Orbital magnetism of an active particle in viscoelastic suspension

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We consider an active (self-propelling) particle in a viscoelastic fluid. The particle is charged and constrained to move in a two-dimensional harmonic trap. Its dynamics is coupled to a constant magnetic field applied perpendicular to its plane of motion via Lorentz force. Due to the finite activity, the generalized fluctuation-dissipation relation (GFDR) breaks down, driving the system away from equilibrium. While breaking GFDR, we have shown that the system can have finite classical orbital magnetism only when the dynamics of the system contains finite inertia. The orbital magnetic moment has been calculated exactly. Remarkably, we find that when the elastic dissipation timescale of the medium is larger (smaller) than the persistence timescale of the self-propelling particle, it is diamagnetic (paramagnetic). Therefore, for a given strength of the magnetic field, the system undergoes a transition from diamagnetic to paramagnetic state (and vice versa) simply by tuning the timescales of underlying physical processes, such as active fluctuations and viscoelastic dissipation. Interestingly, we also find that the magnetic moment, which vanishes at equilibrium, behaves nonmonotonically with respect to increasing persistence of self-propulsion, which drives the system out of equilibrium.

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I. INTRODUCTION

Inertia can have a profound effect on dynamics. It can be a system of particles forming a rigid body or a fluid, and inertia can be equally important in the dynamics of both. From celestial bodies to a spinning top in everyday life, it is evident that the effect of inertia is ubiquitous in rigid body mechanics. In case of fluids, at high Reynolds number [1], inertia plays important role from simpler problems of fluid mechanics, such as inviscid flows, potential flows, laminar flows, etc., all the way to one of the most challenging problems—turbulence [1,2].

However, with respect to the inertial dynamics mentioned above, the dynamics in the world of motile microorganisms (e.g., bacteria, green algae, sperm cells, white and red blood cells, or even smaller-scale objects such as motor proteins) and synthetic microswimmers (e.g., active colloids [3]) are fundamentally different [4,5]. First, unlike the passive particles, the microorganisms can self-propel, consuming energy from their surrounding. They spontaneously generate flow into the system, driving it far from equilibrium [6]. Second, because of their size limitation and the highly viscous medium in which the microorganisms self-propel, the typical Reynolds number is around 10^{-4} or even smaller [7,8]. The typical time required for the microswimmers at low Reynolds number to dissipate their momenta is around 10^{-7} s [8]. Therefore, for all practical purposes, throughout their journey, the momentum of such swimmers remains constant over time. Consequently, the inertia has a negligibly small effect on the dynamics of such self-propellers. Most of the research on *active* systems so far has been done in this low Reynolds number limit where one considers Stokes flow including active stresses (namely, active hydrodynamics [9]) and/or overdamped Brownian motion including self-propelling forces and torques [5].

Clearly, if we push the envelop further considering larger self-propelling objects moving in a medium with lower viscosity, the inertial effect will become prominent. Typically, active particles moving in a low-viscosity medium starting from millimeter size (and onwards) are strongly influenced by inertial forces and torques. Macroscopic particles of granular material with built-in self-propelling or self-vibrating mechanisms (e.g., an internal vibration motor or vibrating plate), miniature robots and the like [10–20]), macroscopic swimmers [21–24], and flying insects [25] are apt examples where inertia can play a significant role in their dynamics, both in the single particle level as well as in the collective level. Recently there have been studies from theoretical as well as experimental perspectives, focusing on inertial effects of self-propelled particles [12,26–30]. Self-propelling robots can be fabricated in the macroscopic length scales [31,32] to explore the inertial effects. An active Langevin model including inertia can describe the dynamics of inertial self-propellers well [33–35]. It is observed that by fine tuning inertia, some of the fundamental properties of active systems are qualitatively modified. For example, it has been shown experimentally as well as theoretically that inertia can induce delay between the orientation dynamics and velocity of active particles, which has a profound influence on their long-time dynamics [12]. In the presence of inertia, different dynamical states are developed by self-propelling particles confined in a trap. The

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transition between these dynamical states is continuous or discontinuous and crucially depends on the inertia present in the system [26]. One of the fundamental features of active Brownian systems is motility-induced phase separation (MIPS). It is strongly influenced and suppressed by the presence of inertia. It has also been shown that due to inertia, the coexisting phases of high and low particle density, obtained with MIPS, have widely different kinetic temperatures, which is in contrast with equilibrium phase separation [36].

