Eikonal formulation of large dynamical random matrix models

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The standard approach to dynamical random matrix models relies on the description of trajectories of eigenvalues. Using the analogy from optics, based on the duality between the Fermat principle (rays) and the Huygens principle (wavefronts), we formulate the Hamilton-Jacobi dynamics for large random matrix models. The resulting equations describe a broad class of random matrix models in a unified way, including normal (Hermitian or unitary) as well as strictly non-normal dynamics. This formalism applied to Brownian bridge dynamics allows one to calculate the asymptotics of the Harish-Chandra-Itzykson-Zuber integrals.

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I. INTRODUCTION

In this work, we study matrices undergoing additive or multiplicative random dynamics [1]. Besides their purely theoretical appeal, such models have proven useful in various problems spanning from quantum mechanics to machine learning. Typical examples are disordered mesoscopic wires where the dynamical time t is identified with the wires' length. In quantum systems with broken time reversal symmetry, t is in turn related to the external magnetic field [2]. In two-dimensional Quantum Chromodynamics, the time parameter corresponds to the area of the loop configuration [3], whereas in quantum gravity, it is interpreted as the size of the string [4]. Remarkably, the variable in which the dynamics takes place is seldom related to the physical time. Other applications share such exotic interpretations as the timelike variable is the depth of the neural network [5] or the strength of noise in signal-plus-noise statistical models [6].

The aim of this work is to describe the classical mechanics perspective on dynamical matrices as the third natural interpretation besides the previously studied hydrodynamical and optical approaches. The hydrodynamical picture is based on the standard approach to dynamical random matrix models due to Dyson and relies on tracing the trajectories of individual eigenvalues via stochastic differential equations of the Langevin type or by the corresponding Smoluchowski-Fokker-Planck (SFP) equations for joint eigenvalue probability distribution functions. In the limit of a large dimension of matrices, $N \to \infty$, the dynamics of random matrices simplifies considerably and attains a hydrodynamical description, with parameter $1/N$ being the viscosity of the flow of the eigenvalue fluid [7,8].

Besides the hydrodynamics, an optical analogy in dynamical matrices was likewise established. In the simplest case of a Gaussian unitary ensemble (GUE), the resolvent evolves according to the complex Burgers equation. It can be easily solved by the method of complex characteristics, in analogy to the real characteristics method applied to the Euler equation. This immediately brings connotations with geometric optics, where rays of light play the role of characteristics. Moreover, fold and cusp diffraction catastrophes in optics [9] seem to have their counterparts in random matrix models, in terms of Airy [10] and Pearcey [7] microscopic universalities. We summarize these findings in Table I.

As a motivation to the present work, we utilize the long-standing ray and wavefront duality in optics between Fermat and Huygens (also known as the geometric vs wave optics), which essentially mirrors the formalisms of Hamilton and Hamilton-Jacobi [11]. In the context of dynamical matrices, our aim is to bring into the forefront the mechanics perspective with special emphasis on the Hamilton-Jacobi (HJ) formalism.

Before we present the full formalism for the HJ equation in random matrix theory, let us explain the main concept on the basis of a matricial additive Brownian walk. We consider the process $Y_t = Y_{t-1} + X_t$, where X_t are independent large ($N \to \infty$) N by N matrices drawn from the GUE. When interested only in the average spectral density, one studies the evolution of the averaged resolvent $G(z,t) = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - Y_t} \rangle$, with the large-$N$ limit taken implicitly. The averaging $\langle \cdot \rangle$ is taken with respect to the random process Y_t. In the above-mentioned limit, the resulting differential equation is the complex inviscid Burgers equation $\partial_t G + G \partial_z G = 0$ [12]. Using the method of complex characteristics [shown as gray dotted arrows in Fig. 1(b)], the solution is given implicitly by the Pastur formula [13] $G = G_0(z - tG)$, where G_0 is the initial resolvent. For a trivial initial condition $X_0 = 0$, $G_0(z) = 1/z$ and the Pastur formula reduces to a quadratic equation for which one of the solutions, $G_-(z,t) = \frac{1}{2t}(z - \sqrt{z^2 - 4t})$, results in the eigenvalue density given by the Wigner semicircle law. In this approach, we omit the wavefronts altogether.

However, following Huygens, the picture will be complete only when we recast the problem in the HJ form. Then the role of the principal Hamilton function is played by a potential-like function of the form $\langle \frac{1}{N} \mathrm{Tr} \ln(z - Y_t)(\bar{z} - Y_t^\dagger) \rangle$, with the

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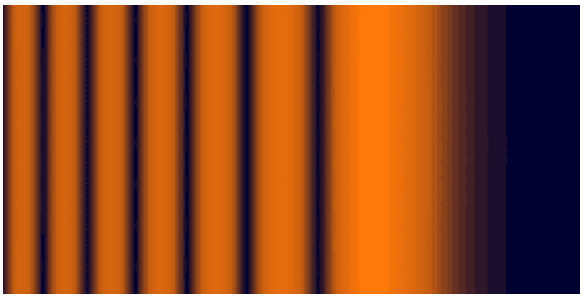
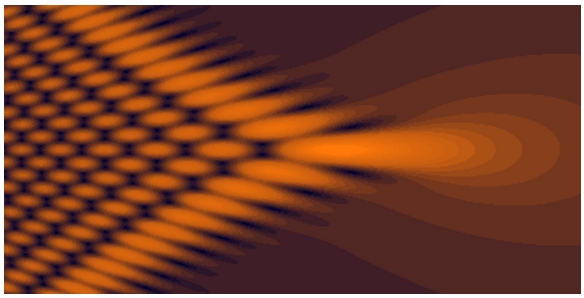
TABLE I. Relations between optics and random matrices.

Airy (fold or edge) singularity	Pearcey (cusp) singularity
	
Optics	**Random matrices**
Wavelength λ	Inverse of matrix size $1/N$
Geometric optics $\lambda = 0$	Global (or macroscopic) scaling
Wave optics $\lambda \to 0$	Local (or microscopic) scaling
Light intensity $E(x, y) \sim \frac{1}{\lambda^{\mu}}\Gamma(\frac{x}{\lambda^{\sigma_x}}, \frac{y}{\lambda^{\sigma_y}})$	Characteristic determinant $D(z, t)$
Fold singularity $\mu = 1/6, \sigma_x = 2/3, \sigma_y = 0$	Edge singularity $D(z, t) \sim N^{1/6}\mathrm{Ai}[(\frac{z}{\sqrt{t}} - 2)N^{2/3}]$
Cusp singularity $\mu = 1/4, \sigma_x = 1/2, \sigma_y = 3/4$	Pearcey singularity $D(z, t) \sim N^{1/4}P(\frac{t-1}{2}N^{1/2}, zN^{3/4})$

large-N limit taken implicitly. Its equipotential surfaces are precisely the omitted wavefronts, shown as black solid lines and surfaces in Fig. 1(b).

Since Y_t is Hermitian, the proposed principal Hamilton function is decomposed as a sum of holomorphic $\phi(z, t) = \langle \frac{1}{N}\mathrm{Tr}\ln(z - Y_t)\rangle$ and its (trivial) antiholomorphic copy $\bar{\phi} = \phi(\bar{z}, t)$, which we omit in what follows. Moreover, the function ϕ is basically a logarithm of the characteristic determinant since $\langle \mathrm{Tr}\ln(z - Y_t)\rangle = \langle \ln\det(z - Y_t)\rangle = \ln\langle \det(z - Y_t)\rangle$, where the last equality holds *only* in the $N \to \infty$ limit. The HJ equation for the principal Hamilton function (modulo its trivial, decoupled antiholomorphic copy) reads

$$\partial_t\phi + H(p = \partial_z\phi, z, t) = 0,$$

where $H(p, z, t) = p^2/2$ is the Hamiltonian. The role of the canonical coordinate q is played by a complex variable z, while the role of the canonical momentum p is the derivative of the principal Hamilton function with respect to coordinate z, i.e., $p = \partial_z\phi$. Note that the momentum p is, by definition, the resolvent G. Surprisingly from the random matrix point of view, the HJ formalism treats the canonical pair $(q, p) \leftrightarrow (z, G)$ as completely *independent*. As we will see in the next section, GUE is the random matrix analog of a free, one-dimensional particle in classical mechanics (see Table IV).

