

## Relating heat and entanglement in strong-coupling thermodynamics

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Explaining the influence of strong coupling in the dynamics of open quantum systems is one of the most challenging issues in the rapidly growing field of quantum thermodynamics. By using a particular definition of heat, we develop an approach to study the thermodynamics in the strong-coupling regime, which takes into account quantum resources such as coherence and entanglement. We apply the method to calculate the time-dependent thermodynamic properties of a system and an environment interacting via the generalized amplitude-damping channel. The results indicate that the transient imbalance between heat dissipated and heat absorbed that occurs in the process is responsible for the generation of system-environment entanglement.

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### I. INTRODUCTION

Classical thermodynamics intrinsically relies on the assumption that the system under analysis is weakly coupled to its surroundings. This is because the energy of the interacting elements of the body's surface is negligible compared to the energy of the bulk. In this regime, one can always treat the states and the energetic properties of the system and environment separately, which allows us, for example, to equate the energy dissipated by the system with the heat absorbed by the reservoir [1–5]. On the other hand, this weak-coupling limit is not in general justified for quantum open systems, since the system-environment interaction involves a large fraction of the system's constituents [6,7]. In this strong-coupling regime, stochastic and quantum effects become important [8], and the usual approach to this problem begins with the decomposition of the total system-environment Hamiltonian in the form

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE}, \quad (1)$$

where the operators  $\hat{H}_S$  and  $\hat{H}_E$  are the bare Hamiltonians of the system and environment, respectively, and  $\hat{H}_{SE}$  is the interaction Hamiltonian, which cannot be neglected.

Some particularly important problems are often pointed out when the coupling is strong. One is how to partition the internal energy into work and heat, as dictated by the first law,  $\Delta U = W + Q$ . In this case, the notion of work is less problematic because, as usual, it only depends on system variables, but the definition of heat has been shown to be more difficult and ambiguous [7,9,10]. Another question is that the reduced density matrix of the open system is not supposed to contain all the information necessary to describe the thermodynamic properties; instead, information originating from the system-environment interaction should be included [11–16]. In this article, we present a framework to study the thermodynamics of open quantum systems in the strong-coupling

regime, based on a recently proposed quantum version of the first law of thermodynamics [17]. The formalism circumvents the limitations indicated above, in the sense that an unambiguous definition of heat is provided, which considers quantum effects such as coherence and entanglement, and the reduced density matrices of the subsystems involved are capable of providing full information about their average thermodynamic behavior. To illustrate the practical implications, we use the method to investigate the thermodynamics of the generalized amplitude-damping channel (GADC). Interestingly, we find that the system-bath entanglement in this case is generated at the cost of the heat asymmetry that naturally emerges during the interaction process.

### II. FIRST LAW OF QUANTUM THERMODYNAMICS

In this section we review the quantum version of the first law of thermodynamics put forward by the author in Ref. [17]. Consider a generic quantum, physical system operating as a working substance, whose Hamiltonian reads  $\hat{H} = \sum_n E_n |n\rangle \langle n|$ , with the  $n$ th energy eigenvalue and eigenstate given by  $E_n = \langle n | \hat{H} | n \rangle$  and  $|n\rangle$ , respectively. We also define the density operator of the system as  $\hat{\rho} = \sum_k \rho_k |k\rangle \langle k|$ , where  $\rho_k = \langle k | \hat{\rho} | k \rangle$  and  $|k\rangle$  are the eigenvalues and eigenkets, respectively. From a statistical perspective, we can define the internal energy of this system as the average of  $\hat{H}$ , i.e.,  $U = \langle \hat{H} \rangle = \text{tr}\{\hat{\rho}\hat{H}\}$ . However, since the trace operation is basis independent, we can calculate  $U$  using either the eigenbasis  $\{|n\rangle\}$  of  $\hat{H}$  or the eigenbasis  $\{|k\rangle\}$  of  $\hat{\rho}$ . In the first case, which we label as C1, we have that  $U = \sum_n P_n E_n$ , where  $P_n = \langle n | \hat{\rho} | n \rangle$  is the occupation probability of the  $n$ th energy level, while in the second case, which we label as C2, one finds  $U = \sum_k \rho_k \epsilon_k$ , with  $\epsilon_k = \langle k | \hat{H} | k \rangle$  as the diagonal elements of  $\hat{H}$  represented in the  $\{|k\rangle\}$  basis.