The physics of the charged, *passive* Brownian particle under an electromagnetic field has been studied previously [37–40]. Being motivated by the aforementioned recent findings on inertial *active* systems, here we will report the magnetic properties of a charged, *active* Brownian particle suspended in a viscoelastic medium and moving under constant external magnetic field. In particular, we will show that if the active particle possesses finite inertia, only then can such a system exhibit orbital magnetism and goes through a transition from the paramagnetic to the the diamagnetic state in a different regime of parameter space from the model. The transition between paramagnetic and diamagnetic states depends crucially on the interplay of different timescales related to the physical processes (e.g., active fluctuations and dissipation) involved in the dynamics of the system.

Before going into the details of our findings, we will introduce here the medium in which the active particle is suspended. We consider the medium to be viscoelastic with transient elasticity. The elastic forces exerted by the medium on the self-propelling particles dissipate within a finite time beyond which particles are dragged only by the viscous forces. Theoretically one can also consider the viscous limit of the problem where elastic forces dissipate very fast and only viscous forces are left to drag the suspended particles. In experiments, usually polymers are added to the viscous fluids to make them transiently viscoelastic [41].

Viscoelasticity can trigger fast transitions of a Brownian particle in a double-well optical potential [42]. It can add remarkable features to the dynamics of the active system suspended in it. For example, it enhances rotational diffusion of the active particles [43,44]. In a viscoelastic environment, the self-propelling colloids exhibit a transition from enhanced angular diffusion to persistent rotational motion beyond a critical propulsion speed [45]. Viscoelasticity can enhance or retard the swimming speed of a helical swimmer, depending on the geometrical details of the swimmer and the fluid properties [46]. Natural active systems are often found in a viscoelastic environment. Hence it is imperative to study viscoelastic effects on their dynamics [47,48]. Theoretically it has been shown that elasticity in a non-Newtonian fluid can suppress cell division and cell motility [49]. In a viscoelastic environment active pulses can reverse the flow [50]. In the case of chemically powered self-propelling dimers, fluid elasticity enhances translational and rotational motion at the single-particle level, whereas in the multiparticle level, it enhances alignment and clustering [51].

In this article we consider a viscoelastic medium in which an inertial particle self-propels in two dimensions. The particle is confined in a two-dimensional (2D) harmonic potential. The particle is charged and subject to a constant magnetic field perpendicular to the plane of its motion [52–55]. We will focus on how the particle responds to the presence of an external magnetic field. We quantify the response by evaluating the classical, orbital magnetic moment of the particle and its characteristic features with respect to the timescales signifying various physical processes occurring within the system, such as (1) the timescale related to the correlation of active fluctuations (due to self-propulsion) and (2) the timescale related to dissipation. The timescale associated with active fluctuations originates from the persistence of the self-propeller to move along a certain direction despite random collisions from surrounding fluid particles. The timescale associated with dissipation signifies the characteristic time of the surrounding viscoelastic fluid *within* which the elastic dissipation dominates and *beyond* which the viscous dissipation dominates. Note that here we consider the elastic dissipation being transient decays within a finite time, whereas the viscous dissipation prevails for a long time. When these two timescales are equal, the generalized fluctuation-dissipation relation (GFDR) [56] holds and the system remains nonmagnetic with zero magnetic moment. Conversely, when the fluctuation and dissipation timescale are unequal, GFDR breaks down and the system shows nonzero magnetic moment under the influence of an external magnetic field. Depending on these two competing timescales, our analysis reveals that the particle can exhibit either paramagnetic or diamagnetic behavior. In particular, we show that when the persistence in active fluctuation dominates over the dissipation, the particle manifests paramagnetism, and when the dissipation dominates over the self-propulsion, the particle exhibits diamagnetism. Therefore, fixing the external magnetic field to a nonzero constant value, when the timescales of these two physical processes, namely, active fluctuations and dissipations (due to elasticity and viscocity of the medium), are tuned, the particle undergoes a transition from diamagnetic states to paramagnetic states and vice versa. Importantly, it has also been shown that all these magnetic characteristics of the system crucially depend on the presence of inertia. If inertia of the particle is negligibly small compared to the dissipative as well as active forces, all the existing forces in the system cancel each other, providing a zero net force, and then the magnetic moment vanishes. As a result the particle loses its magnetic characteristics.

In the next section, we address the model by which the problem is described in detail. The results are systematically detailed in the subsequent sections, and finally we conclude.

