Comment on "Turbulent compressible fluid: Renormalization group analysis, scaling regimes, and anomalous scaling of advected scalar fields"

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Recently, asymptotic scaling behavior of the compressible randomly forced Navier-Stokes equation has been analyzed with the use of field-theoretic renormalization group near four dimensions [Phys. Rev. E **95**, 033120 (2017)]. Two infrared stable nontrivial asymptotic scaling patterns have been found and their parameters determined in the one-loop approximation. Here, it is pointed out that the asymptotic scaling behavior predicted in this way may not be realized in physical fluid systems. This is a consequence of restrictions on viscosities imposed by the energy balance equation which the one-loop fixed-point value of the relative viscosity fails to meet.

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Asymptotic behavior of compressible randomly stirred fluid has been analyzed with the use of renormalization group (RG) during almost three decades [1–3]. As in the case of incompressible randomly stirred fluid [4], the stochastic problem has been set up as the randomly forced Navier-Stokes equation for the velocity $\boldsymbol{v}$ of the compressible fluid in d-dimensional space (notation follows Ref. [3]),

$$\partial_t v_i + (v_j\partial_j)v_i = \nu_0(\delta_{ij}\partial^2 - \partial_i\partial_j)v_j + \nu_0 u_0 \partial_i\partial_j v_j - \frac{1}{\rho}\partial_i p + f_i, \tag{1}$$

where ρ is the density and p the pressure of the fluid. In (1) the unrenormalized (denoted by the subscript) kinematic vicosity ν_0 is assumed constant and is, thus, equal to the ratio of the dynamic vicosity η_0 and the mean density $\overline{\rho}$: $\nu_0 = \eta_0/\overline{\rho}$. The relative viscosity u_0 is a new parameter connected to the bulk (second) viscosity ζ_0 as

$$\frac{\zeta_0}{\overline{\rho}} = \nu_0\left[u_0 - \frac{2(d-1)}{d}\right]. \tag{2}$$

Viscous terms of the Navier-Stokes equation (1) are presented as the sum of the transverse and longitudinal parts for convenience of calculations in perturbation theory.

In textbooks (see, e.g., Ref. [5]), the viscous terms are customarily presented in the form in which the irreducible part of the viscous stress tensor is separated, the coefficient of this term is the (dynamic) viscosity η_0, and the second viscosity ζ_0 is the coefficient of the rest,

$$\eta_0\partial_j\left(\partial_j v_i + \partial_i v_j - \frac{2}{d}\delta_{ij}\partial_l v_l\right) + \zeta_0\delta_{ij}\partial_l v_l = \eta_0(\delta_{ij}\partial^2 - \partial_i\partial_j)v_j + \left[\frac{2(d-1}{d}\eta_0 + \zeta_0\right]\partial_i\partial_j v_j, \tag{3}$$

which leads to (2). Note that in d dimensions the coefficient of the Kronecker symbol in the irreducible tensor is $2/d$; this has been overlooked in Refs. [1,2]; in Ref. [3] the second viscosity has not been discussed separately.

The energy pumping to maintain a steady state is described by a zero-mean Gaussian random force (per unit mass) f_i with the correlation function [3] $[x = (t, \boldsymbol{x})]$,

$$\langle f_i(x) f_j(x')\rangle = \frac{\delta(t-t')}{(2\pi)^d}\int_{k>m} d^d k\, D_{ij}(\boldsymbol{k}) e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}, \tag{4}$$

where m is the infrared cutoff and for purposes of the renormalization-group analysis the spectrum $D_{ij}(\boldsymbol{k})$ is (the last term is generated by renormalization)

$$D_{ij}(\boldsymbol{k}) = g_{10}\nu_0^3 k^{4-d-y}[P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k})] + g_{20}\nu_0^3\delta_{ij}. \tag{5}$$

Here, g_{10}, g_{20} are the unrenormalized coupling constants, $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$ and $Q_{ij}(\boldsymbol{k}) = k_i k_j/k^2$ are transversal and longitudinal projection operators, and y is a regulator of ultraviolet (UV) divergences [6,7]. The coefficient of the longitudinal projection operator α is a free parameter.

In compressible fluid the stochastic problem (1), (4), (5) is augmented with the continuity equation,

$$\partial_t\rho = -\partial_i(\rho v_i), \tag{6}$$

and the equation of state ("isothermal fluid"),

$$(p - \overline{p}) = c_0^2(\rho - \overline{\rho}).$$

Here, $\overline{p}, \overline{\rho}$ denote the mean pressure and the mean density, and c_0 is the speed of sound. In Refs. [2,3] the subsequent De Dominicis–Janssen action of the field theory brought about by the stochastic problem is formulated in terms of the scalar field $\phi = c_0^2 \ln\, \rho/\overline{\rho}$.

