Kardar-Parisi-Zhang equation in a half space with flat initial condition and the unbinding of a directed polymer from an attractive wall

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(Received 11 May 2021; accepted 27 July 2021; published 13 August 2021)

We present an exact solution for the height distribution of the KPZ equation at any time t in a half space with flat initial condition. This is equivalent to obtaining the free-energy distribution of a polymer of length t pinned at a wall at a single point. In the large t limit a binding transition takes place upon increasing the attractiveness of the wall. Around the critical point we find the same statistics as in the Baik-Ben–Arous-Péché transition for outlier eigenvalues in random matrix theory. In the bound phase, we obtain the exact measure for the endpoint and the midpoint of the polymer at large time. We also unveil curious identities in distribution between partition functions in half-space and certain partition functions in full space for Brownian-type initial condition.

DOI: 10.1103/PhysRevE.104.024502

I. INTRODUCTION

The Kardar-Parisi-Zhang (KPZ) equation [1], which describes the growth of the height field of an interface driven by white noise in the continuum, is a paradigmatic example of stochastic nonequilibrium dynamics. It enjoys a remarkable connection to the equilibrium problem of an elastic line in a random potential, also called directed polymer (DP) [2]. In one space dimension, i.e., for the DP in dimension $d = 1 + 1$, some exact solutions for the height distribution at all time t have been found in the last ten years. These finite time solutions are valuable since they allow to study the crossover in time from short times, where the growth is in the Edwards-Wilkinson class [3,4], to the large time asymptotic behavior which is common to a large number of systems in the so-called KPZ class [5–7]. However, they have been found only for a few specific initial conditions (IC), which are the important ones: the droplet IC (point to point DP) [8–11], the flat IC (point to line DP) [12,13] and the Brownian IC (which includes the stationary KPZ) [14–18]. The KPZ equation on the half-line has also been studied, and is related to the DP in a half-space with a wall, with a wall parameter A, which can be repulsive $A > 0$ or attractive $A < 0$. It was found in Ref. [19] that the polymer is bound to the wall for $A < -1/2$ and that it unbinds for $A \geqslant -1/2$ due to the competition with bulk point disorder, a different mechanism from the usual thermal wetting transition [20,21]. It is also different from the full space version of the model with a single columnar defect [22–24] (slow bond problem) where the DP is always pinned, or the case where disorder in only on the column [25–27]. An experimentally feasible realization of half-line KPZ growth in turbulence liquid crystal was obtained in Ref. [28] from a biregional geometry with two different growth rates. In these types of experiments the aforementioned IC can be easily prepared [29,30]. Although the transition at $A = -1/2$ has been studied in details for other models in the KPZ class [31–36], exact finite time solutions for the KPZ equation itself have been obtained until now only for $A \geqslant -1/2$, for droplet IC [37–40], and for stationary IC [41]. Furthermore, although it is expected that the height fluctuations are Gaussian at large time in the bound phase, as was found in Ref. [42] for droplet IC, understanding of the fluctuations of the polymer configuration is still limited, despite the pioneering results of Ref. [19].

In this paper, we obtain the "missing" exact solution for the KPZ equation in the half-space, that is with flat IC. Our solution is valid for any time and any wall parameter A, hence it allows for a complete study of the two phases and of the transition. While the solutions for (i) the flat IC in full space and (ii) the other IC in half-space, are both complicated, the combination of flat IC and half-space geometry leads to a remarkable simplification, and to a simpler solution, in terms of a Fredholm determinant. This unveils curious identities in distribution between partition functions in half-space and certain partition functions in full space for Brownian-type IC. Around the critical point at $A = -1/2$ we find the same statistics for the height fluctuations at large time as for the outlier eigenvalues in the Baik-Ben Arous-Péché transition [43] of random matrices. In the bound phase $A < -1/2$, the fluctuations of the height (i.e., the free energy of the polymer) are Gaussian. To characterize the fluctuations of the polymer configuration we obtain the exact distribution of its endpoint and of its midpoint for long polymers, and explicit formula for their moments. We predict an unbinding transition under a force applied to the endpoint. Interesting connections with the ground state obtained in replica Bethe ansatz studies [19,42] of the half-space delta Bose gas are analyzed.

This paper is organized as follows. In Sec. II we recall the definitions of the KPZ equation on the half-line and of the related model of the continuum directed polymer in the half space. In Sec. III we derive the exact solution for all times of the KPZ equation with flat initial conditions. Some details are provided in the Appendix A. We also obtain the large time asymptotics, the details being provided in Appendix B. In Sec. IV we study the stationary measure of the KPZ equation on the half-line and apply it to obtain a detailed description of the statistics of the endpoint in the polymer problem. The connection with the replica method is given in Appendix E and the calculations of the mean endpoint probability and its correlations using Liouville quantum mechanics are described in Appendix F. In Sec. V we generalize the identity in distribution obtained in Sec. III, relating solutions of half-space KPZ equation to solutions with the full-space KPZ equation with different initial data, some of the details are presented in Appendix C. The matching with full space KPZ equation distributions uses the statistical tilt symmetry recalled in Appendix D. In Sec. VI we point out the features which we believe are universal near the unbinding transition and in relation to the conjectured half-space KPZ fixed point.

II. HALF-SPACE KPZ EQUATION

Let us recall the KPZ equation for the height $h(x,t)$ field of an interface

$$\partial_t h(x,t) = \nu \partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 + \sqrt{D}\xi(x,t), \tag{1}$$

where $\xi(x,t)$ is a space-time white noise. We use space-time units so that $\nu = 1$ and $\lambda = D = 2$. Here we study the problem in a half-line $x \geqslant 0$ with boundary conditions $\partial_x h = A$ depending on a parameter A. From the Cole-Hopf mapping it can be equivalently defined as $h(x,t) = \log Z_A(x,t)$, where $Z_A(x,t)$ satisfies the stochastic heat equation (SHE)

$$\partial_t Z_A(x,t) = \partial_x^2 Z_A(x,t) + \sqrt{2} Z_A(x,t)\xi(x,t), \;\; x \geqslant 0, \tag{2}$$

with boundary condition $\partial_x Z_A(x,t) = A Z_A(x,t)$ at $x = 0$ and as yet unspecified initial condition at $t = 0$.

Let us denote by $Z_A(x,t|y,0)$ the partition function of a continuous DP of length t in a white noise random potential in dimension $d = 1+1$ (at unit temperature), in the half-space $x \geqslant 0$, with endpoints at $(y,0)$ and (x,t); see Fig. 1. In particular $Z_A(x,t|y,0)$ satisfies Eq. (2) with initial condition at $t = 0$ given by a delta mass at the point y.

III. FLAT INITIAL CONDITION

A. Finite-time solution

In this section, we are interested in the solution of the KPZ Eq. (1) with a flat initial condition $h(x,0) = 0$. This is equivalent to studying Eq. (2) with $Z_A(x,0) = 1$, i.e., a polymer with one fixed endpoint at (y,t) and one free endpoint, of partition function

$$Z_A^f(y,t) = \int_0^\infty Z_A(y,t|x,0)dx, \tag{3}$$

the KPZ field being retrieved as $h(y,t) = \log Z_A^f(y,t)$, where the superscript f stands for flat IC.

Consider now the case where the fixed endpoint is at the position of the wall $y = 0$. We will calculate the moments of $Z_A^f(0,t)$ which will allow us to obtain an expression for the Laplace transform of its distribution. From Eq. (3) and

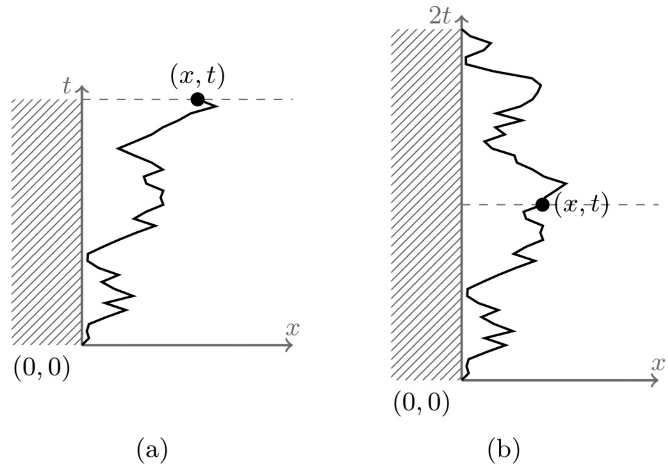

FIG. 1. (a) Directed polymer path in 1 + 1 dimensions in a half-space, pinned at the boundary wall at the point (0,0), with free endpoint (x,t) (point to line problem) in presence of a bulk white noise random potential $\sqrt{2}\xi(x,t)$. We will discuss the fluctuations of the position of the polymer endpoint. By reversing time, this corresponds to the line to point partition function $Z_A^f(y=0,t)$ in Eq. (3), which maps to the KPZ field at time t with flat IC. (b) We will also consider the position of the midpoint of a DP with both endpoints pinned at the boundary.

by symmetry, we can write the nth integer moment of the partition sum as

$$\mathbb{E}\left[Z_A^f(0,t)^n\right] = n! \int_{x_1 \geqslant \cdots \geqslant x_n \geqslant 0} \mathbb{E}\left[\prod_{i=1}^n Z_A(x_i,t|0,0)\right], \tag{4}$$

where here and below, $\mathbb{E}$ denote expectation with respect to the noise ξ. As shown in Ref. [38] the moments appearing in the right-hand side (r.h.s) can be expressed as a multiple contour integral. We have that, for general endpoint positions $x_1 \geqslant \cdots \geqslant x_n \geqslant 0$,

$$\mathbb{E}\left[\prod_{i=1}^n Z_A(x_i,t|0,0)\right] = 2^n \int_{r_1+\mathbf{i}\mathbb{R}} \frac{dz_1}{2\mathbf{i}\pi} \cdots \int_{r_n+\mathbf{i}\mathbb{R}} \frac{dz_n}{2\mathbf{i}\pi} \times \prod_{i=1}^n \frac{z_i}{z_i+A} e^{t z_i^2 - x_i z_i} \times \prod_{1\leqslant a<b\leqslant n} \frac{z_a - z_b}{z_a - z_b - 1}\,\frac{z_a+z_b}{z_a+z_b-1}, \tag{5}$$

where the contours are chosen so that $r_1 > r_2 + 1 > \ldots > r_n + n - 1 > \max\{n-1-A, n-1\}$, i.e., all contours are to the right of $-A$. To obtain the result for the flat initial condition we must now integrate over the endpoints x_i over the positive real axis. To this aim we use the identity

$$\int_{x_1 \geqslant \cdots \geqslant x_n \geqslant 0} \prod_{i=1}^n e^{-x_i z_i} = \prod_{i=1}^n \frac{1}{z_1 + \cdots + z_i}, \tag{6}$$

which is a convergent integral, since all $\mathrm{Re}(z_i) > 0$ from our choice of contour. One thus obtains

$$\mathbb{E}\big[Z_A^f(0,t)^n\big] = n!2^n \int_{r_1+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r_n+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_n}{2\mathbf{i}\pi} \times \prod_{1\leqslant a<b\leqslant n} \frac{z_a - z_b}{z_a - z_b - 1} F(\vec{z}), \tag{7}$$

where

$$F(\vec{z}) = \prod_{1\leqslant a<b\leqslant n} \frac{z_a + z_b}{z_a + z_b - 1} \prod_{i=1}^{n} \frac{e^{t z_i^2}}{z_1 + \cdots + z_i} \frac{z_i}{z_i + A}. \tag{8}$$

