) \rightarrow (x, y - 1)$.

(v) If $N + 1$ is a prime and its last digit is 7, we move to the left in the plane: $(x, y) \rightarrow (x - 1, y)$.

(vi) If $N + 1$ is a prime and its last digit is 9, we move to the right in the plane: $(x, y) \rightarrow (x + 1, y)$.

Note that the last digits of prime numbers are 1, 3, 7, and 9. The only exceptions are primes 2 and 5 at the very beginning of the algorithm. This can be easily implemented in a computer code. For details, see Appendix A.

Of course, the choices above are arbitrary and could be modified, with a permutation of the moves for the different digits. However, it can be easily shown that any permutation necessarily leads to an equivalent result, the path described by

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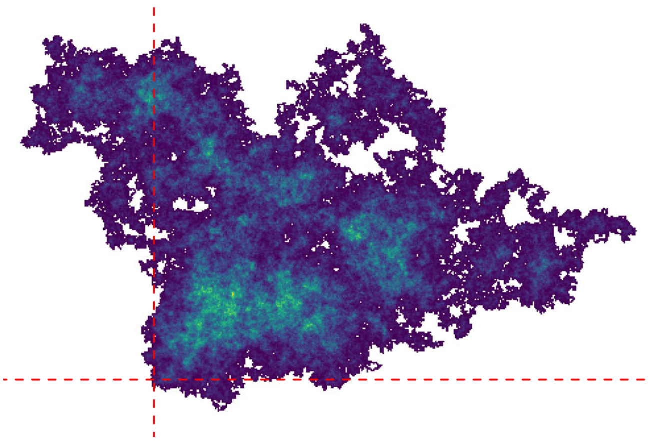

FIG. 1. Example of the PW plot up to $N = 2 \times 10^7$. The color code represents the $z_{\max}$ value. The x and y axes are shown as red dashed lines

the algorithm being a rotation or a mirror symmetry of the one resulting from the choice above. Equivalent algorithms can be ruled out.

III. RESULTS

Following the algorithm, the walker will move through the grid in an erratic way, impossible to predict *a priori*. Figure 1 shows the PW created by the path up to 2×10^7. The color code is interpolated from the first steps (in dark blue) to the final steps (in yellow). The Supplemental Material [8] includes an animation (see Movie 1) showing how the path grows with the increasing number of steps.

Our computation is not large enough to produce a symmetric Gaussian distribution or bell-shaped curve when we look at the structure constructed by the PW algorithm; however, it is natural to expect no preferences for any of the four quadrants of the plane.

We can define the number of points (x, y) visited without repetition as the "area" covered by the path. Since the PW is allowed to pass more than once through the same point in the grid, we can additionally keep track of the number of times that a certain point has been visited and use this value as a third coordinate z in order to visualize a structure in three dimensions. In the Supplemental Material [8], another animation (Movie 2) is presented showing how this area of the path grows with the number of steps in terms of the maximum value of z up to $N = 5 \times 10^{10}$.

Furthermore, in order to help pinpoint patterns in our results, it can be interesting to compare them with those obtained from a random algorithm, in which at every prime N the walker may move up, down, left, or right in a random way (with equal probability). This produces a pseudorandom walk (pRW). Results from this alternative algorithm are presented in following figures along with the results obtained from the main prime walk.

Finally, we also calculate the area covered by the path up to step N. Figure 2 plots this area vs the number of steps. A linear scaling appears, with slope $b = 0.00414 \pm (9 \times 10^{-6})$, up to $N = 10^{11}$ steps. Whether this linear trend will hold for longer or start to saturate when a larger interval is explored is unclear.

