# Origin of dispersionless transport in spite of thermal noise

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The "dispersionless transport" of a weakly damped Brownian particle in a tilted periodic potential is defined by (i) a plateau of the particle's coordinate dispersion extending over a very broad time interval and (ii) by the impossibility to measure the diffusion coefficient within this plateau region. While the first part of this definition has been explained in the literature, the second part has been thought to follow from (i). Here, the impossibility to measure the diffusion coefficient is shown to be actually due to the wild fluctuations of the dispersion itself in the plateau region. An expression for the timescale over which a reliable determination of the diffusion coefficient is possible is derived. A procedure that allows accurate determination of the diffusion coefficient by observing the particle trajectory only within a small part of the plateau region is suggested and shown to be feasible by numerical simulations of a weakly damped Brownian particle in a tilted washboard potential.

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## I. INTRODUCTION

When combined with nonlinearity, noise may produce a number of surprising effects in dynamical systems out of thermal equilibrium. Examples include stochastic resonance [1], ratchet effect [2], absolute negative mobility [3–6], and transport against temperature gradient [7] to name but a few.

To such effects belongs the "dispersionless transport," first reported by Lindenberg *et al.* [8]. They considered a weakly damped Brownian particle in a periodic potential tilted by a constant force. It is textbook knowledge that after an initial transient process, the dispersion of the particle's coordinate should increase linearly in time, and the rate of its increase is the diffusion coefficient multiplied by twice the dimensionality of the system. But instead, it was observed in [8] that the dispersion reached a plateau and stayed constant for a very long time. For some parameter values, the duration of this plateau was much longer than a reasonable computation time, making it impossible to determine the diffusion coefficient from the simulations [8].

Later, the effect of "dispersionless transport" was reported in a ratchet system [9,10] for the time-periodic force of zero mean value [11] and for a combination of dc and ac forces [12], as well as for Brownian motion in a random potential [13]. Although the explanation of this effect proposed in [8] is intuitively appealing, some aspects of the "dispersionless transport" remain unclear. In particular, it is well understood why the plateau value of the dispersion is much larger than the linearly increasing term due to diffusion [8]. But however big, the dispersion plateau value is just a constant, i.e., the dispersion should still increase linearly in time, and the diffusion coefficient should still be obtainable from the dispersion vs time plot by linear fitting in the plateau regime. Yet, this is not the case in practice.

Here we expand the interpretation from [8] and explain this by a combination of two factors: the large value of the initial dispersion in the plateau region and by the fact that in simulations, averaging is necessarily performed over a large but finite number of stochastic trajectories. The latter aspect has been overlooked in the literature but is crucial in understanding the impossibility to measure the diffusion coefficient in the plateau regime. Finally, we introduce a simple modification of the measurement procedure that allows measuring the diffusion coefficient by observing the system's dynamics in the plateau region over a relatively short timescale.

## II. "DISPERSIONLESS TRANSPORT" IN A TILTED PERIODIC POTENTIAL

### A. Dispersion plateau

Consider a weakly damped Brownian particle in a tilted one-dimensional periodic potential $U(x)$ under the action of a constant force $F$. The particle dynamics is governed by the Langevin equation

$$m\ddot{x}_t = -U'(x_t) + F - \gamma\dot{x}_t + \sqrt{2\gamma T}\xi(t),$$
$$U(x) = -\frac{U_0}{2}\cos\frac{2\pi x}{a}, \tag{1}$$

where $x_t$ is the coordinate of the particle at time $t$, and the prime and the overdot denote spacial and time derivatives, respectively. The potential $U(x)$ has corrugation depth $U_0$ and periodicity $a$; $m$ and $\gamma$ are the mass and damping coefficient, respectively, $T$ is the temperature, and $\xi(t)$ is unbiased Gaussian white noise of unit strength: $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. We are interested in the evolution of the particle dispersion, defined as

