Nonequilibrium dynamics in Ising-like models with biased initial condition

Reshmi Roy [*] and Parongama Sen [†]
Department of Physics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata 700009, India

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We investigate the dynamical fixed points of the zero temperature Glauber dynamics in Ising-like models. The stability analysis of the fixed points in the mean field calculation shows the existence of an exponent that depends on the coordination number z in the Ising model. For the generalized voter model, a phase diagram is obtained based on this study. Numerical results for the Ising model for both the mean field case and short ranged models on lattices with different values of z are also obtained. A related study is the behavior of the exit probability $E(x_0)$, defined as the probability that a configuration ends up with all spins up starting with x_0 fraction of up spins. An interesting result is $E(x_0) = x_0$ in the mean field approximation when $z = 2$, which is consistent with the conserved magnetization in the system. For larger values of z, $E(x_0)$ shows the usual finite size dependent nonlinear behavior both in the mean field model and in the Ising model with nearest neighbor interaction on different two dimensional lattices. For such a behavior, a data collapse of $E(x_0)$ is obtained using $y = \frac{(x_0-x_c)}{x_c}L^{1/\nu}$ as the scaling variable and $f(y) = \frac{1+\tanh(\lambda y)}{2}$ appears as the scaling function. The universality of the exponent and the scaling factor is investigated.

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I. INTRODUCTION

Nonequilibrium dynamics associated with spin systems quenched from a high temperature have been extensively studied in the past. Various features associated with the ordering dynamics have been explored for the Ising model defined by the Hamiltonian $H = -J\sum_{ij}\sigma_i\sigma_j$ ($\sigma_i = \pm 1$) [1]. Classical spin models have no intrinsic dynamics; however, one can study the stochastic time evolution using certain dynamical algorithms that maintain the detailed balance [2]. Glauber dynamics is one of the popular choices that reduces to a simple energy lowering scheme at zero temperature. To study the ordering process, the system is taken to be completely disordered (i.e., at a high temperature) initially and suddenly cooled to a lower temperature T; we consider $T = 0$ specifically in this paper. In finite systems, the one dimensional Ising-Glauber model, following such a zero temperature quench, always ends up with all spins up or down irrespective of the initial fraction of up spin x_0. In higher dimensions, striped and blinker states can also be reached when the initial state is completely disordered, i.e., $x_0 = 0.5$ [2–5]. On the other hand, there are a fairly large number of models which use Ising spins but without any energy function associated with them, for example the voter model. In such models, the system evolves by a given dynamical rule.

Various features in the ordering process, for example domain growth, persistence, aging, time evolution of the order parameter, and other relevant quantities, have been studied for quite some time, particularly in the spin models. Exit probability is another feature associated with the nonequilibrium dynamics that has received a fair amount of attention more recently [6–21]. The exit probability $E(x_0)$ is defined as the probability that an all-up configuration is reached starting from x_0 fraction of up spins. $E(x_0)$ is linear in the one dimensional Ising model and the voter model (in all dimensions): $E(x_0) = x_0$ [2]; this occurs due to the conservation of the order parameter. In contrast, in the two dimensional Ising system $E(x_0)$ is nonlinear and shows strong finite size effects [18]. The exit probability as well as the dynamics have also been studied in the recent past for binary opinion dynamics models using mean field and several other analyses simple square lattices [7,8,15,17,22–24].

In this paper, we have considered the dynamics of Ising and Ising-like models where the evolution of the fraction of up spins (x) is studied. A mean field approach leads to the identification of the fixed points. We note that a nontrivial fixed point is $x = 0.5$ which corresponds to a disordered state. The stability of this fixed point is studied by starting from a biased but uncorrelated initial condition where the initial fraction x_0 deviates from 0.5. The results for the mean field Ising model, obtained for different values of z, the coordination number, are compared with the short range model in finite dimensions. The fixed points for the generalized voter model (GVM) are also obtained parametrically, and the stability analysis leads to obtaining the mean field phase diagram in the two parameter plane.

The evolution of x in time helps in understanding the behavior of the exit probability. The exit probability is computed numerically for the Ising-Glauber model in square and triangular lattices and the mean field Ising model. The results for finite sizes show the existence of a scaling function with which two parameters can be associated, as noted in some earlier studies [12,13].

*reshmi.roy80@gmail.com
†parongama@gmail.com; psphy@caluniv.ac.in

II. MEAN FIELD CALCULATION IN THE ISING MODEL

A. Master equation approach

We first consider the Ising model in the mean field approximation. The master equation for the variable $x(t)$, the fraction of up spins at time t, is set up after calculating the spin flip probabilities. The system evolves under the zero temperature Glauber dynamics; i.e., spins are flipped when energy decreases by it and flipped with probability 1/2 when energy does not change by flipping. For a particular spin, a neighboring spin here is simply another spin with which it interacts and the number of such neighbors or the coordination number z is taken as a variable.

1. $z = 2$

We first consider the case $z = 2$. Suppressing the argument t for x, an up spin flips with probability

(i) $(1-x)^2$, when it has two neighboring down spin;

(ii) $2x(1-x)/2$ when it has 1 down neighbor and 1 up neighbor. This can happen in two ways and for each of the cases the spin flips with probability $\frac{1}{2}$.

Denoting P_+ (P_-) as the total probability that an up (down) spin flips, one can therefore write

$$P_+ = (1-x)^2 + x(1-x),$$
$$P_- = x^2 + x(1-x). \tag{1}$$

The evolution equation for $x(t)$ can be expressed in general as

$$\frac{dx}{dt} = (1-x)P_- - xP_+, \tag{2}$$

which reduces to $\frac{dx}{dt} = 0$ using Eq. (1). This implies $x(t) = x_0$; i.e., the dynamics conserve the magnetization $m(t) = 2x(t) - 1$ such that $E(x_0) = x_0$ in this case obviously.

