Nontrivial maturation metastate-average state in a one-dimensional long-range Ising spin glass: Above and below the upper critical range

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Understanding the low-temperature pure state structure of spin glasses remains an open problem in the field of statistical mechanics of disordered systems. Here we study Monte Carlo *dynamics*, performing simulations of the growth of correlations following a quench from infinite temperature to a temperature well below the spin-glass transition temperature T_c for a one-dimensional Ising spin-glass model with diluted long-range interactions. In this model, the probability P_{ij} that an edge $\{i, j\}$ has nonvanishing interaction falls as a power law with chord distance, $P_{ij} \propto 1/R_{ij}^{2\sigma}$, and we study a range of values of σ with $1/2 < \sigma < 1$. We consider a correlation function $C_4(r, t)$. A dynamic correlation length that shows power-law growth with time $\xi(t) \propto t^{1/z}$ can be identified in the data and, for large time t, $C_4(r, t)$ decays as a power law $r^{-\alpha_d}$ with distance r when $r \ll \xi(t)$. The calculation can be interpreted in terms of the *maturation* metastate averaged Gibbs state, or MMAS, and the decay exponent α_d differentiates between a trivial MMAS ($\alpha_d = 0$), as expected in the droplet picture of spin glasses, and a nontrivial MMAS ($\alpha_d \neq 0$), as in the replica-symmetry-breaking (RSB) or chaotic pairs pictures. We find nonzero α_d *even* in the regime $\sigma > 2/3$ which corresponds to short-range systems below six dimensions. For $\sigma < 2/3$, the decay exponent α_d follows the RSB prediction for the decay exponent $\alpha_s = 3 - 4\sigma$ of the *static* metastate, consistent with a conjectured statics-dynamics relation, while it approaches $\alpha_d = 1 - \sigma$ in the regime $2/3 < \sigma < 1$; however, it deviates from both lines in the vicinity of $\sigma = 2/3$.

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I. INTRODUCTION

The low-temperature equilibrium pure-state structure of classical Ising spin glasses has been debated for many years, and is still not well understood. The Ising spin-glass models [1] are defined with discrete two-state spin variables interacting on a d-dimensional hypercubic lattice ($s_i = \pm 1$ for a spin at lattice site $\mathbf{r}_i$ with lattice spacing 1) with the Hamiltonian

$$H(S) = -\sum_{\{i,j\}} J_{ij} s_i s_j, \tag{1}$$

where the bonds $J_{ij} = J_{ji}$ for each undirected edge (unordered pair) $\{i, j\}$ are independent random variables which form a collection $\mathcal{J} \equiv (J_{ij})_{\{ij\}}$. A spin configuration is denoted $S \equiv (s_i)_i$. The edges connect all distinct sites $i \neq j$, $\mathbf{r}_i \in \mathbf{Z}^d$ where d is the dimension of space (for a finite lattice Λ, $\mathbf{r}_i \in \Lambda$ with $\Lambda \subset \mathbf{Z}^d$ with some chosen boundary conditions) of the graph with vertices i and edges $\{i, j\}$. The probability distribution over the bonds $\mathcal{J}$ is denoted $\nu(\mathcal{J})$ and the disorder average over the bonds distribution is then denoted $[\cdots]_{\nu(\mathcal{J})}$.

For an *infinite* system there can be a phase transition which necessarily presents ergodicity breaking, in which the configuration space can be divided into disjoint regions, such that under time evolution the system remains forever in one region, but explores all of it (the dynamics restricted to one region is ergodic). An example of this is the ferromagnetic Ising model defined as for Eq. (1) but with constant couplings $J_{ij} = J > 0$ for edges connecting nearest neighbors ($|\mathbf{r}_i - \mathbf{r}_j| = 1$ with the Euclidean metric). In thermal equilibrium, below some nonzero critical temperature T_c (for spatial dimension $d \geqslant 2$) there is spontaneous symmetry breaking with at least two "pure" (or ordered) states, denoted $\Gamma_\uparrow$ ($\Gamma_\downarrow$) for the "up" ("down") state. Under time evolution of the system, there is zero probability that an initial configuration S drawn from the up pure state will be found in the down pure state, or in any pure state other than the up state (and similarly for initial S in the down state). [Here we restrict the discussion to boundary conditions imposed as $s_i = +1$ for $s_i \in \partial\Lambda$ for the up state and $s_i = -1$ for $s_i \in \partial\Lambda$ for the down state (with $\Lambda \to \infty$).] These two states are the only translationally invariant pure Gibbs states for the ferromagnetic Ising model below the phase transition temperature T_c in any dimension $d \geqslant 2$. Generally, ergodic states for the dynamics are the same as equilibrium pure states.

Unlike for the ferromagnetic Ising model, for the general spin-glass model of Eq. (1), the number and nature of pure states in the low-temperature phase is not clear. This has been debated extensively for short-range spin glasses including the Edwards-Anderson (EA) model [1] in low dimension d, following the early works of Refs. [2–4] which developed the theory of so-called replica-symmetry breaking (RSB) as a mean-field theory in infinite-range models, and of Refs. [5–9]

where the scaling-droplet (SD) theory of finite-range models was developed. In the EA model, only the nearest-neighbor bonds are nonzero, and they are identically distributed Gaussians with vanishing mean and variance unity,

$$[J_{ij}]_{\nu(\mathcal{J})} = 0, \quad \left[J_{ij}^2\right]_{\nu(\mathcal{J})} = 1. \tag{2}$$

Determining the pure-state structure for this model with analytical work is challenging as there is no controlled approach known for low dimension d.

We will consider a dynamical correlation function; in a moment we will further explain its relation to equilibrium properties. The function is defined by

$$C_4(\mathbf{r}_i - \mathbf{r}_j, t) \equiv \left[[\langle s_i(t)s_j(t)\rangle_{S|S_0}]^2_{\eta(S_0)}\right]_{\nu(\mathcal{J})}, \tag{3}$$

where (i) $\langle\cdots\rangle_{S|S_0}$ denotes an average over trajectories $\{S(t') : 0 < t' \leqslant t, S(0) = S_0\}$ of S in time t, with initial value S_0 at $t = 0$, under some stochastic dynamics that has the Gibbs distribution at temperature T as its stationary state (which is unique in a finite-size system; in practice, we will use Monte Carlo evolution), and (ii) $[\cdots]_{\eta(S_0)}$ is an expectation over a distribution $\eta(S_0)$ of initial configurations S_0, which is an infinite temperature state (i.e., the uniform distribution on spin configurations). This corresponds to dynamic evolution of the correlation function following an instantaneous quench from $T = \infty$ to a final temperature T, and we will choose T to be well below the equilibrium transition temperature T_c. Such a correlation function has been studied previously in Refs. [10–14] for the EA model, in Ref. [15] for a one-dimensional diluted long-range interacting model (which we will describe later), and for other models in Ref. [16]. In each case it was expected that $C_4(\mathbf{r}_i - \mathbf{r}_j, t)$ would follow a scaling ansatz

$$C_4(\mathbf{r}_i - \mathbf{r}_j, t) = \frac{1}{r_{ij}^{\alpha_d}} f\left(\frac{r_{ij}}{\xi(t)}\right), \tag{4}$$

where $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$, $f(x)$ is a scaling function [$f(x)$ tends to a constant as $x \to 0$], $\xi(t)$ is a dynamical correlation length, and α_d is the dynamic spatial decay exponent. The correlation length was expected to behave as a power law with time $\xi(t) \propto t^{1/z}$ for large time, where $z < \infty$ is a dynamical exponent. For $r_{ij} \ll \xi(t)$ this gives power-law decay $C_4(\mathbf{r}_i - \mathbf{r}_j, t) \propto 1/r_{ij}^{\alpha_d}$. One would expect that α_d is independent of T for $0 < T < T_c$, while $z = z(T)$ has been found to depend on T [15]. The ansatz was found to hold numerically (for the times and system sizes studied) with varying degrees of accuracy in Refs. [10–15].