It is convenient to deform all the contours to the single contour $r + \mathbf{i}\mathbb{R}$ with $r > \max(-A, 0)$. During the deformation, one encounters many poles of the integrand whose residues need to be taken into account. This is done in a systematic way using Ref. [38, Proposition 5.1] (see also Ref. [44]). We obtain for Eq. (7)

$$\sum_{\ell=1}^{n} \frac{n!2^n}{\ell!} \sum_{\vec{m}:\sum m_i = n} \prod_{i=1}^{\ell} \int_{r+\mathbf{i}\mathbb{R}} \frac{dw_i}{2\mathbf{i}\pi} \det\left(\frac{1}{w_i + m_i - w_j}\right)_{i,j=1}^{\ell} \times E(w_1, w_1+1, \ldots, w_1 + m_1 - 1, \ldots, w_\ell, \ldots, w_\ell + m_\ell - 1), \tag{9}$$

where the $m_i \geqslant 1$ are integers with $\sum_{i=1}^{\ell} m_i = n$, and

$$E(\vec{z}) = \sum_{\sigma\in S_n} \prod_{1\leqslant b\leqslant a\leqslant n} \frac{z_{\sigma(a)} - z_{\sigma(b)} - 1}{z_{\sigma(a)} - z_{\sigma(b)}} F[\sigma(\vec{z})]. \tag{10}$$

This sum over the symmetric group can be simplified. Note that the factor $\prod_{1\leqslant a<b\leqslant n} \frac{z_a+z_b}{z_a+z_b-1}$ in $F(\vec{z})$ is symmetric, so it can be factored out. The remaining symmetrization can be performed as in the solution for the full space flat initial condition [45] (see also Ref. [46]), where it was found that

$$\sum_{\sigma\in S_n} \prod_{1\leqslant b\leqslant a\leqslant n} \frac{z_{\sigma(a)} - z_{\sigma(b)} - 1}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{i=1}^{n} \frac{1}{z_{\sigma(1)} + \cdots + z_{\sigma(i)}} = \prod_{1\leqslant a<b\leqslant n} \frac{z_a + z_b - 1}{z_a + z_b} \prod_{i=1}^{n} \frac{1}{z_i}. \tag{11}$$

Hence, the products over $a < b$ perfectly cancel each other, and we are left with the remarkably simple expression

$$E(\vec{z}) = \prod_{i=1}^{n} \frac{e^{t z_i^2}}{z_i + A}. \tag{12}$$

Due to the product structure of the function E, it can be factored inside the determinant in Eq. (9). This leads to an explicit formula for the integer moments, which we sum over n to obtain the moment generating series, leading to the following Fredholm determinant expression (see Appendix A for details) for $u > 0$

$$\mathbb{E}[e^{-u Z_A^f(0,t) e^{\frac{t}{12}}}] = \det(I - K_{u,t})_{\mathbb{L}^2(0,+\infty)}, \tag{13}$$

where the kernel is given by

$$K_{u,t}(v, v') = \int_{\mathbb{R}} \frac{2u\, dr}{e^{-r} + 2u} \phi_{A,t}(v + r)\psi_{A,t}(v' + r), \tag{14}$$

$$\phi_{A,t}(v) = \int_{a_z+\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} \frac{e^{t\frac{z^3}{3} - vz}}{\Gamma(A + \frac{1}{2} + z)}, \tag{15}$$

$$\psi_{A,t}(v) = \int_{\mathcal{C}_{a_w}} \frac{dw}{2\mathbf{i}\pi} e^{-t\frac{w^3}{3} + vw} \Gamma(A + \tfrac{1}{2} + w). \tag{16}$$

The contour for z is a vertical line with real part $a_z > 0$, and the contour for w, denoted $\mathcal{C}_{a_w}$, is the union of two semi-infinite rays leaving the point $a_w > -(A + \frac{1}{2})$ in the direction $\pm 2\pi/3$ to ensure convergence. This expression is one of our main result and is valid for any value of the wall parameter A and for all time $t > 0$.

B. Large-time limit

From the Laplace transform formula one can extract the probability density function (PDF) of the KPZ height field $h(0,t) = \log Z_A^f(0,t)$ at arbitrary time. Let us now discuss its large time limit, which depends on the value of A. The height takes the form as $t \to +\infty$,

$$h(0,t) = \log Z_A^f(0,t) \simeq v_\infty(A)t + t^\beta \chi, \tag{17}$$

where the free energy per unit length exhibits a transition

$$v_\infty(A) = \begin{cases} -\frac{1}{12} & \text{when } A \geqslant -\frac{1}{2}, \\ -\frac{1}{12} + \left(A + \frac{1}{2}\right)^2 & \text{when } A < -\frac{1}{2}, \end{cases} \tag{18}$$

χ is an $O(1)$ random variable, and β the growth fluctuation exponent

$$\beta = \frac{1}{3} \text{ for } A \geqslant -\frac{1}{2}, \qquad \beta = \frac{1}{2} \text{ for } A < -\frac{1}{2}. \tag{19}$$

Let us turn to the distribution of χ.

1. Case $A > \frac{-1}{2}$

We scale u as $u = e^{-t^{1/3}s}$ with fixed s, so that

$$\lim_{t\to\infty} \mathbb{E}[e^{-u Z_A^f(0,t) e^{\frac{t}{12}}}] = \mathbb{P}(\chi \leqslant s). \tag{20}$$

The limit of the Fredholm determinant in Eq. (13) is obtained by the change of variables $v = t^{1/3}\tilde{v}$, $v' = t^{1/3}\tilde{v}'$ and $r = t^{1/3}\tilde{r}$ in Eq. (14), $z = t^{-1/3}\tilde{z}$ in Eq. (15), and $w = t^{-1/3}\tilde{w}$ in Eq. (16) so that

$$\lim_{t\to\infty} t^{\frac{1}{3}} K_{u,t}(t^{\frac{1}{3}}\tilde{v}, t^{\frac{1}{3}}\tilde{v}') = \int_s^\infty \mathrm{Ai}(\tilde{r} + \tilde{v})\mathrm{Ai}(\tilde{r} + \tilde{v}')d\tilde{r}, \tag{21}$$

leading to

$$\mathbb{P}(\chi \leqslant s) = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(s,+\infty)} = F_2(s), \tag{22}$$

where K_{Ai} is the Airy kernel and $F_2(s)$ is the cumulative distribution function (CDF) of the Tracy-Widom distribution for the largest eigenvalue of a GUE random matrix.

2. Case $A < \frac{-1}{2}$

The condition that $a_w > -(A + \frac{1}{2})$ in Eq. (16) forbids to use the same change of variables. In this case we scale $u = e^{-(A+\frac{1}{2})^2 t - t^{1/2}s}$, and use the change of variables $v = t^{1/2}\tilde{v}$

(and likewise for the variables v'), $r = (A+\frac{1}{2})^2 t + t^{1/2}\tilde{r}$, $z = -(A+\frac{1}{2}) + t^{-1/2}\tilde{z}$ in Eqs. (15) and (16) we evaluate the integral by residues [the residue at $w = -(A+\frac{1}{2})$ is dominant]. We obtain

$$\lim_{t\to\infty} t^{\frac{1}{2}} K_{u,t}(t^{\frac{1}{2}}\tilde{v}, t^{\frac{1}{2}}\tilde{v}') = \frac{1}{\sqrt{4\pi|A+\frac{1}{2}|}} e^{-\frac{(s+\tilde{v})^2}{4|A+\frac{1}{2}|}}, \tag{23}$$

which implies that χ has a Gaussian distribution with variance $2|A+\frac{1}{2}|$, see details in Appendix B.

3. Near the critical point

We may also scale A close to the critical point as $A = \frac{-1}{2} + at^{-1/3}$. The asymptotics are similar as in the case $A > \frac{-1}{2}$, except that (see Appendix B)

$$\mathbb{P}(\chi \leqslant s) = \det(I - K_a^{\rm BBP})_{\mathbb{L}^2(s,+\infty)} = F_a^{\rm BBP}(s), \tag{24}$$

where the CDF $F_a^{\rm BBP}(s)$ was introduced in Ref. [43] and governs the fluctuations of the eigenvalues of spiked Hermitian matrices. It was also found to arise in the context of the KPZ universality class in full-space models for half-Brownian-type IC [14,15,47] and in other contexts [43,48–50]. In particular, for $a = 0$, $F_0^{\rm BBP}(s) = [F_1(s)]^2$ where F_1 is the Tracy-Widom distribution function for the largest eigenvalue of a GOE random matrix.

