Spectrum statistics in the integrable Lieb-Liniger model

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We address the old and widely debated question of the spectrum statistics of integrable quantum systems, through the analysis of the paradigmatic Lieb-Liniger model. This quantum many-body model of one-dimensional interacting bosons allows for the rigorous determination of energy spectra via the Bethe ansatz approach and our interest is to reveal the characteristic properties of energy levels in dependence of the model parameters. Using both analytical and numerical studies we show that the properties of spectra strongly depend on whether the analysis is done for a full energy spectrum or for a single subset with fixed total momentum. We show that the Poisson distribution of spacing between nearest-neighbor energies can occur only for a set of energy levels with fixed total momentum, for neither too large nor too weak interaction strength, and for sufficiently high energy. By studying long-range correlations between energy levels, we found strong deviations from the predictions based on the assumption of pseudorandom character of the distribution of energy levels.

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I. INTRODUCTION

Since the first studies of one-body quantum systems that are strongly chaotic in the classical limit [1–5], the most popular test widely used to distinguish between regular and chaotic systems was the search of the form of the nearest-neighbor-level spacing distribution (LSD) for energy levels. Specifically, it was assumed that for a completely integrable system the LSD has generically the form of a Poisson distribution, $P(s) = \exp(-s)$, characterizing the absence of correlations between energy levels. Contrarily, in the opposite limit of strong chaos, the LSD reflects strong repulsion of close energies, $P(s) \sim s^{\beta}$ (for $s \to 0$) with the repulsion parameter $\beta = 1, 2, 4$ that depends on symmetric properties of the system (see, for example, Ref. [6]). Although it was shown that such a correspondence is not always exact [7–9], the counterexamples are often considered as quite specific ones.

Historically, the interest in the properties of the LSD has been motivated by the experimental studies of quantum spectra of heavy nuclei and multielectron atoms (for references, see, for example, Ref. [10]). One of the first applied studies concerning the form of the LSD is traced back to 1939 [11]. Specifically, in view of the problem of phase transitions in nuclear matter, it was assumed that the LSD has the form of a Poisson distribution. This and other experimental studies of the low energy neutron scattering in nuclear reactions has triggered intensive discussions of the typical form of the LSD (see in Ref. [10]). Finally, it was accepted that according to scaling arguments presented by Wigner [12], the distribution $P(s)$ in application to heavy nuclei might be described by the expression nowadays known as the Wigner surmise (WS). Apart from the specific form of the repulsion of energy levels, the WS suggests even stronger decrease of $P(s) \sim \exp(-Bs^2)$, for $s \to \infty$ as compared with the exponential decrease $\sim \exp(-s)$ of the Poisson distribution. This behavior has been confirmed by experimental data gathered from neutron spectroscopy groups around the world to obtain the first global spacing distribution of s-wave neutron resonances [13].

Later on, following Wigner's studies of random matrices [14], Dyson rigorously derived exact expressions for the tails of $P(s)$ for all values of β [15]. According to these results, for $\beta = 1, 4$, the tails are described by both exponential and Gaussian terms, and only for $\beta = 2$ the exponential term is absent. However, as was noted by Dyson himself, one can correctly resolve the tails of $P(s)$ only when the number of energy levels is very large, i.e., exceeding 10^5. Clearly, this is not possible experimentally, thus the Wigner-Dyson (WD) expression $P(s) = As^{\beta} \exp(-Bs^2)$ (with A, B being normalization constants) can be used as a good approximation in many applications.

Numerous experimental data obtained in the study of energy spectra of heavy nuclei, complex atoms, and molecules have confirmed the emergence of the WD distribution (see, for example, references in Ref. [16]). It was, however, understood that the absence of level repulsion does not necessarily mean absence of strong chaos. The point is that in the analysis of experimental or numerical data one has to be sure that the considered energy spectrum does not contain any subset associated with some specific quantum numbers. Indeed, since such subsets are independent one from each other, the energy levels associated with different quantum numbers are

uncorrelated thus giving rise to the apparent absence of level repulsion. Thus, by superimposing the subsets of energies belonging to different quantum numbers, one can get a LSD which may not show noticeable repulsion, while when fixing all quantum numbers a quite good correspondence to the WD distribution can be found. For the first time, this effect has been discussed in Ref. [17] in application to nuclear reactions for which quantum numbers might not be known in advance.

A famous example is the Bunimovich billiard [18] for which there are four independent energy subsets due to the symmetry of the boundary with respect to reflections in both vertical and horizontal directions. Correspondingly, there are four kinds of eigenstates specified by their symmetric properties in the configuration space. Thus, only by selecting a particular subset of energy levels related to a specific symmetry of the eigenstates, the true WD distribution can be observed. Interesting enough, one can expect that for a N-dimensional Bunimovich billiard with $N \gg 1$ the level spacing distribution $P(s)$ should be very close to the Poisson distribution when considering the total spectrum, and to the WD distribution when analyzing one of the subsets associated with a particular symmetry of the eigenstates. As one can see, the question about the type of $P(s)$ characterizing the spectrum statistics of a given system is, strictly speaking, meaningless, unless all conditions are specified. It should be also noted that, even if the corresponding classical system is completely ergodic and chaotic (as in the case of the Bunimovich billiard) in the lower part of the energy spectrum the quantum effects always suppress chaos that prevents the emergence of the WD distribution. The transition from a Poisson to a WD distribution as a function of the energy and the geometric shape of the billiard have been investigated in Ref. [19] for the slightly perturbed Bunomivich billiard.