II. MODEL AND METHOD

A. Model

We consider an active (self-propelling) particle suspended in a viscoelastic medium at temperature T. The particle is at position $\mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y}$ and at velocity $\dot{\mathbf{r}} = \mathbf{v}$ at time t where $(\hat{x}, \hat{y})$ are the unit vectors along X and Y. The particle has charge $|q|$ and is constrained to move on an X-Y plane. It is confined within a 2D harmonic trap, $U(x, y) = \frac{1}{2}k(x^2 + y^2)$ where k is the spring constant. The particle is subjected to an external constant magnetic field $\mathbf{B} = B\hat{z}$ where $\hat{z}$ is the unit vector along Z. The equation of motion of the particle is given by

$$m\ddot{\mathbf{r}} = -\gamma \int_0^t g(t - t')\mathbf{v}(t')dt' + \frac{|q|}{c}(\mathbf{v} \times \mathbf{B}) - k\mathbf{r} + \sqrt{D}\boldsymbol{\xi}(t), \tag{1}$$

where m is the mass of the particle. To take inertia into account, in the equation of motion of the particle [Eq. (1)], we consider $\ddot{\mathbf{r}} = \dot{\mathbf{v}}$ as the acceleration of the particle.

As the particle is self-propelling, despite the random collision with the particles of the surrounding viscoelastic fluid, it remains persistent to move along a certain direction up to a finite timescale. Moreover the dynamics of the particle contains finite memory due to the elasticity of the fluid, and therefore the dynamics is non-Markovian [57]. The first term in the r.h.s. of Eq. (1) represents the drag force on the particle because of the friction with the surrounding medium. Due to the elasticity present in the medium, the drag at time t not only depends on the velocity of the particle at that particular time t, rather it depends on the weighted sum of all the past velocities within the time interval between 0 and t. As we consider the time evolution of the particle to be stationary, the weight function g should be a function of $(t-t')$, with $t \geqslant t'$. In particular we choose the weight function (in other words, friction kernel) as

$$g(t-t') = \frac{1}{2t_c'} e^{-\frac{(t-t')}{t_c'}}, \qquad t \geqslant t'. \tag{2}$$

The above kernel gives the maximum weight to the current velocity $\mathbf{v}(t)$, whereas the weight to the past velocities decays exponentially with the rate $1/t_c'$. The time $t \sim t_c'$ is the time required for elastic dissipation to decay substantially. Therefore, for time $t > t_c'$, the viscous dissipation dominates. In the limit $t_c' \to 0$, $g(t-t') = \delta(t-t')$ and consequently the system is left with only viscous dissipation. The friction kernel $g(t-t')$ captures the Maxwellian viscoelasticity formalism, where at large enough time the fluid becomes viscous through a transient viscoelasticity, allowing the elastic force to relax down to zero [57].

The second term in the r.h.s. of Eq. (1) represents the Lorentz force [58] caused by the magnetic field which couples X and Y. The third term in the r.h.s. of Eq. (1) represents the harmonic confinement. The term with $\boldsymbol{\xi}$ appeared in Eq. (1) represents active, colored (and thereby athermal) noise. The moments of $\xi_\alpha(t)$ are given by

$$\langle \xi_\alpha(t) \rangle = 0, \qquad \langle \xi_\alpha(t)\xi_\beta(t') \rangle = \frac{\delta_{\alpha\beta}}{2t_c} e^{-\frac{|t-t'|}{t_c}}, \tag{3}$$

where $(\alpha, \beta) \in (X, Y)$. Here t_c is the noise correlation time. The effective noise in the dynamics has finite correlation, and it decays exponentially with time constant t_c, representing the persistence of the self-propeller. Up to $t = t_c$, the self-propeller remain quite persistent to move along a direction that is the same as its previous steps, despite making random collisions with surrounding fluid particles. When $t > t_c$, change of direction with respect to the previous step becomes more probable. In the limit of $t_c \to 0$, with $D = 2\gamma K_B T$, the active fluctuations become thermal and the system becomes passive. Therefore, in the current model for active (self-propelling) particles, finite and nonzero t_c quantifies the activity of the system. Thus we model the dynamics of the particle as an active Ornstein-Uhlenbeck process (AOUP) [59,60]. But with inertia and elastic forces from the medium, the dynamics can be represented with the generalized Langevin's equation [57]. One may note that when $t_c = t_c'$, we get $\langle \xi_\alpha(t)\xi_\beta(t') \rangle = \delta_{\alpha\beta} g(t-t')$, which is the generalized fluctuation dissipation relation (GFDR). Here we will explore both the situations where GFDR holds and where it does not hold (i.e., for both $t_c = t_c'$ and $t_c \neq t_c'$). A special case where the elasticity of the fluid dissipates very fast compared to the active fluctuations (i.e., for $t_c' \to 0$ and $t_c > 0$) will also be discussed in greater detail.