2. $z = 4$

We next consider the case for $z = 4$. In this case, an up spin flips with probability

(i) $(1-x)^4$ if it has 4 neighboring down spins;

(ii) $4x(1-x)^3$ when it has 3 down neighbors and 1 up neighbor which can happen in 4 ways;

(iii) $6x^2(1-x)^2/2$ in the case of 2 up and 2 down neighbors (possible in 6 ways and the spin flips with probability 1/2 in each case).

Therefore,

$$P_+ = (1-x)^4 + 4x(1-x)^3 + 3x^2(1-x)^2,$$
$$P_- = x^4 + 4x^3(1-x) + 3x^2(1-x)^2. \tag{3}$$

At the steady state, we obtain from the master equation

$$\frac{dx}{dt} = -2x^3 + 3x^2 - x = 0, \tag{4}$$

with the solutions $x^* = 0, 0.5, 1$.

To check the stability of the solutions, we consider $x = x^* + \delta(t)$ where $\delta(t)$ is the deviation from the fixed point. For both $x^* = 0$ and 1, considering only up to linear order terms in δ, we get $\frac{d\delta}{dt} = -\delta$, the solution of which is

$$\delta(t) = \delta_0 \exp(-t), \tag{5}$$

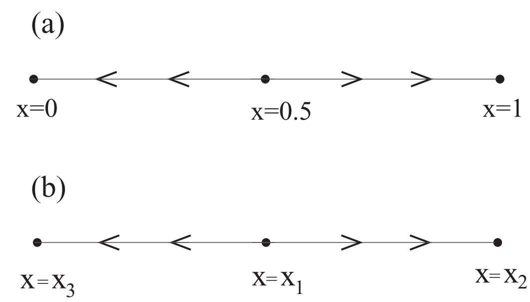

FIG. 1. (a) Flow diagram for the mean field Ising model for $z > 2$. $x = 0$ and 1 are the stable fixed points and $x = 0.5$ is unstable. (b) shows the flow diagram for the generalized voter model. $x_1 = 0.5$ is the unstable fixed point and $x_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{2z_1+z_2-2}{2z_1-z_2}}$ and $x_3 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{2z_1+z_2-2}{2z_1-z_2}}$ are stable fixed points provided that the exponent is positive.

where $\delta_0 \equiv \delta\ (t = 0)$. The negative exponent implies that $x^* = 0$ and 1 are stable fixed points.

For $x^* = 0.5$, one gets

$$\delta(t) = \delta_0 \exp\left(\tfrac{1}{2}t\right). \tag{6}$$

The positive exponent here implies that $x^* = 0.5$ is an unstable fixed point. Of course, δ cannot increase indefinitely and its extreme values are ± 0.5. The stability analysis thus shows that the system ends up with all spins up or down (for δ_0 positive or negative). The magnetization $m(t) = 2\delta(t)$ here.

3. $z = 6$

A similar analysis is done for $z = 6$. Here, P_+ and P_- can be expressed as

$$P_+ = (1-x)^6 + 6x(1-x)^5 + 15x^2(1-x)^4 + 10x^3(1-x)^3,$$
$$P_- = x^6 + 6x^5(1-x) + 15x^4(1-x)^2 + 10x^3(1-x)^3, \tag{7}$$

and therefore the master equation can be written as

$$\frac{dx}{dt} = 6x^5 - 15x^4 + 10x^3 - x. \tag{8}$$

Solving the steady state equation $\frac{dx}{dt} = 0$, one gets $x^* = 0, \frac{1}{2}, \frac{3+\sqrt{21}}{6}, \frac{3-\sqrt{21}}{6}, 1$. The third and fourth solution being unphysical, $x^* = 0, 0.5$, and 1 are the only physical solutions. Considering $x = x^* + \delta(t)$ in Eq. (8), the solution becomes $\delta(t) = \delta_0 \exp(-t)$ for $x^* = 0$ and 1 which are stable fixed points. For $x^* = 0.5$, one obtains

$$\delta(t) = \delta_0 \exp\left(\tfrac{7}{8}t\right), \tag{9}$$

which shows that $x^* = 0.5$ is again an unstable fixed point. The larger value of the exponent in the $z = 6$ case indicates that the dynamics are faster for $z = 6$ compared to that in the $z = 4$ case. Figure 1(a) shows the flow diagram for the mean field Ising model for $z = 4$ and $z = 6$.

From the above studies, we conclude that in the mean field approximation, in general $\delta(t) = \delta_0 \exp(\gamma t)$ where γ increases with z. This behavior is a short time one as the system reaches the stable fixed points at long times, confirmed by the simulation results discussed in the next subsection.

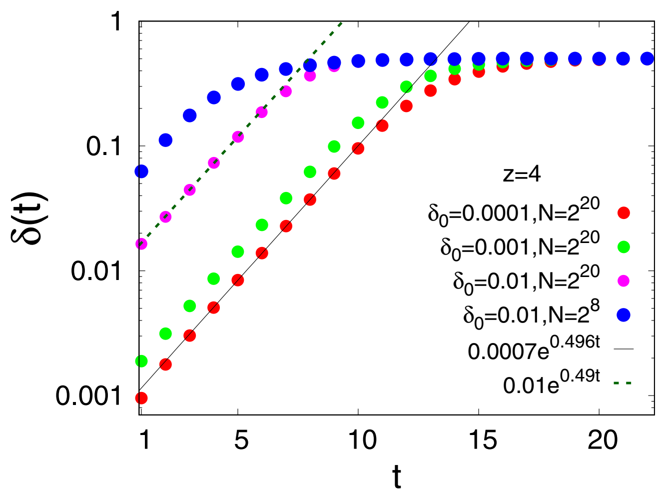

FIG. 2. Variation of $\delta(t)$ is shown with time for $z = 4$ for different δ_0 where mean field approach is used. This plot also shows the data for several system sizes. Data are fitted to the exponential function, mentioned in the key. Maximum number of configuration was 5000. $\delta(t)$ attains the saturation value faster for larger δ_0 and the process is slower for smaller value of δ_0.