Renormalization of UV divergences in the field-theoretic model gives rise to renormalized action with renormalized parameters $g_1,\ g_2,\ \nu,\ u,\ v,\ c$, and renormalization constants

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Z_i $(1 \leqslant i \leqslant 6)$ [3],

$$\mathcal{S}_{\boldsymbol{v}}^R = \frac{1}{2}\int dt \int d\boldsymbol{x} \int d\boldsymbol{x}' v_i' D_{ij}^R v_j' + \int dt \int d\boldsymbol{x}\, v_i' \times[-\partial_t v_i - v_j \partial_j v_i + \nu Z_1(\delta_{ij}\partial^2 - \partial_i\partial_j)v_j + u\nu Z_2 \partial_i\partial_j v_j - Z_4 \partial_i \phi] + \int dt \int d\boldsymbol{x}\, \phi'[-\partial_t\phi - v_j\partial_j\phi + v\nu Z_3 \partial^2\phi - c^2 Z_5 \partial_i v_i], \tag{7}$$

where

$$D_{ij}^R(\boldsymbol{k}) = g_1 \nu^3 k^{4-d-y}[P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k})] + g_2 \nu^3 Z_6 \delta_{ij}.$$

In (7) the term proportional to $\phi'\partial^2\phi$ has been added to ensure multiplicative renormalization [2]. Such a term is generated in the course of renormalization in any case. The physical value of the bare coefficient of this term v_0 is zero, however, because there is no such term the continuity equation (6). In compressible fluid additional divergences appear in the model at four dimensions, therefore, in Ref. [3] the problem is analyzed with two regulators of UV divergences: the parameter y of analytic regularization in the correlation function of the random force and the deviation of the space dimension from the critical value $\varepsilon = 4 - d$. UV divergences show as singularities in y and ε and are collected in renormalization constants.

For consistency of perturbation theory the parameter u_0 and its renormalized counterpart u must be non-negative. There is, however, a stronger restriction on u_0 due to the condition on the bulk viscosity $\zeta_0 \geqslant 0$. The latter is a consequence of the structure of the expression for the rate of energy dissipation in compressible fluid [5],

$$\dot{E}_{\rm kin} = -\frac{\eta_0}{2}\int d\boldsymbol{x}\left(\partial_j v_i + \partial_i v_j - \frac{2}{d}\delta_{ij}\partial_l v_l\right)^2 - \zeta_0 \int d\boldsymbol{x} (\partial_l v_l)^2, \tag{8}$$

from which it follows that the coefficients of viscosity $\eta_0 > 0$ and $\zeta_0 \geqslant 0$. Through the connection (2) a restriction on the relative viscosity u_0 follows as:

$$u_0 \geqslant \frac{2(d-1)}{d}. \tag{9}$$

Relation (2) is a consequence of reorganization (3) of the viscous terms in the Navier-Stokes equation (1). In a multiplicatively renormalized model the basic action (renormalized action (7) with all $Z_i = 1$, see Ref. [6]) is just the same as the bare action generated by the stochastic problem but with renormalized parameters, therefore, connection (2) is immediately transferred to renormalized parameters.

The rate of energy dissipation (8) includes two independent positive definite terms. It is convenient to express the rate of energy dissipation as the sum of terms corresponding to transverse and longitudinal differential operators appearing in (7),

$$\dot{E}_{\rm kin} = -\eta_0 \int d\boldsymbol{x}[(\partial_i v_j)(\partial_i v_j) - (\partial_i v_i)^2] - \eta_0 u_0 \int d\boldsymbol{x} (\partial_i v_i)^2 \equiv -\eta_0 \int d\boldsymbol{x}\, F_1 - \eta_0 u_0 \int d\boldsymbol{x}\, F_2. \tag{10}$$

Here, the terms corresponding to transverse and longitudinal differential operators have been cast into explicitly Galilean-invariant form.

The inequality (9), however, is based on the energy balance equation in the unrenormalized model, which involves—in parlance of quantum field theory—composite operators, i.e., products of fields and their derivatives with coinciding temporal and spatial arguments as well as parameters. In order to make conclusions about restrictions on renormalized viscosity coefficients the energy balance equation must be written in terms of renormalized composite operators [6]. Contrary to parameters of the renormalized model, composite operators mix in renormalization, therefore, conclusions akin to (9) must be performed on the basis of the rate of energy dissipation written in terms of renormalized composite operators. In particular, a straightforward substitution of the definition of the renormalized relative viscosity $u_0 = uZ_u$ is meaningless to this end.