$$\sigma_x^2(t) := \langle x_t^2\rangle - \langle x_t\rangle^2. \tag{2}$$

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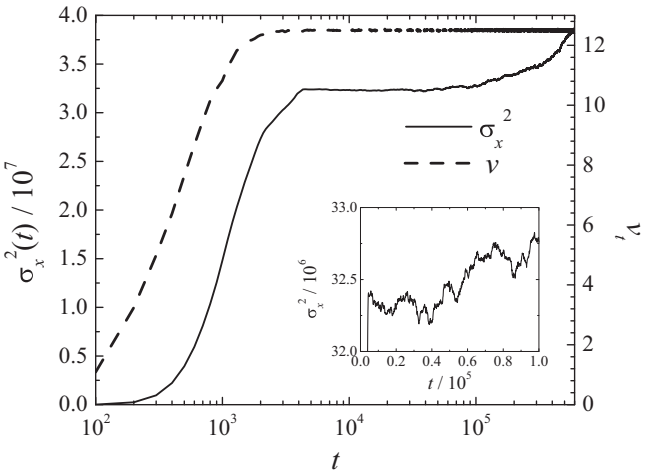


FIG. 1. Dispersion (2) (solid line) and average velocity $v_t = \langle \dot{x}_t \rangle$ (dashed line) of a Brownian particle in a tilted washboard potential (1) with the parameters used in [8]: $m = 1, U_0 = 1, a = 1, \gamma = 0.04$, $T = 0.2$, and $F = 0.5$. The curves are obtained by simultaneously simulating Eq. (1) for $N = 1000$ independent Brownian particles with the initial conditions $x_0 = 0$, $\dot{x}_0 = 0$ and averaging over 1000 trajectories. The inset shows the dispersion vs time curve in the plateau region on the linear scale.

In the long-time limit, it should increase linearly in time,

$$\sigma_x^2(t) = 2Dt, \tag{3}$$

where $D$ is the diffusion coefficient. We performed simulations of Eq. (1) for the parameters specified in the caption to Fig. 1 with the initial conditions

$$x_0 = \dot{x}_0 = 0. \tag{4}$$

It is clear that the dispersion has a plateau, where it practically does not change in time, see Fig. 1. One may suspect that its constancy might be due to the use of the logarithmic scale on the time axis, but using the linear scale does not reveal a linear increase according to (3), see inset in Fig. 1. Hence this plateau is often termed "dispersionless transport" in the literature [8–13].

### B. Interpretation of the dispersion plateau

The interpretation of this effect offered in [8] is as follows. Suppose that initially the particle found itself near some minimum of the total potential $U(x) - Fx$ with $|F| < \max_x |U'(x)|$. Its escape from that potential well is described by Kramers' theory. The mean escape time $\tau_0$ from the potential well depends on the shape of the potential near the minimum, the force, the damping, the temperature, and, most importantly, on the barrier height $\Delta U$ that separates the current potential minimum from the next one [14]. After escape, the particle moves with a constant velocity $F/\gamma$. The distribution of the escape times $\tau$ is exponential and is proportional to $e^{-\tau/\tau_0}$. Hence, the initial width of the spacial probability distribution to find the particle can be estimated as $\sigma^2_{\text{plateau}} \sim (F\tau_0/\gamma)^2$. As the particle moves after the escape, its spacial dispersion is

$$\sigma_x^2(t) \approx \sigma^2_{\text{plateau}} + 2Dt. \tag{5}$$

From this, it is concluded [8] that the duration of the "dispersionless" phase is

$$t_{\text{plateau}} = \sigma^2_{\text{plateau}}/(2D). \tag{6}$$

Because the mean escape time is proportional to the Kramers-Arrhenius factor [14], $\tau_0 \propto e^{\Delta U/T}$ and can be very large for $T \ll \Delta U$, the initial dispersion $\sigma^2_{\text{plateau}} \propto e^{2\Delta U/T}$ may also be very large, implying very long duration of the "dispersionless" phase.

Based on this interpretation, the authors of [8] identified thermal fluctuations, weak damping, periodicity of the potential $U_0(x)$, and the presence of a nonzero bias such that $|F| < \max_x |U'(x)|$ as necessary conditions for the onset of the nondispersive regime.