Hence γ can be interpreted as an inverse timescale over which the exponential growth of $\delta(t)$ can be observed.

B. Simulation results

To check the mean field results we have conducted simulations where a spin interacts with randomly chosen z neighbors. The system consists of N spins and the choice of the random neighbor is made in an annealed manner which implies the interaction can take place with different spins at each step in general.

We defer the discussion on the $z = 2$ case to Sec. IV and consider the cases $z = 4$ and 6 where we expect an unstable point at $x = 0.5$. We have started from a fixed initial fraction of up spin $x_0 = 0.5 + \delta_0$ with $\delta_0 > 0$ and studied how the fraction $\delta(t)$ [$= x(t) - 0.5$] evolves in time. N updates constitute one single Monte Carlo step. Here, we have considered only those configurations for which positive consensus is attained to obtain the exponent γ and compare with the result found in the analytical calculation.

$\delta(t)$ shows an exponential growth with time which shows consistency with the results of Sec. II A as N is increased. The results for $z = 4$ shown in Fig. 2 indicate the exponential growth becomes more noticeable as N increases and that the associated exponent ≈ 0.5 is independent of δ_0 for all practical purposes. A data collapse for different values of δ_0 is obtained by scaling $\delta(t)$ by δ_0, shown in Fig. 3, which is also consistent with the analytical results.

A similar estimation has been done for $z = 6$ by considering the interaction of the selected spin with 6 randomly chosen neighbors. $\delta(t)$ shows an exponential behavior with time again and the exponent is $\sim$0.86. Figure 4 shows the results. The exponent $7/8$ obtained in Sec. II A for $z = 6$ agrees fairly well with the simulation results.

It should be mentioned here that the saturation is obtained very rapidly; the exponential fitting is therefore valid only for a few initial time steps. The saturation is enhanced for larger values of δ_0 and z.

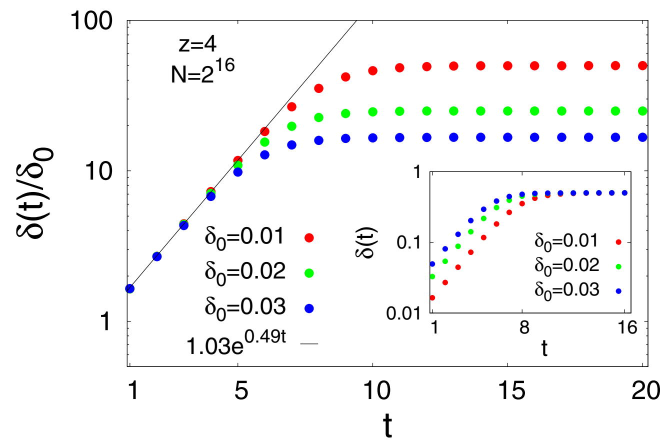

FIG. 3. Data collapse of $\delta(t)$ is shown with time for several δ_0 where $z = 4$ and data are fitted to an exponential form as mentioned in the key. These data are for a system of 2^{16} spins averaged over a maximum of 5000 realizations. Inset shows the unscaled data. In this simulation mean field approach is used.

III. SIMULATIONS FOR SHORT RANGE MODELS ON LATTICES

The simulations for the Ising model are repeated on two dimensional lattices where the spins have short range interactions. We consider the vicinity of the unstable fixed point again, such that $x(0) = 0.5 + \delta(0)$, and study the evolution of $\delta(t)$ where $x(t) = 0.5 + \delta(t)$.

In order to check the dependence on z, we have considered square lattices with nearest and nearest plus next nearest neighbors and triangular lattices with nearest neighbors such that $z = 4$, 8, and 6, respectively.

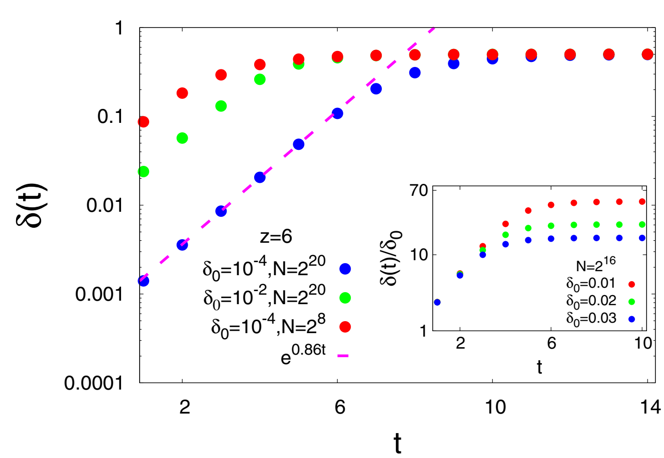

FIG. 4. Plots of $\delta(t)$ are shown against time for several system sizes where $z = 6$. Data for several values of δ_0 are also shown. Inset shows the data collapse for different δ_0 for a system of 2^{16} spins. As δ_0 increases, $\delta(t)$ rapidly saturates. Minimum number of different initial configuration was 2000 and mean field approximation is used for this simulation.