B. Method

Now we solve the model (1) to evaluate the magnetic moment of the particle, $\mathbf{M} = \frac{|q|}{2c} \mathbf{r} \times \mathbf{v}$, in steady states. Introducing the complex variable $z = x + jy$ ($j = \sqrt{-1}$), we rewrite Eq. (1) as

$$\ddot{z}(t) + \int_0^t \Gamma g(t-t')\dot{z}(t')\,dt' - j\,\omega_c \dot{z}(t) + \omega_0^2 z(t) = \varepsilon(t), \tag{4}$$

where the parameters are $\Gamma = \frac{\gamma}{m}$, $\omega_c = \frac{|q|B}{mc}$, $\omega_0 = \sqrt{\frac{k}{m}}$, and $\varepsilon(t) = \sqrt{\frac{2\Gamma K_B T}{m}}[\xi^x(t) + j\,\xi^y(t)]$.

By performing the Laplace transform of the complex variable $z(t)$ and $\dot{z}(t)$ we get $\mathcal{L}\{z\}(s) = \int_0^\infty e^{-st} z(t)\,dt$ and $\mathcal{L}\{\dot{z}\}(s) = s\,\mathcal{L}\{z\}(s)$, where

$$\mathcal{L}\{z\}(s) = \frac{\left(\frac{1}{t_c'} + s\right)\mathcal{L}\{\varepsilon\}(s)}{s^3 + \left(\frac{1}{t_c'} - j\omega_c\right)s^2 + \left(\frac{\Gamma}{t_c'} - \frac{j\,\omega_c}{t_c'} + \omega_0^2\right)s + \frac{\omega_0^2}{t_c'}}. \tag{5}$$

Here we are interested only in the steady-state solutions obtained in $t \to \infty$ limit where the influence of the initial conditions is reduced to zero with time. Therefore we consider $z(0) = 0$ and $\dot{z}(0) = 0$ without losing the generality of the solution.

The denominator in the Eq. (5) can be factorized as follows:

$$D = (s - s_1)(s - s_2)(s - s_3). \tag{6}$$

The roots of Eq. (6) can be found using Cardano's method. Substituting Eq. (6) in Eq. (5) and using the method of partial fraction,

$$\mathcal{L}\{z\}(s) = \sum_{k=1}^{3} \frac{a_k}{s - s_k} \mathcal{L}\{\varepsilon\}(s), \tag{7a}$$

$$\mathcal{L}\{\dot{z}\}(s) = \sum_{l=1}^{3} \frac{b_l}{s - s_l} \mathcal{L}\{\varepsilon\}(s), \tag{7b}$$

where (a_i, b_i) are found solving

$$a_1 + a_2 + a_3 = 0; \quad a_1(s_2 + s_3) + a_2(s_3 + s_1) + a_3(s_2 + s_1) = -1, \quad a_1 s_2 s_3 + a_2 s_3 s_1 + a_3 s_2 s_1 = \frac{1}{t_c'},$$

$$b_1 + b_2 + b_3 = 1; \quad b_1(s_2 + s_3) + b_2(s_3 + s_1) + b_3(s_2 + s_1) = -\frac{1}{t_c'}, \quad b_1 s_2 s_3 + b_2 s_3 s_1 + b_3 s_2 s_1 = 0.$$

By taking the inverse Laplace transform of Eq. (7a) and Eq. (7b), we get $z(t)$ and $\dot{z}(t)$ as

$$z(t) = \sum_{k=1}^{3} a_k \int_0^t e^{s_k(t-t')}\varepsilon(t')\,dt', \tag{8a}$$

$$\dot{z}(t) = \sum_{l=1}^{3} b_l \int_0^t e^{s_l(t-t'')}\varepsilon(t'')\,dt''. \tag{8b}$$

The average orbital magnetic moment of the particle is rewritten as

$$\langle \mathbf{M(t)}\rangle = \frac{|q|}{2c}\langle|\mathbf{r}\times\mathbf{v}|\rangle\hat{z} = M\hat{z} = \frac{|q|}{2c}\mathrm{Im}[\langle z(t)\dot{z}^*(t)\rangle]\hat{z}. \tag{9}$$