The emergence of the Poisson form of the LSD widely treated as an indication of integrability, has attracted much attention from the viewpoint of its mechanism. Indeed, the integrability of a quantum system is closely related to *regular* sequences of energy levels. However, the Poisson distribution itself is known to appear in statistical physics as a strong property of randomness. The source of apparent randomness for the LSD has been studied, for the first time, in Ref. [20], where the Poisson distribution was explained within a semiclassical approach to quantum systems with the well defined classical limit. Note that for one-dimensional systems the LSD is highly nongeneric (and typically far from either the Poisson or the WD). So far, the Berry-Tabor conjecture [20] of the Poisson form of $P(s)$, as a generic property of quantum systems that are integrable in the classical limit, has not yet rigorously proved, in spite of intensive mathematical studies (see, for instance, Refs. [21–23] and references therein).

In more details, the mechanism of pseudorandomness of the LSD was firstly demonstrated in Ref. [24] in the numerical study of the rectangular billiard which is trivially integrable in both classical and quantum description. It was shown that despite the regularity of the energy spectrum, $E_{n,m} = \alpha n^2 + m^2$ (with an irrational value of α and integers n, m) the LSD demonstrates quite good correspondence with the Poisson distribution. This means that the mechanism of apparent pseudorandomness of the energy spacings has here a geometric nature, emerging due to the reduction of the two-dimensional set of the values $E_{n,m}$ to a one-dimensional set of energies E_α. Specifically, in spite of a regular grid of values $E_{n,m}$ on the plane n, m, the number of points $E_{n,m}$ within the area bounded by the curves E and $E + ds$ changes randomly when changing the value of E. However, strong deviations from the Poisson distribution have been detected in the region of very small s spacings [24]. Moreover, other sensitive statistical tests of the randomness of energy levels (such as the absence of correlations between distant energy levels) have shown that the sequence of energy levels cannot be considered truly random. Another interesting example of many-body noninteracting integrable system characterized by a LSD strongly different from Poisson can be found in Ref. [25]. All these data indicate that statistical properties of energy spectra of integrable models have a restricted correspondence to the properties associated with a truly random process at least as far as noninteracting integrable systems are concerned. But what about many-body integrable systems with interacting particles?

In this paper, we focus on the properties of energy spectra of the paradigmatic Lieb-Liniger (LL) model [26–28], to which a huge number of works are devoted (see, for example, Refs. [29–32] and references therein). This model describes one-dimensional (1D) bosons on a circle interacting with a two-body pointlike interaction. The model belongs to a peculiar class of quantum integrable models solved by the Bethe ansatz [33,34]; in particular, it is possible to show that it has an infinite number of conserved quantities. Apart from the theoretical interest, many related problems have been recently discussed in view of various experiments with atomic gases [35–37]. Since this model has no classical counterpart, it is extremely interesting to shed light on the mechanism for the emergence of randomness, if any, in the energy spectrum, and quantify its statistical properties. Moreover, using the Bethe equation it is possible to extract an arbitrary large number of energy levels within an arbitrary small numerical error. This fact renders the LL model really unique in the class of interacting integrable quantum many-body systems.

For a weak inter-particle interaction the LL model can be described in the mean-field (MF) approximation. Contrarily, for a strong interaction, the 1D atomic gas enters the so-called Tonks-Girardeau (TG) regime in which the density of the interacting bosons becomes identical to that of noninteracting fermions (keeping, however, the bosonic symmetry for the wave function) [28]. The crossover from one regime to the other is governed by the ratio between the boson density and the interaction strength. The interaction is inversely proportional to the 1D interatomic scattering length and can be experimentally tuned with the use of the Feshbach resonance (see, for example, Ref. [38] and references therein).

II. THE MODEL

The Hamiltonian of the LL model with N bosons interacting on a ring of length L by a pointlike interaction, has the form

$$H = H_0 + cV = -\sum_i \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i \neq j} \delta(x_i - x_j). \quad (1)$$

Here we have used the units in which $\hbar/2m = 1$ and the key parameter c stands for the strength of the δ-like interaction between bosons.

For the reader's convenience we report here the standard procedure for finding eigenvalues and eigenfunctions; see, for instance, Refs. [29–32]. The solution for the eigenvalue problem can be obtained firstly by restricting the configuration space to the sector $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N$ where the Bose wave function $\Psi(x_1, \ldots, x_N)$ is completely determined. Thus, the system becomes a system of free particles with the interaction playing a role only as a boundary condition, for any $k = 2, \ldots N$,

$$\left[\frac{\partial \Psi}{\partial x_k} - \frac{\partial \Psi}{\partial x_{k-1}}\right]_{x_k = x_{k-1}} = c\Psi.$$

Now, we search the Ψ-function (in the sector defined above) written in the form

$$\Psi(x_1, .., x_N) = \sum_P a_P \exp\left[i \sum_{k=1}^{N} x_k \lambda_{P(k)}\right], \tag{2}$$

where a_P are phase factors and the sum is over the $N!$ permutations of $1, .., N$. Imposing the latter to be a solution of the stationary Schrodinger equation allows one to get the phase factors in terms of the rapidities $\lambda_{P(k)}$.

Then, by fixing periodic boundary conditions on the circle of length L, namely, $\Psi(..., x_k, ...) = \Psi(..., x_k + L, ...)$, one obtains a system of N Bethe equations (see, for example, Ref. [29]), $i = 1, ..., N$,

$$\lambda_i = \frac{2\pi}{L} m_i - \frac{2}{L} \sum_{k \neq i}^{N} \arctan\left(\frac{\lambda_i - \lambda_k}{c}\right) \tag{3}$$

for the rapidities λ_i. Each set of distinct "quantum numbers" $\{m_i\}_{i=1}^N$ is composed by integers (or half integers) for an odd (or even) number of particles N.

Extending the wave function to the whole configuration space, each set of N rapidities $\{\lambda_i\}_{i=1}^N$ defines an eigenstate $|\alpha(\lambda_1, ..., \lambda_N)\rangle$. The set of all eigenstates serves as a complete basis in the completely symmetric many-body Hilbert space. We label with α the N rapidities related to the eigenstates $|\alpha\rangle$ in the following way: $\{\lambda_i^\alpha\}_{i=1}^N$.