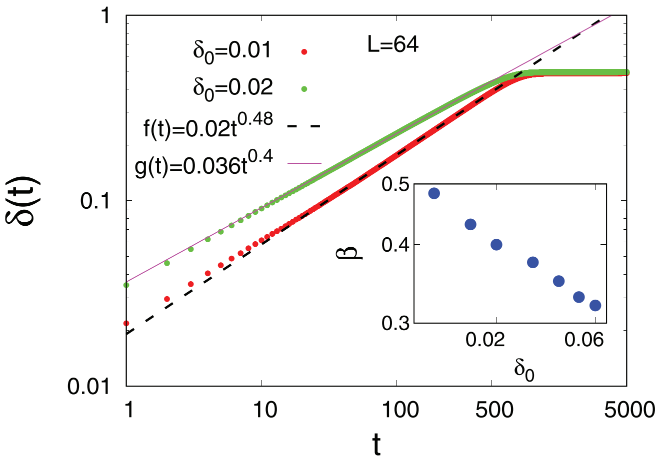

FIG. 5. Variation of $\delta(t)$ with time is shown in square lattice nearest neighbor Ising model for several δ_0. Data are fitted to the power law form; exponents are mentioned in the key. The power law exponent β decreases as the value of δ_0 increases. Inset shows the variation of β with δ_0. These data are for system size $L \times L = 64 \times 64$ averaged over 5000 realizations.

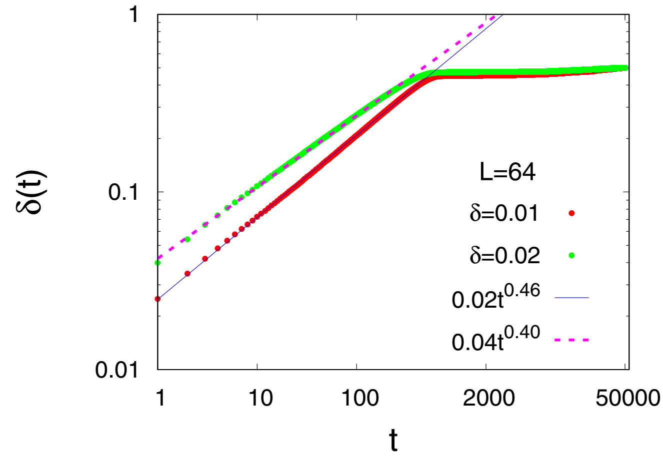

FIG. 6. Variation of $\delta(t)$ with time is shown in triangular lattice for nearest neighbor interaction. Data are fitted to Eq. (10) and the exponents are mentioned in the key. These data are for system size $L \times L = 64 \times 64$ averaged over 5000 realizations.

It is well known that the absolute value of the magnetization grows as t^{β} in the ordering process of the Ising model in all finite dimensions when the initial configuration is completely disordered. This follows from the fact that domains of up or down spins both grow as $t^{\eta d}$ where η is the domain growth exponent and d is the spatial dimension. The magnetization is given by $m = \sum \xi_i$, where $\xi_i \propto \pm t^{\eta d}$ are uncorrelated random variables and the sum is over all domains. The stochastic variable m thus satisfies $\langle m \rangle = 0$ and $\langle m^2 \rangle \propto t^{\eta d}$, leading to the result $|m| \propto t^{\frac{\eta d}{2}}$. One can also derive this from the dynamic scaling obeyed by the correlation function [1]. It is known that $\eta = \frac{1}{2}$ in all dimensions and thus β is dependent on the dimension: in two dimensions $\beta = 0.5$. For $x(t) = 0.5 + \delta(t)$, as mentioned before, magnetization is simply $2\delta(t)$ and the variation of $m(t)$ and $\delta(t)$ would be identical.

It is observed that for any value of z and δ_0, $\delta(t)$ shows a power law behavior with time before reaching the saturation value for all values of δ_0:

$$\delta(t) \sim t^{\beta}. \tag{10}$$

The results for $z = 4$ are shown in Fig. 5. The value of β depends on δ_0; as δ_0 increases (which means the system is more ordered to begin with), it decreases as shown in the inset of Fig. 5 for $z = 4$. This is understandable; in the limit $\delta_0 \to 0.5$, the system is almost static such that the time dependence is weak reflected by a smaller value of β.

In the triangular lattice, where $z = 6$, $\delta(t)$ is also found to show a power law variation with time according to Eq. (10). As δ_0 increases, β decreases as indicated by the data presented in Fig. 6. The values of β are reasonably close to those obtained in the square lattice.

We also consider the the Ising model with a Moore neighborhood where next nearest neighbor interactions are included and $z = 8$. The Hamiltonian of this system is given by

$$H = -J_1 \sum_{\langle i,j \rangle} S_i S_j - J_2 \sum_{\langle i,j' \rangle} S_i S_j, \tag{11}$$

where J_1 and J_2 are the strengths of interaction for nearest neighbor and next nearest neighbor, respectively. We have considered the interactions to be equal in strength, $J_1 = J_2$. Here, $z = 8$ and once again we find a behavior similar to $z = 4, 6$ in two dimensions (see Fig. 7).

It is also interesting to check whether for the same value of z but in a different dimension, the value of β remains the same. For this, simulations have been conducted on a cubic lattice Ising system where $z = 6$ as in the triangular lattice. $\delta(t)$ shows a power law variation in this case also; however, the exponent β is larger compared to the two dimensional case (see Fig. 8).

The above results show that the exponent β is independent of z in two dimensions while for three dimensions, with the same z we find a different value of β when $\delta_0 \neq 0$.