If the power-law form indeed holds asymptotically in an infinite-size system, with $\alpha_d > 0$, it implies that as $t \to \infty$ the system reaches a statistical state described by expectations of the form $\lim_{t\to\infty}[\langle\cdots\rangle_{S|S_0}]_{\nu(S_0)}$ that are independent of t in the limit, with decay of equal-time correlations $\lim_{t\to\infty}[\langle s_i(t)s_j(t)\rangle_{S|S_0}]_{\eta(S_0)}$ to zero with distance, at least in the sense of the disorder average of the square. This differs dramatically from what should occur if the state in the long-time limit (which we assume is stationary, though it is not obvious this must hold; we return to this later) is what we will call a trivial Gibbs state, that is, one that (here again for $T < T_c$) is the equal-weight mixture of two pure states that are related by spin-flip symmetry, as the SD picture [5–9] assumes is the case in equilibrium at zero magnetic field. In the latter case the correlation function $\lim_{t\to\infty} C_4(\mathbf{r}_i - \mathbf{r}_j, t)$ would go to a positive constant as $r_{ij} \to \infty$, which means $\alpha_d = 0$. Thus $\alpha_d > 0$ should imply that there are (infinitely) many pure states, which are accessed by the protocol that defines C_4. We will view the state obtained at long times, averages in which take the form $\lim_{t\to\infty}[\langle\cdots\rangle_{S|S_0}]_{\eta(S_0)}$, as the maturation-metastate–average state, or MMAS (see Ref. [16]; the term metastate is used by analogy with the metastate in statics [17–20]). We further explain some of this in the following section; the value of α_d is quantitative information about the MMAS, and $\alpha_d > 0$ means the MMAS contains many pure states.

In Ref. [15], a one-dimensional diluted long-range interacting model was considered, which makes possible the use of very large (i.e., long) systems in which to consider the correlations. This model is a diluted, non-Gaussian variant (described below) of a well-known one-dimensional model [21] that has independent Gaussian distributions for the bonds $\mathcal{J}$; in both models, the bonds J_{ij} for each pair $\{i, j\}$ have mean zero and variance

$$\left[J_{ij}^2\right]_{\nu(\mathcal{J})} \propto \frac{1}{r_{ij}^{2\sigma}} \tag{5}$$

as $r_{ij} \to \infty$. These models have a transition with $T_c > 0$ for $1/2 < \sigma < 1$, and are sometimes considered as a proxy for a short-range interacting model with σ playing the role of d. Reference [15] considered a single value $\sigma = 0.625$ for which a suggested value of α_d was available (we discuss this in the following section), and obtained excellent agreement with that value.

In this paper we extend the study of Ref. [15] for the diluted long-range one-dimensional model to a wide range of $\sigma < 1$. We find a nontrivial metastate ($\alpha_d > 0$) for all such interactions σ. We also find support, in agreement with Ref. [15], for a conjectured statics-dynamics relation involving α_d in the region $\sigma < 2/3$, and some support for an empirical interpolating form in the complementary regime $2/3 \leqslant \sigma < 1$.

Some additional background material is presented in Sec. II, while the details and results of simulations are given in Sec. III. The Appendix discusses the methods used to obtain the best fits in the scaling collapse plots.

II. BACKGROUND

Here we will briefly review and explain a number of concepts to which we will refer, including states, Gibbs states, pure states, and metastates and metastate-average states, both static and dynamic, along with some of their properties. In general, in this section, systems are assumed to be of infinite size unless stated to be finite.

A. Equilibrium (Gibbs) states and pure states

First, by a state of an Ising spin system we always mean a probability distribution on spin configurations S. In finite-range spin systems, a state of thermal equilibrium is usually assumed to be a Gibbs state. We fix a choice of $\mathcal{J}$ throughout the discussion of Gibbs and pure states. In a *finite*-size system, a Gibbs state $\Gamma_{\mathcal{J}}$ for a given Hamiltonian H as in Eq. (1) and

temperature T is defined by

$$\Gamma_{\mathcal{J}}(S) = e^{-H(S)/T} \Big/ \sum_{S} e^{-H(S)/T}. \tag{6}$$

In an infinite system, this formula cannot be used directly. Instead, a Gibbs state $\Gamma_{\mathcal{J}}$ is defined by the Dobrushin-Lanford-Ruelle conditions [22] which say that, for any finite subset Λ, the conditional probability distribution for the spins $S|_\Lambda = (s_i)_{i\in\Lambda}$ at sites in Λ, conditioned on the remaining spins $S|_{\Lambda^c}$ in the complement Λ^c of Λ, is

$$\Gamma_{\mathcal{J}}(S|_\Lambda \mid S|_{\Lambda^c}) = e^{-H'_\Lambda(S)/T} \Big/ \sum_{S|_\Lambda} e^{-H'_\Lambda(S)/T}, \tag{7}$$

where $H'_\Lambda(S)$ is the sum of only the terms $-J_{ij}s_is_j$ in which at least one of i, j is in Λ. As these conditions never specify what happens at infinity, there may be many distinct Gibbs states that satisfy the same conditions, and the appearance of such nonuniqueness at low temperature describes one possible way in which a phase transition can occur.

From the definition, a convex combination (or "mixture") of Gibbs states is again a Gibbs state. A pure (or extremal) Gibbs state is one that cannot be expressed as a convex combination of other Gibbs states; the pure states form a subset of the set of all Gibbs states. Any Gibbs state $\Gamma_{\mathcal{J}}$, say, can be expressed (or decomposed) uniquely as a convex combination of pure Gibbs states in the form [22]

$$\Gamma_{\mathcal{J}} = \int d\varepsilon\, w_{\mathcal{J}\Gamma_{\mathcal{J}}}(\varepsilon)\Gamma_{\mathcal{J}\varepsilon}, \tag{8}$$

where $w_{\mathcal{J}\Gamma_{\mathcal{J}}}(\varepsilon)$ is a probability distribution on pure states $\Gamma_{\mathcal{J}\varepsilon}$ labeled by ε. $w_{\mathcal{J}\Gamma_{\mathcal{J}}}(\varepsilon)$, which depends on both $\mathcal{J}$ and the chosen Gibbs state $\Gamma_{\mathcal{J}}$, is called the weight of the decomposition; we have written it as an integral for generality, but the decomposition might reduce to a sum of a countable number of terms.

B. Equilibrium (static) metastate

A static, or equilibrium, metastate, denoted $\kappa_{\mathcal{J}}$, is a probability distribution on Gibbs states $\Gamma_{\mathcal{J}}$ in infinite size that is obtained by taking a limit of finite-size systems. There are a couple of different constructions of an equilibrium metastate. For a system of finite size L, we will write $\langle\cdots\rangle$ for an expectation in the unique equilibrium state, which depends on the disorder (the bonds) $\mathcal{J}$. In the Aizenman-Wehr (AW) metastate [17], the metastate average of a quantity is defined by first taking the expectation with respect to ν but only for the bonds J_{ij} with both i, j a distance greater than $M < L$ from the origin (i.e., in the outer region); denote that by $[\cdots]_>$, and the average over the remaining bonds ("in the inner region") by $[\cdots]_<$. On taking the limits $L\to\infty$, then $M\to\infty$, these are denoted by the metastate average $[\cdots]_{\kappa_{\mathcal{J}}}$ (for given $\mathcal{J}$ in the inner region), and by $[\cdots]_{\nu(\mathcal{J})}$, respectively; the inner region is now infinite in size, so we can use the same notation $\mathcal{J}$ for the disorder there, and ν for its distribution. As the equilibrium state depends on the disorder in the outer region, in the $L\to\infty$, $M\to\infty$ limit it becomes a Gibbs state $\Gamma_{\mathcal{J}}$ that, even in a finite region near the origin, may retain some dependence on the disorder in the outer region far away (as well as on $\mathcal{J}$), and if it does, then the metastate is nontrivial (i.e., it is supported on more than one Gibbs state). We write $\langle\cdots\rangle_{\Gamma_{\mathcal{J}}}$ for the thermal expectation in $\Gamma_{\mathcal{J}}$. The Newman-Stein (NS) metastate [18–20] is similar, except that the average over disorder at distance $>M$ from the origin is replaced by an average over a range of system sizes between M and L at given disorder; we will use the same notation for either construction. Both constructions require some further discussion of the limits (e.g., the possible need to take the limit along a subsequence of sizes L and M), for which see [17–20]. Finally, it is useful to define the average of the Gibbs state $\Gamma_{\mathcal{J}}$ over the metastate $\kappa_{\mathcal{J}}$, which produces another Gibbs state, the metastate-averaged state (MAS), denoted $\rho_{\mathcal{J}}$. That is, a MAS thermal correlation function is calculated as

$$\langle\cdots\rangle_{\rho_{\mathcal{J}}} \equiv [\langle\cdots\rangle_{\Gamma_{\mathcal{J}}}]_{\kappa_{\mathcal{J}}}. \tag{9}$$

C. Long-range models

As we are concerned in this paper with long-range spin-glass models, rather than the short-range ones that were implicit in the discussion so far, it needs to be said before going further that the definition of a Gibbs state as in Eq. (7) breaks down in that case. That is because for typical $\mathcal{J}$ the sum in $H'_\Lambda(S)$ does not converge absolutely, and diverges for some S, when $\sigma < 1$ [see the definition in Eq. (5)]. Consequently, pathological states exist in the model, and the definition of a Gibbs state should be modified so that only converging sums occur [23]. Within a metastate construction, such pathologies do not occur, and only Gibbs states in the modified sense arise [24]. The same issues should also be addressed for the maturation metastate, but this has not been carried out so far. For that we will proceed on the assumption that these technical issues do not obstruct what we will discuss. (Of course, simulations are performed in finite systems, for which the issue does not arise.)