The result of the internal energy obtained in the case C1 allows us to write  $dU = \sum_n [E_n dP_n + P_n dE_n]$ . Now, we are in a position to define the work done on the system in an infinitesimal quantum process,  $dW$ . In doing so, we first recall

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the classical concept of work: “The work realized on or by the working substance is the change in the internal energy produced by modifications in the generalized coordinates” [18–20]. In a quantum-mechanical setting, a change in the generalized coordinates of the system, which may include the volume, external electric and magnetic fields, or the gravitational potential, leads naturally to a modification in the energy spectrum  $E_n$ , as for example in quantum dynamics satisfying the adiabatic approximation [21–23], or in the so-called shortcuts to adiabaticity [24–28]. Thus, from the expression of  $dU$  above, we can identify the infinitesimal work as  $dW := \sum_n P_n dE_n$ .

Let us now study the case C2. In this scenario, we have that  $dU = \sum_k [\epsilon_k d\rho_k + \rho_k d\epsilon_k]$ , which permits us to define the heat exchange by the working substance in an infinitesimal quantum process,  $dQ$ . In classical thermodynamics, the concept of heat reads as follows: “The heat exchanged between the working substance and the external environment corresponds to the change in the internal energy that is accompanied by entropy change” [18,20]. In order to generalize this concept to the quantum realm, we first recall that the von Neumann entropy of the system is given by  $S(\hat{\rho}) = -\text{tr}\{\hat{\rho} \log \hat{\rho}\} = -\sum_k \rho_k \log \rho_k$ . Then, for an infinitesimal trace-preserving quantum transformation, we have that  $dS = -\sum_k [\log(\rho_k) d\rho_k]$ , where we used the fact that  $\sum_k d\rho_k = 0$ , because  $\sum_k \rho_k = 1$ . From the results of  $dS$  and  $dU$  obtained in this case C2, if we invoke the above classical definition of heat, we can identify the quantum heat as  $dQ := \sum_k \epsilon_k d\rho_k$ . Indeed, this is the only part of  $dU$  which accompanies entropy change.

Having the definitions of  $dW$  and  $dQ$  above, it can be easily verified that  $dU \neq dW + dQ$ , in apparent contradiction with the first law of thermodynamics. However, as shown in Ref. [17], the missing energetic contribution is given by the infinitesimal quantity  $dC = \sum_n \sum_k (E_n \rho_k) d[|c_{n,k}|^2]$ , where  $c_{n,k} = \langle n|k \rangle$ . In fact, it can be demonstrated that  $dU = dW + dQ + dC$ . Observe that the contribution of  $dC$  only exists if the thermodynamic process in question involves some change in the quantum coherence of the system in the energy basis, i.e., when the coefficients  $|c_{n,k}|$  are time dependent. For this reason, we will refer to the quantity  $C$  here as *coherent energy*. Of course, this is only relevant in specific quantum processes, and the usual form of the first law,  $dU = dW + dQ$ , is recovered in the classical limit.

Overall, the function  $C$ , which is not compatible with the classical definitions of work and heat, expresses the energetic contribution of the dynamics of coherence in the first law. This is why we give it an independent classification. For finite-time processes, the work, heat, and coherent energy can be found by integration of the respective differentials (see details in Ref. [17]):

$$W(t) = \sum_n \sum_k \int_0^t \rho_k |c_{n,k}|^2 \frac{dE_n}{dt'} dt', \quad (2)$$

$$Q(t) = \sum_n \sum_k \int_0^t E_n |c_{n,k}|^2 \frac{d\rho_k}{dt'} dt', \quad (3)$$

$$C(t) = \sum_n \sum_k \int_0^t (E_n \rho_k) \frac{d}{dt'} |c_{n,k}|^2 dt'. \quad (4)$$

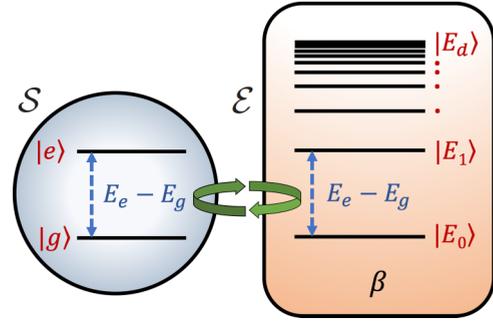


FIG. 1. Schematic representing a two-level atomic system  $S$  interacting with an environment  $\mathcal{E}$ , which is initially in a thermal state at inverse temperature  $\beta$ .