Here Im(...) denotes the imaginary part, and "∗" denotes the complex conjugation. Using Eq. (8) in Eq. (9), we obtain the magnitude of the steady-state magnetic moment of the self-propelling particle at $t \to \infty$ limit as

$$M = \lim_{t\to\infty}\frac{|q|\Gamma K_B T}{m\,c\,t_c}\mathrm{Im}\left(\sum_{k,l=1}^{3} a_k\, b_l^* \int_0^t\int_0^t e^{s_k(t-t')}e^{s_l^*(t-t'')}e^{-\frac{|t'-t''|}{t_c}}\,dt'\,dt''\right)$$

$$= \frac{|q|\Gamma K_B T}{m\,c\,t_c}\mathrm{Im}\left(\sum_{k,l=1}^{3} a_k\, b_l^*\left[\frac{t_c^2}{(1+s_i t_c)(-1+s_j^* t_c)} + \frac{2t_c}{(s_i+s_j^*)\left(-1+s_i^2 t_c^2\right)}\right]\right), \tag{10}$$

which is the main result of the paper. Below various features of M in different parameter spaces will be demonstrated in detail.

III. DISCUSSION

In Fig. 1 we have plotted M as a function of t_c and t_c' with different values of $(\omega_0, \omega_c, \Gamma)$. From this exact result, the following magnetic features of the system become apparent.

First, along the diagonal $t_c' = t_c$, where GFDR holds, the average orbital magnetic moment $M = 0$. Therefore, along the diagonal, the system becomes nonmagnetic even if the dynamics contains memory. It is reminiscent of the exact result stating that the thermal average of magnetization of any classical system in equilibrium is zero [61].

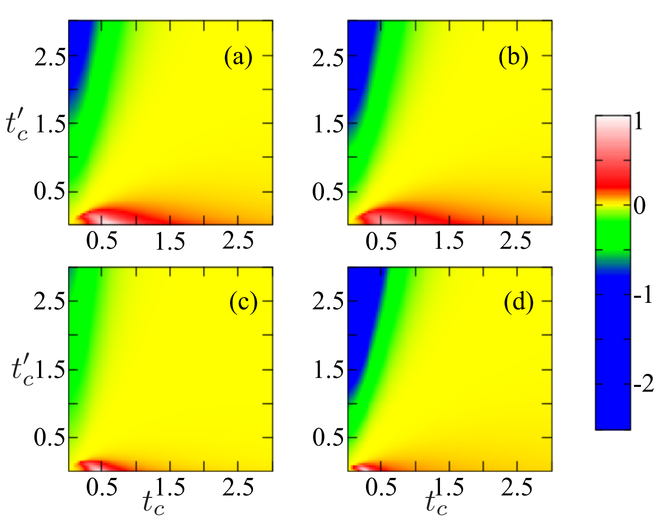

FIG. 1. $\langle M(\infty)\rangle$ [Eq. (10)] as a function of t_c and t_c': (a) with $\omega_c = 1$, $\omega_0 = 1$, and $\Gamma = 1$, (b) with $\omega_c = 1.5$, $\omega_0 = 1$, $\Gamma = 1$, (c) with $\omega_0 = 1.5$, $\omega_c = 1$, $\Gamma = 1$, and (d) with $\Gamma = 3$, $\omega_0 = 1$, $\omega_c = 1$. Both $K_B T$ and $\frac{|q|}{mc}$ are assumed to be unity.

Second, on the t_c'-t_c plane where $t_c' > t_c$, the system becomes diamagnetic as $M < 0$. In this regime (elastic) dissipative forces dominate over the self-propulsion. Moreover, the externally applied magnetic field induces a magnetic field within the system of charged particles which opposes the external magnetic field itself. Therefore the system is diamagnetic when $t_c' > t_c$.

However, on the t_c-t_c' plane where $t_c > t_c'$, the self-propulsion persists for a longer time as compared to the elastic dissipation of the particles. As a result the induced magnetic field within the system follows the external magnetic field. Hence the system becomes paramagnetic.

Therefore, for a given strength of the external magnetic field, the system exhibits a transition from the diamagnetic state to the paramagnetic state and vice versa simply by tuning the rate of the dissipation and the persistence timescale of self-propulsion. Moreover, if the direction of the magnetic field is reversed, the system shows opposite behavior regarding the phases.