## III. DISPERSION FLUCTUATIONS AND DIFFUSION COEFFICIENT MEASUREMENT IN THE PLATEAU REGION

### A. Dispersion plateau of a free Brownian particle

In fact, the above general argument from [8] can be used to predict a dispersion plateau in *any* system where the diffusion proceeds in two stages: a fast initial stage that results in a broad initial dispersion of the coordinate to a large value $\sigma^2_{\text{plateau}}$, and slow normal diffusion that follows the initial spread. For example, a plateau can be observed even if the dynamics (1) is simplified to

$$m\ddot{x}_t = -\gamma \dot{x}_t + \sqrt{2\gamma T}\xi(t), \tag{7}$$

provided the initial conditions are chosen in a special way. Namely, we place initially all particles at $x_0 = 0$, but we assume that the initial temperature of our particles was very high relative to the temperature of the environment: $T_0 \gg T$. Then the initial distribution of the particle velocity $v_0 = \dot{x}_0$ is

$$P(v_0) \propto e^{-mv_0^2/(2T_0)}. \tag{8}$$

Initially, the particles cool down to the environment temperature $T$. In this transient process, the particle's velocity $v_t = \dot{x}_t$ can be decomposed into an exponentially decaying part [the homogeneous solution of Eq. (7)] and the noisy part (the particular solution of this equation): $v_t = e^{-\gamma t/m} v_0 + \Delta v_t$, where $\Delta v_t$ describes the diffusive process with the diffusion coefficient $D = T/\gamma$. As the particle cools down, the coordinate dispersion rapidly increases until it reaches the rather high value

$$\sigma^2_{\text{plateau}} = \int dv_0\, P(v_0) \left( \int_0^\infty dt\; v_0\, e^{-\gamma t/m} \right)^2 = \frac{T_0 m}{\gamma^2}, \tag{9}$$

as follows from the identities $\int_0^\infty dt\, e^{-\gamma t/m} = m/\gamma$ and $\int dv_0\, P(v_0)\, v_0^2 = T_0/m$.

After the transient cool-down process is over, normal diffusion with $d\sigma_x^2(t)/dt = 2D$, $D = T/\gamma$ begins. This expectation is confirmed by the numerical simulations of Eq. (7) with $m = \gamma = T = D = 1$; see Fig. 2 showing the dispersion vs time for two initial temperatures: $T_0 = 10^4$ (lower curve, left inset) and $T_0 = 10^8$ (upper curve, right inset). For the smaller initial temperature, $\sigma^2_{\text{plateau}} = 10^4$ and $t_{\text{plateau}} = 5 \times 10^3$, in full agreement with Eq. (6). For the higher initial temperature, the duration of the plateau predicted by Eq. (6),

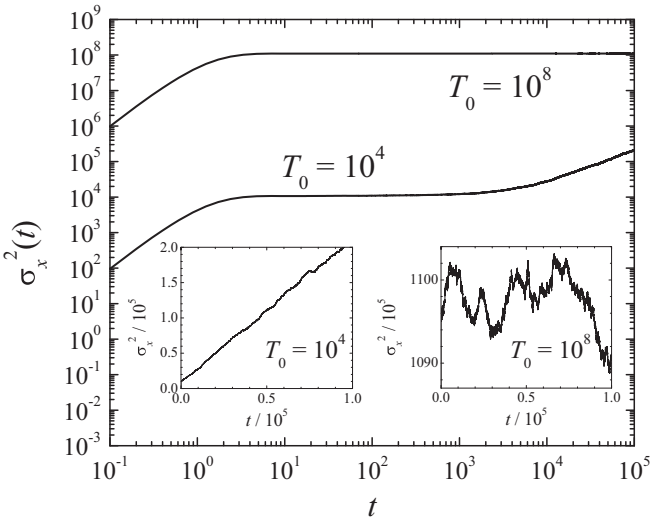


FIG. 2. Dispersion vs time, as obtained from the simulations of Eq. (7) with $m = \gamma = T = 1$ and the initial temperatures $T_0 = 10^4$ (lower curve, left inset) and $T_0 = 10^8$ (upper curve, right inset). The insets show the same plots on the linear scale.

$t_{\text{plateau}} = 5 \times 10^7$, is too long to reach the increasing part of the curve within a reasonable computation time.