IV. STATIONARY ENDPOINT DISTRIBUTION

A. Endpoint distribution

The previous results describe the behavior of the partition function of a polymer of arbitrary length t in a white noise random potential, with one endpoint fixed at $x = 0$ and another endpoint free to move; see Eq. (3). It is natural to ask about the distribution of the distance of the endpoint to the wall. This information is contained in the endpoint PDF $\mathcal{P}_A(x,t)$ in a given disorder realization, i.e.,

$$\mathcal{P}_A(x,t) = \frac{Z_A(x,t|00)}{\int dy Z_A(y,t|00)} = \frac{Z_A(x,t|00)}{Z_A^f(0,t)} \tag{25}$$

and in its average $P_A(x,t) = \mathbb{E}[\mathcal{P}_A(x,t)]$. Direct calculation of this quantity is not available, however one can obtain it in the limit of a large polymer length t in the bound phase $A < -\frac{1}{2}$. For a large class of IC (see below), the distribution of ratios $Z_A(x,t)/Z_A(y,t)$ converges at large time t to a stationary distribution of polymer partition function ratios such that

$$x \mapsto \frac{Z_A(x,t)}{Z_A(0,t)} = x \mapsto e^{\mathcal{B}(x)+(A+\frac{1}{2})x}. \tag{26}$$

It is stationary in the sense that if at a time t the field of partition function ratios is distributed as Eq. (26) where $\mathcal{B}(x)$ is a standard Brownian motion, then at any later time $\tilde{t}$, the field of partition function ratios will still be distributed as Eq. (26), with a new Brownian motion $\tilde{\mathcal{B}}(x)$ depending nontrivially on $\mathcal{B}(x)$ and the disorder $\xi(x,s)$ for $t < s < \tilde{t}$. Therefore, at large time,

$$\lim_{t\to\infty} \mathcal{P}_A(x,t) = p_A(x) := \frac{e^{\mathcal{B}(x)+(A+\frac{1}{2})x}}{\int_0^{+\infty} dy e^{\mathcal{B}(y)+(A+\frac{1}{2})y}}, \tag{27}$$

in the sense that both sides have the same multipoint distribution.

This result allows to obtain formulas for the moments of the endpoint position which become time independent at large t for fixed $A < -1/2$. One denotes the thermal average in a given disorder configuration as $\langle O(x)\rangle = \int_0^{+\infty} dx O(x)\mathcal{P}_A(x,t)$, and the thermal cumulants as usual, e.g., $\langle x^2\rangle_c = \langle x^2\rangle - \langle x\rangle^2$. Interpreting $p_A(x)$ in Eq. (27) as the Gibbs measure of a particle (the endpoint) in a 1d Brownian random potential (at unit temperature), it is natural to introduce [51] $\mathsf{Z}(v) = \int_0^{+\infty} dy e^{\mathcal{B}(y)+vy}$, for $v < 0$, the generating function of the thermal cumulants, such that $\langle x^p\rangle_c = \partial_v^p \log \mathsf{Z}(v)|_{v=A+\frac{1}{2}}$. It is well-known that the random variable $\mathsf{Z}(v)$ is distributed as the inverse of a Gamma variable, $\mathsf{Z}(v) = 1/\Gamma(-2v, \frac{1}{2})$ [52,53]. In particular, using $\mathbb{E}[\log \mathsf{Z}(v)] = \log 2 - \psi(-2v)$, one obtains the disorder averaged thermal cumulants of the polymer endpoint

$$\mathbb{E}[\langle x^p\rangle_c] = -(-2)^p \psi^{(p)}(-2A-1) \tag{28}$$

for $p \geqslant 1$, where $\psi(z)$ is the digamma function, e.g.,

$$\mathbb{E}[\langle x\rangle] = 2\psi'(-2A-1) \simeq \frac{2}{(2A+1)^2}, \tag{29}$$

$$\mathbb{E}[\langle x^2\rangle_c] = -4\psi''(-2A-1) \simeq \frac{8}{|2A+1|^3}, \tag{30}$$

where we indicated the leading behavior for $A \to -1/2^-$, using $\psi(x) \sim \frac{-1}{x}$ at small x. In the bound phase but near the transition, i.e., for $\epsilon = -(A+\frac{1}{2}) > 0$ and small, the endpoint wanders very far. In terms of the rescaled endpoint position $y = \epsilon^2 x$ one has $e^{\mathcal{B}(x)-\epsilon x} = e^{\frac{1}{\epsilon}(\tilde{\mathcal{B}}(y)-y)}$, i.e., ϵ can be interpreted as an effective temperature which tends to zero. The PDF of y thus concentrates around the optimum $y_m = \mathrm{argmax}_{z>0}[\tilde{\mathcal{B}}(z) - z]$. The explicit PDF of y_m is known [54] and reads $P(y) = \sqrt{\frac{2}{\pi y}} e^{-y/2} - \mathrm{Erfc}(\sqrt{\frac{y}{2}})$, which implies the leading behavior of the moments as $A \to -1/2^-$,

$$\mathbb{E}\langle x^n\rangle \simeq c_n\left|A+\tfrac{1}{2}\right|^{-2n}, \qquad c_n = \frac{2^n\Gamma(n+\frac{1}{2})}{(n+1)\sqrt{\pi}}, \tag{31}$$

where $c_1 = \frac{1}{2}$ agrees with Eq. (29). The PDF $P(y)$ is expected to be the limit $a \to -\infty$ of a family of distributions indexed by a, universal within the KPZ class, which describes the endpoint distribution around the critical point (see Sec. VI). As expected, the thermal fluctuations are subdominant as compared to the disorder. Recall that for the polymer in full space, with one endpoint fixed at 0, the cumulants are time dependent with $\mathbb{E}[\langle x^p\rangle_c] = t\delta_{p,2}$ for any t, see Appendix D, a behavior very different from Eq. (28). For the half-space problem, near the transition the first two average cumulants are expected to take the following time dependent scaling form $\mathbb{E}\langle x\rangle \simeq t^{2/3} f_1(a)$ and $\mathbb{E}\langle x^2\rangle_c = t f_2(a)$, where $a = t^{1/3}(A+\frac{1}{2})$ is the critical scaling variable, with the asymptotics $f_1(a) \simeq \frac{1}{2a^2}$ and $f_2(a) \simeq 1/a^3$ for $a \to -\infty$, from Eqs. (29) and (30). Finally, the result Eq. (31) is expected to hold provided $t^{-1/3} \ll -(A+\frac{1}{2}) \ll 1$.

The polymer in the half-space can be also studied by the replica method. It uses the relation between the nth moment of the partition sum and the Lieb-Liniger Hamiltonian $\mathcal{H}_n$ for

n bosons on the half-line, solvable via the Bethe ansatz. This was pioneered by Kardar [19] who proposed an ansatz for the ground state Ψ_0 of $\mathcal{H}_n$, which is a bound state to the wall for $A < -1/2$, and used it to predict $\mathbb{E}[\langle x \rangle]$. This calculation assumes that the limits $n \to 0$ and $t \to +\infty$ commute. We checked that the result of Ref. [19] agrees with Eq. (29) (using $\kappa = 1/2$ and $\lambda = -A$ there), which indicates that this assumption holds in the bound phase (while it *does not* hold in the unbound phase, or in the full space). In Appendix E we provide a detailed comparison of the two methods [replica ground-state dominance, and Brownian stationary measure Eq. (27)] and more details on the replica approach. Note that the full spectrum of $\mathcal{H}_n$ is quite complicated and was obtained recently in Ref. [42], see also Ref. [40], which confirms Ref. [19], and may allow to obtain subleading large time behavior.

It is also possible to obtain exact formula for the m-point averages $\mathbb{E}[p_A(x_1)\dots p_A(x_m)]$ of the Gibbs measure $p_A(x)$, using the Liouville quantum mechanics developed in Refs. [55–59]. The detailed calculations are presented in Appendix F. For instance, in the case $m = 1$, we obtain

$$\mathbb{E}[p_A(x)] = \frac{1}{4\Gamma(2w)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{dz(w^2 - z^2)}{2\mathrm{i}\pi\,\Gamma(2z)\Gamma(-2z)} e^{\frac{-x}{2}(w^2 - z^2)} \times \Gamma(w+z)^2\Gamma(w-z)^2 \times \Gamma(1-w-z)\Gamma(1-w+z), \tag{32}$$

which is valid for $0 < w = -(A + \frac{1}{2}) < 1$ and can be analytically continued to all $w > 0$ (see Appendix F where we also checked that Eq. (32) is normalized to unity and reproduces the first moment Eq. (29)).

B. Midpoint probability and unbinding by a force

Consider now a polymer with both endpoints fixed near the wall at times $t = 0$ and $2t$; see Fig. 1. One may ask about the PDF of the midpoint position $x = x(t)$ for a long polymer, i.e., for large t. In the bound phase $A < -1/2$, it is proportional to (up to a normalization factor)

$$e^{\mathcal{B}_1(x) + \mathcal{B}_2(x) + 2(A + \frac{1}{2})x}, \tag{33}$$

where $\mathcal{B}_1, \mathcal{B}_2$ are independent Brownian motions. Since $\mathcal{B}_1(x) + \mathcal{B}_2(x)$ has the same distribution as $\sqrt{2}\mathcal{B}(x) \overset{(d)}{=} \mathcal{B}(2x)$ ($\mathcal{B}$ being a standard Brownian motion), the PDF of the midpoint equals $2p_A(2x)$, i.e., the midpoint position is distributed as half of the endpoint position [60].

Applying now a force f on the polymer endpoint in Fig. 1 results in the change $Z_A(x,t) \to e^{\mathsf{f}x} Z_A(x,t)$ to its partition function (only at the final time t). In the stationary large time limit Eq. (27) it amounts to shift $A \to A + \mathsf{f}$ in all the above results for the endpoint. An unbinding transition thus occurs at $\mathsf{f} = \mathsf{f}_c = -(A + 1/2)$, with the same behavior as the transition at $\mathsf{f} = 0$ upon varying A (this is also equivalent to tilting the wall). If the force is instead applied on the midpoint in Fig. 1, then the unbinding transition then occurs at $\mathsf{f}_c = -2(A + \frac{1}{2})$.