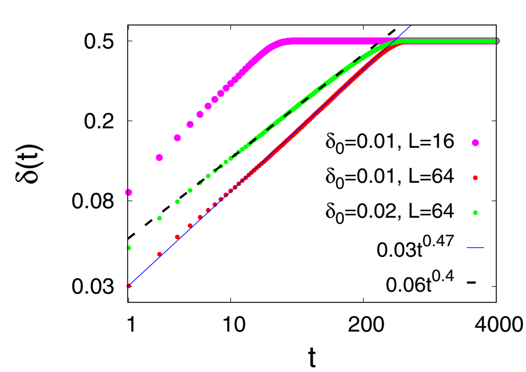

FIG. 7. Variation of $\delta(t)$ with time is shown in square lattice next nearest neighbor Ising model for several system sizes. Data are fitted to the power law form as mentioned in the key.

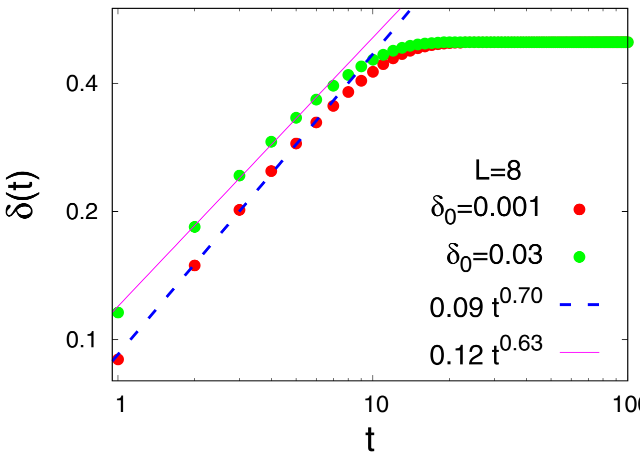

FIG. 8. Variation of $\delta(t)$ with time is shown in cubic lattice nearest neighbor Ising model for different δ_0. β decreases as the value of δ_0 increases. Data are fitted to the power law form; exponents are mentioned in the key. The power law exponent β decreases as the value of δ_0 increases. These data are for system size $L^3 = 8^3$ averaged over 5000 realizations.

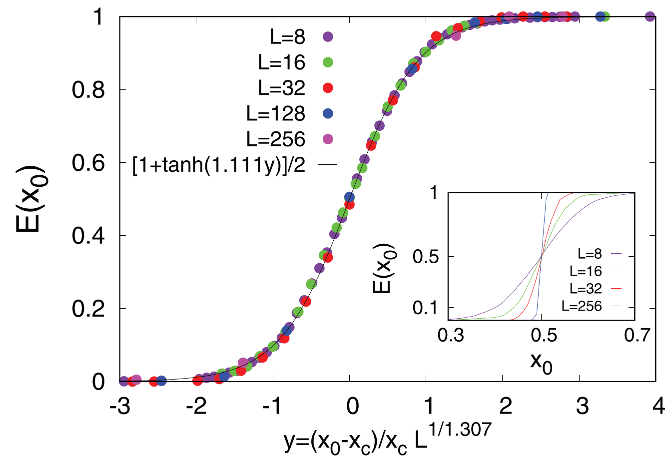

FIG. 9. Data collapse of $E(x_0)$ is shown for different system sizes in square lattice Ising model using $\nu = 1.307$. Data are fitted to the form of Eq. (13) as mentioned in the key. Inset shows the unscaled data. Number of different initial configuration was 5000 for these simulations.

IV. EXIT PROBABILITY

In this section, we present the results for $E(x_0)$, the probability that the system ends up in a state with all spins up, starting from an initial state with x_0 fraction of up spins. Since some results are already known for the short range Ising models, we first discuss that and then continue to report the results for the mean field case.

A. Results for nearest neighbor interactions

Next, we have studied the exit probability for the two dimensional nearest neighbor Ising models. Exit probability $E(x_0)$ is known to have a liner behavior $E(x_0) = x_0$ for a one dimensional Ising Glauber model. In a two dimensional model $E(x_0)$ is nonlinear and shows strong finite size effects [18–20]. As the system size increases the curves become steeper and the data suggest that $E(x_0)$ approaches a step function in the thermodynamic limit. Finite size scaling can be done using the form

$$E(x_0, L) = f\left(\frac{x_0 - x_c}{x_c} L^{\frac{1}{\nu}}\right), \tag{12}$$

as observed in Ref. [12], where $f(y) \to 0$ for $y \ll 0$ and is equal to 1 for $y \gg 0$. Therefore, a data collapse for different system size can be obtained when $E(x_0)$ is plotted against $\frac{x_0 - x_c}{x_c} L^{1/\nu}$ where $x_c = 0.5$. On square lattices, an Ising system freezes into a striped configuration for $x_0 = 0.5$ in 33.9% of cases (an exact result [5]) in the thermodynamic limit. Numerical simulations show that the freezing probability has strong system size dependence [3,4]. However, the dynamical scaling behavior remains intact in spite of the freezing. Very close to $x_0 = 0.5$, such frozen striped states may occur with a nonzero probability in finite systems as shown in Ref. [4]. While calculating $E(x_0)$, such configurations have been discarded.

The data collapse is obtained using eye estimation for the square lattice Ising model when $\nu \approx 1.3$ agreeing with the result of [19,20]. The collapsed data can be fitted to the form

$$f(y) = \frac{1 + \tanh(\lambda y)}{2} \tag{13}$$

as in Ref. [13]. The value of λ turns out to be 1.10 using GNUFIT.

To get a more accurate value of ν required for obtaining the best data collapse, we have employed another method used previously in Ref. [20]. We have calculated $y = \frac{x_0 - x_c}{x_c} L^{1/\nu}$ for the different values of L. As the data collapse is supposed to fit to the form of Eq. (13) we have chosen the range of ν and λ for which the collapse and fitting seem good. We have varied the values of ν and λ in steps of 0.001 and for every pair we have calculated the error ϵ given by

$$\epsilon = \frac{1}{n}\sqrt{\sum_n [f(y) - E(x_0)]^2}. \tag{14}$$

The values of the pair of ν and λ for which the minimum value of ϵ is obtained are the optimal values required for best data collapse and scaling function. The values of ν and λ are 1.307 and 1.111 using this method and the results are shown in Fig. 9.