The interpretation offered in [8] and reiterated above only explains the existence of the plateau region, where the $2Dt$ term in Eq. (5) is much smaller than the first term $\sigma^2_{\text{plateau}}$. This plateau can only be observed if a logarithmic scale is chosen on the time axis, but a linear increase of $\sigma_x^2(t)$ should be seen on the linear scale, see Eq. (5) and the dispersion curves for $T_0 = 10^4$ in the main part and in the left inset of Fig. 2. What remains surprising and unexplained is that the diffusion coefficient cannot be measured by linear fitting of $\sigma_x^2(t)$ with Eq. (5) within the plateau region at large values of $\sigma^2_{\text{plateau}}$ when a linear scale is used, see inset in Fig. 1 and right inset in Fig. 2. In the latter case, the dispersion even shows an overall decrease over a broad timescale $t \sim 10^5$, contrary to the expected linear increase with a known rate $2D = 2T/\gamma$.

Of course, such a decrease must be a spurious effect related to the wild fluctuations of the dispersion in the plateau regime. The diffusion coefficient cannot be established, not because there is a dispersion plateau on the $\sigma_x^2$ vs $\log(t)$ plot, but because within the plateau region the diffusive term $2Dt$ in Eq. (5) is submerged in these fluctuations.

### B. Dispersion fluctuations

Next we try and understand the origin of the dispersion fluctuations. Returning to the main part of Fig. 1, we observe that the plateau of $\sigma_x^2(t)$ begins right after the average velocity of the particle,

$$v_t = \langle \dot{x}_t \rangle, \tag{10}$$

saturates at the value $v_\infty \approx 12.5$.

Hence, we choose the origin of the time axis to be at the beginning of the dispersion plateau, assume the initial velocity of the particle to be $\dot{x}_0 = v_\infty$, and decompose the particle's coordinate into three parts:

$$x_t = x_0 + v_\infty t + y_t, \quad \langle x_0 \rangle = y_0 = \langle \dot{y}_t \rangle = \langle y_t \rangle = 0, \tag{11}$$

where $x_0$ is now a random variable. Without loss of generality, we set its average value to zero by suitably shifting the origin of the $x$ axis. Its dispersion is $\sigma^2_{x_0} = \sigma_x^2(0) = \sigma^2_{\text{plateau}}$. The deviation $y_t$ is the diffusive part of the coordinate. Its average velocity and its average value are both zero, because the average coordinate of the particle is $\langle x_t \rangle = v_\infty t$ for $t > 0$. Its dispersion should behave as

$$\sigma_y^2(t) := \langle y_t^2 \rangle - \langle y_t \rangle^2 = 2Dt \tag{12}$$

for sufficiently large $t$. This follows immediately from the relation between $\sigma_x^2(t)$, given by Eq. (2), and $\sigma_y^2(t)$, namely,

$$\begin{aligned} \sigma_x^2(t) &= \langle (x_0 + v_\infty t + y_t)^2 \rangle - (v_\infty t)^2 \\ &= \sigma_x^2(0) + \sigma_y^2(t) + 2c_t, \end{aligned} \tag{13}$$

where we have defined the correlation function

$$c_t := \langle x_0 y_t \rangle. \tag{14}$$

In view of the initial condition $y_0 = 0$, the initial value $c_0 = 0$. If the potential $U(x)$ is flat, i.e., $U_0 = 0$, then also at later times we should have $c_t = 0$; this is so, because in the frame of reference moving with the average velocity $v_\infty$ of the particle, positive and negative displacements $y_t$ are equally likely. If the potential $U(x)$ is corrugated, i.e., $U_0 > 0$, the correlation function $c_t$ may deviate from zero at $t > 0$. But even in this case, the correlation between $x_0$ and $y_t$ (if any) should very quickly decay to zero on the timescale much shorter than the duration of the plateau from Eq. (6). Keeping this timescale in mind, we can write

$$c_t \to \langle x_0 \rangle \langle y_t \rangle = 0. \tag{15}$$

This means that $c_t$ should play no role in the measurements of the diffusion coefficient. Given that at large times we should have $\sigma_x^2(t) = \sigma_x^2(0) + 2Dt$, we conclude that, indeed, $\sigma_y^2(t) = 2Dt$.