C. Convergence to the stationary distribution

So far we have not justified why the ratios of partition functions converge to Eq. (26) in the bound phase. It was

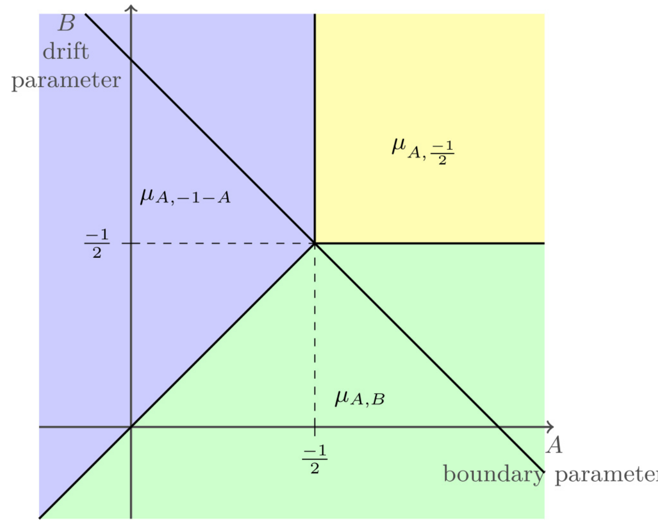

FIG. 2. Domains of attraction of KPZ (conjectural) stationary measures. The horizontal coordinate A is the boundary parameter in the half-space SHE (2). The vertical coordinate means that we start from an initial condition which behaves as a Brownian motion with drift $-(B + \frac{1}{2})$ at infinity. Then, the ratios of partition function $Z(x,t)/Z(0,t)$ converge to one of the stationary measures μ. In the blue region (low density phase in ASEP context) these ratios converge to $\mu_{A,-1-A}$, as stated in the text. In the green region (high density phase) the ratios converge to $\mu_{A,B}$, and in the yellow region (maximal current phase), they converge to $\mu_{A,\frac{-1}{2}}$. Along the antidiagonal line $B = -1 - A$, the ratios always converge to the Brownian stationary measures Eq. (26).

shown in Ref. [41] that this distribution of ratios is indeed stationary. When $A \leqslant \frac{-1}{2}$, we claim that for a large class of initial condition such that the drift at infinity is less than $-(A + \frac{1}{2})$, that is

$$\log Z_A(x+y, 0) - \log Z_A(x, 0) \simeq -\big(B + \tfrac{1}{2}\big)y, \tag{34}$$

for large x and y, with $B \geqslant A$ (and even for more general random initial conditions), then the ratios of partition functions converge to Eq. (26). This class of initial conditions includes the flat IC for the KPZ equation, as well as the fixed endpoint for the DP (equivalently the droplet IC for KPZ). It is important to note that there exist more general stationary distributions of partition function ratios than the ones described in Eq. (26). They can be parametrized by (A, B), and denoted $\mu_{A,B}$, where A is the boundary parameter, and B is a drift parameter meaning that $\log Z_A(x+y,t) - \log Z_A(x,t)$ behaves as a Brownian motion in y with drift $-(B + \frac{1}{2})$ for large x. In the case $B = -1 - A$, the stationary measure $\mu_{A,-1-A}$ is exactly the one described in Eq. (26). Depending on the value of A and the drift at infinity of the initial condition, we expect that the partition function ratios converge to one of these stationary measures according to the phase diagram in Fig. 2, based on a similar analysis performed for the asymmetric simple exclusion process (ASEP) [61,62].

V. IDENTITIES IN DISTRIBUTION

Equation (13), which characterizes the distribution of $Z_A^f(0,t)$, matches with a known formula characterizing the distribution of another quantity. Let $Z^{(A)}(x,t)$ be the solution to the stochastic heat equation

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \sqrt{2} Z(x,t)\xi(x,t), \;\; x \in \mathbb{R}, \quad (35)$$

on the full line, with "half-Brownian" initial condition given by $\mathbb{1}_{x\geqslant 0} e^{\mathcal{B}(x)-(A+\frac{1}{2})x}$, where $\mathcal{B}$ is a standard Brownian motion. It was shown in Ref. [15] (see also Refs. [14,47]) that the Laplace transform of $Z^{(A)}(0,t)$ is given by the same Fredholm determinant as the one that we obtained in Eq. (13). Matching parameters and notations between Ref. [15] and the present paper (see Appendix C) we find the following surprising identity in distribution: for all fixed $t>0$ and any $A\in\mathbb{R}$,

$$Z_A^f(0,t) = 2Z^{(A)}(0,t), \quad (36)$$

where $Z_A^f(0,t)$ was defined in Eq. (3). Remarkably, an identity of a similar flavour as Eq. (36) can be deduced from Ref. [32, Eq. (7.59)] for a model of last passage percolation [63]. We stress that Eq. (36) is also valid in the phase $A < \frac{-1}{2}$, where in the left-hand side (l.h.s.), the polymer is bound to the wall. In the r.h.s., $Z^{(A)}(0,t)$ is the partition function of polymer paths in the full space, weighted by $\mathbb{1}_{x\geqslant 0} e^{\mathcal{B}(x)-(A+\frac{1}{2})x}$ (x being the starting point) which, for $A<-1/2$, is dominated by $x=O(t)$ so that the fluctuations of the Brownian motion are dominant over KPZ-type fluctuations. We do not know whether Eq. (36) extends at several times. Nevertheless, we may generalize Eq. (36) by introducing a spatial parameter, though we cannot simply replace the point 0 by an arbitrary point $X>0$ in Eq. (36). Let us define

$$Z_A^{\text{shifted}}(X,t) = \int_{x\geqslant X} Z_A(0,t|x,0)dx. \quad (37)$$

This corresponds to the value at the origin of the solution to the half-line SHE (2) with initial condition $Z(x,0)=\mathbb{1}_{x\geqslant X}$. Then, we may readily adapt Eqs. (6), (7), (9), (12), and (13) (see details in Appendix C) and match the result with Ref. [15]. We obtain that for any fixed time $t>0$, $X\geqslant 0$ and $A\in\mathbb{R}$, we have the identity in distribution

$$Z_A^{\text{shifted}}(X,t) = 2Z^{(A)}(-X,t). \quad (38)$$

Note that $Z^{(A)}(-X,t)$ has the same law as $e^{-\frac{X^2}{4t}} Z^{(A+\frac{X}{2t})}(0,t)$ from the tilt symmetry (see Appendix D), hence we also have $Z_A^{\text{shifted}}(X,t) = e^{-\frac{X^2}{4t}} Z^{\text{shifted}}_{A+\frac{X}{2t}}(0,t)$ in law.

An even more general identity in distribution holds. Let us denote $Z_{A,B}(x,t)$ the solution to Eq. (2) on the half line $x\geqslant 0$ with initial condition given by $e^{\mathcal{B}(x)-(B+\frac{1}{2})x}$. The moments of $Z_{A,B}(x,t)$ are given in Ref. [41, Sec. 4.4] in a very similar form as in Eq. (5). Thus, we may still apply the same steps: we define

$$Z_{A,B}^{\text{shifted}}(X,t) = \int_{x\geqslant X} Z_{A,B}(x,t)dx, \quad (39)$$

and compute the Laplace transform of $Z_{A,B}^{\text{shifted}}(X,t)$. Then, for $t>0$, $X\geqslant 0$, and parameters A,B such that $A+B+1>0$ and $B>\frac{-1}{2}$, we have the identity in distribution

$$Z_{A,B}^{\text{shifted}}(X,t) = 2Z^{(B|A,B)}(-X,t), \quad (40)$$

where the quantity $Z^{(B|A,B)}(-X,t)$ is again the solution to full-line SHE (35) with some specific IC, that we obtain in Appendix C 3 using exact formulas valid for the exactly solvable log-gamma polymer model. To describe it, let $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ be three independent Brownian motions with respective drifts $-(B+\frac{1}{2})$, $-(A+\frac{1}{2})$ and $-(B+\frac{1}{2})$. Let w be an independent inverse Gamma random variable with parameter $(2B+1)$. Then for $x\leqslant 0$,

$$Z^{(B|A,B)}(x,0) = w e^{\mathcal{W}_1(-x)}, \quad (41)$$

and for $x\geqslant 0$,

$$Z^{(B|A,B)}(x,0) = e^{\mathcal{W}_3(x)}\left(w + \int_0^x e^{\mathcal{W}_2(y)-\mathcal{W}_3(y)}dy\right). \quad (42)$$

When $B\to+\infty$, then $B\,Z_{A,B}^{\text{shifted}}(X,t)$ goes to $Z_A^{\text{shifted}}(X,t)$ and $B\,Z^{(B|A,B)}(X,t)$ goes to $Z^{(A)}(X,t)$ (see Appendix C 4), so that we recover Eq. (38). When $A\to+\infty$, we obtain yet another identity in law,

$$Z_{+\infty,B}^{\text{shifted}}(X,t) = 2w\tilde{Z}^{(B|B)}(-X,t), \quad (43)$$

where the l.h.s. is related to half-line solution to Eq. (2) with $e^{\mathcal{B}(x)-(B+\frac{1}{2})x}$ IC and Dirichlet boundary condition (this solution was studied in Refs. [40,64]), and $\tilde{Z}^{(B|B)}(X,t)$ is the full-line solution to Eq. (35) with $e^{\mathcal{B}(x)-(B+\frac{1}{2})|x|}$ IC, independent on w. This solution was studied in Refs. [16–18] and one checks that the Fredholm determinant obtained here characterizing the law of $Z_{A,B}^{\text{shifted}}(X,t)$ for $A\to+\infty$ matches the one in Ref. [17], see Appendix C 2. Going back to the solution $Z^{(B|A,B)}(-X,t)$ in Eq. (40), its distribution was not obtained in the literature, though its moments can be computed using known methods, and we explain in Appendix C 3 how they match with the moments of $Z_{A,B}^{\text{shifted}}(X,t)$ for generic parameters A,B.

VI. UNIVERSALITY

In this paper we studied the continuum directed polymer model. We found that in the bound phase, taking the limit $t\to+\infty$ first, very near the transition, with $0<\epsilon=-(A+1/2)\ll 1$, the PDF of the scaled endpoint position $y=x(A+1/2)^2$ concentrates around the optimum $y_m = \text{argmax}_{z>0}(\tilde{\mathcal{B}}(z)-z)$. Hence, in that limit this scaled position is distributed with

$$P(y) = \sqrt{\frac{2}{\pi y}}e^{-y/2} - \text{Erfc}\left(\sqrt{\frac{y}{2}}\right), \quad (44)$$

a PDF which behaves as $P(y)\simeq\sqrt{\frac{2}{\pi y}}$ for $y\ll 1$ and as $P(y)\simeq\sqrt{\frac{2}{\pi}}y^{-3/2}e^{-y/2}$ for $y\gg 1$.