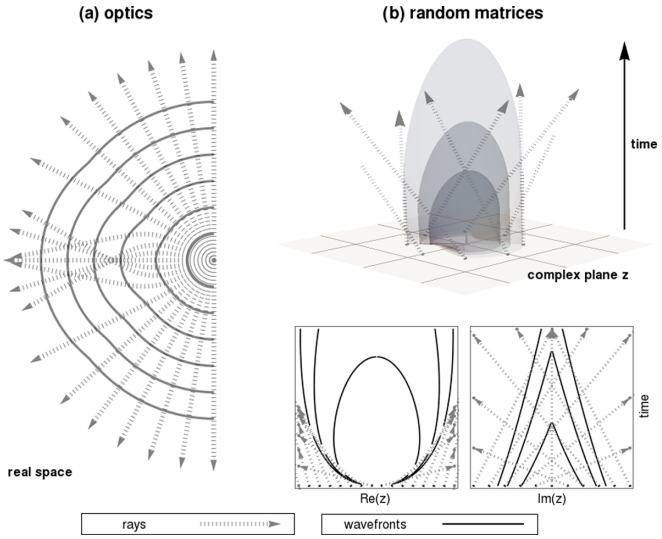


FIG. 1. The construction of the duality between wavefronts (black solid lines) and rays (dotted gray arrows) in (a) optics and (b) dynamical random matrices is highly analogous. Optical wavefronts in real space trace light-ray propagation, while matrix wavefronts $\Phi(z, t) = \text{const}$ trace complex plane propagation of the characteristics. In the latter, the insets show a highly anisotropic wavefront and ray evolution when projected along real and imaginary axes. The presented random matrix dynamics is given by the Gaussian diffusion with zero initial condition giving rise to GUE. Details are given in Appendix A.

Using the formalism of classical mechanics, we write the pair of Hamilton equations $\dot{z} = \frac{\partial H}{\partial p} = p$, $\dot{p} = -\frac{\partial H}{\partial z} = 0$, which, together with the initial conditions $z(0) = z_0$ and $p(0) = p_0$, lead to the solutions $p(t) = p_0$ and $z(t) = p_0 t + z_0$. In accordance with the previous ray-centered approach, the latter equation gives the characteristics. If initial conditions are represented by a set of N points x_i corresponding to the eigenvalues of X_0, then $p_0 = \partial_z\phi(z, t = 0)|_{z=z_0} = \frac{1}{N}\sum\frac{1}{z_0 - x_i}$ and eliminating z_0 from the equations of motion reproduces the Pastur formula $p = p_0(z - pt)$. Alternatively, one can differentiate the HJ equation with respect to z, again recovering the inviscid Burgers equation $\partial_t p + p\partial_z p = 0$.

The main result of the present work is an extension of the above duality to a broader class of dynamical random matrix models, not necessarily Hermitian or Gaussian.

II. ADDITIVE MATRIX DYNAMICS

We define a general additive matrix process Y_t by

$$R_{Y_t}(z, t) = R_{X_0}(z) + tR_X(z), \tag{1}$$

with the R-transform R_X for the increment matrix and an initial matrix X_0. In the above, the large matrix size limit was taken implicitly. Although it is advantageous to introduce this quite involved definition of the matrix process, one can think of it in simpler terms by considering the finite-dimensional case. For Gaussian increments, formulation (1) is equivalent to a limit of the additive process $Y_t = X_0 + X_1 + X_2 + X_3 + \cdots + X_n$ with independent finite-dimensional increments X_i, each with variance δt where the limit $n \to \infty$, $\delta t \to 0$ is taken with $n\delta t = t$ fixed. Beyond Gaussianity, a similar definition is also possible, although it is slightly more involved as the increments contain additional random projections on top of the X_i's [14].

A. Hermitian Hamilton-Jacobi equation

We first look at the case when the increment matrix X is Hermitian. Unless otherwise stated, we work in the limit $N \to \infty$. To derive the Hamilton-Jacobi equation, we introduce several well-known results of free probability applicable to Hermitian matrices.

Basics of free probability. The one-point spectral density $\rho(\lambda) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$ is uniquely given by its Stjelties (Cauchy) transform, also known in the physics literature as the Green's function $G(z) = \int \rho(\lambda)(z-\lambda)^{-1} d\lambda = \langle \frac{1}{N} \mathrm{Tr}(z - X)^{-1} \rangle$, which encodes all its moments. One recovers the spectral density by the Sochocki-Plemelj formula,

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im}\, G(\lambda + i\epsilon).$$

Free probability offers several operational tools to deal with the spectra of asymptotically large matrices. In particular, with the use of freeness (a counterpart of independence in noncommuting random variables), one is able to find an eigenvalue density of a sum of two matrices by knowing their separate densities.

To this end, one introduces a functional inverse of the Green's function, called Blue's function, satisfying

$$B(G(z)) = z, \quad G(B(z)) = z, \tag{2}$$

which is an intermediate step to calculate the R-transform $R(z) = B(z) - 1/z$. Remarkably, the R-transform is additive for two mutually free random variables X and Y, that is [15],

$$R_{X+Y}(z) = R_X(z) + R_Y(z).$$

This additive property explains and enables the definition of the matrix process in terms of the R-transform given in Eq. (1).

Derivation of HJ equation

First, we add $1/z$ to both sides of Eq. (1) resulting in $B(z,t) = B_{X_0}(z) + tR_X(z)$, which we then differentiate with respect to t and obtain

$$\partial_t B(z,t) = R_X(z). \tag{3}$$

Since the matrix Y_t evolves, both the Green's and Blue's functions depend on time, but the relation (2) is satisfied at *any* time,

$$B_{Y_t}(G_{Y_t}(z,t),t) = z. \tag{4}$$

From now on, we skip the subscripts $B_{Y_t} \to B$, $G_{Y_t} \to G$. Differentiating the above definition with respect to t, we get

$$\partial_t B(z,t)|_{z=G} + \left.\frac{\partial B(z,t)}{\partial z}\right|_{z=G} \partial_t G(z,t) = 0. \tag{5}$$

On the other hand, we differentiate Eq. (4) with respect to variable z $\frac{\partial B(z,t)}{\partial z}|_{z=G} \frac{\partial G(z,t)}{\partial z} = 1$ and substitute it to formula (5) to obtain $\partial_t B(z,t)|_{z=G} + (\frac{\partial G}{\partial z})^{-1} \partial_t G(z,t) = 0$. Knowing the time derivative of the Blue's function (3), we finally arrive at

$$\partial_t G(z,t) + R_X(G) \partial_z G(z,t) = 0, \tag{6}$$

which is the Voiculescu equation.

Formal solution of Voiculescu equation. We proceed to formally solve Eq. (6) by the method of characteristics. The result are two equations:

$$\partial_t z = R_X(G), \quad \partial_t G = 0,$$

which we interpret as Hamilton equations $\dot{q} = \partial_p H$, $\dot{p} = -\partial_q H$, where pair (z, G) becomes the coordinate-momentum pair (q, p). Then the Hamiltonian is specified by $\partial_G H = R_X(G)$, $\partial_z H = 0$, which gives

$$H(G,z) = \int_0^G dz R_X(z). \tag{7}$$

The lower integration limit is a convention introduced to fix a constant term in the Hamiltonian. Knowing the Hamiltonian, we write the corresponding Hamilton-Jacobi equation for the Hamilton's principal function S as $\partial_t S = -H(\partial_z S, z)$ and take the derivative with respect to z, which results in the Voiculescu equation (6) with identification $\partial_z S = G$. Hence, we identify the principal function S with the electrostatic potential,

$$\phi(z,t) = \frac{1}{N} \langle \mathrm{Tr} \ln(z - Y_t) \rangle,$$

as $\partial_z \phi = G$. As a result, the sought Hamilton-Jacobi equation for the electrostatic potential reads

$$\partial_t \phi + H(p = \partial_z \phi, z) = 0. \tag{8}$$

Comments. Although Hamiltonians expressed in the most general form via the R-transform (7) are functions of momenta only, the HJ equation (8) holds beyond such cases. Perhaps the simplest instance is the Hamiltonian

$$H_{\mathrm{OU}} = \tfrac{1}{2} p^2 + a(1 - zp), \tag{9}$$

where the coupling between coordinate z and momentum p reproduces the Ornstein-Uhlenbeck process with a drift proportional to a [16].

In Table II, we summarize examples of Hamiltonians considered throughout this work, supplemented with Jacobi and Wishart processes.

B. Non-Hermitian Hamilton-Jacobi equation

We now continue to outline the HJ formalism in the case when the matrix dynamics Y_t is still additive but lacks symmetry constraints. We start off with free probability terms and definitions applicable to this scenario.