Third, while passing from diamagnetic states to the paramagnetic states or vice versa, the system undergoes a considerably large regime of nonmagnetic states with $M = 0$ on the t_c-t_c' plane. This regime includes the diagonal $t_c' = t_c$. However, it also includes a regime across the diagonal where $t_c \neq t_c'$ (but they are comparable). It is to be noted that across this regime, the magnetization of the system still remains zero. This regime grows as one proceeds along the diagonal. It confirms that if GFDR holds good, it implies that the average orbital magnetic moment is always zero; however, the converse is not true. The average magnetic moment can still be zero even if the system is driven out of equilibrium where GFDR does not hold good.

Similarly, we also note that when $t_c \gg t_c'$, GFDR is obviously broken and the system is far from equilibrium conditions. Remarkably, in this regime, the paramagnetic moment gets reduced further and approaches zero. Hence the system becomes nonmagnetic. This is because for large enough t_c, due to high persistence in the dynamics, the self-propulsion of the particles overcomes the influence of the external magnetic field and induces a persistent rectilinear motion into them. This hinders the particles from forming closed current loops which are essential to exhibit finite magnetic moment [62].

Finally, apart from the fluctuation and dissipation timescales, the parameters Γ, ω_0 and ω_c can also alter the profile of $M(t_c, t_c')$. With increasing ω_c, M increases up to a maximum value beyond which M decreases with further increase in ω_c. Therefore M has a nonmonotonic dependence on ω_c. The other two parameters, namely, Γ and ω_0, can only reduce M monotonically.

For further insight, we consider the following special case where the friction kernel is a δ function [note that in the limit $t_c' \to 0$, the exponential friction kernel Eq. (2) becomes a δ function], and the noise correlation is still exponential with correlation time, t_c. In this limit, only viscous dissipation takes place, and the equation of motion [Eq. (1)] reduces to

$$m\ddot{z} = -\gamma \dot{z} + j\frac{|q|B}{c}\dot{z} - kz + \varepsilon(t), \tag{11}$$

where $j = \sqrt{-1}$. This dynamics can be considered as an inertial active Ornstein Uhlenbeck process (IAOUP). The overdamped version of this, namely, the active Ornstein Uhlenbeck process (AOUP), is now commonly used to represent overdamped motion of active Brownian particles and successfully used to describe important features like MIPS [59] and dynamical heterogeneities of active systems at high densities [63]. We have exactly solved the dynamics both analytically as well as using computer simulation. Following a similar procedure as before, the solution of the dynamics (11), $z(t)$ and $\dot{z}(t)$ can be obtained as

$$z(t) = \sum_{k=1,2} a_k \int_0^t e^{s_k(t-t')}\varepsilon(t')\,dt', \tag{12a}$$

$$\dot{z}(t) = \sum_{l=1,2} b_l \int_0^t e^{s_l(t-t'')}\varepsilon(t'')\,dt'', \tag{12b}$$

where s_k are given by

$$s_{(1,2)} = \frac{1}{2}\Big[-(\Gamma - i\,\omega_c) \pm \sqrt{(\Gamma - i\,\omega_c)^2 - 4\omega_0^2}\Big] \tag{13a}$$

and a_k and b_l are given by

$$a_1 = \frac{1}{s_1 - s_2}, \quad a_2 = -a_1, \tag{14a}$$

$$b_1 = \frac{s_1}{s_1 - s_2}, \quad b_2 = -\frac{s_2}{s_1}b_1. \tag{14b}$$

Using the solutions in Eq. (12) and Eq. (9), the average magnetic moment in the long-time limit is given by

$$M_r = M_{t_c'\to 0} = \frac{|q|K_BT}{mc}\left(\frac{t_c^2\omega_c}{\left\{\left[1 + t_c\left(\Gamma + t_c\omega_0^2\right)\right]^2 + t_c^2\omega_c^2\right\}}\right). \tag{15}$$

It is evident from the aforementioned exact result in Eq. (15) that for $t_c = 0$, $M_r = 0$, which is consistent with equilibrium result, namely, the Bohr–van Leeuwen theorem [61]. When $t_c > 0$, the system goes away from equilibrium, and therefore, it is intuitive that M_r also shoots up. When $t_c > 0$ but small, $M_r = (\frac{|q|K_BT\omega_c}{mc})t_c^2$, implying that in this limit M_r increases with t_c^2.