One can argue that this result holds for finite but very large time as well, as long as the critical parameter $a=(A+\frac{1}{2})t^{1/3}$ is very large negative $-a\gg 1$. In fact one can surmise that there is a scaling function which describes the endpoint position $x(t)$ in the critical region as follows:

$$\lim_{t\to+\infty} \text{Prob}\left[x(t)(A+1/2)^2 < y\,\middle|\, A+\tfrac{1}{2} = \frac{a}{t^{1/3}}\right] = \mathcal{P}(y,a). \quad (45)$$

To match the previous result one would need that $\lim_{a\to-\infty}\mathcal{P}(y,a) = \int_0^y dz P(z)$. However, one can conjecture the existence of a (critical) half-space Airy process, denoted

$\mathcal{A}_a(\hat{x})$ continuously depending on the parameter a. It would describe in particular the simultaneous limit $t \to +\infty$ and $A \to -1/2$ of the continuum directed polymer model, with fixed $a = t^{1/3}(A + \frac{1}{2})$

$$\log Z_A(x, t|0, 0) \simeq t^{1/3}(\mathcal{A}_a(\hat{x}) - \hat{x}^2), \quad \hat{x} = \frac{x}{2t^{2/3}}. \tag{46}$$

By universality, this process should be the same as the limit process obtained from half-space last passage percolation, so that the finite dimensional marginals of $\mathcal{A}_a(\hat{x})$ are described in Ref. [35, Theorem 1.7]. The main result of this paper (solution for the flat IC) can be stated in terms of this process $\mathcal{A}_a(\hat{x})$,

$$\max_{\hat{x}>0}[(\mathcal{A}_a(\hat{x}) - \hat{x}^2)] \overset{(d)}{=} \mathrm{BBP}_a, \tag{47}$$

where $\overset{(d)}{=}$ denotes the equality of distributions and BBP_a denotes the BBP distribution defined in Eq. (24). More generally, from the identity Eq. (37) obtained in this paper, one would conclude that, for fixed $\hat{X} \geqslant 0$,

$$\max_{\hat{x}>\hat{X}}[(\mathcal{A}_a(\hat{x}) - \hat{x}^2)] \overset{(d)}{=} a^2 + 2a\hat{X} \tag{48}$$

$$+ [\mathcal{A}_{2\to BM}(-\hat{X} - a) - (\hat{X} + a)^2], \tag{49}$$

where $\mathcal{A}_{2\to BM}$ was introduced in Ref. [65] (see also Refs. [14,66]). For Eq. (46) to match our results on the stationary large time limit requires that for $\hat{x} \ll 1$,

$$\mathcal{A}_a(\hat{x}) \simeq \sqrt{2}B(\hat{x}) + 2a\hat{x}. \tag{50}$$

Finally, the endpoint PDF scaling function $\mathcal{P}(y, a)$ defined in Eq. (45) would be obtained from this process as the PDF of $y = \mathrm{argmax}\{[\mathcal{A}_a(\hat{x}) - \hat{x}^2]\}$. It is then natural to conjecture the universality of the above distributions at a half-space KPZ fixed point.

In the context of last passage percolation, an identity reminiscent of Eq. (36) relating the distribution of the point to point energy in a full-space model and the point to line energy in a half-space model, was stated as Ref. [32, Eq. (7.59)]. In the large scale limit (studied in Ref. [33]) both distributions converge to the BBP distribution (the limiting distribution function was denoted $F^{\boxtimes}(x; w)$ in Ref. [33], it coincides with the BBP distribution defined later in Ref. [43]). Our asymptotic results at large time for the KPZ equation in Sec. III B thus confirm universality predictions. Let us stress, however, that the identity in distribution from Ref. [32] cannot be scaled to the KPZ equation: one cannot deduce from it our finite time identities in distribution Eqs. (36), (38), (40), and (C17).

VII. CONCLUSION

We obtained the solution for all times to the KPZ equation on a half-line with flat IC, i.e., the distribution of the height at the origin for any wall parameter A. Thanks to remarkable algebraic cancellations it is simpler than the solution for flat IC on the full line. In fact, we find that it is related to the half-Brownian IC on the full line, and uncover further curious relations between full and half-line problems. Equivalently it gives the free energy of a DP of any length t in a half-space, with one free endpoint and the other pinned at the wall. We showed that its critical behavior at the unbinding transition at $A = -1/2$ is identical to the BBP critical behavior for outliers of GUE random matrices. For $A < -1/2$ the polymer is bound to the wall and at large t its endpoint position fluctuates as a particle at equilibrium in a one-sided Brownian plus linear confining potential. This considerably extends early predictions within the replica Bethe ansatz. These results open questions such as generalization of the aforementioned identities to several points or times, studying the rate of convergence to the stationary measure μ_A that we determined, possibly in relation to excited states within the replica Bethe ansatz, and properties of the non Gaussian stationary measures $\mu_{A,B}$ (their analogues in finite volume were recently studied in Ref. [67]). Our results near criticality are part of a larger universal KPZ fixed point structure in half-space, yet to be fully characterized.

ACKNOWLEDGMENTS

We thank Alexandre Krajenbrink for many useful conversations related to half-space solutions of the KPZ equation. We thank Christophe Texier for an interesting discussion. G.B. was partially supported by ANR Grant No. ANR-19-CE40-0012 MicMov. P.L.D. acknowledges support from ANR Grant No. ANR-17-CE30-0027 RaMaTraF.

APPENDIX A: FLAT INITIAL CONDITION: DETAILS

1. Explicit formula for the moments

Computing the factor $E(w_1, w_1 + 1, \dots, w_1 + m_1 - 1, \dots, w_\ell, \dots, w_\ell + m_\ell - 1)$ in Eq. (9) using Eq. (12) one finds

$$\mathbb{E}\big[Z_A^f(0, t)^n\big] = n! 2^n \sum_{\ell=1}^{n} \frac{1}{\ell!} \sum_{\vec{m}: \sum_{i=1}^{\ell} m_i = n} \prod_{i=1}^{\ell} \int_{r+\mathrm{i}\mathbb{R}} \frac{dw_i}{2\mathrm{i}\pi} \det\left(\frac{1}{w_i + m_i - w_j}\right)_{i,j=1}^{\ell} \prod_{i=1}^{\ell} \frac{\Gamma(A + w_i)}{\Gamma(A + w_i + m_i)} e^{t(G(w_i+m_i) - G(w_i))}, \tag{A1}$$

where we used that $\sum_{k=0}^{m-1}(w + k)^2 = G(w + m) - G(w)$ and $G(w)$ is defined as $G(w) := \frac{w^3}{3} - \frac{w^2}{2} + \frac{w}{6}$. We recall that the real part of integration contours is such that $r > -A$. One can check that the integrals over w_j are convergent.

2. Laplace transform

Let us now consider the generating function $1 + \sum_{n=1}^{+\infty} \frac{(-u)^n}{n!} \mathbb{E}[Z_A^f(0, t)^n]$. The summation over n allows to eliminate the constraint $\sum_{i=1}^{\ell} m_i = n$ in the sum over the variables m_i in Eq. (A1). Although it is a divergent series, after rearrangements of the terms and use of a Mellin Barnes representation of the sums, it yields an expression for the Laplace transform $\mathbb{E}[e^{-uZ_A^f(0,t)}]$ which

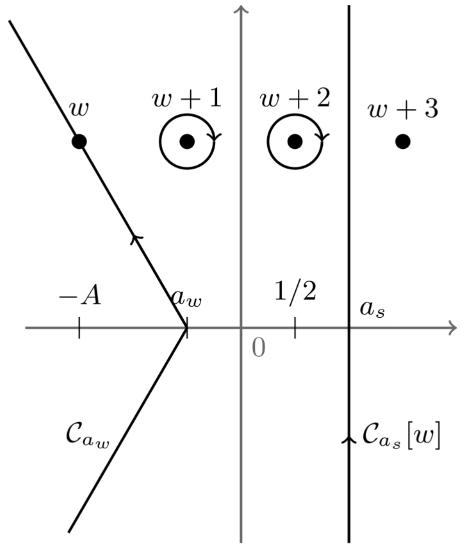

FIG. 3. The contours $\mathcal{C}_{a_w}$ and $\mathcal{C}_{a_s}[w]$ are shown in the figure. The contour $\mathcal{C}_{a_s}[w]$ depends on the location of w. For the w depicted in the figure, the contour consists of the union of the vertical line with real part a_s and small negatively oriented circles around $w+1$ and $w+2$, since $w+1$ and $w+2$ lie to the left of the vertical line with real part a_s.

in all known cases has given the correct result. The Mellin-Barnes representation replaces the sum over the integer variables m_i by integrals, for each i

$$\sum_{m=1}^{+\infty}(-1)^m f(m) = \int_{\mathcal{C}} \frac{dz}{2\mathbf{i}\pi}\frac{\pi}{\sin(-\pi z)} f(z), \tag{A2}$$

where $\mathcal{C} = a + \mathbf{i}\mathbb{R}$ with $0 < a < 1$, oriented from bottom to top. Introducing a variable z_j for each m_j and performing the change from z_j to $s_j = w_j + z_j$, this leads to

$$\mathbb{E}[e^{-uZ_A^f(0,t)}] = \sum_{\ell=0}^{+\infty}\frac{1}{\ell!}\int_{\mathcal{C}_{a_w}}\frac{dw_1}{2\mathbf{i}\pi}\cdots\int_{\mathcal{C}_{a_w}}\frac{dw_\ell}{2\mathbf{i}\pi}\int_{\mathcal{C}_{a_s}[w_1]}\frac{ds_1}{2\mathbf{i}\pi}\cdots\int_{\mathcal{C}_{a_s}[w_\ell]}\frac{ds_\ell}{2\mathbf{i}\pi}\det\left(\frac{1}{s_i - w_j}\right)_{i,j=1}^{\ell}$$
$$\times\prod_{j=1}^{\ell}\left[\frac{e^{tG(s_j)}}{e^{tG(w_j)}}\frac{\pi}{\sin(-\pi(s_j - w_j))}(2u)^{s_j - w_j}\right]\prod_{j=1}^{\ell}\frac{\Gamma(A+w_j)}{\Gamma(A+s_j)}. \tag{A3}$$