However, this kind of reasoning only applies to an idealized numerical experiment, in which an infinite number of stochastic trajectories are simulated. But in the real-life numerical simulations, averaging is performed over a large but finite number $N$ of stochastic processes $x_t^{(i)}, y_t^{(i)}$, $i = 1, \ldots, N$. Correspondingly, the *numerical* correlation function, denoted with a tilde,

$$\tilde{c}_t = \frac{1}{N} \sum_{i=1}^{N} x_0^{(i)} y_t^{(i)}, \tag{16}$$

is a random variable which goes to zero only in the limit $N \to \infty$, namely,

$$\tilde{c}_t \xrightarrow[N\to\infty]{} c_t \to 0. \tag{17}$$

But at a finite number $N$ of trajectories, the fluctuating part of the dispersion does not go to zero at any time $t > 0$. In fact, the size of the dispersion fluctuations increases in time.

To see this, we calculate the variance of $\tilde{c}_t$ based on the definition (16). We consider an infinite number of independent replicas of the numerical experiment, in each of which $N$

stochastic trajectories are simulated, and perform averaging over those replicas, denoted as $\langle \ldots \rangle$ in the derivation below:

$$\sigma_{\tilde{c}}^2(t) := \langle \tilde{c}_t^2 \rangle = \frac{1}{N^2} \sum_{i,j=1}^{N} \langle x_0^{(i)} y_t^{(i)} x_0^{(j)} y_t^{(j)} \rangle$$
$$= \frac{1}{N^2} \sum_{i=1}^{N} \langle x_0^{(i)2} y_t^{(i)2} \rangle = \frac{\langle x_0^{(1)2} y_t^{(1)2} \rangle}{N}$$
$$\to \frac{\langle x_0^2 \rangle \langle y_t^2 \rangle}{N} = 2Dt \frac{\sigma_x^2(0)}{N}. \tag{18}$$

In the second line, we used independence of the replicas, which implies that only the terms with $j = i$ survive in the double sum. For $i \neq j$, the summands split into $\langle x_0^{(i)} y_t^{(i)} \rangle \langle x_0^{(j)} y_t^{(j)} \rangle = c_t^2 \to 0$. Because all trajectories are statistically the same, the sum in the second line contains $N$ identical terms, each of which equals the first one. In the third line we removed the superfluous superscript (1) and focused on the timescale sufficient for the correlation between $x_0$ and $y_t$ to be lost, allowing us to replace $\langle x_0^2 y_t^2 \rangle$ with $\langle x_0^2 \rangle \langle y_t^2 \rangle$. In the last step we used the identities $\langle x_0^2 \rangle = \sigma_x^2(0)$ and $\langle y_t^2 \rangle = 2Dt$.

Thus in the realistic numerical simulations, the dispersion (13) contains a linearly increasing part $\sigma_y^2(t) = 2Dt$ and a fluctuating part $2\tilde{c}_t$. The standard deviation of the dispersion can be estimated as $2\sigma_{\tilde{c}}(t) = 2\sqrt{2Dt/N}\sigma_x(0)$, as follows from Eq. (18). Note that $\sigma_y^2(t) \sim t$ increases faster than $2\sigma_{\tilde{c}}(t) \sim \sqrt{t}$.

In order to measure the diffusion coefficient with a reasonable accuracy, the former must be bigger than the latter by a large number $A$ inversely proportional to the desired accuracy of $D$; e.g., $A = 10$ corresponds to the accuracy of 10%. This means that the waiting time, $t_D$, necessary to reliably measure $D$, can be estimated as

$$\sigma_y^2(t_D) = 2Dt_D = 2A\sigma_{\tilde{c}}(t_D), \quad t_D = 2A^2 \frac{\sigma_x^2(0)}{ND}. \tag{19}$$

With the parameters used to build the graphs in Fig. 1 ($\sigma^2_{\text{plateau}} = 3.2 \times 10^7$, $D = 5$, $N = 1000$), the inequality $t > t_D$ implies that the diffusion coefficient becomes measurable for the waiting times exceeding $t > 10^6$, in agreement with the simulation results. Alternatively, one would need to simulate $N \sim 10,000$ trajectories over the timescale of $t_D \sim 10^5$ to measure $D$ with a 10% accuracy, corresponding to $A = 10$.