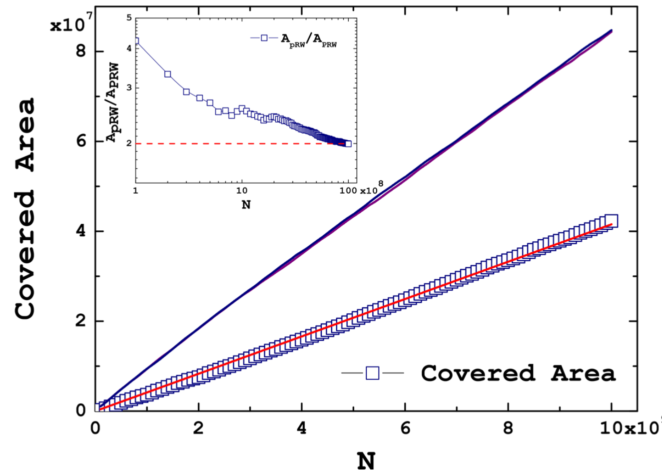

FIG. 2. Total area covered by the PW vs number of steps N. We can see that a linear trend (red line) is followed up to 10^{11} (linear fit: $y = bx$). The purple and dark blue lines represent the results of two pseudorandom walks (pRWs). The inset shows the ratio of the pRW area to the PW area vs N; note the log scale.

The first thing one notices in Fig. 2 is that, clearly, the area covered by the PW is smaller than the one covered by the pRWs. The difference is a *factor of* 2 when N is large enough (see inset).

It seems clear that the randomness of the prime number sequence produces an exact half-spread-path when the PW is compared with the pRW. This more compact random walk (RW) is in contrast with the recently defined concept of the maximum entropy random walk (MERW). As opposed to generic random walks (GRWs), which maximize entropy locally (neighbors are chosen with equal probabilities), the MERW does it globally (all paths of given length and endpoints are equally probable) [9].

The fact that, in a geometrical sense, the PW is more convoluted, spreading at a slower pace than the pRW, is also reflected in the maximum value of the z coordinate, $z_{\max}$. This value can be computed, both in a cumulative way and within separate intervals. The differences between the values obtained from the PW and the pRWs are again clear in this case, $z_{\max}$ being higher for the PW by a factor that tends approximately to 1.6 (see Fig. 3).

IV. CONJECTURES

Euclid's theorem tells us that there exists an infinite amount of primes; however, does this necessarily imply that the area covered by our PW path will turn out to be infinite?

It is natural to assume from the start that the area covered by the path after N steps must be related in some way to the number of primes $\pi(N)$. In particular, we expect

$$\pi(N) \sim \frac{N}{\log N} \quad \text{when } N \to \infty. \tag{1}$$

In our results, within the explored range, the covered area follows an approximately linear trend (see Fig. 2), and a more

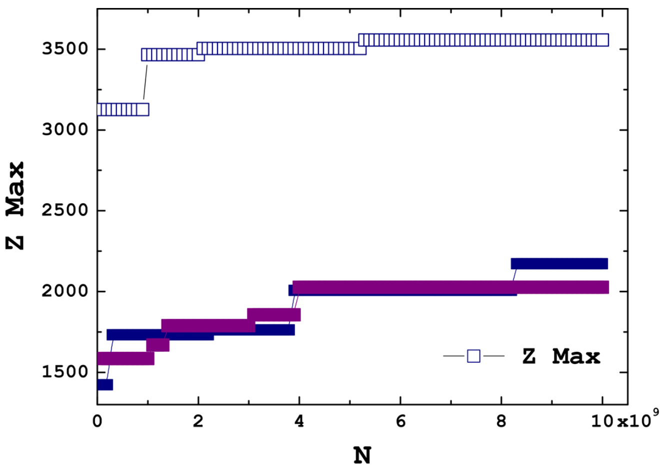

FIG. 3. Maximum z value in the structure created by the PW vs number of steps N. Full symbols are values corresponding to two pseudorandom walks.

careful analysis shows it to be proportional to the number of primes by a certain constant value $\psi \simeq 1/10$ (see Fig. 4). Apparently, the ratio

$$\frac{\pi(N)}{A_{\text{pRW}}(N)} \to 10 \tag{2}$$

when N is big enough. Will this be the case for even larger values of N?

We present here a plausible conjecture derived from our analysis, and two corollaries will follow straightforwardly. One of the main questions that our results invite us to ponder is whether there is an infinite number of points which are *never* visited by the PW, or whether the whole 2D plane is visited at least once.

Consequently, we propose the following.