The contour for variables w_i, denoted $\mathcal{C}_{a_w}$, is the union of two semi-infinite rays leaving the point $a_w > -A$ in the direction $\pm 2\pi/3$, oriented from bottom to top. The contour for variables s_i, denoted $\mathcal{C}_{a_s}[w]$ is formed by the union of the vertical line $a_s + \mathbf{i}\mathbb{R}$ and the union of negatively oriented circles around the poles at $w+1, w+2,\ldots$ when these lie to the left of the vertical line (see Fig. 3). The vertical line is oriented from bottom to top. Furthermore, a sufficient condition for the integrals over s_i to be convergent is that $a_s - 1/2 > 0$. We choose the real numbers a_s and a_w so that

$$-A < a_w < a_s. \tag{A4}$$

Since from the choices of integration contours one has $\mathrm{Re}(s_i - w_j) > 0$ in Eq. (A3), we can use the representation

$$\frac{1}{s_i - w_j} = \int_0^{+\infty} dv e^{-v(s_i - w_j)} \tag{A5}$$

inside the determinant. After some simple manipulations one recognizes the expansion of a Fredholm determinant

$$\mathbb{E}[e^{-uZ_A^f(0,t)}] = \mathrm{Det}(I + \overline{K}_{u,t})_{\mathbb{L}^2(\mathbb{R}^+)}, \tag{A6}$$

with the kernel

$$\overline{K}_{u,t}(v,v') = \int_{\mathcal{C}_{a_w}}\frac{dw}{2\mathbf{i}\pi}\int_{\mathcal{C}_{a_s}[w]}\frac{ds}{2\mathbf{i}\pi}\frac{\pi}{\sin[-\pi(s-w)]}(2u)^{s-w}e^{-vs+v'w}\frac{e^{tG(s)}}{e^{tG(w)}}\frac{\Gamma(A+w)}{\Gamma(A+s)}. \tag{A7}$$

This provides an expression of the generating function in terms of a kernel involving two contour integrals. We will now transform this formula to obtain an alternative expression in terms a second kernel, as given in the main text in Sec. III. Let us use the identity for $\Re\lambda > 0$,

$$(2u)^{\lambda}\frac{\pi}{\sin(-\pi\lambda)} = -\int_{\mathbb{R}} dr \frac{2u}{e^{-r}+2u}e^{-\lambda r}, \tag{A8}$$

and perform the shift $w_j \to w_j + 1/2$ and change of variables $s_j = z_j + 1/2$. This leads to our final formula

$$\mathbb{E}[e^{-uZ_A^f(0,t)e^{\frac{t}{12}}}] = \det(I - K_{u,t})_{\mathbb{L}^2(0,+\infty)}, \tag{A9}$$

with the kernel

$$K_{u,t}(v,v') = \int_{\mathbb{R}} dr \frac{2u}{e^{-r}+2u}\phi_{A,t}(v+r)\psi_{A,t}(v'+r), \tag{A10}$$

$$\phi_{A,t}(v) = \int_{a_z+\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi}\frac{e^{t\frac{z^3}{3}-vz}}{\Gamma(A+\frac{1}{2}+z)}, \tag{A11}$$

$$\psi_{A,t}(v) = \int_{\mathcal{C}_{a_w}} \frac{dw}{2\mathbf{i}\pi}e^{-t\frac{w^3}{3}+vw}\Gamma(A+\tfrac{1}{2}+w), \tag{A12}$$

where the contour for z is a vertical line such that $a_z > 0$ and the contour for w, denoted $\mathcal{C}_{a_w}$, is the union of two semi-infinite rays leaving the point $a_w > -(A+\frac{1}{2})$ in the direction $\pm 2\pi/3$.

APPENDIX B: ASYMPTOTICS: DETAILS

1. Case $A > \frac{-1}{2}$

Using the rescalings indicated in Sec. III, we have the limits

$$\lim_{t\to\infty}\Gamma\left(A+\frac{1}{2}\right)t^{1/3}\phi_{A,t}(t^{1/3}\tilde{v}) = \lim_{t\to\infty}\frac{t^{1/3}\psi_{A,t}(t^{1/3}\tilde{v})}{\Gamma(A+\frac{1}{2})} = \mathrm{Ai}(\tilde{v}) = \int_{1+\mathbf{i}\mathbb{R}}\frac{dz}{2\mathbf{i}\pi}e^{\frac{z^3}{3}-\tilde{v}z}. \tag{B1}$$

We also have, with $u = e^{-t^{1/3}s}$ and $r = t^{1/3}\tilde{r}$,

$$\frac{2u}{e^{-r}+2u} = \frac{2}{2+e^{t^{1/3}(s-\tilde{r})}} \xrightarrow[t\to\infty]{} \theta(\tilde{r}-s), \tag{B2}$$

which leads to Eqs. (21) and (22) in the main text.

2. Case $A < \frac{-1}{2}$

When $A < \frac{-1}{2}$, we use the following change of variables in Eqs. (A10)–(A12):

$$r = \left(A+\frac{1}{2}\right)^2 + t^{1/2}\tilde{r}, \qquad v = t^{1/2}\tilde{v}, \qquad z = -\left(A+\frac{1}{2}\right) + t^{-1/2}\tilde{z}, \tag{B3}$$

so that

$$\phi_{A,t}\left[\left(A+\frac{1}{2}\right)^2 t + t^{1/2}(\tilde{v}+\tilde{r})\right] \simeq \frac{1}{t}e^{\frac{2t(A+\frac{1}{2})^3}{3}}e^{(A+\frac{1}{2})t^{1/2}(\tilde{v}+\tilde{r})}\int\frac{d\tilde{z}}{2\mathbf{i}\pi}\tilde{z}e^{-(A+\frac{1}{2})\tilde{z}^2-(\tilde{v}+\tilde{r})\tilde{z}}. \tag{B4}$$

To take the asymptotics of $\psi_{A,t}$, we first shift the contour to the left of the pole at $w = -(A+\frac{1}{2})$ and we obtain

$$\psi_{A,t}(v) = e^{\frac{t(A+\frac{1}{2})^3}{3}-v(A+\frac{1}{2})} + \int_{\mathcal{C}_{a'_w}}\frac{dw}{2\mathbf{i}\pi}e^{-t\frac{w^3}{3}+vw}\Gamma(A+\tfrac{1}{2}+w), \tag{B5}$$

where now, $\mathcal{C}_{a'_w}$ is the union of two semi-infinite rays in direction $\pm\phi$ where $\phi \in (\frac{\pi}{2},\frac{5\pi}{6})$ (so that $\mathrm{Re}[w^3] > 0$), which intersect the horizontal axis at a'_w with $-A-\frac{3}{2} < a'_w < -(A+\frac{1}{2})$. We obtain

$$\psi_{A,t}\left[\left(A+\frac{1}{2}\right)^2 t + t^{1/2}(\tilde{v}'+\tilde{r})\right] \simeq e^{-\frac{2t(A+\frac{1}{2})^3}{3}}e^{-(A+\frac{1}{2})t^{1/2}(\tilde{v}'+\tilde{r})}\left[1+\int_{\mathcal{C}_{a_{\tilde{w}}}}\frac{d\tilde{w}}{2\mathbf{i}\pi}\frac{e^{(A+\frac{1}{2})\tilde{w}^2+(\tilde{v}'+\tilde{r})\tilde{w}}}{\tilde{w}}\right], \tag{B6}$$

where we have chosen $\phi \in (\frac{3\pi}{4},\frac{5\pi}{6})$ (so that $\mathrm{Re}[\tilde{w}^2] > 0$), and we have used the change of variables $w = -(A+\frac{1}{2}) + t^{-1/2}\tilde{w}$ and we scale $a'_w = -(A+\frac{1}{2}) + a_{\tilde{w}}t^{-1/2}$ with $a_{\tilde{w}} < 0$. Notice that in the integral in the r.h.s. of Eq. (B6), the contour can be freely shifted to the left to $-\infty$, so that the integral is zero. We now set $u = e^{-(A+\frac{1}{2})^2 t - st^{1/2}}$, so that

$$\frac{2u}{e^{-r}+2u} = \frac{2}{2+e^{t^{1/2}(s-\tilde{r})}} \xrightarrow[t\to\infty]{} \theta(\tilde{r}-s) \tag{B7}$$

and

$$\mathbb{E}[e^{-uZ_A^f(0,t)e^{\frac{t}{12}}}] \xrightarrow[t\to\infty]{} \mathbb{P}\left[\frac{\log Z_A^f(0,t)+\frac{t}{12}-(A+\frac{1}{2})^2 t}{t^{1/2}} \leqslant s\right]. \tag{B8}$$

Putting all terms together, there are cancellations of the prefactors from $\psi_{A,t}$ and $\phi_{A,t}$, so that $t^{1/2}K_{u,t}(v,v') \xrightarrow[t\to\infty]{} \widetilde{K}_s(\tilde{v},\tilde{v}')$ and we obtain that

$$\lim_{t\to\infty} \det(I-K_{u,t})_{\mathbb{L}^2(0,+\infty)} = \det(I-\widetilde{K}_s)_{\mathbb{L}^2(0,+\infty)}, \tag{B9}$$

where

$$\widetilde{K}_s(\tilde{v},\tilde{v}') = \int_s^{+\infty} d\tilde{r} \int \frac{d\tilde{z}}{2\mathbf{i}\pi} \tilde{z} e^{-(A+\frac{1}{2})\tilde{z}^2-(\tilde{v}+\tilde{r})\tilde{z}} = \int \frac{d\tilde{z}}{2\mathbf{i}\pi} e^{-(A+\frac{1}{2})\tilde{z}^2-(\tilde{v}+s)\tilde{z}} = \frac{1}{\sqrt{4\pi|A+\frac{1}{2}|}} e^{-\frac{(s+\tilde{v})^2}{4|A+\frac{1}{2}|}}, \tag{B10}$$

so that, at large time

$$\log Z_A^f(0,t) \simeq \left[-\frac{1}{12}+\left(A+\frac{1}{2}\right)^2\right]t + t^{1/2}\chi, \tag{B11}$$

with

$$\mathbb{P}(\chi < s) = \det(I-\widetilde{K}_s)_{\mathbb{L}^2(0,+\infty)} = 1 - \int_0^\infty \widetilde{K}_s(\tilde{v},\tilde{v})d\tilde{v} = \mathbb{P}(G \leqslant s), \tag{B12}$$

where G is a centered Gaussian random variable with variance $2|A+\frac{1}{2}|$, as announced in the main text. In Eq. (B12) the Fredholm determinant is simple to evaluate since $\widetilde{K}_s$ is a rank one kernel (i.e., a projector). Note that the variance of the random variable χ has the same value as for the droplet IC, as found by a more heuristic method in Ref. [42].