The change in the internal energy is given by  $\Delta U(t) = W(t) + Q(t) + C(t)$ . In what follows, we apply these results to study the energy exchanges that occur in a particular strong system-environment interaction.

### III. PHYSICAL MODEL

In order to study the evolution of an open quantum system  $S$ , one usually considers it as part of a larger closed system, which also includes the environment  $\mathcal{E}$ , undergoing a unitary transformation  $\hat{U}$  that depends on the total Hamiltonian as that of Eq. (1). Following this reasoning, we illustrate our description of strong-coupling thermodynamics by examining a model based on the generalized amplitude-damping channel (GADC) [29]. This model is a useful tool to describe the dynamics of a qubit system in contact with a thermal bath with finite temperature. For instance, the GADC has been used to characterize a spin-1/2 system coupled to an interacting spin chain at nonzero temperature [30,31], the influence of noise in superconducting-circuit-based quantum computing [32], and the finite-temperature thermal noise in linear optical systems [33]. Here, we consider a particular approach consisting of an open two-level atomic system  $S$  interacting with two levels of a finite environment  $\mathcal{E}$  initially in a thermal state, as depicted in Fig. 1.

The ground and excited states of the system,  $|g\rangle$  and  $|e\rangle$ , have energies  $E_g$  and  $E_e$ , respectively, so that the free Hamiltonian of the system is given by  $\hat{H}_S = E_g |g\rangle \langle g| + E_e |e\rangle \langle e|$ . On the other hand, we assume that the environment  $\mathcal{E}$  has a small number of states,  $d + 1$ , and that only the transitions between the states  $|E_0\rangle$  and  $|E_1\rangle$  are capable of causing some influence in  $S$ . The free environmental Hamiltonian and initial thermal state are given by  $\hat{H}_\mathcal{E} = \sum_{i=0}^d E_i |E_i\rangle \langle E_i|$  and  $\hat{\rho}_\mathcal{E}(0) = e^{-\beta \hat{H}_\mathcal{E}} / \mathcal{Z}_\mathcal{E}$ , respectively, where  $\mathcal{Z}_\mathcal{E} = \text{tr}[e^{-\beta \hat{H}_\mathcal{E}}]$  is the partition function, and  $\beta$  the inverse temperature. As a matter of fact, since we are assuming that  $S$  interacts with  $\mathcal{E}$  only via transitions that occur between  $|E_0\rangle$  and  $|E_1\rangle$ , the environment can be effectively represented as a qubit (which we call  $E$ ), with initial state  $\hat{\rho}_E(0) = w_0 |E_0\rangle \langle E_0| + w_1 |E_1\rangle \langle E_1|$ , where the probabilistic weights obey  $w_1 = w_0 e^{-\beta(E_1 - E_0)}$  and  $w_0 + w_1 = 1$ . We also see from Fig. 1 that  $E_1 - E_0 = E_e - E_g$ .

To describe the system-environment dynamics, we use the joint  $\mathcal{SE}$  basis  $\{|g, E_0\rangle, |g, E_1\rangle, |e, E_0\rangle, |e, E_1\rangle\}$  and establish an (probabilistic) unitary evolution such that

$$|g, E_0\rangle \rightarrow |g, E_0\rangle, \quad (5)$$

$$|g, E_1\rangle \rightarrow \sqrt{1-p}|g, E_1\rangle + \sqrt{p}|e, E_0\rangle, \quad (6)$$

$$|e, E_0\rangle \rightarrow \sqrt{1-p}|e, E_0\rangle + \sqrt{p}|g, E_1\rangle, \quad (7)$$