D. Correlations in the equilibrium MAS

Reference [25] introduced a correlation function in the MAS,

$$C(\mathbf{r}_i - \mathbf{r}_j) \equiv \left[\langle s_is_j\rangle^2_{\rho_{\mathcal{J}}}\right]_{\nu(\mathcal{J})} \tag{10}$$

$$= \left[[\langle s_is_j\rangle_{\Gamma_{\mathcal{J}}}]^2_{\kappa_{\mathcal{J}}}\right]_{\nu(\mathcal{J})}. \tag{11}$$

Note that, by Eq. (9), both the thermal expectation and the metastate average are performed *before* the square is taken, which, for example for the AW metastate, differs from the traditional average over all disorder at once (for which, see below). The large-distance behavior of this correlation function in the low-temperature phase was predicted to be

$$C(\mathbf{r}_i - \mathbf{r}_j) \sim \frac{1}{r_{ij}^{\alpha_s}}, \tag{12}$$

as $r_{ij}\to\infty$, up to a constant factor, with a decay exponent $\alpha_s \geqslant 0$. $\alpha_s > 0$ implies that there are many pure states in the decomposition of the MAS $\rho_{\mathcal{J}}$, and (presumably) that the metastate is nontrivial, as we will explain; as for α_d, one expects the value to be the same for all $0 < T < T_c$. (This form also holds in some of the models in Ref. [16].) It was

further shown in Ref. [25] that the Landau-Ginzburg field theory of RSB in a finite-range spin glass leads to a description with a nontrivial metastate for $T < T_c$, and that

$$\alpha_s = d - 4 \tag{13}$$

for $d > 6$ where the calculation can be done. For the one-dimensional power-law models mentioned at the end of the preceding section, this formula becomes

$$\alpha_s = 3 - 4\sigma \tag{14}$$

for $1/2 < \sigma \leqslant 2/3$ [15]. The latter region, to which we may refer as σ below the upper critical range, is also that in which the critical exponents at $T = T_c$ take their mean-field values (as in a short-range system for d above the upper critical dimension, which is $d = 6$ for spin glasses), while the region $\sigma > 2/3$ is above the critical range, and some of the critical exponents for $2/3 < \sigma < 1$ differ from their mean-field values. It is natural to expect similar phenomena for α_s and α_d, even though they are defined for $T < T_c$, because the perturbative field theory approach for correlations runs into (so far unresolved) difficulties for $T < T_c$ when $d < 6$ or $\sigma > 2/3$ that are more severe than those for $T = T_c$ (where the renormalization group allows calculation of the exponents).

In the SD picture of spin glasses, the metastate is tacitly assumed to be trivial, and

$$\lim_{r_{ij}\to\infty} C(\mathbf{r}_i - \mathbf{r}_j) = q^2, \tag{15}$$

where q is the order parameter, so α_s is then defined to be zero. The order parameter would be defined in general as the limit of the spin-glass correlation function

$$\lim_{r_{ij}\to\infty} \left[\left[\langle s_i s_j\rangle^2_{\Gamma_\mathcal{J}}\right]_{\kappa_\mathcal{J}}\right]_{\nu(\mathcal{J})} = q^{(2)}. \tag{16}$$

Note the crucial difference from the MAS correlation function C in Eq. (10); the square in Eq. (16) is taken *before* the metastate average, and for the AW metastate $[[\cdots]_{\kappa_\mathcal{J}}]_{\nu(\mathcal{J})}$ corresponds simply to the traditional average over all disorder. If the metastate is nontrivial, then the left-hand side of Eq. (16) (without the $r_{ij} \to \infty$ limit) is different from C. In terms of RSB, $q^{(2)} = \int_0^1 q(x)^2\, dx$ [4], while in the SD picture $q(x)^2 = q^2$ is constant. In RSB, C tends to $q(0)^2$ [25], which is zero in zero magnetic field, and $q(0)^2 \leqslant q^{(2)}$ because $q(x)^2$ is an increasing function of x. In general, we can *define* $q(0)^2$ by $q(0)^2 = \lim_{r\to\infty} C(r)$, and then $q(0)^2 \leqslant q^{(2)}$ always, but $q(0)^2$ is not necessarily zero (see below for further discussion of this point). Then

$$q(0)^2 < q^{(2)} \tag{17}$$

always implies a nontrivial metastate. The SD picture of spin glasses is the only scenario with a trivial metastate and trivial Gibbs state. The chaotic pairs picture [18–20] has a nontrivial metastate supported on trivial Gibbs states; in that case $q(x)^2$ is a constant, larger than $q(0)^2$, for all $x > 0$ [25], and the power-law form with C tending to zero is valid in some cases [16] there also, though possibly not always. An accurate and reliable calculation of the static MAS correlation function would then partially resolve the debate about the low-temperature structure for a spin-glass model. Calculating the exponent α_s for low dimension d, however, remains difficult but there has been recent numerical progress with a Monte Carlo study of the EA model in Ref. [26].

E. Maturation MAS

There is an evident similarity or analogy between the dynamical MMAS defined by expectations $\lim_{t\to\infty}[\langle\cdots\rangle_{S|S_0}]_{\eta(S_0)}$ and the static MAS defined by $\langle\cdots\rangle_{\rho_\mathcal{J}}$, and between their corresponding correlation functions C_4 and C, respectively. First, if the MMAS exists as a limit, it is plausible that it must be a stationary state, and stationary states are believed to be necessarily Gibbs states (this has been proved in the translation-invariant case; see, e.g., Ref. [27]). For example, consider one picture of the evolution of the state from a given random initial condition S_0, and assume the validity of the SD picture. At long times there will be domains, within each of which the state locally can be approximated by one of the two pure states, and the domains will be separated by domain walls where the state changes to the other pure state; the scale of the domains increases with time as $\xi(t)$. [In the SD theory, $\xi(t)$ is expected to diverge as a power of $\ln t$, not as a power of t [28].] The domain walls should be sparse [9], so the probability that one separates a given $\mathbf{r}_i$ from a given $\mathbf{r}_j$ at time t for given S_0 eventually goes to zero as $t \to \infty$. Hence within this picture we expect that the $t \to \infty$ limit of the $\eta(S_0)$-average state *is* a stationary state, which is the trivial Gibbs state. Note, however, that the state for given S_0, for example in any fixed finite region, does not tend to a limit, but keeps switching.

Second, the static MAS is an average of the state (correlations) of the spins near the origin with respect to either the disorder far away, or the finite system size, and we will show that this average may reveal that there are many pure states of the infinite system, even when a single Gibbs state drawn from $\kappa_\mathcal{J}$ only involves a smaller number (as in the RSB and chaotic pairs pictures). Similarly, the dynamic MMAS is the long-time limit of the average of the equal-time correlations with respect to the initial conditions, and this average too may show that there are many pure states; the initial configuration can affect the state far from the region of interest at long times, somewhat like the distant disorder. The MAS and the MMAS may thus be very closely related, or possibly identical (a similar remark appears in Ref. [29]). To sharpen the analogy, we denote the MMAS by $\rho^{\mathrm{M}}_\mathcal{J}$, and so $\langle\cdots\rangle_{\rho^{\mathrm{M}}_\mathcal{J}} = \lim_{t\to\infty}[\langle\cdots\rangle_{S|S_0}]_{\eta(S_0)}$.