The rapidities completely specify the energy of the eigenstate $|\alpha\rangle$

$$E\left(\lambda_1^\alpha, ..., \lambda_N^\alpha\right) = \sum_{i=1}^{N} \left(\lambda_i^\alpha\right)^2, \tag{4}$$

the momentum

$$P\left(\lambda_1^\alpha, ..., \lambda_N^\alpha\right) = \sum_{i=1}^{N} \lambda_i^\alpha, \tag{5}$$

and also the infinite set of conserved charges with $k > 2$,

$$Q_k\left(\lambda_1^\alpha, ..., \lambda_N^\alpha\right) = \sum_{i=1}^{N} \left(\lambda_i^\alpha\right)^k. \tag{6}$$

In the following for simplicity we restrict our study by taking an odd value of N and setting $L = 2\pi$.

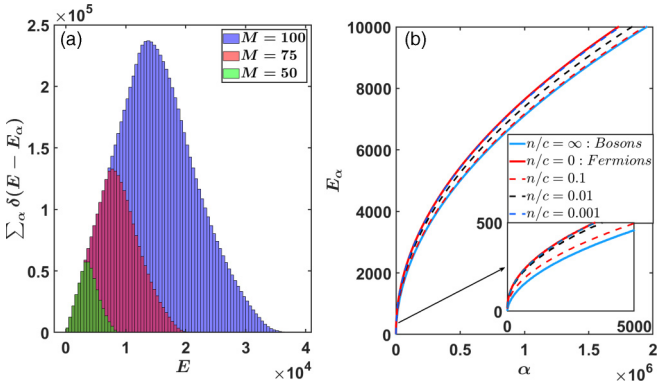

FIG. 1. (a) Density of many-body states for different values of the cutoff $M = 50, 75, 100$ for interaction $n/c = 0.01$. (b) Eigenvalues E_α as a function of the index label α for different interactions strengths n/c as indicated in the caption, together with the analytical solutions for the infinite and zero interactions. The data are obtained for $N = 5$ particles and fixed total momentum $P = 2$.

The goal of this paper is to find all possible eigenvalues in a finite energy region and to study in a very accurate way their statistics. To do that we first fix a large integer number M and consider all possible sets of N different integers $|m_j| < M$. For each set we compute all possible sets of rapidities (that determine the energies) satisfying Eq. (3). Let us call $E_{\max}(M)$ the maximal value of the energy for a given value M of the cut-off. After, we consider another cut-off number $M' > M$ and compute again all possible rapidities and energies. In this way we obtain more energy levels in the region with the maximal value $E_{\max}(M') > E_{\max}(M)$, but, most importantly, we obtain many missing energy levels in the interval $[0, E_{\max}(M)]$. We continue the procedure of increasing M up to the complete filling of the interval $[0, E_{\max}(M)]$, meaning that a further increase of M does not produce any new eigenvalue in the specified energy range. Typically, we have found that $M' \sim 2M$ is enough to find all eigenvalues in the interval $[0, E_{\max}(M)]$.

To solve the nonlinear Eq. (3), we have used the standard Newton solver with the precision $\epsilon = 10^{-15}$ in finding the rapidities. As initial conditions we provided an educated guess taking into account that we know explicitly the solution in two simple limit cases:

- infinite interaction (fermionization, free fermions),

$$c = \infty, \quad \lambda_j^\alpha = \frac{2\pi}{L} m_j^\alpha; \tag{7}$$

- no interaction (free bosons),

$$c = 0, \quad \lambda_j^\alpha = \frac{2\pi}{L}\left(m_j^\alpha - j + \frac{N+1}{2}\right). \tag{8}$$

Since any eigenenergy is the sum of N squared distinct integer numbers [see Eq. (4)] it is obvious that the distribution $P(s)$ is dramatically different from a Poisson in both limits above.

As an example of spectrum, we compute all energy eigenvalues by taking three different values of M. Results for density of states are shown in Fig. 1(a). As one can see, for this N value, the spectrum is linear in energy, with the bell shape

which is simply due to the cutoff M. It is clear that in the infinite case ($M \to \infty$) only the linear unbounded spectrum remains. Needless to say, we consider the eigenvalues taken from the linear part of the energy spectrum only, in which the values of energies are not suffered by the cutoff. An example of the spectrum, as a function of the label α and for different interaction strengths n/c is given in Fig. 1(b). In this figure and below, the density of bosons is $n = N/L = N/2\pi$. Here and below we use the rescaled parameter n/c of the interaction, which was found to be the key characteristic for the crossover from to the mean-field to the Tonks-Girardeau regime.

III. INDEPENDENT SPECTRA

Before starting to analyze the spectral properties, let us consider the fundamental role played by the total momentum P. Since the arctan in Eq. (3) is an odd function of its argument, the total momentum for our choice ($L = 2\pi$ and N odd) is an integer number,

$$P = \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} m_i.$$

Arranging the eigenvalues according to (i) its momentum and for each fixed momentum according to its growing energy, one can see that the Hamiltonian matrix has an infinite block diagonal structure. Each block matrix has been obtained by bracketing eigenstates having the same total momentum P, and it is disconnected from any other block (due to the momentum conservation there are no matrix elements connecting states with different total momenta). Moreover, a careful analysis has led to the unexpected result indicating that the energy levels in different energy subsets corresponding to different values of the total momentum P, are strongly correlated. To show that let us first start from Eq. (3) written as

$$\lambda_i = m_i + \sum_{k \neq i} f(\lambda_i - \lambda_k), \tag{9}$$

with $f(x) \equiv \arctan(x/c)/\pi$. Here, each set of different integers $\{m_i\}_{i=1}^{N}$ determines a set of rapidities $\{\lambda_i\}_{i=1}^{N}$ characterizing completely an eigenstate with the energy $E(\lambda_1, ..., \lambda_N)$ and the momentum $P(\lambda_1, ..., \lambda_N)$, see Eq. (4).