Non-Hermitian free probability. Eigenvalues of diagonalizable (not necessarily normal) random matrices form a subset

TABLE II. Hamiltonians for HJ equation (8) describing Hermitian additive matrix dynamics. Besides the bridge scenario discussed in Sec. IV where phase space is comprised of two complex pairs (z, p), (α, p_α) and the Hamiltonian is nonstationary, all remaining examples are defined for a single complex pair (z, p) and do not depend explicitly on time t.

Dynamics type	Hamiltonian	Details
GUE	$p^2/2$	
R-based	$\int_0^p R_X(z)dz$	A general R-transform
Ornstein-Uhlenbeck	$\frac{1}{2}p^2 + a(1 - zp)$	a is the drift parameter
Bridge	$\frac{1}{2}p^2 + \frac{1}{1-t}[1 - zp - (\alpha - 1)p_\alpha]$	Details in Sec. IV
Wishart	$(1 - r)p + rzp^2$	r is the rectangularity parameter [56]
Jacobi	$\lambda\theta z(1 - z)p^2 + p[\theta(1 - \lambda) - (1 - 2\lambda\theta)z]$	θ, λ are defined in [57]

of the complex plane. In order to work with such objects, we use the following representation of the Dirac δ [17–20]:

$$\delta^{(2)}(z) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon^2}{(z\bar{z} + \epsilon^2)^2}.$$

In the spirit of the electrostatic analogy, one introduces the potential

$$\Phi(z, w, t) = \left\langle \frac{1}{N}\mathrm{Tr}\ln[(z - Y_t)(\bar{z} - Y_t^\dagger) + |w|^2]\right\rangle. \tag{10}$$

The limiting spectral density can be recovered from the Poisson law,

$$\rho(z) = \lim_{w\to 0}\frac{1}{\pi}\partial_{z\bar{z}}\Phi(z, w, t).$$

Using known identity $\mathrm{Tr}\ln = \ln\det$, the determinant in (10) can be rewritten in block form,

$$\Phi(z, w, t) = \left\langle \frac{1}{N}\ln\det(\mathbb{Q} - \mathbb{X})\right\rangle,$$

where

$$\mathbb{Q} = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z}\end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger\end{pmatrix}.$$

$\mathbb{Q}$ is a 2×2 matrix representation of the real quaternion. In direct analogy to Hermitian matrices, one constructs the Green's function of a quaternion argument, which is now a 2×2 matrix,

$$\mathbb{G}(\mathbb{Q}) = \mathbb{D}_\mathbb{Q}\Phi = \begin{pmatrix} \partial_z\Phi & \partial_w\Phi \\ -\partial_{\bar{w}}\Phi & \partial_{\bar{z}}\Phi\end{pmatrix}, \tag{11}$$

with a quaternionic derivative $(\mathbb{D}_\mathbb{Q})_{ij} \equiv \frac{\partial}{\partial\mathbb{Q}_{ji}}$ $(i, j = 1, 2)$. In direct analogy, the inverse of the quaternionic Green's function is the non-Hermitian analog of the Blue's function $\mathbb{B}(\mathbb{G}(\mathbb{Q})) = \mathbb{Q} = \mathbb{G}(\mathbb{B}(\mathbb{Q}))$. This directly leads to the quaternionic $\mathbb{R}$-transform $\mathbb{R}(\mathbb{Q}) = \mathbb{B}(\mathbb{Q}) - \mathbb{Q}^{-1}$. As previously shown, it is additive under the addition of free non-Hermitian matrices [21],

$$\mathbb{R}_{X+Y}(\mathbb{Q}) = \mathbb{R}_X(\mathbb{Q}) + \mathbb{R}_Y(\mathbb{Q}). \tag{12}$$

A generalized resolvent $\mathbb{G}$ was previously proposed to solve non-Hermitian problems in the past [18–21], although without any link to the underlying Hamilton dynamics.

1. Deriving the HJ equation

We use indices $\alpha, \beta = 1, 2$ to specify matrix elements. Non-Hermitian additive dynamics is defined by Eq. (1) with straightforward substitutions $R \to \mathbb{R}, z \to \mathbb{Q}$ motivated by additive property (12). We add $\mathbb{Q}^{-1}$ to the resulting equation so that

$$\mathbb{B}_{\alpha\beta}(\mathbb{Q}, t) = \mathbb{B}^0_{\alpha\beta}(\mathbb{Q}) + t\mathbb{R}^X_{\alpha\beta}(\mathbb{Q}),$$

where $\mathbb{B} = \mathbb{B}^{Y_t}$ is the Blue's function for the matrix Y_t while $\mathbb{R}^X$ is the $\mathbb{R}$-transform of the increment matrix X with standard variance. We again calculate the time derivative,

$$\partial_t\mathbb{B}_{\alpha\beta}(\mathbb{Q}, t) = \mathbb{R}^X_{\alpha\beta}(\mathbb{Q}). \tag{13}$$

It is convenient to treat quaternionic objects not as 2×2 matrices but as column vectors with four components in, say, lexicographic order: $\mathbb{Q}_\alpha = (\mathbb{Q}_{11}, \mathbb{Q}_{12}, \mathbb{Q}_{21}, \mathbb{Q}_{22})^T$. Now, $\alpha = 1, 2, 3, 4$. Such vector representation makes derivation less convoluted. The quaternionic Blue's $\mathbb{B}_\alpha(\mathbb{Q}, t)$ and Green's $\mathbb{G}_\alpha(\mathbb{Q}, t)$ functions are, by definition, related as

$$\mathbb{B}_\alpha(\mathbb{G}(\mathbb{Q}, t), t) = \mathbb{Q}_\alpha.$$

As previously shown, we differentiate the above definition with respect to time to get

$$\partial_t\mathbb{B}_\alpha(\mathbb{Q}, t)|_{\mathbb{Q}=\mathbb{G}} + \sum_{\beta=1}^4\left.\frac{\partial\mathbb{B}_\alpha(\mathbb{Q}, t)}{\partial\mathbb{Q}_\beta}\right|_{\mathbb{Q}=\mathbb{G}}\frac{\partial\mathbb{G}_\beta(\mathbb{Q}, t)}{\partial t} = 0, \tag{14}$$

and with respect to the quaternionic element $\mathbb{Q}_\beta$,

$$\sum_{\beta=1}^4\left.\frac{\partial\mathbb{B}_\alpha(\mathbb{Q}, t)}{\partial\mathbb{Q}_\beta}\right|_{\mathbb{Q}=\mathbb{G}}\frac{\partial\mathbb{G}_\beta(\mathbb{Q}, t)}{\partial\mathbb{Q}_\gamma} = \delta_{\alpha\gamma}.$$

We see that the above matrices of derivatives are mutual inverses. By multiplying (14) on the left by $\partial\mathbb{G}_\gamma/\partial\mathbb{Q}_\alpha$, summing over repeated indices, and substituting the expression (13), we are led to

$$\partial_t\mathbb{G}_\alpha + \sum_{\beta=1}^4\mathbb{R}^X_\beta(\mathbb{G})\frac{\partial\mathbb{G}_\alpha}{\partial\mathbb{Q}_\beta} = 0.$$

Finally, we restore the quaternionic structure to arrive at a generalized Voiculescu-type equation,

$$\partial_t\mathbb{G}_{\alpha\beta} + \sum_{\mu,\nu=1}^2\mathbb{R}^X_{\mu\nu}(\mathbb{G})\frac{\partial\mathbb{G}_{\alpha\beta}}{\partial\mathbb{Q}_{\mu\nu}} = 0, \tag{15}$$

which is a direct generalization of Voiculescu equation (6).