It is tempting to go further and try to define a maturation metastate [16] $\kappa^{\mathrm{M}}_\mathcal{J}$, a distribution on Gibbs states $\Gamma_\mathcal{J}$, such that $\langle\cdots\rangle_{\rho^{\mathrm{M}}_\mathcal{J}} = [\langle\cdots\rangle_{\Gamma_\mathcal{J}}]_{\kappa^{\mathrm{M}}_\mathcal{J}}$. We are not aware of a formal treatment of such a construction, but the initial steps (similarly to the equilibrium metastate [17,19]), might be to consider (in infinite size) the joint distribution of the state (not the spins), the bonds, and the initial configuration, take the $t \to \infty$ limit (possibly using a subsequence), sum over initial conditions, and then condition on the bonds to obtain $\kappa^{\mathrm{M}}_\mathcal{J}$. One would then want to know that the states drawn from $\kappa^{\mathrm{M}}_\mathcal{J}$ are Gibbs states. If so, the analogs of the general statements above for the static metastate, such as Eq. (17), would also hold for the maturation metastate. Variations of this construction can also be considered; for example, the random variables involved in the dynamics up to a time $t^* > 0$ with $t^* < t$

[in other words, $S(t')$ for $0 \leqslant t' \leqslant t^*$] can be treated as part of the initial conditions along with S_0, with only the subsequent evolution producing the state. In these constructions, the maturation metastate average $[\cdots]_{\kappa^{\rm M}_{\mathcal{J}}}$ of a quantity is the average over the initial segment $S(t')$ for $0 \leqslant t' \leqslant t^*$ (including S_0), with suitable limits taken, analogously to the AW static metastate, and exactly as stated informally in the preceding paragraph.

Instead of the long-time limit, the literature generally focuses on the state in a finite region at a finite time after the quench, and attempts to describe how the limit $t \to \infty$ is approached. In particular, we can ask whether, conditioned on S_0 and on the dynamical randomness up to time $t^* < t$, the subsequent time evolution to $t \to \infty$ (with $t^* \to \infty$ as some function of t) produces a *pure* state (for a more precise discussion, see Ref. [29]); in that case, the behavior described above, in which any fixed finite region switches infinitely often from one pure state to another (the phenomenon of "local non-equilibration" [29]), is excluded. If that holds, then a theorem of NS (Theorem 2 in Ref. [29]) shows that the number of pure states in the decomposition of $\rho^{\rm M}_{\mathcal{J}}$ (for which see below also) must be uncountable, and it also follows from their result that $\lim_{t\to\infty} C_4(r,t) \to 0$ as $r \to \infty$, ruling out $\alpha_d = 0$. (Stated differently, NS's result shows that for the SD picture, the state that evolves from a given S_0 and given dynamical randomness up to t^* must exhibit local nonequilibration, no matter how t^* diverges as t does.) When the hypothesis holds, the corresponding $\kappa^{\rm M}_{\mathcal{J}}$ becomes a distribution on *pure* Gibbs states, but again the general statements remain valid.

F. Correlations in the MMAS

In the remainder of this paper, we will consider only the MMAS, which is simpler to define and study numerically, and which has a close relation with the static MAS. We will use the terms trivial or nontrivial for the MMAS at $T < T_c$ in the following way: $\alpha_d = 0$ is considered the trivial case, and occurs if there is a finite or countably infinite number of pure states in the MMAS that each have nonzero weight, while $\alpha_d > 0$ is considered nontrivial, and implies that (i.e., occurs only if) there is an uncountably infinite number of pure states involved and no one pure state has nonzero weight. To explain this, first, the term "weight" refers to the decomposition of the MMAS, which we assume is a Gibbs state, into pure states:

$$\rho^{\rm M}_{\mathcal{J}} = \int d\varepsilon\, \mu^{\rm M}_{\mathcal{J}}(\varepsilon)\, \Gamma_{\mathcal{J}\varepsilon}, \tag{18}$$

where again $\Gamma_{\mathcal{J}\varepsilon}$ is a pure state for the given $\mathcal{J}$, and the probability measure $\mu^{\rm M}_{\mathcal{J}}(\varepsilon)$ on the pure states ε could be continuous, or could consist solely of δ functions so that the integral reduces to a simple sum of weights on a countable collection of pure states, or could be a combination of both. (There is a completely parallel analysis for the static MAS $\rho_{\mathcal{J}}$, with corresponding weight $\mu_{\mathcal{J}}$.) Next, as $\lim_{t\to\infty} C_4$ is supposed to tend to a limit as $r \to \infty$, it will make no difference to the value of that limit if we average both $\mathbf{r}_i$ and $\mathbf{r}_j$ over a hypercubic box Λ_W of W^d sites, and take $W \to \infty$. Due to the factorization (or clustering) property of pure states [22], the position-averaged product of correlations $\langle s_i s_j\rangle_{\Gamma_{\mathcal{J}\varepsilon}} \langle s_i s_j\rangle_{\Gamma_{\mathcal{J}\varepsilon'}}$ for two pure states $\varepsilon, \varepsilon'$ tends to the square of the overlap,

$$q_{\varepsilon\varepsilon'} = \lim_{W\to\infty} \frac{1}{W^d} \sum_{\mathbf{r}_i \in \Lambda_W} \langle s_i\rangle_{\Gamma_{\mathcal{J}\varepsilon}} \langle s_i\rangle_{\Gamma_{\mathcal{J}\varepsilon'}}, \tag{19}$$

of the pure states, and by translation invariance of the joint distribution of $(\mathcal{J}, \varepsilon, \varepsilon')$ and the ergodic theorem for translation averages, the limit exists and is translation invariant. Then

$$\lim_{r\to\infty} \lim_{t\to\infty} C_4(r,t) = \left[\int d\varepsilon \int d\varepsilon' \mu^{\rm M}_{\mathcal{J}}(\varepsilon) \mu^{\rm M}_{\mathcal{J}}(\varepsilon')\, q^2_{\varepsilon\varepsilon'} \right]_{\nu(\mathcal{J})}, \tag{20}$$

and if there is at least one δ function in $\mu^{\rm M}_{\mathcal{J}}$, putting nonzero weight on one pure state, say ε_0 (and another for its global spin flip), the nonvanishing of the self-overlap $q_{\varepsilon_0\varepsilon_0}$ of that pure state when $T < T_c$ implies that $\lim_{t\to\infty} C_4(r,t)$ tends to a nonzero constant as $r \to \infty$, which is all we needed to show. If there are no such δ functions, then the limit will be zero if the overlaps of distinct pure states drawn independently from $\mu^{\rm M}_{\mathcal{J}}$ are almost always zero. That is what occurs under the hypothesis in Theorem 2 of NS [29], and in that case $\kappa^{\rm M}_{\mathcal{J}} = \mu^{\rm M}_{\mathcal{J}}$. [It is also what occurs for C in the equilibrium case in the RSB theory [25] where, as we mentioned already, $C(r) \to q(0)^2 = 0$, while it is believed that the spin-glass correlation function in Eq. (16) tends to $q^{(2)} > 0$ in RSB because the pure-state decomposition of each Gibbs state $\Gamma_{\mathcal{J}}$ is countable.] It might appear that the statements about the pure-state structure of the MMAS could depend on $\mathcal{J}$; however, because of *translation* ergodicity of $\nu(\mathcal{J})$, the total weight of the δ functions is the same for almost every $\mathcal{J}$, and so the character of the pure-state structure is the same for $\nu(\mathcal{J})$-almost every $\mathcal{J}$. What we term trivial pure-state structure of the MMAS is of course not necessarily a completely trivial pure-state decomposition of the MMAS, but our use of the term is the most natural one for the behavior of the MMAS correlation function, and includes the SD case.

G. Statics-dynamics relation

It was proposed in Refs. [13,15] that the trivial or nontrivial nature of the pure-state structure can be probed with Monte Carlo dynamics using the dynamically generated MMAS correlation function in Eq. (3). The system is evolved in time with Monte Carlo dynamics: for a given time step $t \to t+1$, each lattice site of the total N sites is visited, and a Metropolis accept or reject move is made for a single spin-flip proposal according to the finite-size Gibbs distribution Eq. (6). Based on the relations between static and maturation metastate averages already discussed, Ref. [15] conjectured a *statics-dynamics* relation (see also Ref. [16])

$$\alpha_s = \alpha_d, \tag{21}$$

and found empirically that, in the one-dimensional model with $\sigma = 0.625$ (note this is less than 2/3), α_d is in quantitative agreement with the value $\alpha_s = 1/2$ that would be expected on the basis of the preceding statements and conjecture. While it is unknown whether this conjectured equality always holds, α_s (and α_d) will be zero for trivial pure-state structure of the (M)MAS, and is expected to be nonzero for nontrivial.