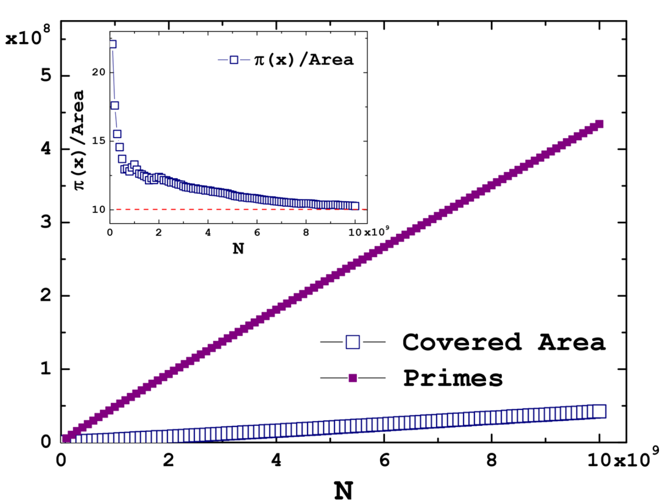

FIG. 4. Dark blue symbols, area covered by the path up to 10^{10}. Purple symbols, number of primes in the same interval, for comparison. The inset shows the ratio of the number of primes to the area covered by the path vs N.

Conjecture 1. The number of points within the area's perimeter which go unvisited by the PW increases homogeneously with N.

Corollary 1.1. If Conjecture 1 is true, then there is an infinite number of points (x, y) which are never visited by the PW.

Corollary 1.2. If Conjecture 1 is true, then the area visited by the PW continues to grow indefinitely, ultimately becoming infinite in the limit $N \to \infty$.

In the explored range (up to 5×10^{10}) we observe an almost perfectly linear growth, but it seems clear that subsequently, with the primes becoming less frequent, the area will foreseeably continue growing at a lower rate. However, we conjecture that its growth will not stop (this is what Conjecture 1 supports). The exact function describing this asymptotic growth is beyond the scope of this paper.

V. DISCUSSION

The covered area, as well as the value of $z_{\max}$, is governed partially (in a nontrivial way) by the gaps between primes. In the algorithm, every time the walk reaches a point (x, y) of the grid, it stays there until the next prime is drawn and it can move to the next point. Since z keeps track of the number of times that a certain point is visited, it is by construction a (nonrandom) sum of prime (but not necessarily consecutive) gaps.

It is to be noted, though, that in an unlimited simulation, the "structure" created by the PW would not be bounded in the z direction by any upper limit, since the gaps between primes can become arbitrarily large and thus for any given bound b, a gap $g > b$ would eventually appear within the infinite sequence. Nevertheless, this is just half of the problem, since for any given point (x, y), it is impossible to know *a priori* how many times it will have been revisited by the PW after a given number of steps N. Additionally, there is also the question of whether or not the PW is confined between some upper and lower values in x and/or y.

Regarding the gaps between primes, it is known that gaps between consecutive prime numbers cluster on multiples of 6 [10,11]. Because of this, 6 is frequently called the "jumping champion," and it is conjectured that it holds this title all the way up to about 10^{35}. Beyond 10^{35}, and until 10^{425}, the jumping champion then becomes 30 $(= 2 \times 3 \times 5)$, and beyond that, the most frequent gap is 210 $(= 2 \times 3 \times 5 \times 7)$ [11]. Further important results on some statistical properties between gaps have been recently discovered [12,13]. However, all of the aforementioned numerical observations, despite revealing intriguing properties about the prime sequence, cannot be easily applied to our problem to help figure out whether or not the PW will acquire definite boundaries.

On the other hand, according to Ares and Castro [14], the apparent regularities previously observed [15–17] reveal no structure in the sequence of primes; *au contraire*, those regularities are precisely a consequence of its randomness. This is, however, a highly controversial topic. Recent computational work points out that "after appropriate rescaling, the statistics of spacings between adjacent prime numbers follows the Poisson distribution" [18]. See Refs. [18,19] and