3. Critical case $A = \frac{-1}{2} + at^{-1/3}$

We use the same scalings as indicated in Sec. III for $A > \frac{-1}{2}$, and we obtain

$$\lim_{t\to\infty} t^{2/3}\phi_{A,t}(t^{1/3}\tilde{v}) = \phi_a(\tilde{v}) := \int_{1+\mathbf{i}\mathbb{R}} \frac{d\tilde{z}}{2\mathbf{i}\pi}(\tilde{z}+a)e^{\frac{\tilde{z}^3}{3}-\tilde{v}\tilde{z}}, \tag{B13}$$

$$\lim_{t\to\infty} \psi_{A,t}(t^{1/3}\tilde{v}) = \psi_a(\tilde{v}) := \int_{\mathcal{C}_{a_{\tilde{w}}}} \frac{d\tilde{w}}{2\mathbf{i}\pi} \frac{e^{\frac{-\tilde{w}^3}{3}+\tilde{v}\tilde{w}}}{\tilde{w}+a}, \tag{B14}$$

where the contour $\mathcal{C}_{a_{\tilde{w}}}$ in Eq. (B14) is the union of two semi-infinite rays in direction $\pm 2\pi/3$ which intersect the horizontal axis at $a_{\tilde{w}}$ is such that $a_{\tilde{w}} > -a$. Putting all together we obtain, as announced in the main text,

$$\mathbb{P}(\chi \leqslant s) = \det\left(I - K_a^{\mathrm{BBP}}\right)_{\mathbb{L}^2(s,+\infty)} = F_a^{\mathrm{BBP}}(s), \tag{B15}$$

with the kernel

$$K_a^{\mathrm{BBP}}(v,v') = \int_0^{+\infty} dr \phi_a(v+r)\psi_a(v'+r), \tag{B16}$$

where the functions ϕ_a and ψ_a are defined in Eqs. (B13) and (B14). This distribution was introduced in Ref. [43] and the form that we obtained in Eq. (24) [i.e., Eq. (B15)] can be matched with the original definition from Ref. [43] using, e.g., Ref. [47].

APPENDIX C: GENERALIZATION TO THE SHIFTED PARTITION FUNCTION WITH $X \geqslant 0$ AND ARBITRARY PARAMETERS A, B

1. Moment formulas

The starting point is the following formula from Ref. [41, Sec. 4.4] for the moments of $Z_{A,B}(x,t)$, that is the solution to the half-space SHE (2) with Brownian IC $e^{\mathcal{B}(x)-(B+\frac{1}{2})}$. For $B > n-1$, $A+B > n-1$, and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$,

$$\mathbb{E}\left[\prod_{i=1}^n Z_{A,B}(x_i,t)\right] = 2^n \frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)} \int_{r_1+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r_n+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_n}{2\mathbf{i}\pi} \prod_{1\leqslant a<b\leqslant n} \frac{z_a-z_b}{z_a-z_b-1}\frac{z_a+z_b}{z_a+z_b-1}$$

$$\times \prod_{i=1}^n \frac{z_i}{z_i+A}\frac{1}{B^2-z_i^2} e^{tz_i^2-x_iz_i}, \tag{C1}$$

where the contours are chosen so that $B > r_1 > r_2 + 1 > \ldots > r_n + n - 1 > \max\{n-1-A, n-1\}$, i.e., all contours are to the right of $-A$ and to the left of B. Recall the definition of $Z_{A,B}^{\text{shifted}}(X,t)$ from Eq. (39). Using the same steps as in the main text around Eq. (6), we have

$$\mathbb{E}\big[Z_{A,B}^{\text{shifted}}(X,t)^n\big] = n!2^n \frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)} \int_{r_1+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r_n+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_n}{2\mathbf{i}\pi} \prod_{1\leqslant a<b\leqslant n} \frac{z_a - z_b}{z_a - z_b - 1} F(\vec{z}), \tag{C2}$$

where now,

$$F(\vec{z}) = \prod_{1\leqslant a<b\leqslant n} \frac{z_a + z_b}{z_a + z_b - 1} \prod_{i=1}^{n} \frac{e^{t z_i^2 - X z_i}}{z_1 + \cdots + z_i} \frac{z_i}{z_i + A} \frac{1}{B^2 - z_i^2}. \tag{C3}$$

As in the main text, we may use Ref. [38, Proposition 5.1] to obtain that Eq. (C1) becomes

$$n!2^n \frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)} \sum_{\ell=1}^{n} \frac{1}{\ell!} \sum_{\vec{m}:\sum m_i = n} \int_{r+\mathbf{i}\mathbb{R}} \frac{dw_1}{2\mathbf{i}\pi} \cdots \int_{r+\mathbf{i}\mathbb{R}} \frac{dw_\ell}{2\mathbf{i}\pi} \det\left(\frac{1}{w_i + m_i - w_j}\right)_{i,j=1}^{\ell}$$
$$\times\, E(w_1, w_1+1, \ldots, w_1+m_1-1, \ldots, w_\ell, \ldots, w_\ell + m_\ell - 1), \tag{C4}$$

where now, the real part of contours is such that $\max\{-A, 0\} < r < B$ and the function E is given in terms of the function F in Eq. (C3) by the same formula as in as in Eq. (10). It is computed as in the main text using the same symmetrization Eq. (11), and we obtain that

$$E(\vec{z}) = \prod_{i=1}^{n} \frac{e^{t z_i^2 - X z_i}}{z_i + A} \frac{1}{B^2 - z_i^2}. \tag{C5}$$

At this point, we may use Ref. [38, Proposition 5.1] backwards, and obtain that the moments of $Z_{A,B}^{\text{shifted}}(X,t)$ are given by the relatively simple nested contour formula:

$$\mathbb{E}\big[Z_{A,B}^{\text{shifted}}(X,t)^n\big] = 2^n \frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)} \int_{r_1+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r_n+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_n}{2\mathbf{i}\pi} \prod_{1\leqslant a<b\leqslant n} \frac{z_a - z_b}{z_a - z_b - 1} \prod_{i=1}^{n} \frac{z_i}{z_i + A} \frac{1}{B^2 - z_i^2} e^{t z_i^2 - X z_i}, \tag{C6}$$

where the contours are chosen so that $B > r_1 > r_2 + 1 > \ldots > r_n + n - 1 > \max\{n-1-A, n-1\}$.

2. Laplace transform

We may also use Eq. (C4) to form the moment generating series as in Appendix A 2 and obtain a Fredholm determinant formula. The main difference with Appendix A is that there is a prefactor $\frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)}$ in the moment Eq. (C4). This is the reason why it is convenient to introduce an inverse gamma random variable W with parameter $A+B+1$, i.e., of PDF $P(w) = \frac{1}{\Gamma(A+B+1)} w^{-A-B-2} e^{-1/w} \theta(w)$ and moments $\mathbb{E}[W^n] = \frac{\Gamma(A+B+1-n)}{\Gamma(A+B+1)}$, independent from $Z_{A,B}^{\text{shifted}}(X,t)$, so that the moments of $W Z_{A,B}^{\text{shifted}}(X,t)$ satisfy the same formula as Eq. (C4) without the ratio of Gamma functions. At this point we may reproduce the steps detailed above in Appendix A 2. We use $\prod_{i=0}^{m-1} \frac{1}{w+i+A} = \frac{\Gamma(A+w)}{\Gamma(A+w+m)}$ and $\prod_{i=0}^{m-1} \frac{1}{B^2-(w+i)^2} = \frac{\Gamma(B+w)\Gamma(B-w-m+1)}{\Gamma(B-w+1)\Gamma(B+w+m)}$. We obtain

$$\mathbb{E}[e^{-uW Z_{A,B}^{\text{shifted}}(X,t)}] = \sum_{\ell=0}^{+\infty} \frac{1}{\ell!} \int_{\mathcal{C}_{a_w}} \frac{dw_1}{2\mathbf{i}\pi} \cdots \int_{\mathcal{C}_{a_w}} \frac{dw_\ell}{2\mathbf{i}\pi} \int_{\mathcal{D}_{a_s}[w_1]} \frac{ds_1}{2\mathbf{i}\pi} \cdots \int_{\mathcal{D}_{a_s}[w_\ell]} \frac{ds_\ell}{2\mathbf{i}\pi} \det\left(\frac{1}{s_i - w_j}\right)_{i,j=1}^{\ell}$$
$$\times \prod_{j=1}^{\ell} \left[\frac{e^{tG(s_j) - X\frac{(s_j-1/2)^2}{2}}}{e^{tG(w_j) - X\frac{(w_j-1/2)^2}{2}}} \frac{\pi}{\sin(-\pi(s_j - w_j))} (2u)^{s_j - w_j} \right] \prod_{j=1}^{\ell} \frac{\Gamma(A+w_j)\Gamma(B+w_j)\Gamma(B-s_j+1)}{\Gamma(A+s_j)\Gamma(B+s_j)\Gamma(B-w_j+1)}. \tag{C7}$$

The contours are chosen similarly as in Appendix A 2. More precisely, the contour for variables w_i, denoted $\mathcal{C}_{a_w}$, is the union of two semi-infinite rays leaving the point $a_w > \max\{-A, -B\}$ in the direction $\pm 2\pi/3$. The contour for variables s_i, denoted $\mathcal{D}_{a_s}[w]$ is formed by the union two parts: (1) a wedge shaped contour, that is the union of two semi-infinite rays leaving the point a_s with $a_w < a_s < B+1$, and (2) the union of negatively oriented circles around the poles at $w+1, w+2, \ldots$ when these lie to the left of the wedge; see Fig. 4. All infinite contours are oriented from bottom to top. The moment Eq. (C7) leads to

$$\mathbb{E}[e^{-uW Z_{A,B}^{\text{shifted}}(X,t) e^{\frac{t}{12}}}] = \det\left(I - K_{u,t}^{A,B}\right)_{\mathbb{L}^2(0,+\infty)}, \tag{C8}$$