Formal solution. As previously shown, the obtained equation is amenable to solving by the method of characteristics which result in first-order Ordinary Differential Equations:

$$\dot{\mathbb{Q}}_{\mu\nu} = \mathbb{R}^X_{\mu\nu}(\mathbb{G}), \quad \dot{\mathbb{G}}_{\mu\nu} = 0,$$

where $\mu, \nu = 1, 2$. The above equations are again in Hamilton form, where the pair $(\mathbb{Q}, \mathbb{G})$ is identified with a set of coordinate-momentum pairs $(\mathbb{Q}, \mathbb{P}^T)$. Transposition is indispensable to align the Green's function (11) as a derivative of the potential with respect to quaternion $\mathbb{Q}$. As a consequence, the first equation reads $\dot{\mathbb{Q}}_{\mu\nu} = \mathbb{R}^X_{\nu\mu}(\mathbb{P})$ following from $\mathbb{R}(\mathbb{P}^T) = \mathbb{R}(\mathbb{P})^T$. Hamiltonian $H(\mathbb{P}, \mathbb{Q})$ is found from equations $\partial_{\mathbb{P}_{\nu\mu}} H = \mathbb{R}^X_{\mu\nu}(\mathbb{P})$ and $\partial_{\mathbb{Q}_{\mu\nu}} H = 0$. We integrate out each one separately so that the result is a sum of integrals,

$$H(\mathbb{P}, \mathbb{Q}) = \int_0^{\mathbb{P}} d\mathbb{Q}_{11}\mathbb{R}^X_{11}(\mathbb{Q}) + \int_0^{\mathbb{P}} d\mathbb{Q}_{12}\mathbb{R}^X_{21}(\mathbb{Q}) + \int_0^{\mathbb{P}} d\mathbb{Q}_{21}\mathbb{R}^X_{12}(\mathbb{Q}) + \int_0^{\mathbb{P}} d\mathbb{Q}_{22}\mathbb{R}^X_{22}(\mathbb{Q}), \quad (16)$$

where the lower limit is again introduced to fix an arbitrary additive constant in the Hamiltonian. Instead of the multiple terms present in Eq. (16), in what follows we introduce a succinct notation,

$$H(\mathbb{P}, \mathbb{Q}) = \int_0^{\mathbb{P}} \mathrm{Tr}[\mathbb{R}^X(\mathbb{Q})d\mathbb{Q}]. \quad (17)$$

The newfound Hamiltonian admits the following Hamilton-Jacobi equation for Hamilton's principal function S:

$$\partial_t S + H[\mathbb{P} = (\mathbb{D}_{\mathbb{Q}} S)^T, \mathbb{Q}] = 0.$$

Lastly, we identify the principal function with a known matrix object. To this end, we take the derivative $\mathbb{D}_{\mathbb{Q}}$ so that $(\mathbb{D}_{\mathbb{Q}})_{kl} H(\mathbb{P}, \mathbb{Q}) = \sum_{ij} \mathbb{R}^X(\mathbb{P})_{ij} (\mathbb{D}_{\mathbb{Q}})_{kl} \mathbb{P}_{ji}$ and

$$\partial_t (\mathbb{D}_{\mathbb{Q}})_{kl} S + \sum_{ij} \mathbb{R}^X(\mathbb{D}_{\mathbb{Q}} S^T)_{ij} (\mathbb{D}_{\mathbb{Q}})_{ij} (\mathbb{D}_{\mathbb{Q}})_{kl} S = 0.$$

We again use $\mathbb{R}(\mathbb{P}^T) = \mathbb{R}(\mathbb{P})^T$ so that

$$\partial_t (\mathbb{D}_{\mathbb{Q}})_{kl} S + \sum_{ij} \mathbb{R}^X(\mathbb{D}_{\mathbb{Q}} S)_{ji} (\mathbb{D}_{\mathbb{Q}})_{ij} (\mathbb{D}_{\mathbb{Q}})_{kl} S = 0.$$

Since $(\mathbb{D}_{\mathbb{Q}})_{ij} = \partial_{\mathbb{Q}_{ji}}$, we recreate the Voiculescu-type equation (15) when the principal function S is identified with electrostatic potential Φ (10) so that $\mathbb{D}_{\mathbb{Q}} S = \mathbb{G}$.

As a result, we have derived the main result of this paper, a Non-Hermitian Hamilton-Jacobi equation,

$$\partial_t \Phi + H[\mathbb{P} = (\mathbb{D}_{\mathbb{Q}} \Phi)^T, \mathbb{Q}] = 0 \quad (18)$$

that matches Eq. (15). The Hamilton equations are readily solved as $\mathbb{P} = \mathbb{P}_0[\mathbb{Q} - t\mathbb{R}(\mathbb{P})^T]$ with initial condition $\mathbb{P}_0 = \mathbb{D}_{\mathbb{Q}} \Phi^T$. Coordinates $\mathbb{Q}, \mathbb{P}$ comprise a set of action-angle variables casting the problem as fully integrable and stable with respect to small perturbations according to the seminal Kolmogorov-Arnold-Moser theorem [11].

2. Examples

In this section, we consider a few examples of descriptions of non-Hermitian additive matrix dynamics in terms of Hamilton-Jacobi equations. We provide a summary in Table III, where we present examples of the discussed Hamiltonians for non-Hermitian HJ equation (18).

TABLE III. Hamiltonians for HJ equation (18) describing non-Hermitian additive matrix dynamics.

Dynamics type	Hamiltonian	Details
Ginibre	$-\lvert p_w\rvert^2$	
R-based	$\int_0^{\mathbb{P}} \mathrm{Tr}[\mathbb{R}^X(\mathbb{Q})\mathrm{d}\mathbb{Q}]$	A general non-Hermitian $\mathbb{R}$-transform
Elliptic	$\frac{\tau}{2}(p_z^2 + p_{\bar{z}}^2) - \lvert p_w\rvert^2$	τ is the interpolation parameter
R-diagonal	$\int_0^{-\lvert p_w\rvert^2} A(x)dx$	A is a cumulant generating function

a. Reduction to Hermitian case. The non-Hermitian formalism presented in this section is not disjoint from the Hermitian dynamics considered in Sec. II A. Now we show that in fact, it is contained within a non-Hermitian framework, at least in part. In the case of Hermitian matrices, the quaternionic embedding is redundant and one can set w to zero from the very beginning, projecting the quaternion to a complex number. In this way, both the potential $\Phi \to \phi + \bar{\phi}$ and the quaternion $\mathbb{R} = \mathrm{diag}[R_X(z), \overline{R_X(z)}]$ decouple into holomorphic and antiholomorphic copy. The Hamiltonian (17) likewise decouples and reads

$$H = \int_0^{p} R_X(z)dz + \int_0^{\bar{p}} \overline{R_X(z)}d\bar{z},$$

so the dynamics of each part separately is equivalent; the holomorphic part of the Hamiltonian exactly recreates Eq. (7) found in the Hermitian scenario.

b. Non-normal increment matrix X. The crucial difference between Hermitian and non-Hermitian models comes from the fact that the separability into holomorphic and antiholomporhic parts breaks down, since the support of the spectra represents the nonholomorphic region. This was known in the literature [22] and the variable $|w|^2 \equiv \epsilon$ was kept nonzero before the large-N limit was taken. In such a case, the spectral density follows from the two-dimensional (2D) Gauss law $\rho = \frac{1}{\pi}\partial_{\bar{z}} g$, where $g = \partial_z \Phi$ plays the role of the electric field. Considering ϵ only as an infinitesimal regularizer is too reductive, as it is responsible for the crucial dynamics of eigenvectors, which, contrary to the Hermitian case, do not decouple from the eigenvalues during the evolution. This is perhaps best visible when we diagonalize Y_t in terms of left and right eigenvectors $Y_t = \sum_i |R_i\rangle\lambda_i\langle L_i| = R\Lambda L^\dagger$. Then the potential Φ explicitly reads

$$\Phi(z, w, t) = \frac{1}{N}\left\langle \ln\det \begin{pmatrix} z - \Lambda & -\bar{w}L^\dagger L \\ wR^\dagger R & \bar{z} - \Lambda^\dagger \end{pmatrix} \right\rangle.$$

Since the N by N blocks in the determinant do not commute, eigenvalues are correlated with eigenvectors. In the large-N limit, the off-diagonal momenta in $\mathbb{G}$ are responsible for the diagonal part of the Chalker-Mehlig correlator [23–25],

$$O(z, t) = \frac{1}{N^2}\left\langle \sum_i O_{ii}\delta^{(2)}(z - \lambda_i) \right\rangle = -\frac{1}{\pi}|p_w|^2_{w=0},$$

where O_{ii} is the diagonal part of the overlap matrix [25] $O_{ij} = \langle L_i|L_j\rangle\langle R_j|R_i\rangle$ (see [26]). This quantity is also related to the Petermann factor [27] and the eigenvalue condition number in the stability theory [28]. One can therefore see that during the evolution parameters, z and w need to be treated *on an equal footing*.