Schwinger equations for the renormalized model (7) with respect to auxiliary fields $\boldsymbol{v}'$ and ϕ' lead to the energy balance equation in the form

$$\partial_t \int d\boldsymbol{x} \frac{1}{2}\overline{\rho}\, \exp\left(\frac{\phi}{Z_5 c^2}\right) v^2 = \int d\boldsymbol{x}\, \overline{\rho}\, \exp\left(\frac{\phi}{Z_5 c^2}\right) v_i D_{ij}^R v_j' + Z_4 Z_5 c^2 \int d\boldsymbol{x}\, \overline{\rho}\left[\exp\left(\frac{\phi}{Z_5 c^2}\right) - 1\right]\partial_i v_i + \eta \int d\boldsymbol{x}\, \exp\left(\frac{\phi}{Z_5 c^2}\right) \times\left(-Z_1 F_1 - u Z_2 F_2 + \frac{v Z_3}{2c^2 Z_5} v^2 \partial^2\phi\right) = 0. \tag{11}$$

Here, the density has been expressed using variables of model (7). In (11) the dependence of the dissipative terms (three last terms on the right side) on the density fluctuation field ϕ is an artifact of the approximation used in (7) where the renormalized kinematic viscosity ν is the ratio of the dynamic viscosity and the *mean* density instead of the variable density. It should be noted that the rate of energy dissipation $\dot{E}_{\rm kin}$ (10) is independent of the density under the usual assumption of constant *dynamic* viscosities. Therefore, in the framework of this approximation only leading terms of the exponential of ϕ should be retained. Moreover, the coupling constant v/c^2 of the last term on the right side of (11) has canonical dimension -2, and, thus, in the RG analysis of critical behavior is irrelevant [6]. Thus, the energy balance equation consistent with model (7) is of the form

$$\partial_t \int d\boldsymbol{x} \frac{1}{2}\overline{\rho} v^2 = \int d\boldsymbol{x}\, \overline{\rho} v_i D^R_{ij} v'_j + Z_4 \int d\boldsymbol{x}\, \overline{\rho}\phi\, \partial_i v_i$$
$$-\eta \int d\boldsymbol{x}(Z_1 F_1 + u Z_2 F_2) = 0. \qquad (12)$$

The composite operators F_1 and F_2 appearing in the rate of energy dissipation (10) and in the energy balance equation (12) are Galilean invariant scalar operators of canonical dimension four. Analysis of such operators in the case of incompressible fluid was first carried out in Ref. [8], whose basic line of argument is followed here. In the minimal scheme used in Refs. [2,3] these operators only mix with Galilean invariant scalar operators of the same canonical dimension [4,6]. There are numerous such operators: $F_1,\ F_2,\ \phi\partial_i v_i,\ \phi'\partial_i v_i,\ \phi\phi,\ \phi\phi',\ \phi'\phi',\ \partial_i v'_i,\ \partial^2\partial_i v_i,\ \partial^2\phi,$ and $\partial^2\phi'$. In what follows the composite operators are enumerated in the order of appearance in this list. The structure of interaction in (7) is such that the field ϕ appears in counterterms as a derivative $\partial_i\phi$. The corresponding extracted wave vector renders counterterms of operators $\phi\partial_i v_i$, $\phi\phi'$, and $\phi\phi$ to F_1 and F_2 superficially finite. Inspection of Feynman rules reveals that operators $\phi'\partial_i v_i$, $\phi'\phi'$, and $\partial^2\phi'$ cannot admix into F_1 and F_2 because there are no UV-divergent counterterms with these structures. Therefore, renormalization of F_1 and F_2 is described in terms of the renormalized counterparts F_1^R and F_2^R of these operators and, possibly, composite operators constructed from single fields $\partial_i v'_i$, $\partial^2\partial_i v_i$, and $\partial^2\phi$,

$$F_1 = Z_{11}F_1^R + Z_{12}F_2^R + Z_{18}F_8^R + Z_{19}F_9^R + Z_{1\,10}F_{10}^R,$$
$$F_2 = Z_{21}F_1^R + Z_{22}F_2^R + Z_{28}F_8^R + Z_{29}F_9^R + Z_{2\,10}F_{10}^R. \qquad (13)$$

In the integral relation (12), however, contributions of renormalized operators F_8^R, F_9^R, and F_{10}^R vanish. Therefore, for the purposes of the present Comment, the actual values of their coefficients are not needed.

What is needed is the fact that operators F_1 and F_2 do not admix into any of the operators $F_3,\dots,F_{11}$. They do not admix into F_3, F_5, and $F_8,\dots,F_{11}$ due to the absence of UV-divergent counterterm graphs. Into F_4, F_6, and F_7 they do not admix because the counterterm graphs contain closed loops of retarded propagators and vanish. From this, it follows that the expression of $F_3 = \phi\partial_i v_i$ [which appears in (12)] in terms of renormalized composite operators does not contain F_1^R and F_2^R.