Let us now consider another (shifted) set of quantum numbers $m_i' = m_i + k$ with k a positive or negative integer number. It is clear that the shifted rapidities $\lambda_i' = \lambda_i + k$ satisfy the same Eq. (9),

$$\lambda_i' = m_i' + \sum_{k \neq i} f(\lambda_i' - \lambda_k'), \tag{10}$$

but with a shifted momentum and energy given, respectively, by

$$P\{\lambda_i'\} = \sum_i (\lambda_i + k) = P + kN \tag{11}$$

and

$$E\{\lambda_i'\} \quad = \textstyle\sum_i (\lambda_i + k)^2 = E\{\lambda_i\} + \nu, \tag{12}$$

where $\nu = 2kP + k^2 N$ is an integer number.

This means that for a given number N of particles, all the energies corresponding to some fixed momentum P, turn out to be shifted by the same constant integer number ν (and thus the levels statistics for the energy subset with fixed total momentum, will be the same). In particular, let us note that for $k = -2P/N$ (consider here that only k integer is valid) we have $\nu = 0$ and $P\{\lambda_i'\} = -P\{\lambda_i\}$. Thus, we recover the fact that P and $-P$ are related to the same eigenenergy.

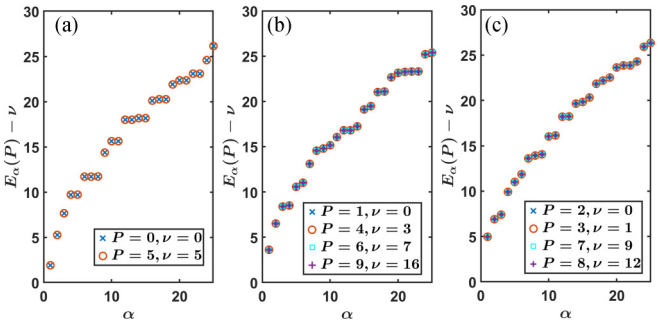

FIG. 2. (a), (b) First 25 energies rescaled by the factor ν for the interaction $n/c = 1$ and $N = 5$ particles, for different total momenta: (a) $P = 0$, (b) $P = 1$, and (c) $P = 2$.

Let us now analyze Eqs. (11) and (12) in more detail. Setting, for instance, $k = 1$ the energy spectrum for P and $P + N$ is simply shifted by the factor $\nu = 2P + N$. This suggests that, at most, only the spectra for $P = 0, 1, ...N - 1$ might be independent. However, it is not the case. Actually, the energy subsets for $P = 1$ and $P = N - 1$ have the same spectrum (with a constant shift). To see this, one has to simply take $k = 1$ and $P = -1$ in Eq. (12) and to observe that $P = 1$ and $P = -1$ give the same spectrum. In the same way the energy subsets for $P = 2$ and $P = N - 2$ are the same with respect to a constant shift, and so on. The bottom line is that for an odd number of particles N only the spectra with the momentum values $P = 0, 1, ..., (N - 1)/2$ are independent, all the other being simply shifted by a constant. This is a quite unexpected property of the energy spectra since it is completely independent of the interaction strength c.

A numerical verification of the above mathematical proof is shown in Fig. 2. There, we present the first 100 energy levels for the LL model with $N = 5$ bosons, rescaled due to the shift ν for the interaction $n/c = 1$. The energy levels have been obtained by solving the Bethe equations for few values of the total momentum, namely, for $0 \leqslant P \leqslant 9$, and plotted together with the energies corresponding to the momenta $P = 0, 1, 2$. The data demonstrate that all eigenvalues, properly shifted by factor ν, are exactly the same.

IV. STATISTICS OF CLOSE ENERGIES

As we explained above, one can exactly compute an amount of eigenvalues in a given energy region by choosing a large enough value of the cutoff M. In this way we can explore the statistical properties of energy spectra without the influence of spectrum truncation, at least for a not very large number of particles and for not very high energies.

One of the conclusions from our study is that it is meaningless to expect the Poisson form of the LSD when speaking of the total energy spectrum. Namely, from our analysis it is clear that there is strong clustering of energy levels that does

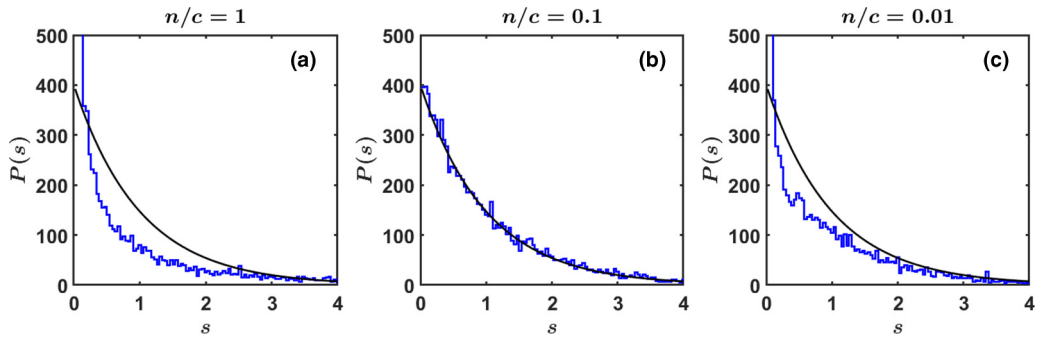

FIG. 3. LSD distribution for $N = 5$ particles, fixed momentum $P = 2$ and $n/c = 1$ (a), $n/c = 0.1$ (b), and $n/c = 0.01$ (c). The data are obtained taking the first 10^4 different energy levels starting from the beginning of the energy spectrum at fixed momentum. Black curves indicate the Poisson distribution.

not allow to speak about the Poisson distribution. There are two mechanisms of such clustering. The first is the influence of a high degeneracy of energy levels in two limit cases (either zero or infinite inter-particle interaction). And the second mechanism is due to very strong correlations between the subsets of energy levels, belonging to different values of the total momentum (see the proof above).