$$|e, E_1\rangle \rightarrow |e, E_1\rangle. \quad (8)$$

These interaction rules can be interpreted as follows: (i) Equation (5) says that if  $\mathcal{S}$  starts out in the ground state and  $\mathcal{E}$  has no excitation (e.g., zero temperature), no transition occurs; (ii) Eq. (6) indicates that if  $\mathcal{S}$  is in the ground state and  $\mathcal{E}$  in the first excited state, after a given time interval  $\tau$ , there is a probability  $p$  that  $\mathcal{S}$  becomes excited and  $\mathcal{E}$  decays to the fundamental state; (iii) Eq. (7) says that if  $\mathcal{S}$  is in the excited state and  $\mathcal{E}$  in the fundamental one, after the time  $\tau$ ,  $\mathcal{S}$  decays to the ground state with probability  $p$ , and  $\mathcal{E}$  is led to the first excited state; (iv) Eq. (8) tells us that, if  $\mathcal{E}$  is in the first excited state, the state  $|e\rangle$  of  $\mathcal{S}$  has a comparatively longer lifetime, so that no transition is expected during the time  $\tau$ . The longer lifetime of  $|e, E_1\rangle$  in comparison with  $|e, E_0\rangle$  can be justified by the fact that, in some strong-coupling cases, the metastability of quantum states is sensitive to the environmental conditions [34–36].

According to Eqs. (5)–(8), the matrix representation of the evolution of  $\mathcal{SE}$  is given by

$$\hat{\mathcal{U}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & \sqrt{p} & 0 \\ 0 & \sqrt{p} & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (9)$$

with  $p \in [0, 1]$ . Note that, in the limit where  $p = 1$ ,  $\hat{\mathcal{U}}$  reduces to the SWAP gate [37]. Now, if we assume that  $\mathcal{S}$  and  $\mathcal{E}$  are initially uncorrelated,  $\hat{\rho}_{SE}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$ , we can describe the evolution of  $\mathcal{S}$  through the quantum channel  $\hat{\rho}_S(0) \rightarrow \Phi[\hat{\rho}_S(0)] = \text{tr}_E[\hat{\mathcal{U}}(\hat{\rho}_S(0) \otimes \hat{\rho}_E(0))\hat{\mathcal{U}}^\dagger]$ , where  $\text{tr}_E$  denotes trace over the environment states  $|E_0\rangle$  and  $|E_1\rangle$ . It can be shown that  $\Phi[\hat{\rho}_S(0)] = \sum_{ij=0,1} \hat{K}_{ij} \hat{\rho}_S(0) \hat{K}_{ij}^\dagger$  is a completely positive trace preserving (CPTP) map with Kraus operators  $\hat{K}_{ij} = \sqrt{w_i} \langle E_j | \hat{\mathcal{U}} | E_i \rangle$  given by  $\hat{K}_{00} = \sqrt{w_0}(|g\rangle \langle g| + \sqrt{1-p}|e\rangle \langle e|)$ ,  $\hat{K}_{01} = \sqrt{w_0}(\sqrt{p}|g\rangle \langle e|)$ ,  $\hat{K}_{10} = \sqrt{w_1}(\sqrt{p}|e\rangle \langle g|)$ , and  $\hat{K}_{11} = \sqrt{w_1}(\sqrt{1-p}|g\rangle \langle g| + |e\rangle \langle e|)$ , which satisfy  $\sum_{ij=0,1} \hat{K}_{ij}^\dagger \hat{K}_{ij} = \mathbb{I}_S$  [37,38]. This is the GADC [29].

We next turn to investigating the dynamics of  $\mathcal{E}$ . This is dictated by the channel  $\hat{\rho}_E(0) \rightarrow \Lambda[\hat{\rho}_E(0)] = \text{tr}_S[\hat{\mathcal{U}}(\hat{\rho}_S(0) \otimes \hat{\rho}_E(0))\hat{\mathcal{U}}^\dagger]$ , where  $\text{tr}_S$  denotes trace over the system states,  $|g\rangle$  and  $|e\rangle$ , which yields  $\Lambda[\hat{\rho}_E(0)] = \sum_{k=0,1} \hat{L}_k \hat{\rho}_E(0) \hat{L}_k^\dagger$ . For simplicity, we assume that the system is prepared in the pure state  $|\psi(0)\rangle = \alpha|g\rangle + \sqrt{1-\alpha^2}|e\rangle$ , with  $\alpha \in \mathbb{R}$  (this assumption does not invalidate the generality of the results, i.e., a general mixed state could be equally used). In this case, the two Kraus operators are given by  $\hat{L}_0 = |g\rangle \langle \hat{\mathcal{U}} |\psi(0)\rangle = \alpha|E_0\rangle \langle E_0| + \sqrt{p(1-\alpha^2)}|E_1\rangle \langle E_0| + \sqrt{1-p\alpha^2}|E_1\rangle \langle E_1|$ , and  $\hat{L}_1 = \langle e | \hat{\mathcal{U}} | \psi(0) \rangle = \sqrt{(1-p)(1-\alpha^2)}|E_0\rangle \langle E_0| +$