It is useful to illustrate this democracy of dynamics of eigenvalues and eigenvectors in the case of the elliptic ensemble [29], corresponding to the matricial measure $P(X) \sim \exp(-\frac{N}{1-\tau^2}\{\mathrm{Tr}XX^\dagger - \frac{\tau}{2}\mathrm{Tr}[X^2 + (X^\dagger)^2]\})$. Parameter τ allows for continuous interpolation between GUE ($\tau = 1$) and the Ginibre ensemble ($\tau = 0$). The generalized $\mathbb{R}$-transform for the elliptic ensemble reads [23,24]

$$\mathbb{R}^X(\mathbb{Q}) = \begin{pmatrix} \tau z & -\bar{w} \\ w & \tau\bar{z} \end{pmatrix}.$$

The application of the HJ formula (17) leads to

$$\begin{aligned} H_{\text{elliptic}} &= \int_0^{\mathbb{P}} (\tau z dz + \tau\bar{z}d\bar{z} - wd\bar{w} - \bar{w}dw) \\ &= \frac{\tau}{2}\left(p_z^2 + p_{\bar{z}}^2\right) - |p_w|^2, \end{aligned} \tag{19}$$

with a pair of momenta $p_z = \mathbb{G}_{11}$, $p_w = \mathbb{G}_{12}$ comprising the quaternionic resolvent $\mathbb{G}$. Indeed, setting $\tau = 1$ reproduces the GUE case as the "eigenvector part" vanishes in the large-N limit. Although eigenvector and eigenvalue parts in the Hamiltonian are decoupled, they are coupled by the initial condition. The presence of the τ part is actually spoiling the rotational symmetry of the Ginibre ensemble and reproduces the ellipse, as easily seen from solving the corresponding HJ equations. The signs in front of the "kinetic" terms are also important. In the Hermitian limit $\tau = 1$, the positive kinetic term in the Hamiltonian is responsible for the Airy oscillations at the wavefront. When Hermiticity is broken, the term $-|p_w|^2$ shapes the critical behavior at the edge and is the source of smooth decay of the Erfc type [30,31].

Along the solution of the HJ equation, H, $\frac{\tau}{2}p_z^2$, $\frac{\tau}{2}p_{\bar{z}}^2$ and $|p_w|^2$ are constants of motion since the corresponding Poisson brackets vanish. We stress here the crucial dynamics of eigenvectors, which is a generic feature of non-normal random matrix models as argued recently in [32,33].

In the Ginibre case $\tau = 0$, the entire evolution of eigenvalues and eigenvectors is solely driven by the w dynamics, and in this simplest non-normal case, by the Chalker-Mehlig eigenvector correlator. Explicitly, the HJ equations read $\dot{p}_w = 0$, $\dot{w} = -\overline{p}_w$ and form equations along characteristic lines reproducing the recent result [30].

c. R-diagonal matrices. If we consider a random complex number, its probability distribution function (PDF) can take, in general, a complicated form. One can consider simplified PDF's, which are effectively one dimensional. One of the examples is the isotropic random variables, defined as follows. Any complex number can be written in a polar form $z = re^{i\varphi}$. A complex random variable is said to be isotropic if its PDF depends only on r. In such a case, the PDF for a phase φ is uniform on a unit circle, yet r and φ are independent.

In the analogy to isotropic complex random variables, one considers a class of non-Hermitian random matrices which we call isotropic. Any matrix X possesses a polar decomposition $X = HU$, where H is Hermitian positive definite and U is unitary. U plays a role of the "phase" of a matrix; therefore, if X was to be isotropic, U has to be distributed uniformly on the $U(N)$ group. Such a probability distribution function exists and is called the Haar measure. Moreover, U and H have to be mutually free. In the literature, such matrices belong to the bi-unitary ensembles because the probability density for their elements is invariant under multiplication by two independent unitary matrices from both sides. Mathematically, $P(X) = P(UXV)$ for $U, V \in U(N)$.

In this case, the spectral properties of the isotropic matrix are completely determined by the spectral distribution of the "squared modulus" $XX^\dagger$. The precise relation is given by the Haagerup-Larsen theorem [34] (or the single-ring theorem [35–37]). Recently, this theorem was extended to also describe the eigenvector correlation function [24].

The only nonvanishing cumulants are of the form $\alpha_k = \frac{1}{N}\mathrm{Tr}(XX^\dagger)^k$. Let us define a generating function for all cumulant of such matrices,

$$A(x) := \sum_{k=1}^{\infty} \alpha_k z^{k-1},$$

which is also known as a generating sequence. The quaternionic $\mathbb{R}$-transform of such matrices assumes a remarkably simple form [38],

$$\mathbb{R}^X(\mathbb{Q}) = A(-|w|^2)\begin{pmatrix} 0 & -\bar{w} \\ w & 0 \end{pmatrix}.$$

By direct substitution to (17), we calculate the R-diagonal Hamiltonian as

$$H_{\text{R-diag}} = \int_0^{-|p_w|^2} A(x)dx. \tag{20}$$

III. MULTIPLICATIVE MATRIX DYNAMICS

The HJ equation also shows up for multiplicative matrix dynamics of the form

$$Z_t = \prod_{j=1}^{M} \exp\sqrt{\delta t}X_j,$$

where X_j are independent random matrices. A continuous version of such random walk is defined in the limit $\sqrt{\delta t} \to 0$, $M \to \infty$, $M\delta t = t$ fixed. In general, symmetries of X_j induce two natural classes of such dynamics:

(1) $X_j = iH_j$, with H_j Hermitian. Z_t is unitary, its eigenvalues lie on the unit circle.

(2) X_j is non-Hermitian. Z_t has complex eigenvalues.

Below we provide examples for each type of dynamics.

A. Hermitian multiplicative dynamics

Let H_j be a Gaussian Hermitian matrix. Since unitary matrices are normal, eigenvectors and eigenvalues decouple. As eigenvalues of P_t lie on the unit circle, it is convenient to investigate their phases $\lambda_i(t) = \exp i\theta_i(t)$ and consider a

potential which respects the 2π periodicity of the phase,

$$\phi(\theta,t)=\frac{1}{N}\left\langle\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}}\ln[\theta-\theta_i(t)+2k\pi]\right\rangle.$$

In this case, the evolution resembles an additive case, modulo that the principal Hamilton function has to take into account the periodicity of the angular variable. The conjugate momentum $\partial_\theta\phi\equiv J$ is obtained by noticing the series expansion of the cotangent,

$$J(\theta)=\frac{1}{2}\int_{-\pi}^{\pi}\cot\frac{(\theta-\varphi)}{2}\rho(\varphi)d\varphi.$$

The Burgers equation reads $\partial_t J+J\partial_\theta J=0$ [39]. Equivalently, $\phi(\theta,t)$ evolves according to the HJ equation with the Hamiltonian $H=\frac{J^2}{2}$. This example, where the unitary evolution is represented by the canonical pair (angle θ, angular momentum J), is a free rotator.

The same problem can be formulated in the $z=e^{i\theta}$ variable[40], where the principal Hamilton function is given again by the log of the characteristic determinant, but the resulting Hamiltonian is less trivial and reads

$$H=-\tfrac{1}{2}z^2p^2+\tfrac{1}{2}zp. \tag{21}$$

Hermitian multiplicative dynamics is also investigated in pure mathematics [41].

B. Non-Hermitian multiplicative evolution

The non-normal evolution is highly nontrivial as eigenvectors enter nontrivially into the evolution process. Like its unitary analog, such evolution also develops a structural phase transition manifested by the change of topology in the support of complex eigenvalues [42–44]. This topological phase transition does not depend on the type of X_j and appears in both Hermitian (GUE) and non-Hermitian (Ginibre) cases. Although the shape of the boundary was explicitly calculated for the Ginibre case in [42,43], understanding of the spectral density was beyond the reach of mathematical methods available at that time. Only very recently, explicit spectral formulas were calculated by [45,46], using the formalism of the partial differential equations of the HJ type. Somewhat conservatively, the authors concentrated on the spectral evolution, but their Hamiltonian, when rephrased in our language of (z,w) variables, explicitly reads

$$H=\frac{r}{2}p_r\left(1+\frac{|z|^2-r^2}{2r}p_r-zp-\bar{z}\bar{p}\right),$$

where $r=|w|$ is the radial coordinate and p_r its conjugate momentum. Clearly, the dynamics is driven primarily by the w evolution (eigenvectors), coupled nontrivially to the z evolution (eigenvalues).

Interestingly, the HJ equation can be applied to the singular-value problem of this non-Hermitian evolution with X_j drawn from the Ginibre ensemble, where the spectra are real and decoupled from the eigenvectors, with the result $H_{Z_tZ_t^\dagger}=z^2p^2-zp$, i.e., identical to the Hamiltonian for the unitary diffusion (21), modulo factor $-1/2$. The corresponding HJ equations for both ensembles are related by replacing time t in unitary diffusion by $t\to -t/2$ for singular-value evolution, pointing at some *a priori* unexpected dualities between these two models. Such model also has practical applications, in particular in the study of trainability of residual neural networks [47].