H. Other recent work

In a recent paper [30], it was suggested that the results for C_4 may be affected by a crossover from RSB-like to SD behavior as r and t increase. It was further suggested [30], with reference also to [31], that when the SD picture is correct for equilibrium, and so applies at $r \ll \xi(t)$, the value of the exponent is given by $\alpha_d = 2\theta$, where θ is the stiffness exponent of SD theory (in the notation of Ref. [9]). This value was obtained [30] from a calculation within SD theory of the decay exponent of a *connected* (or truncated) correlation function that describes the nonlinear susceptibility and which in SD theory decays to zero as $r \to \infty$ [9]. However, the equilibrium correlation function $[\langle s_i s_j \rangle^2]_{\nu(\mathcal{J})}$ (for trivial metastate) should tend to $q^{(2)}$ as $r_{ij} \to \infty$ (with power-law correction at finite r), where $q^{(2)}$ is not small when T is well below T_c, and it is not clear in Ref. [30] why this constant has been dropped; hence we do not believe that this argument establishes a relation between α_d (or α_s) and θ.

III. SIMULATION RESULTS

Our simulations are performed with a one-dimensional model introduced in Ref. [32] where, on average, a site i only has z_b neighbors. The basic idea is to use diluted bonds such that, for an edge $\{i, j\}$, the bond J_{ij} is nonzero (or the edge is occupied) with probability $P_{i,j} \propto 1/R_{ij}^{2\sigma}$ where $R_{ij} = (N/\pi)\sin(\pi|i-j|/N)$ is the chord distance between sites i and j (whereas the lattice distance is $r_{ij} = |i-j|$ for $|i-j| < N/2$ and $r_{ij} = N - |i-j|$ otherwise), and the occupation numbers, which are 0 or 1 for each edge, are independent. The coefficient in $P_{i,j} \propto 1/R_{ij}^{2\sigma}$ is chosen so that the expected number of occupied edges is $Nz_b/2$ ($P_{ij} > 1$ can be avoided by softening the dependence on R_{ij} at short distance, but keeping the asymptotic form at large R_{ij}). Given the set of occupied edges, those edges are then assigned values J_{ij} independently from a Gaussian distribution with mean zero and variance unity (thus they are indeed nonzero with probability one), while unoccupied edges have $J_{ij} = 0$. Note that the bonds J_{ij} then satisfy Eq. (5) for all i, j, and are independent random variables, as in the model of Ref. [21]. In practice, as in Refs. [32,15], we will use a modified definition that is much less costly to implement at large lattice sizes. The interactions $\mathcal{J} = (J_{ij})_{\{ij\}}$ for a given disorder realization, lattice size N, and coordination number z_b are determined with the following procedure: (i) A site i is chosen uniformly at random from the N lattice sites. (ii) A site j is then selected with probability $\widetilde{P}_{i,j} \propto 1/R_{ij}^{2\sigma}$, where now $\sum_j \widetilde{P}_{i,j} = 1$ for the given lattice size N. (iii) If the edge $\{i, j\}$ does not already have a nonzero bond, then we select one for this edge independently with a Gaussian distribution with mean zero and variance unity. If the edge already has a nonzero bond, then we return to step (i) without modifying that bond. (iv) This process is repeated until there are $Nz_b/2$ nonzero bonds. In the resulting model, the bonds again have mean zero and variance as in Eq. (5) asymptotically, as stated there, and are uncorrelated but not strictly independent, because the occupation probabilities are no longer independent; in particular, the number of occupied edges is fixed, not random. We made comparisons of the results from the two models for some parameter values, and found that the differences were very small.

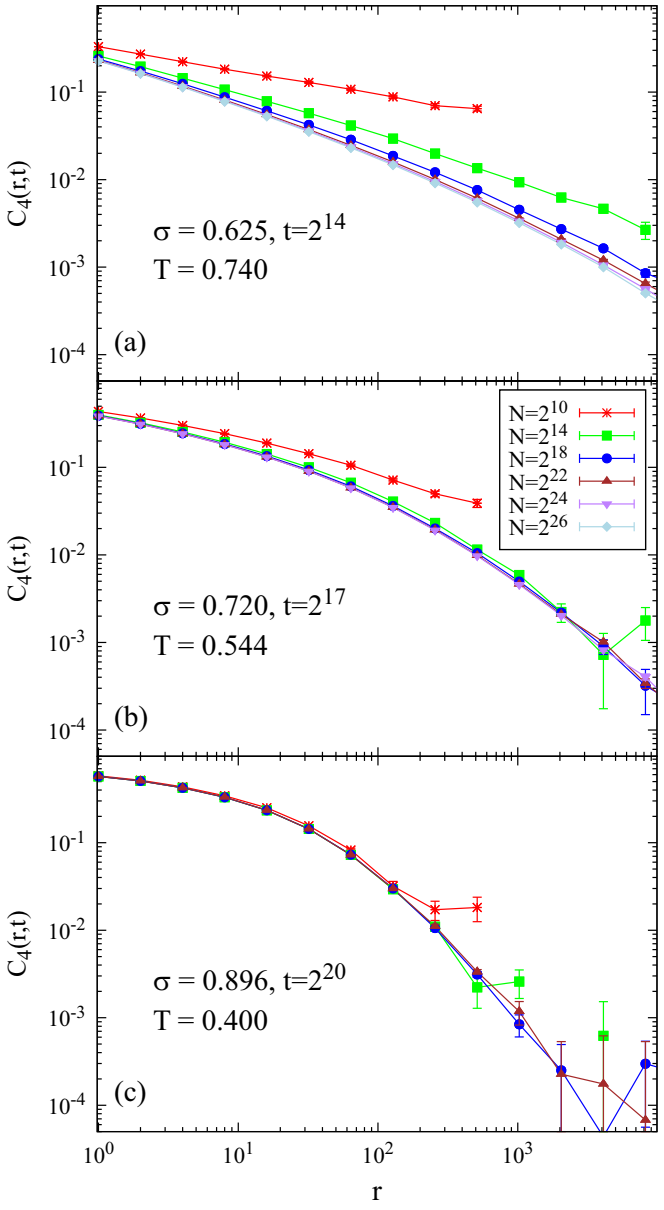

FIG. 1. $C_4(r, t)$ as a function of distance along the lattice r for multiple lattice sizes N for the largest times reached in the simulations for a few representative interaction parameters (a) $\sigma = 0.625$, (b) $\sigma = 0.720$, and (c) $\sigma = 0.896$. We see that finite-size effects are well controlled already at $N = 2^{14}$ for the largest σ values but we require lattice sizes $N = 2^{26}$ for $\sigma = 0.625$.

The simulations were performed with coordination number $z_b = 6$ for eight power-law interaction exponents ranging from $\sigma = 0.61$ to $\sigma = 0.896$. These simulations reach times of at least $t = 2^{14}$ in all cases and $t = 2^{20}$ for some interactions and lattice sizes. For each lattice size and coupling constant, we used between $N_s = 80$ and $N_s = 2000$ disorder realization samples with $N_r = 2$ real replicas for each realization initialized for $t = 0$ with spin configurations drawn independently and randomly.

The temperatures used in the simulations were well below T_c. For the largest interaction exponent, $\sigma = 0.896$, simulations were performed for $T = 0.400$ given the critical

TABLE I. Exponents α_d and dynamic exponents z for the best-fit collapse results of Fig. 4. We also list the lattice size N used in the collapse analysis and the maximum time $t_{\max}$ reached in the simulations. $r_{\min}$ values are listed as used for the best-fit analysis. The fitting procedure is discussed in detail in the Appendix.