On the other hand, it is also evident from the aforementioned exact result in Eq. (15) that for $t_c \to \infty$, $M_r = 0$. From the point of view of equilibrium physics, it is counterintuitive, because in this limit the system is very far from equilibrium, and hence the equilibrium result ($M_r = 0$) should not hold good. However, on closer inspection one may note that in this limit the persistence of the self-propulsion is so high that it is not allowing the particles to form a close current loop under the influence of the magnetic field, which is essential to build up a finite magnetic moment within the realm of classical equilibrium physics. Therefore the magnetic moment vanishes in the $t_c \to \infty$ limit. But for finite and large t_c, $M_r = (\frac{|q|K_BT\omega_c}{mc\omega_0^4})t_c^{-2}$ signifying that for large t_c, M_r decreases with t_c^{-2}.

Taking all this together, we find M_r has a nonmonotonic dependence on t_c. We exactly determine the optimum t_c ($\equiv \tau_0$) at which M_r is maximum by solving

$$\left.\frac{dM_r}{dt_c}\right|_{t_c=\tau_0} = \omega_0^4\tau_0^4 + \Gamma\omega_0^2\tau_0^3 - \Gamma\tau_0 - 1 = 0. \tag{16}$$

Here $\tau_0 = \frac{1}{\omega_0}$ solves the equation, which implies the optimum t_c or τ_0 is independent of ω_c (i.e., independent of magnetic field), but it is inversely proportional to ω_0 (i.e., varies inversely with the strength of the harmonic trap). The value of the magnetic moment at τ_0 is given by $M_r(\tau_0) = \frac{|q|K_BT}{mc}[\frac{\omega_c}{\omega_c^2 + (\Gamma + 2\omega_0)^2}]$. Clearly $M_r(\tau_0)$ can only decrease with ω_0, whereas it is nonmonotonic with ω_c. [For small ω_c, $M_r(\tau_0)$ increases with ω_c but for large enough ω_c it decreases and finally $\lim_{\omega_c\to\infty} M_r(\tau_0) = 0$.]

In Fig. 2 we show the behavior of M_r with t_c for different values of ω_0 and ω_c from both theoretical calculation as well as simulation. Numerical simulation of the dynamics has been carried out using Heun's method algorithm. We perform the simulation with a time step of 0.001 s and run the simulation up to 10^4 s. For each realization the results are taken after ignoring the initial 10^3 transients in order for the system to approach steady state. The averages are taken over 10^4 realizations.

From Eq. (15), one can also analyze the behavior of M_r with respect to Γ, ω_0, and ω_c. As $\Gamma \to \infty$, it is evident from the expression that $M_r = 0$. It implies that in high viscous limit where inertia is negligibly small, the magnetic moment decreases to zero, and it vanishes as $M_r \sim \Gamma^{-2}$. Therefore the active particles can exhibit nonzero magnetic moment

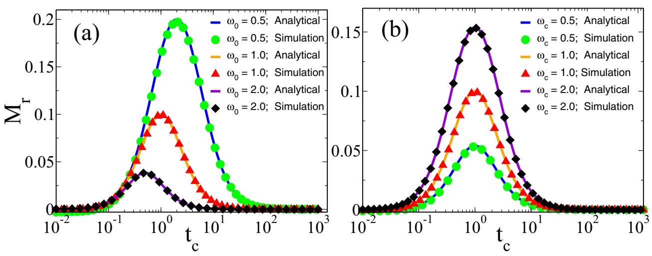

FIG. 2. M_r [Eq. (15)] is plotted with t_c (at $t_c' \to 0$ limit) from analytics (solid lines) as well as from numerical simulation (solid points) for (a) various values of ω_0, keeping ω_c and Γ to be fixed as unity, and (b) for various values of ω_c, keeping ω_0 and Γ as fixed to unity. Both K_BT and $\frac{|q|}{mc}$ are also assumed to be unity.

only when the dynamics of the system contains considerable inertia. Similarly from Eq. (15), it is evident that for large ω_0, M_r approaches zero as $M_r \sim \omega_0^{-4}$. Physically this occurs because, as ω_0 increases, the particles are constrained to move in a smaller area, and therefore the area of the current loop formed by the particles decreases. This eventually leads to a zero magnetic moment. The dependence of M_r on ω_c is nonmonotonic. When $\omega_c = 0$, $M_r = 0$, and for small but finite ω_c, M_r increases with ω_c (for small ω_c, $M_r \simeq \frac{|q|K_BT}{mc} t_c^2 \omega_c$). On the other hand, for large ω_c, $M_r \simeq (\frac{|q|K_BT}{mc})\frac{1}{\omega_c}$. Therefore, M_r decreases with large ω_c, and eventually it vanishes as $\omega_c \to \infty$. This nonmonotonic dependence of M_r on ω_c for different Γ and t_c is depicted in Fig. 3, from both analytics and simulation.