These results were already available from previous studies, although for smaller system sizes. We repeat these simulations as our aim is to determine whether any universality in the scaling behavior exists in two dimensional Ising systems. Hence we have studied the exit probability in a triangular lattice (number of nearest neighbors $z = 6$). To obtain the best data collapse the least squares method has been employed in this case also and is graphically illustrated in Fig. 10. The data collapse of $E(x_0)$ for different system sizes is obtained with $\nu = 1.204$ using the above method and the scaled data are fitted according to Eq. (13). Figure 11 shows the results. The value of ν is close but not exactly equal to the value obtained for the square lattice. $\lambda = 0.857$ is definitely different.

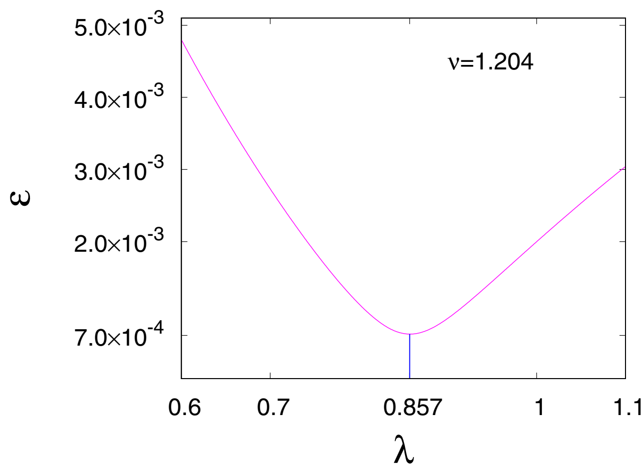

FIG. 10. Variation of the least square error ϵ with λ is shown for $\nu = 1.204$ in triangular lattice. The minimum of the curve is at $\lambda = 0.857$.

B. Results for mean-field-like model

Here we present the results for the exit probability $E(x_0)$ using a mean field approach where the z neighbors are chosen randomly.

For $z = 2$, the exit probability shows a linear behavior $E(x_0) = x_0$ (see Fig. 12). This is consistent with the conservation we noted for x in Sec. II A. It may seem surprising that the mean field result with $z = 2$ gives the exact result known for the one dimensional Ising model. We attempt to justify why this happens in the following way.

We note that for the voter model, the ith spin σ_i flips with a probability

$$w(\sigma_i) = \frac{1}{2}\left(1 - \sigma_i \sum_j \sigma_j / z\right), \tag{15}$$

where j is a neighbor of i. This probability is valid in any dimension. Thus the above dynamics in the voter model conserve the total spin in any dimension. It is well known that in

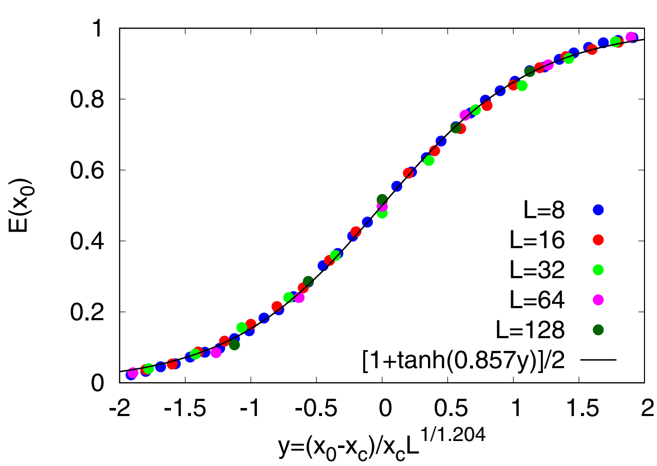

FIG. 11. Data collapse of $E(x_0)$ is shown for different system sizes in triangular lattice using $\nu = 1.204$ and data are fitted to Eq. (13). Number of different initial configuration was 5000 for these simulations.

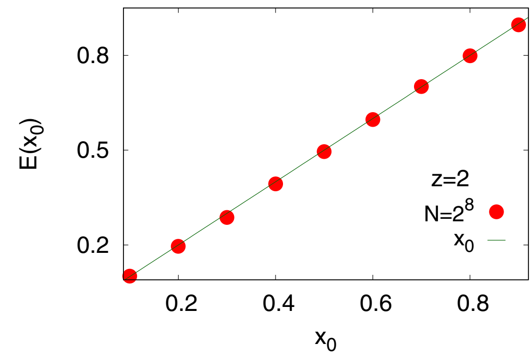

FIG. 12. $E(x_0)$ is shown against x_0 for $z = 2$ where mean field approach is used. $E(x_0)$ shows a linear variation with x_0. Number of different initial configuration was 5000 for these simulations.

one dimension, the voter model dynamics coincide with the Ising dynamics where $z = 2$. In the mean field calculations for $z = 2$ it is evident that the voter model dynamics are being used precisely and since in the latter, conservation is valid always, we get a result which is the exact one for the one dimensional Ising model too. It is interesting to note that hence for the $z = 2$ case, it does not matter whether one picks up randomly any two neighbors or strictly the nearest neighbors as far as conservation is concerned. We have also checked that if the choice of neighbors is done randomly in a quenched manner, the results remain the same.