σ	α_d	$\delta\alpha_d$	z	δz	T	N	$r_{\min}$	$t_{\max}$	$\xi(t_{\max})$
0.610	0.571	0.026	1.095	0.145	0.740	2^{26}	2^{8}	2^{14}	3210
0.625	0.501	0.009	1.439	0.052	0.740	2^{26}	2^{7}	2^{14}	1415
0.667	0.418	0.009	2.489	0.055	0.500	2^{24}	2^{6}	2^{17}	335
0.685	0.367	0.009	2.529	0.047	0.544	2^{24}	2^{6}	2^{16}	290
0.720	0.327	0.011	2.739	0.057	0.544	2^{22}	2^{6}	2^{17}	210
0.784	0.202	0.013	3.464	0.102	0.544	2^{18}	2^{5}	2^{20}	150
0.840	0.157	0.011	4.413	0.082	0.450	2^{18}	2^{4}	2^{20}	65
0.896	0.127	0.017	5.316	0.120	0.400	2^{18}	2^{4}	2^{20}	35

temperature $T_c \simeq 0.795$ of Ref. [33] where finite-size scaling of the static spin-glass susceptibility was used to determine T_c. For $\sigma = 0.784$ we used $T = 0.544$ given $T_c \simeq 1.36$ of the same work, Ref. [33]. For $\sigma = 0.625$, $T_c = 1.85(2)$ from Ref. [15] and, following this work, we performed simulations for $T = 0.740$. The expected monotonic increase in critical temperature with decreasing interaction exponent guided our choices for the remaining couplings simulated (see Table I for each temperature simulated and a summary of the simulation parameters and findings).

For an unbiased estimate of $C_4(r,t)$ of Eq. (3) ($r \equiv r_{ij}$ by translation invariance) two replicas (an Ising spin is now denoted as s_i^γ where $\gamma \in \{a, b\}$ labels the replica a or b) of each disorder realization are independently simulated. Each replica has the same quenched couplings $\mathcal{J}$ but independent random initial spin configurations which are then independently evolved in time. For each of the N_s disorder realization samples, we calculate the estimator

$$C_4(r,t) = \left[\frac{1}{N}\sum_{i=1}^{N} s_i^a(t)s_{i+r}^a(t)s_i^b(t)s_{i+r}^b(t)\right]_{\nu(\mathcal{J})}. \qquad (22)$$

In Fig. 1 we show $C_4(r,t)$, calculated as in Eq. (22), as a function of lattice size for three representative interactions $\sigma \in \{0.625, 0.720, 0.896\}$. For each coupling multiple lattice sizes $N \in \{2^{10}, 2^{14}, 2^{18}, 2^{22}, 2^{24}, 2^{26}\}$ were simulated to investigate finite-size effects. We see from panel (c) of Fig. 1 that finite-size effects are controlled for $\sigma = 0.896$ already with lattice size $N = 2^{14}$ for the time shown $t = 2^{20}$ (results only up to lattice size $N = 2^{22}$ were obtained for this time). For the smallest interaction exponent shown, $\sigma = 0.625$, the finite-size effects are much more substantial even for $t = 2^{14}$ requiring a lattice size $N = 2^{26}$ as previously found in Ref. [15]. The reduction of finite-size effects with increasing σ (for a given simulation time t), due to the faster power-law decay of the probability of the presence of a bond for $\{i, j\}$ with increasing r, has allowed for multiple $C_4(r,t)$ results with $\sigma > 2/3$. However, the larger σ values require much larger times to achieve appreciable correlation lengths $\xi(t)$ and so similar computational effort was required for each interaction exponent σ.

With the finite-size effects controlled for each interaction exponent, the time dependence of $C_4(r,t)$ is analyzed. The results are plotted in Fig. 2. The large-time and short-distance behavior for each interaction is fit with a power law (dashed line) in each panel for (a) $\sigma = 0.625$ with lattice size $N = 2^{26}$ through (f) $\sigma = 0.84$ with lattice size $N = 2^{18}$. The large-time and large-distance behavior is also fit by a power law (dash-dotted line) showing the expected $C_4(r,t) \propto 1/r^{2\sigma}$ dependence for the long-range model for $r \gg \xi(t)$ [15]. We observe a crossover from short-range to long-range

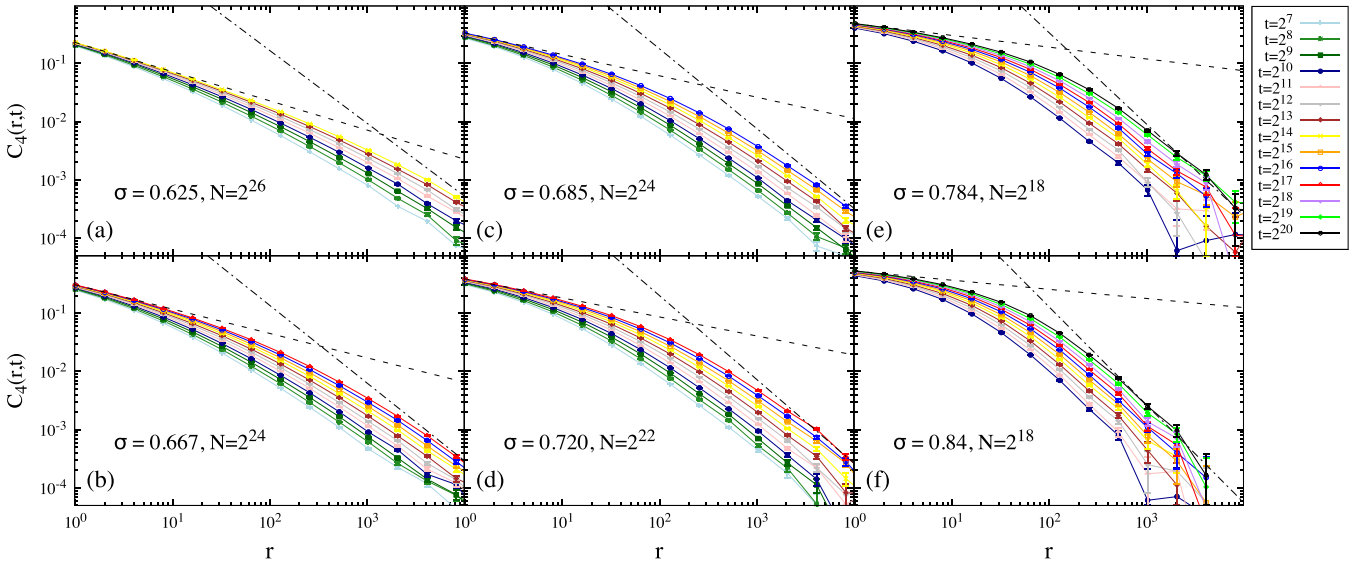

FIG. 2. $C_4(r,t)$ as a function of distance along the lattice r for a given lattice size (with controlled finite-size effects) for varying times. Panel (a) shows interaction $\sigma = 0.625$ for lattice size $N = 2^{26}$ up to time $t = 2^{14}$. Panel (b) shows results at $\sigma = 2/3$ for lattice size $N = 2^{24}$ up to time $t = 2^{17}$. Panels (c)–(f) are for interactions in the regime $\sigma > 2/3$, which is above the upper critical range. We also show the short-distance power-law behavior for each interaction σ (dashed lines) and the large-distance power-law behavior (dash-dotted lines) for the largest available value of t.