Much more interesting is the question about the statistical properties of energy spectra for a fixed total momentum. In this situation there is only the first mechanism of clustering for a relatively weak or very strong interaction. Let us consider for instance the LSD for the first 10^4 energy levels for three different interaction strengths. Results are shown in Fig. 3. As one can see, the clustering of energy levels for small values of s persists both for a relatively weak interaction, $n/c = 1$, and for a very strong interaction, $n/c = 0.01$. In both cases one can see that the LSD has a very pronounced peak at the origin, at variance with the Poisson distribution shown for comparison. Also, a clear similarity between these two cases is seen. As for the intermediate interaction strength, one can see that, formally, the LSD can be treated as the Poisson one. We did not explore this distribution more carefully (for example with the use of the χ^2 test), instead, we have used other famous tests for the check of the absence of correlations between nearest energy levels. As a result, concerning the Poisson form of the LSD in the Lieb-Liniger model, one can conclude that it can be observed under specific conditions only (for fixed total momentum, plus not weak or strong inter-particle interaction).

Let us now discuss another test recently suggested for the discriminating the Poisson statistics from the Wigner-Dyson one. Specifically, we focus on the ratio of consecutive level spacings that is used in the literature (see, for instance, Refs. [39–42] and references therein) when analyzing the repulsion between nearest levels. This test has the advantage of not requiring the unfolding of the spectrum since it involves the two closest energy levels only. Following this approach, we introduce the variables

$$\xi_\alpha = \frac{s_\alpha}{s_{\alpha-1}}, \quad s_\alpha = E_\alpha - E_{\alpha-1}, \tag{13}$$

and create the quantity of our interest,

$$\chi_\alpha = \mathrm{Min}(\xi_\alpha,\ 1/\xi_\alpha). \tag{14}$$

Even if the spectrum unfolding is not required, this test is useful when an average over disorder is performed, focusing on a particular part of the energy spectrum. According to the theory, supported by various numerical studies, for the Poisson distribution one gets, $\langle \chi_\alpha \rangle = 0.386$.

In Fig. 4 we summarize our study of the above quantity χ_α for different interaction strengths n/c, various values of the total momentum P and for different parts of the energy spectrum. Our results (light blue dots) are compared to the theoretical value related to the Poisson distribution (red line). As one can see, the variable χ_α undergoes huge fluctuations which can hardly give a precise answer about the kind of

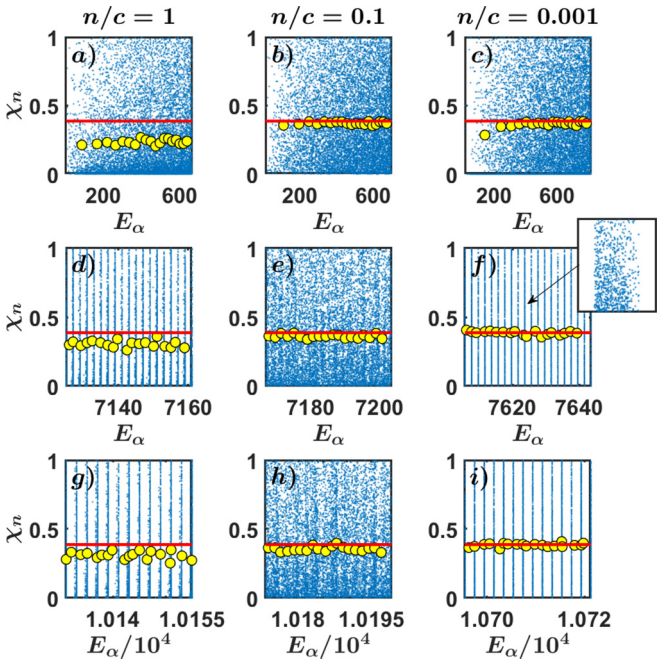

FIG. 4. Blue dots: 10^4 values of $\min(\chi_\alpha,\ 1/\chi_\alpha)$ for $N = 5$ particles and $P = 2$ of the total momentum. Panels in different rows stand for different parts of the energy spectrum (see x axis), while different columns indicate different values of the interaction strength (as indicated in the legend). Yellow dots are shown for the average over 500 close values, while the continuous red line corresponds to the value 0.386 obtained from a completely random set of energies. In panel (f) a blow up of a single blue vertical line is shown.

the LSD. To facilitate the comparison we average χ_α over 500 consecutive energy values (yellow circles). The averaged data show that, in general, there is a good agreement with the value corresponding to the Poisson distribution, obtained for the intermediate interaction between bosons. Moreover, as one can see from the middle-bottom right panels (high energy and strong interaction), even if the yellow circles are well fitted by the red line, an extreme clustering of the levels indicates a complete absence of randomness as one should expect.

V. Δ_3 STATISTICS

More carefully, the randomness of the energy spectrum can be verified by analyzing not only short-range correlations, e.g., nearest-neighbor statistics, but also long-range ones. According to this test, known as the Δ_3-statistics, one can reveal the so-called *rigidity* (or stiffness) of the energy spectrum and discriminate between regular and chaotic motion in the corresponding classical systems [43]. The strongest rigidity can be associated with that given by the equidistant energy levels. As was shown by Dyson and Mehta [15,44], the spectrum of full random matrices reveals a kind of rigidity which is due to correlations between distant energy levels. Such a rigidity of energy spectrum can be compared with a slightly melted crystal, an analogy which has been used by Dyson to derive many statistical properties of random matrices.