The composite operator of dimension four $\partial_t v^2$ is not renormalized and is completely decoupled from renormalization of other composite operators [4,6]. Galilean invariant composite operators may admix into the noninvariant operator of dimension four $v_i D^R_{ij} v'_j$ appearing in (12), but F_1 and F_2 do not admix into it because the counterterms vanish due to closed cycles of retarded propagators. Therefore, the operators F_1^R and F_2^R are absent in the expression of the operator $v_i D^R_{ij} v'_j$ in terms of renormalized composite operators.

The main conclusion from this analysis is that when the energy balance equation (12) is written in terms of renormalized composite operators (i.e., finite quantities), the operators F_1^R and F_2^R appear only in the expression for the rate of energy dissipation and the coefficients of these two independent renormalized operators must be finite separately. In the minimal scheme used in Refs. [2,3] the finite part of a renormalization coefficient of a parameter or a field as well as a diagonal renormalization coefficient of composite operators is equal to unity, whereas a nondiagonal renormalization coefficient has only the divergent part [4,6]. Therefore, we arrive at the conclusion,

$$\eta Z_1 Z_{11} + \eta u Z_2 Z_{21} = \eta,$$
$$\eta Z_1 Z_{12} + \eta u Z_2 Z_{22} = \eta u. \qquad (14)$$

Substitution of relations (13) and (14) in presentation (10) of the rate of energy dissipation yields (we recall that $\nu_0 = \nu Z_1$ and $\nu_0 u_0 = \nu u Z_2$, Ref. [3]),

$$\begin{aligned}\dot{E}_{\rm kin} &= -\eta_0 \int d\boldsymbol{x}\, F_1 - \eta_0 u_0 \int d\boldsymbol{x}\, F_2\\ &= -\eta \int d\boldsymbol{x}\, F_1^R - \eta u \int d\boldsymbol{x}\, F_2^R\\ &= -\eta \int d\boldsymbol{x}\left[F_1^R + \frac{2(d-1)}{d}F_2^R\right]\\ &\quad -\eta\left[u - \frac{2(d-1)}{d}\right]\int d\boldsymbol{x}\, F_2^R,\end{aligned}$$

where the two integrals in the rightmost expression are positive definite and, thus, impose on the renormalized parameters the conditions,

$$\eta > 0, \qquad u \geqslant \frac{2(d-1)}{d}. \qquad (15)$$

It should be emphasized here that, contrary to u_0 in the physical relation (9), the parameters η and u in (15) are not numbers but functions of the bare parameters and the renormalization scale μ, defined (implicitly) by renormalization constants [e.g., $\nu_0 = \nu Z_1(u, g_1, g_2)$ and $\nu_0 u_0 = \nu u Z_2(u, g_1, g_2)$]. For asymptotic analysis solutions of differential equations of the RG are expressed in terms of “'invariant variables” $(\overline{\nu}, \overline{u}, \overline{g}_1, \overline{g}_2)$. The invariant variables $\overline{\nu}, \overline{u}$ are related to bare parameters just in the same way as the renormalized parameters (scaling variables are different, however): $\nu_0 = \overline{\nu} Z_1(\overline{u}, \overline{g}_1, \overline{g}_2)$ and $\nu_0 u_0 = \overline{\nu}\,\overline{u} Z_2(\overline{u}, \overline{g}_1, \overline{g}_2)$ [4,6] so that restrictions (15) are imposed on the invariant variables as well.

As a result of the standard RG analysis of a multicharge model three infrared stable fixed points have been found in Ref. [3] to describe the large-scale asymptotic behavior of the model for different (small) values of the regulators y and ε. For the argument of the present comment the fixed-point value of the renormalized relative viscosity u^* is crucial, therefore, details of fixed-point values of other parameters are not quoted here. At the trivial fixed point (FP I of Ref. [3]) the renormalized coupling constants vanish $g_1^* = g_2^* = 0$, the relative viscosity u^* is arbitrary and, thus, may be chosen to satisfy conditions for the bulk viscosity.

However, at the two other (nontrivial) IR stable fixed points (FP II and FP III of Ref. [3]) the leading-order (in an expansion in y and ε) fixed-point value of the relative viscosity $u* = 1$ does not satisfy condition (15). In the one-loop approximation the fixed points FP II and FP III lie in the unphysical region of the parameter space and, thus, do not describe the large-scale asymptotic behavior of model [7]. For any practical purposes, however, the expansion parameters ε and y—on which multiloop corrections to $u* = 1$ depend, see Ref. [9]—are prescribed finite values which may give rise to values of u^* meeting condition (15). Two-loop results have been published for the model in generic space dimension $d > 2$ with the single regulator y [9], therefore, similar progress in the two-regulator model [3] is called for to clarify the situation.

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