$\sqrt{p\alpha^2}|E_0\rangle \langle E_1| + \sqrt{1-\alpha^2}|E_1\rangle \langle E_1|$ . This is also a CPTP map, in which  $\sum_{k=0,1} \hat{L}_k^\dagger \hat{L}_k = \mathbb{I}_E$ .

As a result of the application of the maps, the states of  $\mathcal{S}$  and  $\mathcal{E}$  become

$$\Phi[\hat{\rho}_S(0)] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \Lambda[\hat{\rho}_E(0)] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (10)$$

respectively. The entries of  $\Phi[\hat{\rho}_S(0)]$  are given by  $A_{11} = [\alpha^2 + (1-\alpha^2)p]w_0 + \alpha^2(1-p)w_1$ ,  $A_{12} = A_{21} = \alpha\sqrt{(1-\alpha^2)(1-p)}$ , and  $A_{22} = (1-\alpha^2)(1-p)w_0 + [(1-\alpha^2) + \alpha^2p]w_1$ , whereas the entries of  $\Lambda[\hat{\rho}_E(0)]$  are  $B_{11} = [\alpha^2 + (1-\alpha^2)(1-p)]w_0 + \alpha^2pw_1$ ,  $B_{12} = B_{21} = \alpha\sqrt{(1-\alpha^2)p}$ , and  $B_{22} = (1-\alpha^2)pw_0 + [(1-\alpha^2) + \alpha^2(1-p)]w_1$ . It can be seen that, although the unitary (reversible) evolution of  $\mathcal{SE}$ , the partial trace operations used to construct the above maps lead to nonunitary (irreversible) evolution of  $\mathcal{S}$  and  $\mathcal{E}$ , individually.

To express the time evolution of  $\mathcal{S}$  and  $\mathcal{E}$ , we will assume that the probability of a quantum transition event per unit time is  $\Gamma$ , in such a way that  $p = \Gamma\Delta t \ll 1$  for a short-time interval  $\Delta t$ . Then, the evolution of the system and the environment after a time  $t = n\Delta t$  is a result of the application of the respective maps  $n$  times in succession. This assumption is equivalent to assuming that the evolution of  $\mathcal{S}$  and  $\mathcal{E}$  are *Markovian*, i.e., the influence of the quantum channels acting on  $\mathcal{S}$  and  $\mathcal{E}$  are completely determined by the respective quantum states at each time step [39–41]. Accordingly, these transformations can be implemented as  $\hat{\rho}_S(t) = \Phi^n[\hat{\rho}_S(0)]$  and  $\hat{\rho}_E(t) = \Lambda^n[\hat{\rho}_E(0)]$ , which make the probabilistic factors change based on the rule  $(1-p) \rightarrow (1-p)^n = \lim_{n \rightarrow \infty} (1 - \frac{\Gamma t}{n})^n = e^{-\Gamma t}$ , where we assumed  $\Delta t \rightarrow 0$  [38]. In this form, we can write

$$\hat{\rho}_S(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}, \quad \hat{\rho}_E(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}. \quad (11)$$