To imply this test for physical systems, it is necessary to proceed with the unfolding of the sequence of energy levels due to the dependence of the mean level spacing on the energy. The idea of the unfolding is to pass from a given sequence of levels to that having a constant level spacings, however, with the same correlations between the levels, both short and long-range ones. In this approach we start with the function.

$$\eta(E) = \sum_{\alpha=1}^{K} \Theta(E - E_\alpha), \tag{15}$$

which counts the number of levels with the energy less than or equal to E and is usually referred to as the staircase function. Specifically, the unfolding consists in mapping the sequence $\{E_1, E_2, ..., E_K\}$ onto the numbers $\{\xi_1, \xi_2, ..., \xi_K\}$ in such a way that the function $\xi(E)$ is the smooth part of $\eta(E)$ and $\hat{\eta}_{fl}(E)$ is the fluctuating part: $\eta(E) = \xi(E) + \hat{\eta}_{fl}(E)$. Thus, we define

$$\Delta_3 = \min_{A,B} \frac{1}{L} \int_{\xi_s}^{\xi_s+L} [\hat{\eta}(\xi) - A\xi - B]^2 d\xi, \tag{16}$$

where $\hat{\eta}(E)$ counts the number of levels in the interval $[\xi_s, \xi_s + L]$. Minimizing Eq. (16) we obtain the following relations:

$$\begin{cases} \frac{d\Delta_3}{dA} = -\frac{2}{L}\int_{\xi_s}^{\xi_s+L} \xi[\hat{\eta}(\xi) - A\xi - B]d\xi = 0 \\ \frac{d\Delta_3}{dB} = -\frac{2}{L}\int_{\xi_s}^{\xi_s+L} [\hat{\eta}(\xi) - A\xi - B]d\xi = 0 \end{cases}, \tag{17}$$

from which one finds

$$\begin{cases} A = \dfrac{px_1 - 2q}{x_1^2 - 2x_2} \\ B = \dfrac{qx_1 - px_2}{x_1^2 - 2x_2} \end{cases}, \tag{18}$$

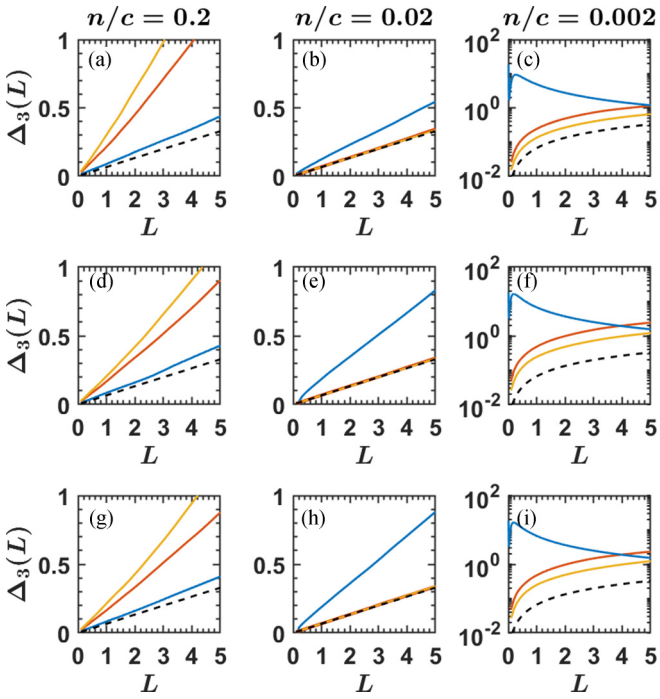

FIG. 5. The Δ_3 statistics obtained for different values of the total momentum, $P = 0, 1, 2$ (respectively, upper, middle, and lower row), averaged over 10^4 consecutive initial energies. Each column represents different interaction strengths: $n/c = 0.2$ (left), $n/c = 0.02$ (middle), $n/c = 0.002$ (right). Different colors used for different energy regions: low energy region with ξ_α starting from $\alpha = 1$ in blue, middle energy region with ξ_α starting from $\alpha = 10^6$ in red, and ξ_α starting from $\alpha = 2\times10^6$ in yellow, Dashed black line stands for the Poisson statistics. In computing spectra for the case $P = 0$ the accidental degeneracy of few eigenvalues has been eliminated.

where

$$x_1 = \frac{2}{L}\int_{\xi_s}^{\xi_s+L} \xi d\xi = 2\xi_s + L, \tag{19}$$

$$x_2 = \frac{2}{L}\int_{\xi_s}^{\xi_s+L} \xi^2 d\xi = \frac{2}{3}\left(L^2 3\xi_s^2 + 3\xi_s L\right), \tag{20}$$

$$p = \frac{2}{L}\int_{\xi_s}^{\xi_s+L} \hat{\eta}(\xi)d\xi \quad q = \frac{2}{L}\int_{\xi_s}^{\xi_s+L} \xi\hat{\eta}(\xi)d\xi, \tag{21}$$

and

$$t = \frac{2}{L}\int_{\xi_s}^{\xi_s+L} \hat{\eta}^2(\xi)d\xi. \tag{22}$$

With these parameters the expression for Δ_3 takes the form,

$$\Delta_3 = \tfrac{1}{2}t + \tfrac{1}{2}A^2x_2 + B^2 - Aq - Bp + ABx_1. \tag{23}$$

In case of a Poisson statistics the relation $\Delta_3(L) = L/15$ holds. In Fig. 5 we show the Δ_3 statistics for $N = 5$ and all possible values of total momentum $P = 0, 1, 2$ giving rise to independent spectra (one line for each fixed momentum). Moreover, we considered three different values of the interparticle interaction, $n/c = 0.2; 0.02; 0.002$ (from the left to the right column). In each panel we show the dependence of Δ_3 on L for not too large values $L < 5$, obtained in three