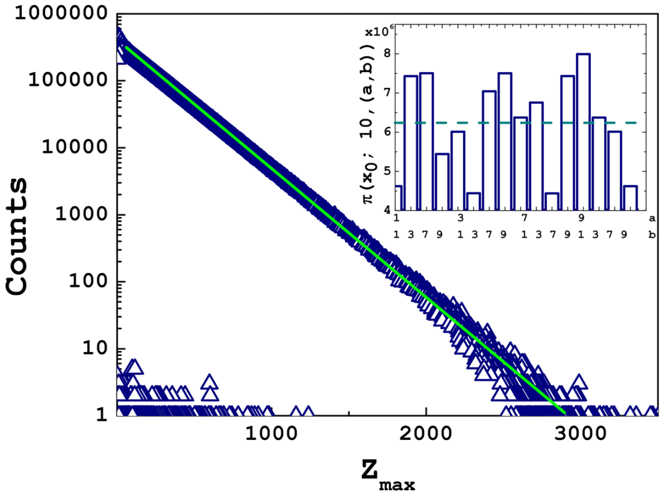

FIG. 5. Histogram of the values of $z_{\max}$ for $N = 10^{10}$. Note the log scale of the vertical axis. The green line shows the linear fit of the data (see inset). Among the first 100 000 000 primes (modulo 10), there is substantial deviation from the prediction that each of the 16 pairs (a, b) should have about 6.25×10^6 occurrences [24].

references therein for more on the statistics of the gaps between consecutive prime numbers.

In Fig. 5 we show the histogram $C(z_{\max})$, which behaves almost as $\log C(z_{\max}) = b - az_{\max}$ with $a = 0.0019$ and $b = 5.5 \times 10^5$ for $N = 10^{10}$ (fit done removing low x, low y points in the graph). This is similar to the histogram for differences between primes (prime gaps); see Fig. 1 a of Ref. [14].

However, we can be rather sure of a certain fact: These gaps, despite all of the complexity they present, lead to an absolutely clear Benford's law behavior for $z_{\max}$ (see Appendix B for extra analysis). This result cannot be coincidental.

The second part of the problem concerns the last digit of the primes, a relatively unexplored topic. Many papers have been published about the first digits in the sequence of prime numbers [2,20,21], but much less work has been devoted to studying the last digits. Firstly, it is important to note that in view of Chebyshev's bias [22,23] it may not be so obvious that the last digits of primes are distributed with equal probability (an assumption made in this paper for simplicity). This is not a hard fact, however, since some of the results regarding this phenomena are only proved assuming strong forms of the Riemann hypothesis.

Additionally, if we look at a pair of consecutive prime numbers (a, b), assuming a purely random distribution, one would expect it to be just as likely that these consecutive primes end in 1 and 1, or in 3 and 7, or in 3 and 9, and so on. However, intriguing irregularities (or biases) are actually observed in the distribution of consecutive primes. For example, it is a known fact that among the first 100 000 000 primes (modulo 10), there is substantial deviation from the prediction that each of the 16 pairs (a, b) should have about 6.25 000 000 occurrences. The inset in Fig. 5 shows the results according to Oliver and Soundararajan [24]. Note that in our study we explored a little bit further since up to 10^{10} the number of primes is around 4.34×10^8.

This result must ultimately be the cause of the factor 2 observed in Fig. 2. The bias observed in Ref. [24], which essentially means that the prime numbers' last digits are usually not repeated when taken in consecutive pairs, seems to be the cause of the properties of the PW, and it contributes even more to its complexity when compared with the pRW. It actually does so in a very precise way, resulting in the covered area being exactly half the pRW value.

We know that the set of primes is algorithmically decidable since one can always find an effective primality test or a sieve to separate all primes. The problem of the same set having an underlying order or pattern is of a different nature. Specifically, measures such as Shannon's entropy may be insufficient in that they only count collective symbol occurrences. However, we know that certain periodic as well as disordered patterns can give the same probabilities inside a whole multinomial set and, hence, the exact same entropy value. For this reason a different measure of complexity as opposed to "randomness" was proposed by Kolmogorov [25] in terms of the shortest formula or "program" that can reproduce a given sequence. Although in principle an incalculable quantity, it can be approximated with data compression theory [26], which searches for redundancies, this then being the equivalent of the shortest string able to reproduce the original. One can record in memory sufficiently large chunks of the last digit sequence and pass them through standard algorithms such as the Lempel-Ziv algorithm [27] to find large compression ratios. Yet any such subset does not suffice for discovering a complete set of rules of fixed length for all primes.

Finally, we note that the algorithm for the PW is built onto the number system in base 10. In different bases it will not be possible to obtain a walk with four directional steps, and random walks could be produced in higher dimensions (the number of dimensions being dictated by the number of possible last digits of the primes in the corresponding representation). Further research could go in this direction; in fact, this work could be extended in many ways, considering as well different types of grids (tessellations) to define different walks.