In this case, the entries of  $\hat{\rho}_S(t)$  are given by  $A_{11}(t) = [\alpha^2 + (1-\alpha^2)\delta(t)]w_0 + \alpha^2\gamma(t)w_1$ ,  $A_{12}(t) = A_{21}(t) = \alpha\sqrt{1-\alpha^2}[\gamma(t)]^{1/2}$ , and  $A_{22}(t) = (1-\alpha^2)\gamma(t)w_0 + [(1-\alpha^2) + \alpha^2\delta(t)]w_1$ . In turn, the entries of  $\hat{\rho}_E(t)$  are  $B_{11}(t) = [\alpha^2 + (1-\alpha^2)\gamma(t)]w_0 + \alpha^2\delta(t)w_1$ ,  $B_{12}(t) = B_{21}(t) = \alpha\sqrt{1-\alpha^2}[\delta(t)]^{1/2}$ , and  $B_{22}(t) = (1-\alpha^2)\delta(t)w_0 + [(1-\alpha^2) + \alpha^2\gamma(t)]w_1$ . In these equations we have that  $\gamma(t) = e^{-\Gamma t}$  and  $\delta(t) = 1 - e^{-\Gamma t}$ . The consistency of the states in Eq. (11) in this physical scenario relies on the tacit assumption that the characteristic timescale of the system-environment interaction,  $\tau_{\text{int}} = 1/\Gamma$ , is much faster than the characteristic time of the other transitions involving the environment states  $|E_0\rangle$  and  $|E_1\rangle$ . It is worth mentioning that the positive constant  $\Gamma$ , which can also be interpreted as a decoherence rate of  $\mathcal{S}$ , is a characteristic of the Markovian dynamics assumed here [39–41]. As a consequence, there is a flow of information from the system to the environment before they reach a steady state. As we shall see, this behavior is indicated in the graphs of the thermodynamic quantities shown in Fig. 2.

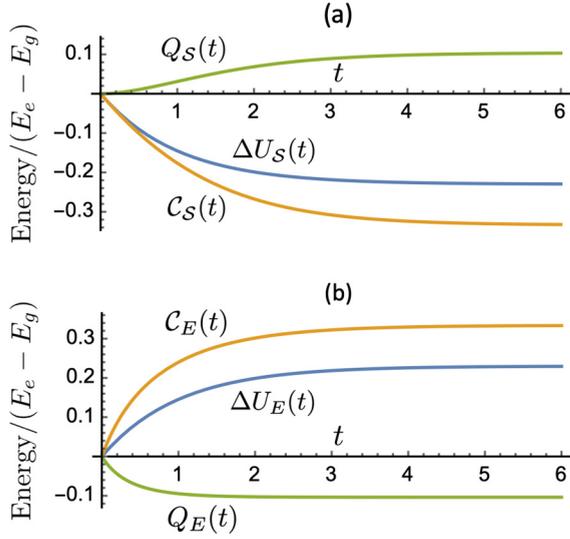


FIG. 2. Time evolution of the heat exchange, coherent energy, and internal energy of the (a) system  $S$  and (b) environment qubit  $E$ , interacting via the GADC. Initially,  $S$  is in a maximally coherent state,  $|+\rangle = 1/\sqrt{2}(|g\rangle + |e\rangle)$ , and  $E$  is at thermal equilibrium with inverse temperature  $\beta = 1/(E_e - E_g)$ . For simplicity, we assumed  $\Gamma = 1$  in both panels.

#### IV. THERMODYNAMICS OF THE MODEL

Having found the density operators of the system  $S$  and the environment qubit  $E$ ,  $\hat{\rho}_S(t)$  and  $\hat{\rho}_E(t)$ , and considering the respective Hamiltonians,  $\hat{H}_S = E_g |g\rangle \langle g| + E_e |e\rangle \langle e|$  and  $\hat{H}_E = E_0 |E_0\rangle \langle E_0| + E_1 |E_1\rangle \langle E_1|$ , we can calculate the eigenvalues and eigenstates of these four operators. This information allows us to calculate the thermodynamic properties of  $S$  and  $E$  as a function of time during the interaction process, according to Eqs. (2)–(4). As the energy eigenvalues of  $S$  and  $E$  are time independent, it is straightforward to see from Eq. (2) that no work is done on the system and the environment, i.e.,  $W_S(t) = W_E(t) = 0$ . However, in order to calculate the heat exchange and the coherent energy of  $S$  and  $E$ , we first need to fix some parameters. As can be seen from the off-diagonal elements of  $\hat{\rho}_S(t)$  and  $\hat{\rho}_E(t)$ , the quantum coherence of these states varies only if  $\alpha \neq 0, 1$  [42,43]. Thus, an interesting case to study is when  $\alpha = 1/\sqrt{2}$ . Let us also consider that the inverse temperature of the environment is  $\beta = 1/(E_e - E_g)$ . In this form, we have that  $w_0 \approx 0.73$  and  $w_1 \approx 0.27$ .