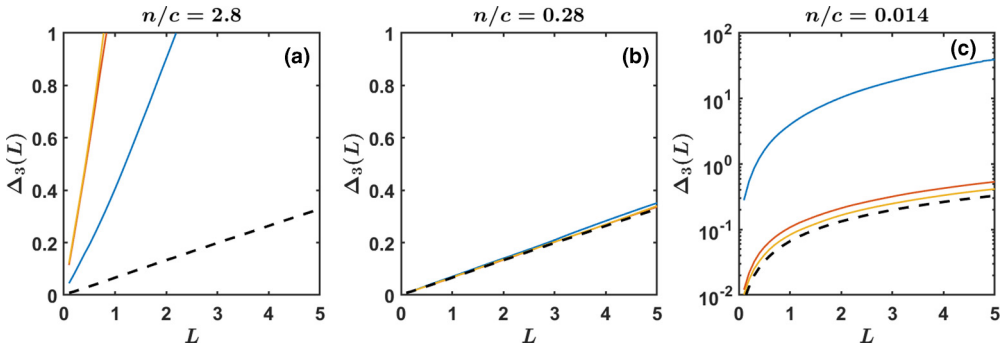

FIG. 6. The Δ_3 statistics obtained for total momentum $P = 1$ and $N = 7$ particles, averaged over 10^4 consecutive initial energies. Panels: (a) $n/c = 2.8$ (left), (b) $n/c = 0.28$ (middle), (c) $n/c = 0.014$ (right). Different colors refer to different energy regions: low energy region with ξ_α starting from $\alpha = 1$ in blue, middle energy region with ξ_α starting from $\alpha = 10^6$ in red, and ξ_α starting from $\alpha = 2\times10^6$ in yellow, dashed black curve stands for the Poisson statistics.

different energy regions of the spectrum: a low energy region close to the ground state (in blue), a middle (red), and a high one (orange). Below we extend our analysis to the larger L values. As one can see from the comparison with the analytical prediction (indicated by a dashed line), the Δ_3 test shows some similarity with the Poisson statistics apparently only for intermediate values of the interaction (middle column) and sufficiently high energy values (not too close to the ground state).

To check whether the global properties of the Δ_3 statistics obtained for $N = 5$ are generic and not sensitive to the model parameters, we have slightly increased the number of particles (with approximately the same values of the inter-particle interaction). Note that we cannot increase N strongly due to a dramatic increase of the computation time. However, we believe that the studied values $N = 5, 7$ can be treated as typical ones for the situation with a finite number of particles, not too small and not very large. The data for $N = 7$ are presented in Fig. 6 which should be compared with the vertical central panel of Fig. 5. As one can see, the conclusion we draw is the same: the closest relation to the theoretical result for the linear dependence, $\Delta_3 \propto L$, emerges for the intermediate values of the perturbation strength and for large enough energies E_α of the eigenstates.

To check quantitatively the deviations from the prediction for random spectrum (straight line with the slope $1/15$) we fit $\Delta_3(L)$ to the dependence,

$$\Delta_3(L) = \gamma_0 L + \gamma_1, \tag{24}$$

in the range $0 \leqslant L \leqslant 5$. Then we plot in Fig. 7 the slope γ_0 as a function of the interaction strength n/c for different energy regions in the spectrum (see different colors). From this picture it is clear that from one side one can say that for any energy range a suitable range of values of the interaction can be found where the Δ_3 test indicates a rather good agreement with the value $\gamma_0 = 1/15$ in the region $0 < L < 5$). However, for any interaction strength one can find an energy range where the Δ_3 statistics indicates strong deviations from the Poisson predictions.

Now, we extend our study of the Δ_3 statistics on a scale much larger than $L = 5$. As was found in Ref. [24] for the (integrable) rectangular billiard, with an increase of L the deviation from the prediction $\gamma_0 = 1/15$ is increasing, even if both the LSD and Δ_3 tests have shown quite good correspondence to the theoretical predictions. Specifically, it was found that a kind of stiffness of the energy levels occurs when very large values of L are taking into account. Thus, here we ask the question whether the correlations between the energy levels not seen in our study for the intermediate values of interaction, will emerge on a larger L scale. To this end we study the Δ_3 statistics for very large L values for those interaction strength values for which the data seem to confirm the theoretical predictions (see, for instance, the middle column of Fig. 5). Results for this study are shown in Fig. 8. As one can see, the linear increase of Δ_3 lasts approximately up to $L \lesssim 200$ (we considered 10^4 different energy levels). Beyond this scale, we have discovered a dramatic increase for the slope of the Δ_3

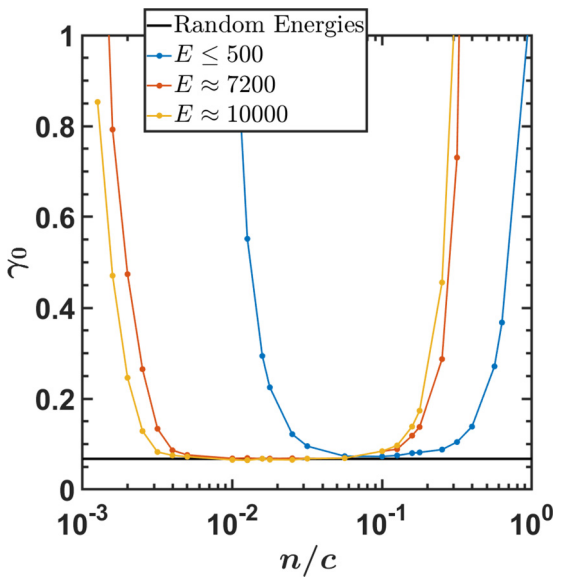

FIG. 7. Slope of $\Delta_3(L)$ for different values of the interaction and different regions of the spectrum as in previous figures, compared with the slope obtained with a completely random sequence (see the black line). The data are presented for $P = 2$ total momentum, and the average is done over a set of 10^4 energies around the energy indicated in the legend. The log-scale in x axis shows a nice symmetry with respect to a particular interaction strengths n/c in dependence on the chosen energy range.