VI. CONCLUSIONS

In this paper we have intensively used simple numerical representations of prime numbers in two and three dimensions to investigate the distribution of primes along the natural numbers.

Our mathematical experiment shows some important, unexpected, and rather remarkable results. Within the explored range, the following was found:

(i) The area covered by the PW is smaller than the one covered by the pRWs, the difference being "exactly" a factor of 2 when N is big enough.

(ii) The number of primes up to N is 10 times the area covered by the PW, $A(N)$. In other words, the covered area is $1/10$ the number of primes $\pi(N)$.

(iii) We showcase a remarkable match between the first-digit count of the $z_{\max}$ values and Benford's law.

The results presented here highlight the important role of large-scale computer calculations as a way to discover *possible* new properties of prime numbers. "Possible" needs to be stressed since we cannot prove that the results we observed will hold for any given larger range.

With the availability of increasing computational power, in a few years it will be possible to explore further orders of magnitude. This, however, will clearly never be enough. We need to turn our conjectures into demonstrated theorems.

It is interesting to note that the approach described both here and in our previous paper [28] can be easily applied to any infinitely large sequence of numbers, such as, for example, the decimal digits of π, e, γ, or any other irrational number (either working in base 4, or selecting an arbitrary set of four final digits to build an algorithm). Some interesting studies have been published about the randomness of π [29,30]; however, a plethora of questions remain open still. Could similar insights be extracted for them to some extent? We believe this to be an open question deserving of our attention.

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APPENDIX A: IMPLEMENTATION

PYPY implementation

To satisfy the speed demands and reach the larger prime numbers, primality-testing and prime-number-generating algorithms play a crucial role. The most crucial criteria in the analysis of the prime number generators are the number of probes, the number of generated primes, and the average time required in producing each prime. For our study we used the simple and efficient Sequences Containing Primes algorithm, which employs the function $m = 6k + 1$ or $m = 6k - 1$. The algorithm can be easily implemented in a code.

We used PYPY, which is an implementation of the PYTHON programming language written in RPYTHON, a subset of the PYTHON language, with its own interpreter [31–33]. It implements PYTHON 2.7.10 and passes the PYTHON test suite with some minor modifications [34]. PYPY is intended to perform faster than CPYTHON by employing a tracing just-in-time (JIT) compiler. A JIT compiler, as the name suggests, compiles code during execution rather than before, as an ahead-of-time (AOT) compiler would do. The JIT used in PYPY is a metatracing JIT compiler. It does not encode any language semantics or profile in the execution of the program. Instead, it profiles the execution of the interpreter running the program. PYPY uses several optimizing techniques in its compiler: constant folding, common subexpression elimination, function inlining, and loop invariant code motion, among others [35]. The trace also contains guards for each point in the recorded code that could branch off in another direction, for example, in an IFstatement. When the trace is compiled to machine code, each guard is compiled into a check that the execution is still correct. If it is not, the interpreter once again takes over execution. If a guard failure occurs more times than a certain limit, PYPY will attempt to compile the new execution branch as well.

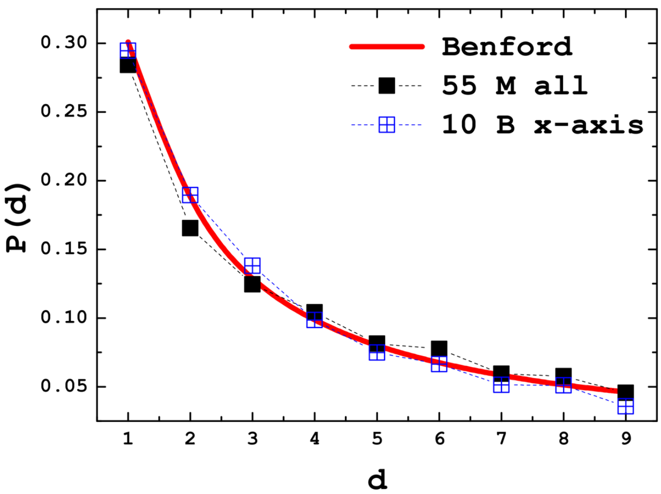

FIG. 6. Leading-digit histogram of the z_{max} values (PW up to 5.5×10^7). The black squares show the proportion of each of the z_{max} values. The blue squares show another example, here up to 10^9, but the (x, y) points considered are only those on the x axis. The expected values according to Benford's law are shown by the red curve. P, proportion; d, leading digit; M, $\times 10^6$; B, $\times 10^9$.