For other values of z, $E(x_0)$ becomes steeper in the mean field case than that was obtained using nearest neighbor interactions. Here, a data collapse is obtained by plotting $E(x_0)$ against $\frac{x_0 - x_c}{x_c} N^{1/\nu'}$ (where N is the total number of spins) using $\nu' = 2$ for both $z = 4$ and $z = 6$. The scaled data are fitted to the form of Eq. (13) where $y = \frac{x_0 - x_c}{x_c} N^{1/\nu'}$. The value of λ obtained for $z = 4$ is $\approx$0.48 and $\lambda \approx 0.58$ when $z = 6$. Data collapse of $E(x_0)$ is shown in Fig. 13 for $z = 4$ and $z = 6$.

Table I shows the values of ν, ν', and λ obtained numerically for the short ranged and mean field Ising models. Note that for the short ranged systems, $\nu' = 2\nu$. These results are discussed in the last section.

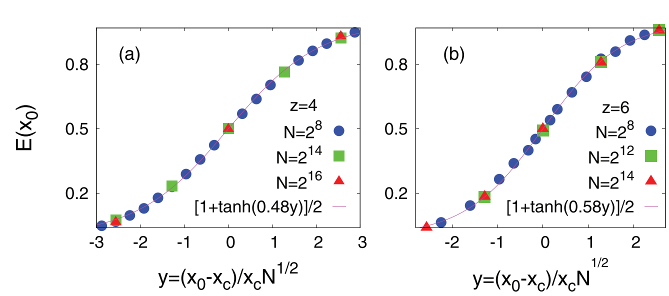

FIG. 13. (a) Data collapse of $E(x_0)$ is shown for different system sizes using $\nu' = 2$ where $z = 4$. (b) shows the data collapse of $E(x_0)$ for different system sizes with $\nu' = 2$ for $z = 6$. Mean field approximation is used in these simulations where number of different initial configuration was 5000. Data are fitted to the functions as mentioned in the key.

TABLE I. ν, ν', and λ obtained for Ising model using numerical simulations.

	Nearest neighbor (NN) interaction		Mean field	
Quantity	Square	Triangular	$z = 4$	$z = 6$
ν, ν' ($\nu' = 2\nu$ for NN models)	$\nu = 1.307(1)$	$\nu = 1.204(1)$	$\nu' \sim 2$	$\nu' \sim 2$
λ	1.111(1)	0.857(1)	~0.48	~0.58

V. GENERALIZED VOTER MODEL

We have considered next the generalized voter model. We first describe the model on a square lattice where a spin variable $\sigma_i = \pm 1$ is associated with every site of the lattice. The time evolution is governed by a single spin flip stochastic dynamics; the spin flip probability $w_i(\sigma)$ for the ith spin is given by [25],

$$w_i(\sigma) = \tfrac{1}{2}[1 - \sigma_i f_i(\sigma)], \tag{16}$$

where $f_i(\sigma) = f(\Sigma_\delta\, \sigma_{i+\delta})$, a function of the sum of the nearest neighbor spin variables. The model is defined taking $f(0) = 0$, $f(2) = -f(-2) = z_1$, and $f(4) = -f(-4) = z_2$, where z_1 and z_2 are restricted to $z_1 \leqslant 1$ and $z_2 \leqslant 1$. The original voter model corresponds to $z_1 = 0.5$ and $z_2 = 1$, and the Ising model is recovered for $z_1 = 1$, $z_2 = 1$.

In the mean field approximation to obtain the master equation, the above dynamical rule is followed which means that $z = 4$ is taken and the parameters defined as above. For an up spin, flipping probabilities P_i ($i = 0$ to 4), when there are i neighboring spins in the up state, are given by

$$P_1 = \tfrac{1}{2}(1 - z_2)x^4, \quad P_2 = 2(1 - z_1)x^3(1 - x),$$
$$P_3 = 3x^2(1 - x)^2, \quad P_4 = 2(1 + z_1)x(1 - x)^3,$$
$$P_0 = \tfrac{1}{2}(1 + z_2)(1 - x)^4.$$

The total probability P_+ that an up spin flips is $P_+ = \sum_{i=0}^{4} P_i$ such that

$$P_+ = \tfrac{1}{2}(1 - z_2)x^4 + 2(1 - z_1)x^3(1 - x) + 3x^2(1 - x)^2 + 2(1 + z_1)x(1 - x)^3 + \tfrac{1}{2}(1 + z_2)(1 - x)^4. \tag{17}$$

Similarly the probability P_- that a down spin flips is

$$P_- = \tfrac{1}{2}(1 - z_2)(1 - x)^4 + 2(1 - z_1)x(1 - x)^3 + 3x^2(1 - x)^2 + 2(1 + z_1)x^3(1 - x) + \tfrac{1}{2}(1 + z_2)x^4. \tag{18}$$

Therefore, the master equation $\frac{dx}{dt} = (1 - x)P_- - xP_+$ reduces to

$$\frac{dx}{dt} = x^3(-4z_1 + 2z_2) + x^2(6z_1 - 3z_2) + x(-2z_1 + 2z_2 - 1) + \frac{1}{2} - \frac{z_2}{2}. \tag{19}$$

Putting the values $z_1 = 0.5$ and $z_2 = 1$ in Eq. (19), one gets $\frac{dx}{dt} = 0$, consistent with the voter model result that there is conservation in any dimension. On the other hand, by taking $z_1 = 1$ and $z_2 = 1$ in Eq. (19), Eq. (4) is recovered for the Ising model with $z = 4$.

For general values of z_1 and z_2, the steady state condition leads to three fixed points x_1, x_2, x_3 where $x_1 = 0.5$, $x_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{2z_1+z_2-2}{2z_1-z_2}}$, and $x_3 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{2z_1+z_2-2}{2z_1-z_2}}$. Now, let us take $x(t) = x_1 + \delta(t)$, i.e., the behavior close to the fixed point $x_1 = 0.5$. Considering up to linear term in δ only, $\delta(t)$ is found to be

$$\delta(t) = \delta_0 \exp\left(z_1 + \tfrac{1}{2}z_2 - 1\right)t, \tag{20}$$

where $\delta_0 \equiv \delta(t = 0)$. The exponent is thus $z_1 + \frac{1}{2}z_2 - 1$.