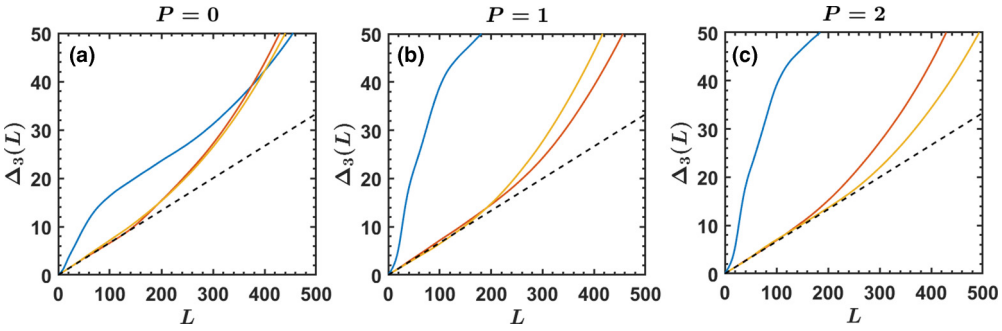

FIG. 8. Average Δ_3 statistics for large L values. The parameters are the same as in Fig. 5, central column.

statistics. This effect is similar to that found numerically in Ref. [24]. However, to our surprise the deviations for the slope occur in the opposite direction. Namely, at variance with the results of Ref. [24] demonstrating a kind of a decrease of the rigidity, our data clearly indicate an *increase of the rigidity* of the energy spectrum with an increase of L. Specifically, the slope of Δ_3 statistics increases with L [we remind that for the Wigner-Dyson statistics, $\Delta_3(L) \approx \ln(L)$] and not decreases as found in Ref. [24]. This fact that $\Delta_3 \propto L^{\alpha}$ with $\alpha > 1$ for $L \gg 1$ which we have observed numerically, may have a strong impact for the dynamical properties of the Lieb-Liniger model, and still awaits for additional clarifications.

VI. SUMMARY

Our analysis shows that the properties of energy spectra of the LL-model dramatically depends on whether we consider the total energy spectrum or only a subset of energy levels associated with a fixed value of the total momentum (which is a physical constant of motion besides the energy). By considering the total energy spectrum, we found very strong clustering of energy levels for any strength of the interparticle interaction. This means that it is meaningless to speak of the Poisson form of the level spacing distribution unless the total momentum is fixed. The conservation of total momentum is a distinctive property of the LL model, that has to be taken into account when quantifying the statistical properties of energy spectrum.

Due to the above peculiarity of the LL model, our main attention was paid to the properties of energy spectra for fixed total momentum, in dependence on other model parameters, such as the strength of interparticle interaction and the energy of many-body eigenstates. The rigorous analysis has revealed a quite unexpected property of energy spectra. It was found that some of the subsets of energy levels associated with specific values of the total momentum, can be obtained from other values of total momentum, with an appropriate shift along the energy spectrum. Therefore, such energy subsets are strongly correlated and this fact does not depend on the interparticle interaction. This property of energy spectra may have strong impact for the dynamical properties of the LL-model.

By studying the level spacing distribution (LSD) for fixed values of the total momentum, it can be shown that for zero and infinite inter-particle interaction there is a strong degeneracy of the energy levels, that gradually disappears moving away from the limits of either zero or infinite interaction. We have found that depending on the model parameters one may speak of the Poisson form of LSD, however, the residual effect of the clustering of levels for $s = 0$ may remain. Still, the question about what happens with the LSD with an increase of the number of particles or with an increase of the energy, awaits for further clarification.

We have also used another test to explore the absence of correlations between nearest energy levels based on the ratio of consecutive level spacings. In contrast with the LSD test, it does not need the unfolding of the energy spectrum. However, the considered ratios of spacings reveal very strong fluctuations that can be only washed out by a further averaging over a large numbers of spacings. Nevertheless, we have numerically overcame this problem and found a quite good confirmation of the results obtained by studying the LSD when the interaction between particles is neither very weak nor very strong.

Our further analysis is due to the so-called Δ_3 statistics, a test which measures the correlations between distant energy levels and gives information about the rigidity of the spectrum. This test is much more sensitive if compared to the LSD test which detects the correlations between the nearest energy levels only. To this end, we have carefully studied the Δ_3 statistics in the LL-model and found that the agreement with Poisson lasts for small values of L only. However, with an increase of L we typically observe strong deviations from the theoretical predictions obtained for truly random sequences. It is interesting that these deviations are in contrast to those found in Ref. [24], and seem to be influenced by the rigidity of energy spectrum. This rigidity strongly depends on the model parameters, as we found when changing the number of interacting particles.

We would like to stress that our numerical study is free from the effects of the cutoff of the energy spectrum. The control of the accuracy of numerically obtained energy levels is a typical problem in any numerical approach (see, for instance, Refs. [30,32,45–47]). The peculiarity of our approach is that the accuracy of the computation of energy levels is determined by the numerical solver of the equations Eq. (3) determining the rapidities λ only. Since the accuracy in solving these equations is extremely high, one can treat our computation as *exact*.

In conclusion, our study manifests that the common belief of the pseudorandom character of energy spectra of integrable many-body systems cannot be taken as general, at least when the number of particles is finite. Second, we have revealed the details of how pseudorandom properties of energy spectra

of the LL-model depend on the energy and the interaction strength, for fixed values of the total momentum. We hope that our results may be important for the study of quench dynamics, and can stimulate the mathematical community in proving rigorous results in this direction.

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