On the other hand, for the parameters used to produce the plot at $T_0 = 10^8$ in Fig. 2 ($\sigma^2_{\text{plateau}} = 10^8$, $D = 1$, $N = 1000$), we must wait over $2 \times 10^7$ time units to reliably measure $D$. The timescale of $10^5$ time units, see the right inset in Fig. 2, is way too short to reveal an increase of $\sigma_x^2(t)$. Over this timescale, the time evolution of $\sigma_x^2(t)$ may even exhibit an overall decrease, which is entirely due to the small sampling size.

Finally, for the initial temperature $T_0 = 10^4$ we have $t_D = 2 \times 10^3$, much shorter than the simulation time in Fig. 2. It is for this reason that the coordinate dispersion increases linearly with time on the linear scale (Fig. 2, left inset), in spite of the plateau observed on the logarithmic scale.

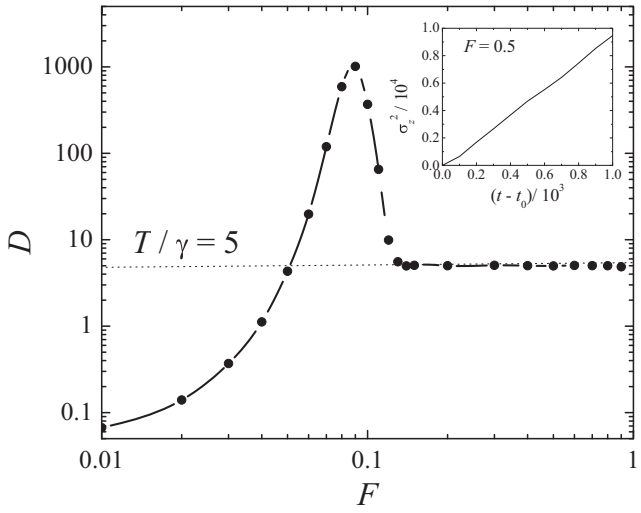


FIG. 3. Diffusion coefficient $D$ vs applied bias $F$ for the same parameters as in Fig. 1. All data points were obtained from the slope of the dispersion $\sigma_z^2(t)$ line in the plateau region, see inset showing this line for $F = 0.5$. For convenience, the time is measured from the moment $t_0 = 10^4$ inside the plateau of $\sigma_x^2(t)$.

### C. Determination of the diffusion coefficient in the plateau region

Fortunately, to accurately determine the diffusion coefficient from the simulations, it is not necessary to wait for such a long time or to simulate a huge number of trajectories. Instead, one may adopt the following numerical procedure.

First, the system is simulated to the point $t_0$ when the initial transient process has decayed and the dispersion plateau has started. The time $t_0$ does not have to be very large; for instance, for the parameters from Fig. 1, it may be $t_0 = 10^4$. At the time $t_0$, the coordinate of the particle $x_{t_0}$ is a random variable characterized by a very broad distribution. Hence, one needs to measure the dispersion of an auxiliary variable,

$$z_t = x_t - x_{t_0} = y_t - y_{t_0} + v_\infty(t - t_0),$$
$$\sigma_z^2(t) = \langle z_t^2 \rangle - \langle z_t \rangle^2 = 2D(t - t_0), \tag{20}$$

see the discussion above. Again, neither the simulation time nor the number of trajectories $N$ have to be very large; as Fig. 3 exemplifies, a very small portion of the total duration of the plateau phase is sufficient to get a linear increase of $\sigma_z^2(t)$ with very small fluctuations for $N = 1000$.