Random implementation

For the random implementation, we used probably the most widely known tool for generating random data in PYTHON, its random module library. PYTHON uses the Mersenne Twister pseudorandom number generator (PRNG) algorithm [36] as its core generator. The Mersenne Twister (MT) was proposed for generating uniform pseudorandom numbers. For a particular choice of parameters, this algorithm provides a superastronomical period of $2^{19\,937} - 1$ and a 623-dimensional equidistribution up to 32-bit accuracy, while using a working area of only 624 words. The underlying implementation in C is both fast and threadsafe. MT is one of the most extensively tested random number generators in existence.

APPENDIX B: EXTRA ANALYSIS

Benford's law

Figures 1 and 2 in this paper invite us to ask: Are these z values randomly distributed, or do they possibly follow some kind of distribution? Could they actually follow Benford's law [37]? Figure 6 seems to indicate this to be the case (although a proof is beyond the scope of this paper) by plotting the z_{max} values even for small numbers from a statistical point of view.

For instance, when the PW reaches just 5.5×10^7, we have $z_{max} = 155\,802$ points, and sorting these values according to the leading digit and comparing with Benford's law, the match is remarkable (black squares in Fig. 6). Even if we take just the z_{max} values along some given line (the x axis, for example), the match is obvious (blue empty squares in Fig. 6).

It is worth noting that the construct presented here resembles Jacob's ladder [28] (the description of the equivalence is beyond the scope of this paper), so this result seems to point

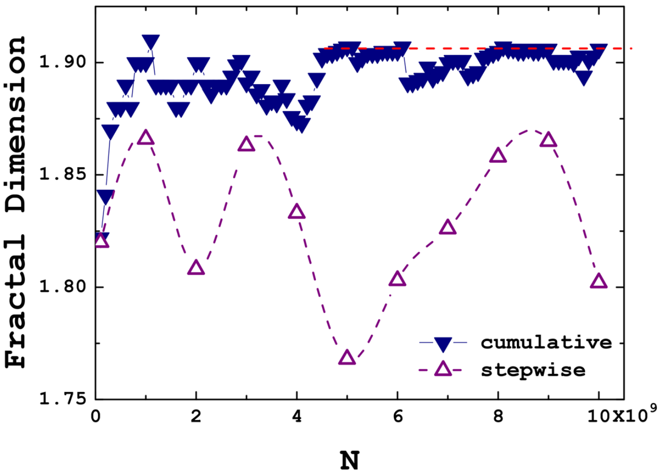

FIG. 7. Fractal dimension vs N calculated in a cumulative way (blue inverted triangles) and in steps of 10^8 (magenta empty triangles).

towards a behavior of the zeros in Jacob's ladder following Benford's law as well when N is large enough.

Fractal dimension

Fractal analysis is a contemporary method of applying nontraditional mathematics to describe patterns that defy understanding according to traditional Euclidean concepts. Recently, fractal analysis has been used to study a wide variety of complex patterns, such as those of many types of biological cells [38], tree and tumor growth [39], gene expression [40], forest fire progression [41], economic trends, and cellular differentiation in space and time [42].

In fractal analysis, complexity refers to the change in detail that comes with a change in scale. Many metrics of complexity can be defined, but the main parameter to capture them is the fractal dimension D_F defined as a scaling rule comparing how detail in a pattern changes with the scale at which it is measured. Formally, each iteration driving the change in detail introduces new pieces into the fractal construct. The number of pieces N at every step is related to the corresponding scale ϵ by

$$N \propto \epsilon^{D_F}. \tag{B1}$$

Figure 7 shows results for the fractal dimension D_F as calculated with IMAGEJ. The data points represent the result of the covered area vs N. We can safely conclude that D_F tends to a value of 1.91 ($\pm$0.01).

On the other hand, we can see that if steps of 10^8 are considered separately, larger oscillations are observed, and the shorter the length of intervals under examination, the larger the oscillations may be. These oscillations are expected [43,44], and similar "chaotic behavior" is observed in other properties such as the area. However, when taken in a cumulative way, these properties show a smooth trend as we have seen.

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