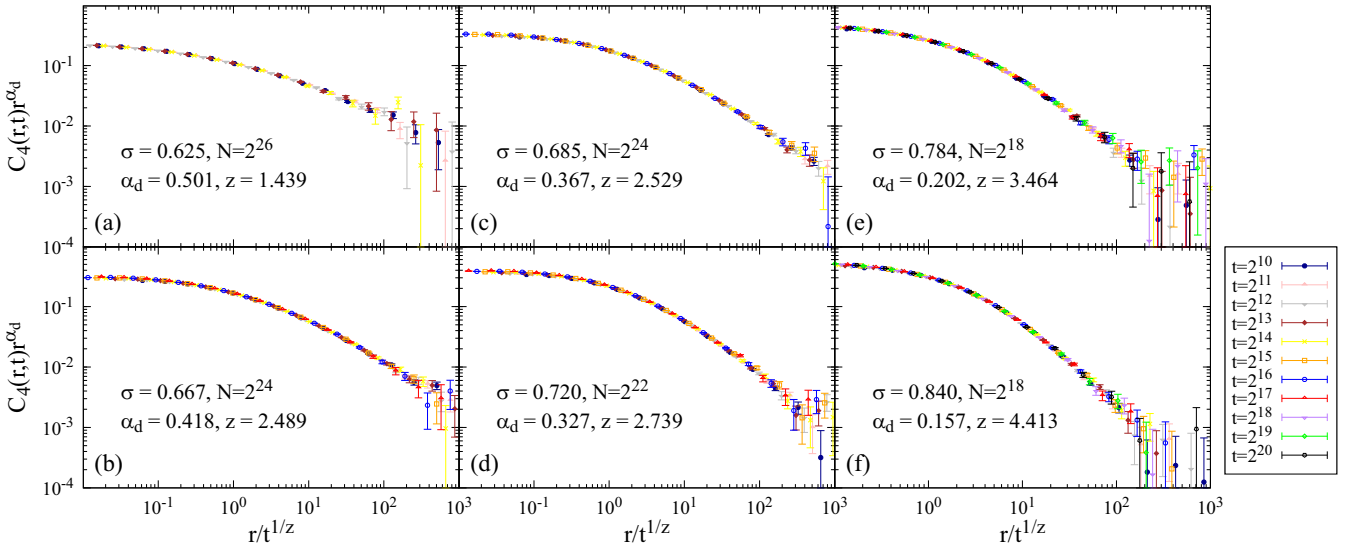

FIG. 3. Data collapse results for multiple interaction parameters from (a) $\sigma = 0.625$, below the upper critical range, to (f) $\sigma = 0.840$, well above the upper critical range. The collapse is performed with the ansatz $C_4(r,t) = \frac{1}{r^{\alpha_d}} f(\frac{r}{\xi(t)})$ for the listed lattice sizes where $\xi(t) \propto t^{1/z}$. The best-fit values used in the collapse are determined as discussed in the Appendix.

power-law dependence. The crossing of the two power-law curves for the maximum simulated time is used to give a rough estimate of the dynamic correlation length $\xi(t)$ reached (see Table I for these estimates). From the raw data of Fig. 2 we perform a data collapse based on the scaling ansatz of Eq. (4) with the product of $C_4(r,t)$ and r^{α_d} on the y axis and $r/t^{1/z}$ on the x axis and present the best-fit collapse results in Fig. 3. The collapse for each value of σ supports the scaling ansatz. We discuss our method for extracting the best-fit collapse parameters and the associated errors in detail in the Appendix.

The final results of the analysis are listed in Table I. In Fig. 4 we show the best-fit collapse values for α_d as a function of interaction power-law exponent σ. We show the prediction obtained using RSB theory together with the conjectured statics-dynamics relation, $\alpha_d = 3 - 4\sigma$ [15,25] for $\sigma \leqslant 2/3$ (blue dashed line). We also show an upper bound $\alpha_d = 2 - 2\sigma$ (purple dash-dotted line); strictly speaking, this bound is obtained [24] from an upper bound $\alpha_s' \leqslant 2 - 2\sigma$ on the scaling, $\ln \mathcal{N}(W) \sim W^{\alpha_s'}$, of the logarithm of the number $\mathcal{N}$ of pure states that could be seen in a window of size W in any Gibbs state, and α_s' was conjectured to equal α_s [25], as it does also in some of the models in Ref. [16]. The same bound can also be obtained in another way: $\alpha = 2 - 2\sigma$ is the decay exponent for the spin-glass correlation function at $T = T_c$ in this model in equilibrium [21], and one would expect a slower decay for the (M)MAS correlations below T_c. Finally, we show the line $\alpha_d = 1 - \sigma$ (green dotted line), which interpolates between the expected value $1/3$ at $\sigma = 2/3$ and 0, which might be expected as $\sigma \to 1$ and which agrees with the upper bound. We see strong agreement of the best-fit collapse results with $\alpha_d = 3 - 4\sigma$ from the two values ($\sigma = 0.625$ and $\sigma = 0.61$) in the expected regime below the upper critical range $2/3$. This is in agreement with the previous study of Ref. [15] and supports both the statics-dynamics conjecture and the result from RSB for $\sigma < 2/3$. As σ is increased beyond $\sigma = 2/3$ we find that the best-fit values for α_d remain nonvanishing and the line $\alpha_d = 1 - \sigma$ is a good fit to the three highest values of σ. This supports the theories with nontrivial MASs in the regime beyond the upper critical range $2/3$ also, and is in contrast to the expected result $\alpha_d = 0$ of the SD picture. The data are not as close to either line where they intersect at $2/3$, and instead suggest a smooth curve. By analogy with critical phenomena, this might be due to logarithmic corrections at the boundary value $\sigma = 2/3$.

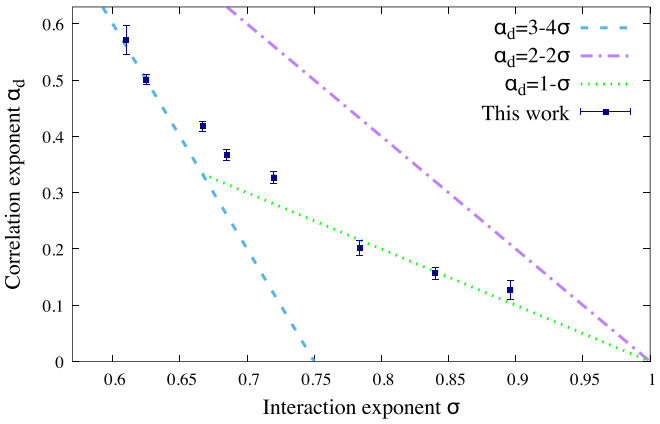

FIG. 4. Exponent α_d (dark-blue squares) describing the decay of $C_4(r,t)$ as a function of interaction power-law exponent σ. $\alpha_d > 0$ for all interactions studied, suggesting a nontrivial metastate at low T for $\sigma < 1$. We clearly see the predicted behavior $\alpha_d = 3 - 4\sigma$ (light-blue dashed line) for $\sigma < 2/3$. The dependence of α_d on σ approaches $\alpha_d = 1 - \sigma$ (light-green dotted line) for $\sigma > 2/3$. We also show the bound $\alpha_d \leqslant 2 - 2\sigma$ [24] (purple dash-dotted line) which is satisfied by the simulation results.

The small statistical errors notwithstanding, our confidence in the results decreases as σ approaches 1, where we were only able to reach rather short correlation lengths, and C_4 does not decrease much as r increases before $\xi(t)$ is reached, due to the small α_d. This means that systematic errors could be

much more significant as $\sigma \to 1$ (e.g., because of corrections to scaling), and the results could change for larger t. It is difficult to say with confidence that we rule out the SD picture for σ close to 1 unless we can see that $\lim_{t\to\infty} C_4(r,t) \ll q^{(2)}$ at sufficiently large r; this is not the case for $r < \xi(t)$ at our largest σ.

IV. CONCLUSION

In this work we have extended the study of Ref. [15] for a 1D diluted long-range model to interaction exponents σ other than $\sigma = 0.625$, considered in that work. We have performed dynamics simulations following a quench to temperatures below the critical temperature for interactions with range exponent σ both below ($\sigma < 2/3$) and above ($\sigma > 2/3$) the upper critical range. For all interactions considered we determined the best-fit scaling exponents z and α_d. The best-fit collapse value of α_d is found to be nonzero for all interactions considered indicating that the pure-state structure is nontrivial (i.e., not the scaling-droplet picture) even above the upper critical range. We found evidence with multiple interaction exponents for the statics-dynamics equality conjecture $\alpha_d = \alpha_s$ ($\alpha_d = 3 - 4\sigma$) for $\sigma < 2/3$, which previously was quantitatively addressed only for $\sigma = 0.625$ in Ref. [15]. Further, we found empirically that the correlation exponent approaches an interpolation curve $\alpha_d = 1 - \sigma$ as $\sigma \to 1$.

It remains to determine in future studies if the statics-dynamics equivalence conjecture which is supported in this study can be strengthened or ruled out, including for the region $2/3 < \sigma < 1$. Both analytic and numerical approaches will be useful to address this. However, we emphasize again that both α_s and α_d are expected to be nonzero for nontrivial pure-state structure and vanish for trivial. It will also be interesting to perform a similar numerical study in the presence of a magnetic field, where the phase diagram and the existence of an Almeida-Thouless line [34] remains uncertain [33,35–37].