In this way we were able to obtain the force-dependent diffusion coefficient of the system (1) with the parameters (other than force) specified in the captions to Fig. 1, also covering the range of forces $0.1 < F < 1$ which was unaccessible for the procedure adopted in [8]. At these forces, the diffusion coefficient is practically constant and has the value $T/\gamma = 5$, see Fig. 3. This value corresponds to the so-called running state of the Brownian particle. At low forces, $F < 0.1$ in Fig. 3, the diffusion proceeds via thermally activated hopping over one or more barriers that separate the adjacent minima of the total potential $U(x) - Fx$; since those barriers decrease with force, $D(F)$ increases until it reaches a maximum. This maximum is due to the high frequency of switching between the running and the locked states, which leads to the fastest

temporal growth of the particle coordinate dispersion in the frame of reference which moves with the average velocity of the particle.

## IV. CONCLUSIONS

The "dispersionless transport" phenomenon is actually more ubiquitous than suggested in the original paper [8]. It can be expected in any noisy system, in which there is an initial transient process leading to a broad spreading of the coordinate. After the decay of this process, a plateau in the coordinate dispersion can be expected for some time.

Even though the numerical examples discussed in this paper involve a Brownian particle with finite mass and relatively low damping, one can easily think of other possible scenarios that result in a dispersion plateau even in the overdamped limit, see Eq. (1) with $m$ formally set to 0. For instance, consider the free diffusion of an overdamped Brownian particle: $\dot{x}_t = \sqrt{2T/\gamma}\,\xi(t)$. One may start with a high temperature $T = T_H$ and then abruptly change the temperature to a much lower value $T = T_L \ll T_H$ at some time $t_0 > 0$. Then, for $t \gg t_0$, the dispersion will increase according to Eq. (5), with $D = T_L/\gamma$ and $\sigma^2_{\text{plateau}} = 2t_0 T_H/\gamma$. On the logarithmic scale, the dispersion will have a plateau of duration $t_{\text{plateau}} = t_0 T_H/T_L \gg t_0$, which can be made arbitrarily long by adjusting the ratio $T_H/T_L$.

Another question is why the diffusion coefficient cannot be extracted from the dispersion vs time plot in the plateau region. This has to do with both the large initial value of the dispersion and with the fact that averaging is performed over a finite number of stochastic trajectories.

In practice, the existence of the dispersion plateau does not pose a problem in a numerical determination of the diffusion coefficient. The only modification that is required to perform this measurement is to account for the distribution of the initial position in the beginning of the plateau phase. The deviation of the coordinate from the initial position does exhibit the standard linear increase characteristic of the regular diffusive process.

Because the linear increase of the dispersion within the plateau region is measurable, we believe that the term "dispersionless transport" is actually a misnomer and use it in quotation marks throughout this paper.

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[1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[2] P. Reimann, Phys. Rep. **361**, 57 (2002).
[3] R. Eichhorn, P. Reimann, and P. Hänggi, Phys. Rev. Lett. **88**, 190601 (2002).
[4] B. Cleuren and C. Van den Broeck, Phys. Rev. E **67**, 055101(R) (2003).
[5] A. Sarracino, F. Cecconi, A. Puglisi, and A. Vulpiani, Phys. Rev. Lett. **117**, 174501 (2016).
[6] J. Cividini, D. Mukamel, and H. A. Posch, J. Phys. A: Math. Theor. **51**, 085001 (2018).
[7] J. Wang, G. Casati, and G. Benenti, Phys. Rev. Lett. **124**, 110607 (2020).
[8] K. Lindenberg, J. M. Sancho, A. M. Lacasta, and I. M. Sokolov, Phys. Rev. Lett. **98**, 020602 (2007).
[9] W. L. Reenbohn, S. Saikia, R. Roy, and M. C. Mahato, Pramana - J. Phys. **71**, 297 (2008).
[10] S. Saikia and M. C. Mahato, Phys. Rev. E **80**, 062102 (2009).
[11] S. Saikia and M. C. Mahato, J. Phys.: Condens. Matter **21**, 175409 (2009).
[12] W. Guo, L.-C. Du, and D.-C. Mei, J. Stat. Mech. (2014) P04025.
[13] M. S. Simon, J. M. Sancho, and K. Lindenberg, Eur. Phys. J. B **87**, 201 (2014).
[14] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).