According to Eqs. (3) and (4), we are now able to calculate the heat exchange and coherent energy of the system,  $Q_S(t)$  and  $C_S(t)$ , and the environment,  $Q_E(t)$  and  $C_E(t)$ , as the GADC proceeds. Although these quantities can be computed analytically, the expressions are too cumbersome to be shown here. Instead, we plot the results as a function of time, as shown in Fig. 2. It can be verified that  $\Delta U_S(t) = -\Delta U_E(t) = \text{tr}\{\hat{H}_S(\hat{\rho}_S(t) - \hat{\rho}_S(0))\}$ ,  $\forall t$ . However, we call attention to the difference between  $Q_S(t)$  and  $-Q_E(t)$ , and between  $C_S(t)$  and  $-C_E(t)$ , especially for  $t < 4$ . This result confirms that the weak-coupling condition, in which the relation  $Q_S(t) = -Q_E(t)$  is applicable, is not fulfilled in the present model. In order to quantify the discrepancy between the heat released by  $E$  and the heat absorbed by  $S$ , we introduce the

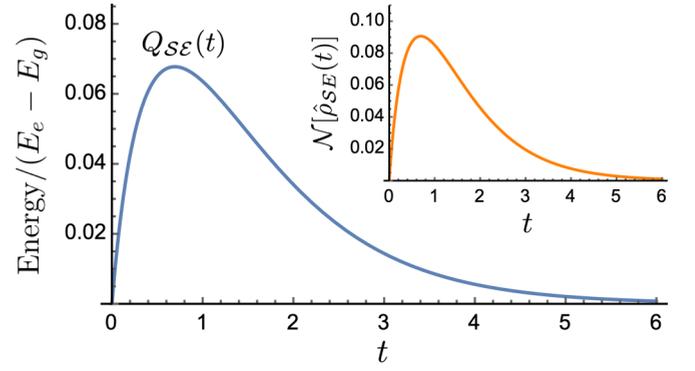


FIG. 3. Time evolution of the heat asymmetry  $Q_{SE}(t)$  between the system and environment, which interact via the GADC (we used the same parameters of Fig. 2). If compared with the values of  $Q_S(t)$  and  $Q_E(t)$  shown in Fig. 2, we observe that the heat asymmetry is high for short times ( $t < 2$ ). The inset presents the time evolution of the entanglement negativity  $\mathcal{N}[\hat{\rho}_{SE}(t)]$  created between  $S$  and  $E$ . Curiously,  $Q_{SE}(t)$  and  $\mathcal{N}[\hat{\rho}_{SE}(t)]$  are proportional in this model.

quantity  $Q_{SE}(t) = |Q_S(t) + Q_E(t)|$  that we call *heat asymmetry*, whose time dependence is shown in Fig. 3. Note that, since  $\Delta U_S(t) = -\Delta U_E(t)$  and  $W_S(t) = W_E(t) = 0$ , the event  $Q_{SE}(t) \neq 0$  is a consequence of the difference between the rates of entropy change of the system and the environment, i.e.,  $dS(\hat{\rho}_S)/dt \neq -dS(\hat{\rho}_E)/dt$  [43]. Meantime, the quantity  $Q_{SE}(t)$  approaches zero for long times.

We also see from Eq. (4) that the coherent energy  $\mathcal{C}(t)$  can be physically interpreted as the energy transfer to or from a system accompanied by coherence change. Based on this interpretation, we can say that the amount of coherent energy entailed in a quantum process depends both on the initial coherence of the system and, of course, the nature of the interaction with the environment. In general, quantum transformations that involve large amounts of coherent energy are those in which the system is initially in a high-coherence state, and the coupling with the environment causes strong dissipation and decoherence [44–46]. In this context, we observe that the coherent energy flow may become more prominent when the system is coupled to low-temperature environments, as in the spin-spin model [47,48], and high-temperature environments, as in the Caldeira-Leggett model [49,50]. Actually, we note that coherent energy, as well as heat exchange, is not a quantity that is determined by the coupling strength.