We will now analyze the sign of the exponent and thus the stability of the fixed point $x_1 = 0.5$ which corresponds to a completely disordered state. Since the magnetization m is given by $2x - 1$, $|m| = (\frac{2z_1+z_2-2}{2z_1-z_2})^{\frac{1}{2}}$ for x_2 and x_3. m can have nonzero values for x_2 and x_3, provided $\frac{2z_1+z_2-2}{2z_1-z_2} \geqslant 0$ and also we require $|m| \leqslant 1$. The first criterion is satisfied ($|m| > 0$) when either (i) $2z_1 + z_2 - 2 \geqslant 0$ and $2z_1 - z_2 > 0$ or when (ii) $2z_1 + z_2 - 2 \leqslant 0$ and $2z_1 - z_2 < 0$. We note in the first case the first condition implies the second and in the next case the second condition implies the first one. Hence, for $|m| > 0$, $2z_1 + z_2 - 2$ and $2z_1 - z_2$ can in principle be either both positive or both negative. However, the condition that $|m| \leqslant 1$ is violated for case (ii) since $z_1, z_2 \leqslant 1$ and hence $m = 0$ is the only possible solution when $2z_1 + z_2 - 2 < 0$. Thus the only condition for an ordered region to exist is that the quantity $2z_1 + z_2 - 2$ must be positive. This is consistent with the fact that the exponent (which is an identical expression in z_1, z_2), has to be positive to make the $x_1 = 0.5$ (i.e., $m = 0$) fixed point unstable. On the other hand, when it is stable, i.e., $2z_1 + z_2 - 2 < 0$, $m = 0$ is the only solution. Figure 1(b) shows the flow diagram of the generalized voter model.

Hence the phase boundary between the ordered and disordered phases is given by $2z_1 + z_2 - 2 = 0$. We have plotted

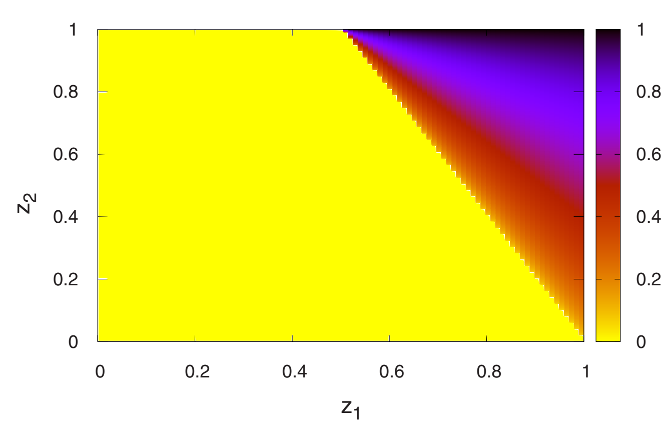

FIG. 14. Magnetization $|m|$ is shown as a function of z_1 and z_2.

the phase diagram in Fig. 14, where the magnitude of the magnetization is also indicated. Obviously, the mean field phase diagram shows a larger region that is ordered compared to the two dimensional case.

VI. SUMMARY AND CONCLUSIONS

We have studied the dynamics in zero temperature Ising-like systems with up-down symmetry with different coordination number z. Using the mean field approximation, it is observed that the dynamics always lead to one unstable fixed point which corresponds to the disordered state for $z > 2$. This fixed point is precisely $x = 0.5$ where x is the fraction of up spins. The stability of this fixed point has been considered by introducing a small deviation δ_0 from 0.5 in x. This essentially means we have a biased initial condition in the system with unequal fractions of up and down spins. The initial bias is generally considered to be small such that the system does not have any appreciable correlation.

For the unstable fixed points we obtain an initial exponential growth of $\delta(t)$ with time which strongly depends on the coordination number z. The growth is characterized by an exponent γ that increases with z. These results have been checked by numerical simulations for the mean field Ising model.

The simulations of the short ranged Ising model in two dimensions on the other hand showed that the behavior of $\delta(t)$ is a power law with time. The power law exponent is nonuniversal and depends on δ_0. The exit probability has also been calculated which for the two dimensional Ising model shows the expected nonlinear behavior. The exponent ν and the scaling factor λ related to the finite size behavior have been calculated. It appears that ν shows a weak dependence on the lattice structure (i.e., z) while for λ the values are appreciably different (Table I). The exit probability study for the mean field model on the other hand shows ν' is independent of z while λ again shows strong dependence.

We have also conducted a similar study for the two parameter generalized voter model. In this case, we find that the stability of the disordered fixed point depends on the parameter values and it is possible to obtain a phase diagram based on this analysis.

Our studies show that the behavior of $\delta(t)$ which is related to magnetization for the fixed point $x = 0.5$ is different in the mean field case and the short range model. However, when the number of neighbors $z = 2$, the mean field result that the dynamics conserve the ensemble magnetization is the same as that of the one dimensional Ising model or the voter model. We have justified this result on the basis of the voter model dynamics. Hence an important conclusion is that for $z = 2$, the results are independent of the range of the interaction.

The instability of the $x = 0.5$ fixed point for the higher values of z indicates the exit probability should be a step function in the mean field case in the thermodynamic limit. This behavior is found to be true for the short range models as well in which the exit probability for larger lattice sizes shows the tendency to approach a step function. However, the exponents associated with the finite size scaling analysis are quite different quantitatively. In particular, ν' is independent of z in the mean field case and less compared to the value obtained for the model on two dimensional lattices.

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