ACKNOWLEDGMENTS

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APPENDIX: BEST-FIT COLLAPSE

We do not know the scaling function $f(x)$ of Eq. (4) for the data collapse of Fig. 3 of the main text but determine the best fit by introducing the quality S quantitatively following the methods of Refs. [38,39]. For a fitting window of times $t \in (t_{\min}, t_{\max})$ and lattice distances $r > r_{\min}$, and for given collapse parameters α_d and z, we have an estimate for the collapsed correlation function, $C_4(r,t)r^{\alpha_d}$ denoted $y_{r,t}$ for each distance r and time t and a statistical error for this value $dy_{r,t}$. An estimate of the master curve function at $r/t^{1/z}$, denoted $Y_{r,t}$, can be made from the data $y_{r,t}$. For each point $y_{r,t}$ this is done by fitting a cubic polynomial to the three nearest scaled distances above, i.e., $r'/t'^{1/z} > r/t^{1/z}$, and the three nearest scaled distances below, $r'/t'^{1/z} < r/t^{1/z}$. The statistical error on the interpolated estimate is denoted $dY_{r,t}$. We define the collapse quality S with

$$S = \frac{1}{N_L} \sum_{r,t} \frac{(y_{r,t} - Y_{r,t})^2}{dy_{r,t}^2 + dY_{r,t}^2}, \tag{A1}$$

where N_L is the number of terms in the sum fixed by the size of the fitting window. Larger values of the quality indicate poor fits while the fit is considered good for $S \approx 1$. We show the quality S as a function of α_d and z for $\sigma = 0.685$ in Fig. 5 with $t_{\min} = 2^{13}$ and $t_{\max} = 2^{16}$ with $r_{\min} = 2^6$.

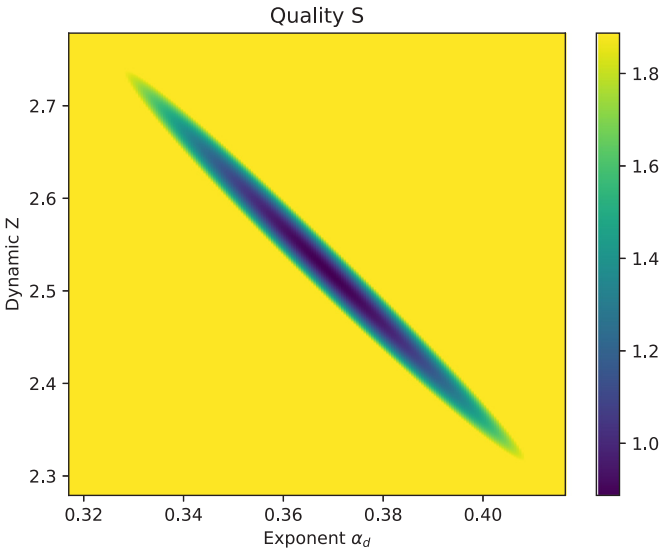

FIG. 5. Collapse quality S heatmap for $\sigma = 0.685$. The quality is considered good for $S \approx 1$ though, as discussed in the main text, it can be much smaller due to the highly correlated nature of the data. The quality shown was evaluated with data for $t_{\min} = 2^{13}$ to $t_{\max} = 2^{16}$ for $N = 2^{24}$ and lattice distances $r_{\min} = 2^6$.

The data considered here is highly correlated so the quality S can be much smaller, e.g., the best fit $S_{\min} \approx 0.1$ for $\sigma = 0.896$. The best-fit quality $S_{\min}$ can also be significantly larger if the interpolation is not sufficiently accurate as the statistical errors are relatively small on the correlation function $C_4(r,t)$ (this is why we use the cubic polynomial instead of the straight-line fit interpolation in Ref. [39]). To determine a final estimate and error on the estimates for both z and α_d with a given fitting window, specified by the parameters ($r_{\min}$, $t_{\min}$, $t_{\max}$), we perform a bootstrap analysis with $N_b = 1000$ bootstrap samples [40]. For each bootstrap sample drawn from the underlying N_s disorder realizations data, we determine the parameters z and α_d which give the smallest fit value $S_{\min}$ of Eq. (A1).

We show the bootstrap sample distributions in Fig. 6 which gave the best-fit values and error bars for the parameters used in the collapse of Fig. 3 of the main text. This was done for each interaction exponent σ with $t_{\max}$ as the largest time reached for the lattice size N and multiple values for $r_{\min}$. $t_{\min}$ was chosen to be $t_{\max}/2^3$ for each interaction exponent. We found the estimates to vary substantially for small $r_{\min}$. As

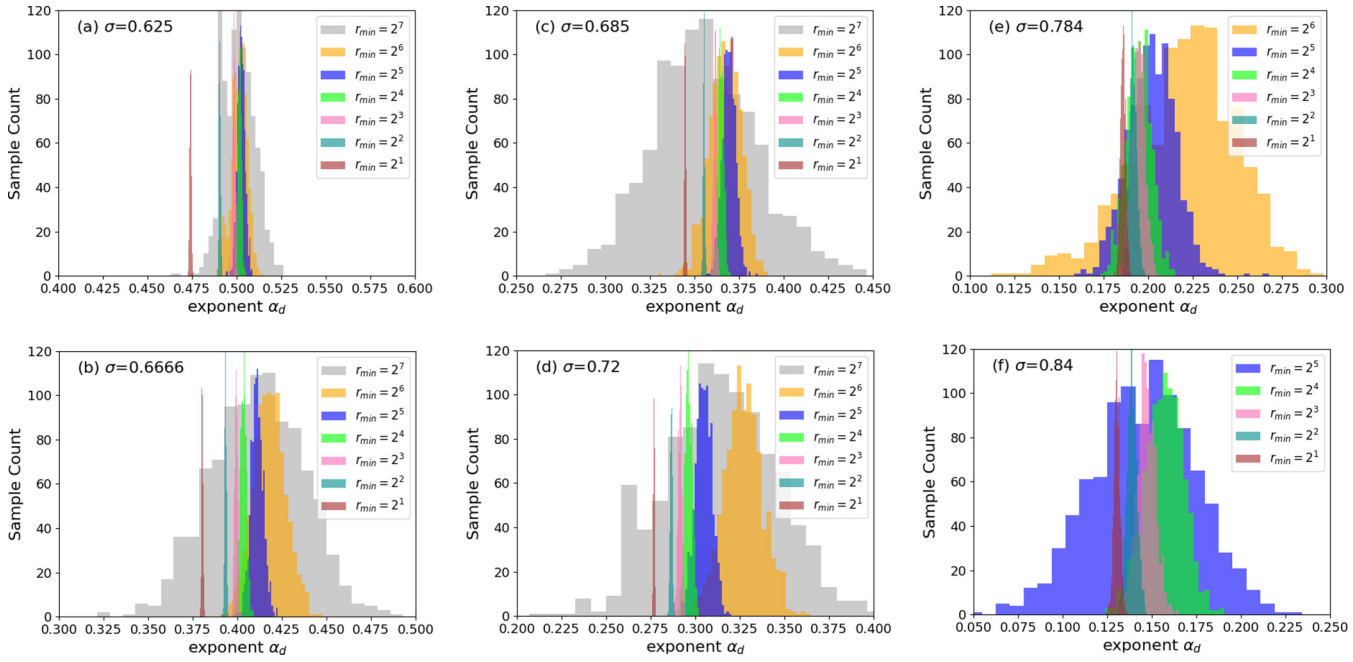

FIG. 6. Histograms for the best-fit collapse exponent α_d with $N_b = 1000$ bootstrap samples with interaction exponents (a) $\sigma = 0.625$, (b) $\sigma = 0.667$, (c) $\sigma = 0.685$, (d) $\sigma = 0.720$, (e) $\sigma = 0.784$, and (f) $\sigma = 0.840$. Each panel shows the bootstrap histogram for varying $r_{\min}$. We see that the best estimate value for α_d varies significantly for small $r_{\min}$, e.g., for $\sigma = 0.625$ we observe variations up to $r_{\min} \approx 2^4$. The best-fit collapse value of $\alpha_d = 0.501$ (used in Fig. 3) and the error $\delta\alpha_d = 0.009$ are determined with $r_{\min} = 2^7$. The values for each interaction exponent are given in Table I.

an example consider $\sigma = 0.685$ (top middle panel). The best estimate value for α for this value of σ varies for small $r_{\min}$ up to $r_{\min} \approx 2^4$. The value of $r_{\min}$ used for the final collapse parameters was selected by the requirement that $S_{\min}$ saturates as a function of $r_{\min}$. The best-fit collapse value of $\alpha_d = 0.367$ (used in Fig. 3) and the error $\delta\alpha_d = 0.009$ were determined with $r_{\min} = 2^6$. For each value of the interaction exponent, the final collapse parameters found, and the value of $r_{\min}$ used for the final estimates, are given in Table I of the main text.

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