IV. CONCLUSION

In this work, we consider a system of noninteracting charged particles, self-propelling in two dimensions, being suspended in a Maxwellian viscoelastic medium, with considerable inertia. They are confined in a harmonic trap and subject to a Lorentz force due to an externally applied constant magnetic field, perpendicular to the plane of their motion. Due to the imbalance between elastic as well as viscous dissipation (due to the medium) and active fluctuations (due to self-propulsion), the system goes out of equilibrium. The magnetic moment of the system can become nonzero, and the system undergoes an interesting transition from diamagnetic phase to paramagnetic phase and vice versa. The transition depends on the interplay between the timescales involved in the dissipative processes and active fluctuations. In other words, for a given magnetic field, the transition depends on the interplay between two characteristic dimensionless numbers: the Weissenberg number (the ratio of elastic and viscous forces of the suspension, which in our case is $\Gamma t_c'$) and the Péclet number (the ratio of advective transport due to activity and diffusive transport in the suspension, which in our case is $\omega_0 t_c$) of the system.

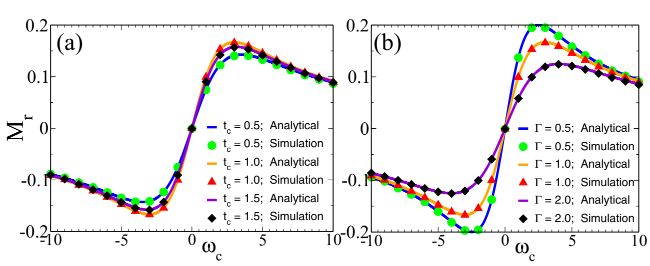

FIG. 3. M_r [Eq. (15)] is plotted with ω_c (a) for different t_c and (b) for different Γ, keeping other parameters fixed at unity.

We have also determined how the magnetic moment of the system depends on parameters like the cyclotron frequency ω_c related to the strength of the magnetic field, the natural frequency ω_0 related to the strength of the harmonic trap, and the friction coefficient Γ. As $\Gamma \to \infty$, the magnetic moment vanishes, suggesting that the orbital magnetism of the active viscoelastic suspension is exclusive for the active system with significant inertia. Interestingly, we also find that even if the system remains under deep nonequilibrium conditions (in particular, when the particles are self-propelling with large persistence timescale), still it can have zero magnetic moment, as in the case of equilibrium.

The model we consider to show the aforementioned results is a generalized Langevin equation with exponentially correlated colored noise. From the fundamental point of view, the following issues regarding the model used here are important to note. First, in general, active fluctuations are athermal, and therefore its strength D is not related to the temperature T of the medium in which the active particles are suspended. It is rather proportional to the square of the self-propulsion speed of the particle [64–66]. In certain limits (for example, at $t_c' \to 0$ together with $|t - t'| \gg t_c$), one can define an effective temperature with the self-propulsion speed of the active particle and relate it to D [67]. However, in general, dynamics of active particles cannot be mapped always to an equilibrium dynamics with an effective temperature which is different from the actual temperature of the system [68]. Here we consider $D = 2\Gamma K_BT$ (the fluctuation dissipation relation or FDR) simply to have a transparent equilibrium limit of the problem [59,69,70]. Moreover, it is evident that our main result, the steady-state magnetic moment M in Eq. (10), will not change qualitatively if ΓK_BT in the expression of M is replaced by the generic noise strength D.

Second, in addition to the active fluctuations, the dynamics of an active particle suspended in a medium at temperature T should also contain thermal fluctuations [64,67]. Thermal fluctuations are Gaussian and delta correlated and maintain FDR. Here in Eq. (15) with limit $t_c \to 0$ and also [54,55], it has been shown explicitly that the Langevin dynamics with only thermal fluctuation results in equilibrium where the classical orbital magnetic moment is zero. Hence, in the current analysis, the additive thermal noise is excluded.

It is important to explore how the results qualify for relatively denser system with significant interparticle interactions. Work in this direction is in progress. We believe all the aforementioned theoretical results are amenable to suitable experiments, and they are important to implement magnetic control on the dynamics of an active suspension by fine tuning its physical properties together with the external magnetic field.

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