#### V. QUANTUM CORRELATIONS

We next focus on the study of the entanglement created due to the coupling between  $S$  and  $E$ . The quantification of entanglement for bipartite mixed states is not a trivial task. Nevertheless, since we reduced our problem to a qubit-qubit interaction, we can use the concept of negativity to characterize the system-environment entanglement. The negativity is an entanglement monotone given by [51,52]

$$\mathcal{N}(\hat{\rho}_{SE}) = \frac{\|\hat{\rho}_{SE}^{T_S}\| - 1}{2}, \quad (12)$$

where  $\hat{\rho}_{SE}$  denotes the density matrix of the composite system, comprising  $S$  and  $E$ , and  $\hat{\rho}_{SE}^{T_S}$  the partial transpose of

$\hat{\rho}_{SE}$  with respect to the system. The trace norm of an operator  $\hat{O}$  is defined as  $\|\hat{O}\| = \text{tr}\{\sqrt{\hat{O}\hat{O}^\dagger}\}$ . The negativity is also given by the sum of the absolute values of the negative eigenvalues of  $\hat{\rho}_{SE}^T$ , which vanishes for unentangled states. Previous studies have used the negativity to quantify system-environment quantum correlations [53,54]. The time evolution of the operator  $\hat{\rho}_{SE}$  can be calculated from the initial uncorrelated state  $\hat{\rho}_{SE}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$  as  $\hat{\rho}_{SE}(t) = \hat{U}\hat{\rho}_{SE}(0)\hat{U}^\dagger$ , with  $\hat{U}$  given in Eq. (9), along with the probabilistic rules considered here,  $p \rightarrow 1 - e^{-\Gamma t}$ . This, in combination with Eq. (12), permits us to calculate the system-environment entanglement as a function of time,  $\mathcal{N}[\hat{\rho}_{SE}(t)]$ , for the GADC [43]. The result is displayed in the inset of Fig. 3. Interestingly, our calculations showed that  $\mathcal{N}[\hat{\rho}_{SE}(t)]$  and  $Q_{SE}(t)$ , which were obtained from two completely different theories, are proportional to each other. This can be observed in the time-dependent profiles shown in Fig. 3.

With the results of the energy exchanges and quantum correlations between  $S$  and  $E$  established, we can investigate the energy cost to generate entanglement in this model, which is an essential resource for many quantum information tasks [37]. From Eqs. (5)–(8) we see that the energy of the composite system is clearly conserved in the GADC, as confirmed by the result  $\Delta U_S(t) = -\Delta U_E(t)$ . Then, we pose the question of where the energy used to create this entanglement comes from. In Fig. 2 we observe that, when the interaction begins,  $E$  releases an amount of heat that is not totally absorbed by  $S$ , and, according to Fig. 3, this is exactly when entanglement is created. After that (mostly in the interval  $1 < t < 3$ ),  $E$  still releases heat, but  $S$  absorbs a larger amount. This process continues until all heat released by  $E$  is absorbed by  $S$ ; for long times we have  $Q_S = -Q_E \approx 0.104(E_e - E_g)$ . In this form, we can assign the energy used to generate entanglement to the transient imbalance between the released and absorbed heat

involving  $S$  and  $E$ . This justifies the proportionality between  $Q_{SE}(t)$  and  $\mathcal{N}[\hat{\rho}_{SE}(t)]$  along the entire process, and sheds a different light on the idea that entanglement is created at the cost of work [55–58].

## VI. CONCLUSION

We have presented a framework to study the thermodynamics of an open quantum system strongly coupled to a heat bath, which takes into account the energetic aspects of quantum-mechanical resources as coherence and entanglement. The method was used to provide a thermodynamic description of the generalized amplitude-damping channel (GADC), from the point of view of both system and environment. We demonstrated that, when the interaction begins, an asymmetry between the heat released by the environment and the heat absorbed by the system emerges, while a quantum correlation is established. More specifically, it was found that the heat asymmetry in this example is proportional to the entanglement negativity during the complete time evolution. This important finding suggests that the creation of quantum correlations does not come necessarily at the price of doing work on the interacting systems. This development opens up another venue for exploring thermodynamics at strong coupling.

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