IV. ASYMPTOTICS OF HARISH-CHANDRA-ITZYKSON-ZUBER INTEGRAL

Finally, the presented formalism offers an appealing way to study the asymptotics of the celebrated Harish-Chandra-Itzykson-Zuber (HCIZ) [48,49] integral,

$$I_{\mathrm{HCIZ}}=\int dU e^{\frac{\beta}{2}N\mathrm{Tr}UAU^\dagger B},$$

for fixed matrices A,B and parameter β encoding whether the integral is taken over unitary $\beta=2$ or orthogonal $\beta=1$ matrices.

Following [50], the main asymptotic contribution in the $N\to\infty$ limit reads

$$I_{\mathrm{HCIZ}}\sim e^{-\frac{\beta}{2}N^2S},$$

where the Euler-type hydrodynamic action reads

$$S=\frac{1}{2}\int_0^1 dt\int dx\rho\left[\mu^2+\frac{\pi^2}{3}\rho^2\right], \tag{22}$$

with "fluid" density ρ and momentum profile μ. Action is evaluated on the fluid trajectory such that the initial $\rho(x,t=0)=\rho_A(x)$ and final $\rho(x,t=1)=\rho_B(x)$ densities are specified by matrices A and B, respectively. The proper density is found by solving the following Euler equations:

$$\partial_t\mu+\mu\partial_x\mu=\frac{\pi^2}{2}\partial_x(\rho^2), \tag{23}$$

$$\partial_t\rho+\partial_x(\rho\mu)=0. \tag{24}$$

One possible way, pursued in [50], is to compose a complex solution $h=\mu+i\pi\rho$ for which we recreate a well-known Burgers' equation,

$$\partial_t h+h\partial_z h=0. \tag{25}$$

Its solutions should then obey boundary conditions $\mathrm{Im}\,h(t=0)=\pi\rho_A$ and $\mathrm{Im}\,h(t=1)=\pi\rho_B$. Such an approach, while elegant, was of limited use to solve either special cases [51] or as a method of the indirect generation of solutions [50].

Our approach to finding proper solutions h is slightly different and has two stages—first we look for a proper density ρ based on the solution of bridge-type matrix dynamics and then match a velocity function μ such that the Euler equations (23) and (24) close.

A. Brownian bridge matrix dynamics

Finding a proper density consists of constructing matrix dynamics Y_t starting at $Y_0=A$ and reaching matrix $Y_1=B$. We first recall basic facts for the one-dimensional stochastic process with such properties known as the *bridge*.

We start from a Brownian bridge dynamics for a single variable [52] for which the SFP equation starting at $x_0,t=0$ and ending in $x=x_f,t=t_f$ reads

$$\partial_t P(x,t)=D\partial_x[\partial_x P-2\partial_x\ln Q(x,t)P],$$

where Q is a solution to the inverse SFP equation $\partial_t Q = -D\partial_{xx}P$ with $x = x_f$ for $t = t_f$ and reads $Q(x,t) = [4\pi D(t_f - t)]^{-1/2} e^{-\frac{(x_f - x)^2}{4D(t_f - t)}}$. Therefore, the SFP equation for the Brownian bridge reads

$$\partial_t P = D\partial_{xx}P - \partial_x\left(\frac{x_f - x}{t_f - t}P\right).$$

Besides the usual diffusive term $\sim\partial_{xx}P$, there is a linear restoring force vanishing at $x = x_f$ and singular at $t = t_f$, ensuring that the trajectory ends up at the prescribed final point of the trajectory.

We move on to a multidimensional generalization of the bridge process. We first decompose Hermitian matrix $Y_{kl} = x_{kl} + iy_{kl}$ for $k \neq l$, $Y_{kk} = x_{kk}$ and, from the above equation, form a set of SFP formulas for each matrix element:

$$\partial_t P(x_{ii},t) = \frac{1}{2N}\partial^2_{x_{ii}}P(x_{ii},t) - \partial_{x_{ii}}\left[\frac{x^f_{ii} - x_{ii}}{t_f - t}P(x_{ii},t)\right],$$

$$\partial_t P(v_{ij},t) = \frac{1}{4N}\partial^2_{v_{ij}}P(v_{ij},t) - \partial_{v_{ij}}\left[\frac{v^f_{ij} - v_{ij}}{t_f - t}P(v_{ij},t)\right], \quad i \neq j,$$

where $v = x, y$ and v^f denote final matrix elements. Joint PDF $\mathcal{P}(Y,t) = \prod_i P(x_{ii},t)\prod_{i<j}P(x_{ij},t)P(y_{ij},t)$ satisfies a joint SFP equation $\partial_t\mathcal{P} = \mathcal{A}\mathcal{P}$, with

$$\mathcal{A} = \sum_{k=1}^{N}\left[\frac{1}{2N}\partial^2_{x_{kk}}P(x_{kk},t) - \partial_{x_{kk}}\frac{x^f_{kk} - x_{kk}}{t_f - t}\right] + \frac{1}{4N}\sum_{i<j}\left(\partial^2_{x_{ij}} + \partial^2_{y_{ij}}\right) - \sum_{i<j}\left(\partial_{x_{ij}}\frac{x^f_{ij} - x_{ij}}{t_f - t} + \partial_{y_{ij}}\frac{y^f_{ij} - y_{ij}}{t_f - t}\right).$$

B. Hamiltonian for the Brownian bridge matrix dynamics

To find the Hamiltonian for the HJ equation of the matrix bridge process, we first define a deformed characteristic polynomial,

$$\hat{U}(z,\alpha,t) = \langle\det(z - Y_t + \alpha B)\rangle.$$

Deformation consists of an arbitrary addition of auxiliary parameter α, which has a role in what follows. Using a standard approach of [16,53], we derive an exact dynamical equation for $\hat{U}$,

$$\partial_t\hat{U} = \frac{1}{t_f - t}[z\partial_z\hat{U} + (\alpha - 1)\partial_\alpha\hat{U} - N\hat{U}] - \frac{1}{2N}\partial_{zz}\hat{U}.$$

Parameter α is indispensable to close the above equation. We transform it through half of the Cole-Hopf transform $\hat{\phi} = \frac{1}{N}\ln\hat{U}$ and take the large-N limit,

$$\partial_t\hat{\phi} = -\frac{1}{2}(\partial_z\hat{\phi})^2 - \frac{1}{t_f - t}[1 - z\partial_z\hat{\phi} - (\alpha - 1)\partial_\alpha\hat{\phi}]. \quad (26)$$

By the self-averaging property of the large-N limit, $\ln\hat{U} = \ln\langle\det(\cdots)\rangle \sim \langle\ln\det(\cdots)\rangle$, and thus $\hat{\phi}$ becomes an effective potential $\hat{\phi}(z,\alpha,t) = \frac{1}{N}\langle\ln\det(z - Y_t + \alpha B)\rangle$. Equation (26) is in the Hamilton-Jacobi form, from which we read the Hamiltonian,

$$H_{\text{bridge}} = \frac{1}{2}p^2 + \frac{1}{t_f - t}[1 - zp - (\alpha - 1)p_\alpha], \quad (27)$$

where, besides the usual pair z, $p = \partial_z\hat{\phi}$, an auxiliary coordinate α and momentum $p_\alpha = \partial_\alpha\hat{\phi}$ are present.

Solving HJ equation. The Hamilton equations read

$$\dot{z} = p - \frac{z}{t_f - t}, \quad \dot{\alpha} = -\frac{\alpha - 1}{t_f - t},$$
$$\dot{p} = \frac{p}{t_f - t}, \quad \dot{p}_\alpha = \frac{p_\alpha}{t_f - t}.$$

To continue, we first set $t_f - t = e^{-\tau}$ so that $(t_f - t)\frac{d}{dt} = \frac{d}{d\tau}$ and $t = 0$ corresponds to $\tau = -\ln t_f$, while $t = t_f$ is transformed to $\tau \to \infty$. The Hamilton equations are then

$$\dot{z} = e^{-\tau}p - z, \quad \dot{\alpha} = 1 - \alpha, \quad \dot{p} = p, \quad \dot{p}_\alpha = p_\alpha,$$

with the overdot now denoting $d/d\tau$. In the newly introduced time variable, the latter two equations for momenta are readily solved,

$$p = t_f e^\tau p_0 = \frac{t_f}{t_f - t}p_0, \quad p_\alpha = t_f e^\tau p_{\alpha,0} = \frac{t_f}{t_f - t}p_{\alpha,0},$$

where $p_0 = p(\tau = -\ln t_f) = p(t = 0)$, $p_{\alpha,0} = p_\alpha(\tau = -\ln t_f) = p_\alpha(t = 0)$. We plug the above solutions into the remaining Hamilton equations for z, α,

$$\frac{dz}{t_f p_0 - z} = d\tau, \quad \frac{d\alpha}{1 - \alpha} = d\tau, \quad (28)$$

and find the remaining solutions,

$$z = p_0(t_f - e^{-\tau}) + \frac{z_0}{t_f}e^{-\tau} = p_0 t + z_0\frac{t_f - t}{t_f},$$
$$\alpha = 1 - \frac{1}{t_f}e^{-\tau} + \frac{\alpha_0}{t_f}e^{-\tau} = \frac{t}{t_f} + \alpha_0\frac{t_f - t}{t_f}.$$

Next we introduce an initial condition $p_0(z_0,\alpha_0) = [\partial_z\hat{\phi}]_{\alpha=\alpha_0,z=z_0} = \frac{1}{N}\text{Tr}\frac{1}{z_0 - A + \alpha_0 B}$, which couples together α_0, p_0, and z_0. We invert $p_0 = p(t_f - t)/t_f$ and calculate $\alpha_0 = 1 + (\alpha - 1)t_f/(t_f - t)$, $z_0 = zt_f/(t_f - t) - pt$ so that the solution is given by

$$p = t_f e^\tau p_0\left[z_0 = \frac{zt_f}{t_f - t} - pt, \alpha_0 = 1 + \frac{(\alpha - 1)t_f}{t_f - t}\right],$$

or, with explicitly plugging the initial condition,

$$p = \frac{1}{N}\text{Tr}\frac{1}{z - p\frac{t(t_f - t)}{t_f} - \frac{t_f - t}{t_f}A + \left(\alpha - \frac{t}{t_f}\right)B}.$$

In what follows, we set $t_f = 1, \alpha = 0$ and reintroduce Green's function $G = p$. Define the bridge function $G_{\text{Br}}(z,t) = \frac{1}{N}\text{Tr}\frac{1}{z - (1-t)A - tB}$ and the resolvent is implicitly given by

$$G = G_{\text{Br}}[z - t(1-t)G, t]. \quad (29)$$

By construction, the limits $t \to 0, 1$ recreate correct boundary Green's functions $G(z, t = 0) = \frac{1}{N}\text{Tr}(z - A)^{-1}$ and $G(z, t =$

TABLE IV. Dictionary for optics, classical mechanics, and random matrices.

Optics	Classical mechanics	(Hermitian or non-Hermitian) random matrices
Real space	Real space + time	Complex or quaternionic space + time
Huygens principle	Hamilton-Jacobi equation	Hamilton-Jacobi equations (8)/(18)
Ray	Trajectory	Characteristic curve
Geodetic distance (Eikonal)	Action	Electrostatic potential (10)
Slowness vector of the wavefront	Momentum	Green's function
Refractive index	Hamiltonian	Integral of R-transform (17)

$1) = \frac{1}{N}\mathrm{Tr}(z-B)^{-1}$. At the same time, one can explicitly check that G is itself not the sought solution to Burgers' equation (25). This discrepancy can be understood by decomposing G into real and imaginary parts, $G = i\pi\rho + \mathcal{H}\rho$, related by the Hilbert transform and ultimately dependent on the single function ρ. On the other hand, the complex h *a priori* consists of two independent (i.e., not related through any transform) functions ρ, μ. We therefore assume that the densities ρ are calculated correctly, while the velocity profile needs further specification. Hence, in the second step, we plug the density found from (29) into the Euler equation (23) and calculate a matching velocity profile μ. Once both ρ and μ are identified, we evaluate the corresponding action (22) and find the asymptotics of the HCIZ integral.

An example. As a demonstration of the method, we calculate the simplest example of vanishing matrices $A = B = 0$. The boundary resolvent reads $G_{\mathrm{Br}}(z_0) = 1/z_0$ and the density is simply a semicircle law, $\rho_{\mathrm{sem}}(x,t) = \frac{1}{2\pi t(1-t)}\sqrt{4t(1-t)-x^2}$, with the appropriately rescaled size vanishing at both $t = 0, 1$. Plugging it into (23) results in

$$\partial_t \mu + \mu\partial_x \mu = -\frac{x}{4t^2(1-t)^2}, \tag{30}$$

since $\frac{\pi^2}{2}\partial_x(\rho_{\mathrm{sem}}^2) = -\frac{x}{4t^2(1-t)^2}$. We solve (30) using the method of characteristics,

$$\frac{d}{d\beta}\mu(\alpha,\beta) = -\frac{x}{4t^2(1-t)^2},$$
$$\frac{d}{d\beta}x(\alpha,\beta) = \mu, \quad \frac{d}{d\beta}t(\alpha,\beta) = 1,$$

with initial conditions $t(\alpha,0) = 1/2$, $x(\alpha,0) = \alpha$, and $v(\alpha,0) = 0$. We readily find $t = \beta + 1/2$, plug into the remaining equations, and combine them in matrix form,

$$\frac{d}{d\beta}\begin{pmatrix}\mu\\ x\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{2(1-4\beta^2)^2}\\ 1 & 0\end{pmatrix}\begin{pmatrix}\mu\\ x\end{pmatrix}.$$

Solution is found by diagonalization:

$$\mu(x,t) = \frac{2t-1}{2t(1-t)}x.$$

This example recreates the results of [51].

Although the above example only recreates the results of [51], the approach by the HJ equation is general and does not require any guesswork. It provides a principled way of studying the asymptotics of the Berezin-Karpelevich integrals where a similar hydrodynamic description was found [54] as well as non-Hermitian analogues of (generally unknown) HCIZ-type integrals.

V. SUMMARY

We have proposed to apply the Hamilton-Jacobi dualism between the Lagrange-Euler description (based on trajectories) and the Hamilton description (based on wavefronts) in the context of large-N dynamical random matrix models. As shown in Table IV, we have successfully transferred the optical analogy between the Fermat principle and the Huygens principle to the realm of large random matrices. Such a scheme offers an inspiring perspective for merging several physical concepts from classical mechanics, optics, hydrodynamics, statistical physics and even quantum mechanics with advanced mathematical methods of random matrix theory.

As a result of this approach, we derive several formulas. First, we present HJ equations (8) and (18) for general (i.e., Hermitian and non-Hermitian as well as non-Gaussian) dynamics. Second, we deliver concrete Hamiltonians for Ornstein-Uhlenbeck (9), elliptic (20), bi-unitary (20) matrix processes, and several others, as summarized in Tables II and III. Last, we enlarge the matrix HJ formalism to a matrix bridge scenario (27) enabling recreation of the asymptotics of the Harish-Chandra-Itzykson-Zuber integral.

We believe that since the most interesting phenomena in random matrix models (e.g., new classes of universalities) occur mostly at places where wavefronts change their behavior (boundaries of the spectral support, corresponding to gradient catastrophes), the formalism which focuses on such objects is indeed promising.

In particular, in future work, we plan to apply quantization of the HJ equation [55] as a fresh approach to the universality achieved directly from large-N asymptotics. Another advantage of this formulation is visible at the level of non-normal models, where the proper identification of canonical "coordinates" and "momenta" leads to complete treatment of the evolution of both eigenvalues and the eigenvectors. The resulting HJ equations represent a dimensional reduction of large-N problems. The proposed formalism allows also for rephrasing several open questions, such as the issue of large deviations in non-normal matrix models, which we plan to expose in the sequel to this work.

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APPENDIX A: FIGURE 1(B)—CHARACTERISTICS AND WAVEFRONTS FOR GUE DYNAMICS

We find the potential function ϕ by solving HJ equation (8),

$$\partial_t \phi + H(p = \partial_z \phi) = 0, \tag{A1}$$

for the simplest case of GUE dynamics, $H(p) = p^2/2$. Next, we use the obtained formulas to plot wavefronts alongside characteristics as their natural counterparts and comment on how Fig. 1(b) in the main text was obtained.

The Hamiltonian is not explicitly dependent on time so that the solution is of the form

$$\phi(z, \alpha, c, t) = W(z, \alpha, c) - \alpha t, \tag{A2}$$

where α is the first conserved quantity and c is an overall additive constant. HJ equation (8) reads

$$\alpha = \tfrac{1}{2}(\partial_z W)^2, \tag{A3}$$

so that $W = \sqrt{2\alpha} z + c$ and the full solution reads

$$\phi(z, \alpha, t, c) = \sqrt{2\alpha} z(t) - \alpha t + c. \tag{A4}$$

Canonical transformations give the transformed constant position β and "old" momentum p:

$$\beta = \partial_\alpha \phi = \frac{z}{\sqrt{2\alpha}} - t, \quad p = \partial_z \phi = \sqrt{2\alpha}.$$

Position z and momenta p are given in terms of constants of motion α, β as

$$z(t) = \sqrt{2\alpha}(\beta + t), \quad p(t) = \sqrt{2\alpha}.$$

Constants of motion are reformulated in terms of initial position $z(0) = z_0 = \beta\sqrt{2\alpha}$ and momentum $p(0) = p_0 = \sqrt{2\alpha}$ or $\alpha = p_0^2/2$, $\beta = z_0/p_0$. We plug it back to the solution (A4):

$$\phi = p_0 z_0 + c + \frac{p_0^2}{2} t.$$

In terms of initial conditions, position and momentum describe a one-dimensional free particle,

$$z(t) = z_0 + p_0 t, \quad p(t) = p_0.$$

In our case, the initial positions and momenta z_0, p_0 are related by the initial condition $\phi(z_0, t = 0) = f(z_0)$ and $p_0 = \partial_z \phi(t = 0)_{|z=z_0} = f'(z_0)$. These relations between z_0, p_0 are

$$p_0 z_0 + c = f(z_0), \quad p_0 = f'(z_0),$$

which results in the final form of the potential function,

$$\phi[z(t), t] = f(z_0) + \frac{[z(t) - z_0]^2}{2t},$$

where $z(t) = z_0 + f'(z_0)t$. It gives only the holomorphic part of the function, while the total potential reads

$$\Phi[z(t), t] = \phi + \bar{\phi} = 2\mathrm{Re} f(z_0) + \frac{1}{t}\mathrm{Re}[z(t) - z_0]^2. \tag{A5}$$

Obtaining Fig. 1(b). The figure in the main text is found by plotting wavefronts as equipotential surfaces $\Phi[z(t), t] =$ const, while complex characteristics form a family of curves given by $z(t) = z_0 + f'(z_0)t$, where z_0 is the parameter labeling the curves. Both were found for a special case, $f(z) = \ln z$.

APPENDIX B: CHARACTERISTICS AND WAVEFRONTS FOR GINIBRE DYNAMICS

In this section, we solve HJ equation,

$$\partial_t \Phi + H[\mathbb{P} = (\mathbb{D}_\mathbb{Q} \Phi)^T, \mathbb{Q}] = 0, \tag{B1}$$

for Ginibre dynamics with $H = -|p_w|^2$. We essentially follow the same steps as in Appendix A on GUE dynamics. As a result, in Fig. 2 we present both wavefronts and characteristics in a restricted quaternionic space spanned by moduli $|z|, |w| = r$ and time t.

In polar coordinates $w = re^{i\alpha}$, the Ginibre Hamiltonian reads $H(r, p_r, t) = -\frac{1}{4}p_r^2$. Notice that despite the variable z missing in the Hamiltonian, the dynamics still takes place in both z and w.

Since the Hamiltonian does not depend on time, we start off by setting

$$\Phi(r, \alpha, c, t) = W(r, \alpha, c) + \alpha t,$$

where α is the first conserved quantity and c is an overall additive constant. We plug in the ansatz to HJ equation and find

$$\alpha = \tfrac{1}{4}(\partial_r W)^2$$

so that the solution reads $W = \sqrt{4\alpha} r + c$. Full potential Φ reads

$$\Phi(r, \alpha, t) = \sqrt{4\alpha} r(t) + \alpha t + c, \tag{B2}$$

where we explicitly write r's relation on time. Canonical transformations give the transformed constant position β and "old" radial momentum p_r,

$$\beta = \partial_\alpha \Phi = \frac{r}{\sqrt{\alpha}} + t, \quad p_r = \partial_r \Phi = \sqrt{4\alpha}. \tag{B3}$$

What results are equations of motion,

$$r(t) = \sqrt{\alpha}(\beta - t), \quad p_r(t) = 2\sqrt{\alpha},$$

given in terms of constants of motion α, β (or new position and momenta). Next, we evaluate these constants in terms of interpretable quantities such as initial position and momentum, $r(t = 0) = r_0$, $p_r(t = 0) = p_{r,0}$. We set $t = 0$ in both equations and obtain

$$r_0 = \sqrt{\alpha}\beta, \quad p_{r,0} = 2\sqrt{\alpha},$$

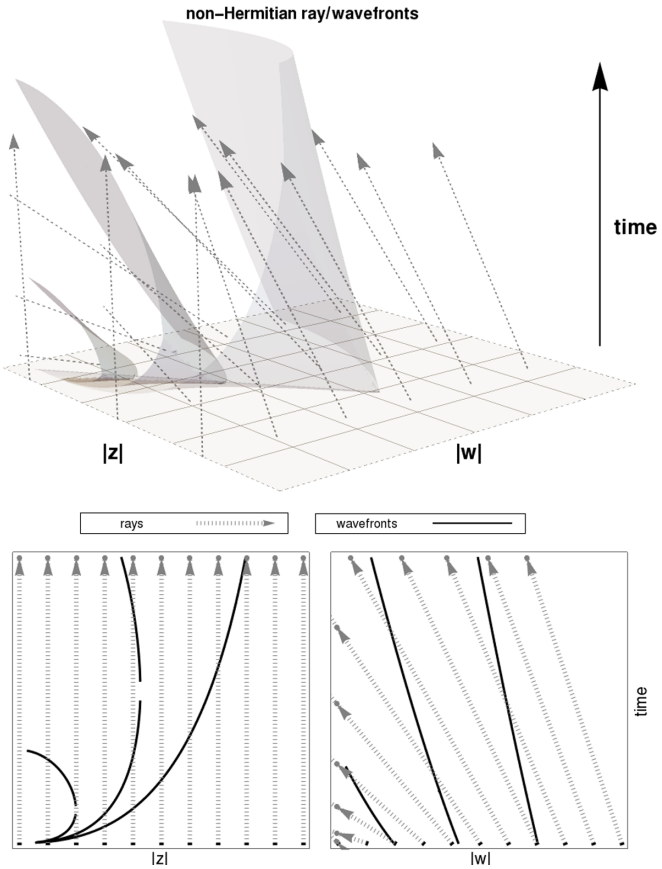

FIG. 2. Construction of duality between wavefronts (black solid lines) and rays (dotted gray arrows) in non-Hermitian dynamical random matrices. Propagation takes place in restricted quaternionic space spanned by modules $|z|$ and $|w| = r$. Wavefronts are defined by condition $\Phi(r(t), t) = \text{const}$, with the potential given by Eq. (B7) and characteristics $z(t) = z_0$, $r(t) = r_0 - r_0/(|z_0|^2 + r_0^2)t$ by Eq. (B5). The z variable is trivially added despite the absence of interesting dynamics. Similarly to Fig. 1, propagation is likewise anisotropic. The presented evolution happens for the Ginibre dynamics

and solve for α, β to obtain $\alpha = \frac{1}{4}p_{r,0}^2$, $\beta = \frac{2r_0}{p_{r,0}}$. We plug those back into (B3) and solve them to obtain

$$r(t) = r_0 - \frac{p_{r,0}}{2}t, \quad p_r(t) = p_{r,0}.$$

In this parametrization, the Hamilton principal function (or potential) reads

$$\Phi = p_{r,0}r + \tfrac{1}{4}p_{r,0}^2 t + c. \tag{B4}$$

Again, the problem we aim to solve couples the initial momentum and position through the initial value of the potential $t = 0\Phi(r_0, \alpha, t = 0) = F(r_0)$ and its derivative $\partial_r\Phi(r_0, \alpha, t = 0) = F'(r_0)$. These two conditions translate to the relation between initial r_0, p_0 and constant c:

$$\Phi(t = 0) = F(r_0) \;\rightarrow\; r_0 p_{r,0} + c = F(r_0),$$
$$\partial_r\Phi(t = 0) = F'(r_0) \;\rightarrow\; p_{r,0} = F'(r_0).$$

This in turn renders the underlying r, p dynamics dependent only on r_0:

$$r(t) = r_0 - \frac{F'(r_0)}{2}t, \tag{B5}$$
$$p(t) = F'(r_0), \tag{B6}$$

and sets the constant $c = F(r_0) - F'(r_0)r_0$. Last, we plug all newfound quantities into Φ given by (B4),

$$\Phi[r(t), t] = F(r_0) - \frac{[r(t) - r_0]^2}{t}, \tag{B7}$$

where r_0 is expressed through $r(t) = r_0 - \frac{F'(r_0)}{2}t$.

We consider the simplest initial value $F(r_0) = \ln(|z|^2 + r_0^2)$. Both wavefronts and characteristics are plotted in Fig. 2.

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