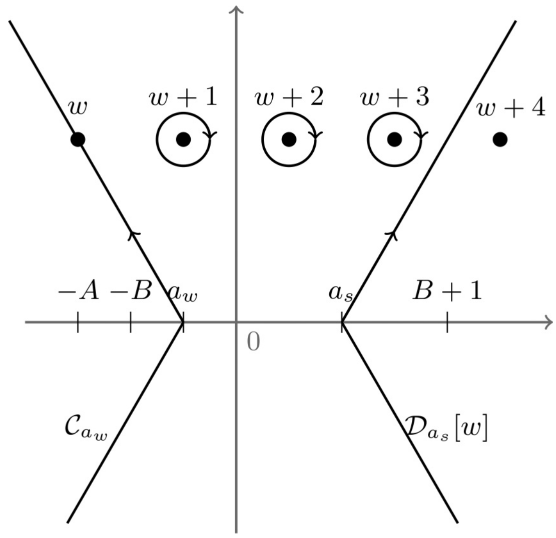

FIG. 4. The contours $\mathcal{C}_{a_w}$ and $\mathcal{D}_{a_s}[w]$ are shown in the figure. The contour $\mathcal{D}_{a_s}[w]$ depends on the location of w. For the w depicted in the figure, the contour consists of the union of the wedge crossing the real axis at the point $a_s < B+1$ and small negatively oriented circles around $w+1$, $w+2$ and $w+3$, since $w+1$, $w+2$ and $w+3$ lie to the left of the wedge.

with the kernel

$$K_{u,t}^{A,B}(v,v') = \int_{\mathbb{R}} dr \frac{2u}{e^{-r}+2u} \phi_{A,B,t}(v+r)\psi_{A,B,t}(v'+r), \tag{C9}$$

$$\phi_{A,B,t}(v) = \int_{\mathcal{D}_{a_z}} \frac{dz}{2\mathbf{i}\pi} e^{t\frac{z^3}{3} - X\frac{z^2}{2} - vz} \frac{\Gamma(B+\frac{1}{2}-z)}{\Gamma(A+\frac{1}{2}+z)\Gamma(B+\frac{1}{2}+z)}, \tag{C10}$$

$$\psi_{A,B,t}(v) = \int_{\mathcal{C}_{a_w}} \frac{dw}{2\mathbf{i}\pi} e^{-t\frac{w^3}{3} + X\frac{w^2}{2} + vw} \frac{\Gamma(A+\frac{1}{2}+w)\Gamma(B+\frac{1}{2}+w)}{\Gamma(B+\frac{1}{2}-w)}, \tag{C11}$$

where the contours for z, denoted $\mathcal{D}_{a_z}$ is the union of two semi-infinite rays leaving the point $a_z < B + \frac{1}{2}$ in the direction $\pm\pi/3$, and the contour for w, denoted $\mathcal{C}_{a_w}$, is the union of two semi-infinite rays leaving the point $a_w > \max\{-(A+\frac{1}{2}), -(B+\frac{1}{2})\}$ in the direction $\pm 2\pi/3$.

Limit as $A \to +\infty$. In this limit, $Z_{A,B}(x,t)$ converges to $Z_{Dir,B}(x,t)$, the solution to the half-space SHE (2) with Brownian IC $e^{\mathcal{B}(x)-(B+\frac{1}{2})x}$ and Dirichlet boundary condition, that is it satisfies the boundary condition $Z_{Dir,B}(0,t) = 0$ for all $t > 0$. Then, defining

$$Z_{Dir,B}^{\text{shifted}}(X,t) = \int_{x \geqslant X} Z_{Dir,B}(x,t)dx, \tag{C12}$$

as in Eq. (39), Eqs. (C8) and (C9) become

$$\mathbb{E}[e^{-uZ_{Dir,B}^{\text{shifted}}(X,t)e^{\frac{t}{12}}}] = \det(I - K_{u,t}^{\infty,B})_{\mathbb{L}^2(0,+\infty)}, \tag{C13}$$

where

$$K_{u,t}^{\infty,B}(v,v') = \int_{\mathbb{R}} dr \frac{2u}{e^{-r}+2u} \phi_{\infty,B,t}(v+r)\psi_{\infty,B,t}(v'+r), \tag{C14}$$

$$\phi_{\infty,B,t}(v) = \int_{a_z+\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} e^{t\frac{z^3}{3} - X\frac{z^2}{2} - vz} \frac{\Gamma(B+\frac{1}{2}-z)}{\Gamma(B+\frac{1}{2}+z)}, \tag{C15}$$

$$\psi_{\infty,B,t}(v) = \int_{\mathcal{C}_{a_w}} \frac{dw}{2\mathbf{i}\pi} e^{-t\frac{w^3}{3} + X\frac{w^2}{2} + vw} \frac{\Gamma(B+\frac{1}{2}+w)}{\Gamma(B+\frac{1}{2}-w)}. \tag{C16}$$

Note that when performing the limit $A \to +\infty$ the prefactor $\frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)}$ in the moment Eq. (C4) is replaced by A^n which is compensated by the total factor A^{-n} from the Gamma functions inside the integrals. Hence there is no need anymore for the variable W and one obtains the finite limit in Eq. (C13).

This kernel $K^{\infty,B}$ already appeared in Ref. [17, Prop. 1] and Ref. [18, Th. 2.9] in the context of the full space KPZ with two sided Brownian IC. This implies that we have the equality in distribution, for fixed X, t and $B > \frac{-1}{2}$,

$$Z_{Dir,B}^{\text{shifted}}(X,t) = 2Z^{(B|B)}(-X,t) = 2w\tilde{Z}^{(B|B)}(-X,t), \tag{C17}$$

where $Z^{(B|B)}(X,t)$ is the solution to the full-space SHE (35) with initial condition $we^{\mathcal{B}(x)-(B+\frac{1}{2})|x|}$ where $\mathcal{B}(x)$ is a two-sided Brownian motion with $\mathcal{B}(0)=0$ and w is an independent inverse Gamma random variable with parameter $2B+1$. The last identity is trivial since w can be put in factor at all x, t and $\tilde{Z}^{(B|B)}(-X,t)$ has the same IC but without the w factor, as defined in the main text. As we explained in the main text, the identity in law Eq. (C17) can be seen as the limit of Eq. (40) as A goes to infinity.

3. Mapping to full-space KPZ with specific initial condition

In this section, we explain the identity in law Eq. (40) (which in particular implies the identity in distribution Eq. (38) after letting B go to $+\infty$). Recall the definition of $Z^{(B|A,B)}(X,t)$, that is the solution to full-line SHE (35) with IC depending on parameters A, B and specified by Eqs. (41) and (42). It seems that for this quantity, moment formulas have not been written down previously, nor a Fredholm determinant representation for the moment generating function. Thus, we cannot immediately compare its distribution with Eq. (C6) or Eq. (C8).

Nevertheless, the moments of $Z^{(B|A,B)}(X,t)$ can be obtained from methods that are available in the literature. For this, we first need to establish a moment formula for certain partition functions of the log-gamma polymer, a directed polymer model on the lattice $\mathbb{Z}^2$ introduced in Ref. [68]. Then we will take the continuous limit to the KPZ equation along the lines of Ref. [41], and find that the moments match with Eq. (C6), up to a factor 2 that accounts for the factor 2 in Eq. (40).

Remark: Note that for the same log-gamma polymer model, Fredholm determinant formulas are available in Refs. [18,69,70], and after taking the limit to the KPZ equation, this should allow to match with Eq. (C8) but this route is more technical and we will not pursue it here.

We need to briefly define the log-gamma polymer partition function that we will be working with, and we refer to Refs. [68,71] for details. Consider a sequence of random variables $(w_{i,j})_{i,j\geqslant 1}$ distributed as independent inverse Gamma random variables with parameter $\alpha_i+\beta_j$, where α_i and β_j are arbitrary sequences of real numbers such that $\alpha_i+\beta_j>0$. We define the partition function

$$Z(n,m) = \sum_{\pi:(1,1)\to(n,m)} \prod_{(i,j)\in\pi} w_{i,j}, \tag{C18}$$

where the sum runs over upright paths π in $\mathbb{Z}^2$ going from (1,1) to (n,m). For $n_1 \geqslant \cdots \geqslant n_k \geqslant 1$ and $1 \leqslant m_1 \leqslant \cdots \leqslant m_k$, we have

$$\mathbb{E}\left[\prod_{i=1}^k Z(n_i,m_i)\right] = \oint \frac{\mathrm{d}w_1}{2\mathbf{i}\pi}\cdots\oint\frac{\mathrm{d}w_k}{2\mathbf{i}\pi}\prod_{a<b}\frac{w_a-w_b}{w_a-w_b-1}\prod_{j=1}^k\left(\prod_{i=1}^{n_j}\frac{1}{\alpha_i-1/2+w_j}\prod_{i=1}^{m_j}\frac{1}{\beta_i-1/2-w_j}\right), \tag{C19}$$

where all integration contours are positively oriented and enclose the $-\alpha_i+1/2$ but not the $\beta_j-1/2$, and are nested such that for $i<j$, the w_i-contour encloses the w_j-contour shifted by 1. These conditions can be satisfied only for small enough k. The contours may be taken as closed curves or be deformed to become infinite vertical lines. It seems that Eq. (C19) has not been written anywhere in the literature (though Fredholm determinant formulas for the Laplace transform are given in Refs. [18,69,70]). This moment formula can be obtained by taking appropriate specializations and limits in Ref. [72, Theorem 4.6] (the appropriate specializations and limits that one needs to take are explained in many references, see, e.g., Ref. [44, Secs. 4 and 5.3]).

Now, we are ready to take the continuous limit. The fact that the partition function $Z(n,m)$ converges to the solution to the SHE (35) was originally proved in Refs. [73,74] for general directed polymer models (see also Ref. [75] for the application to the log-gamma polymer), but we will follow the arguments from the physics work [41, Sec. 4]. Assume that we scale α_i and β_j such that

$$\beta_1 = \tfrac{1}{2}+B, \quad \beta_i = \tfrac{1}{2}+\sqrt{n}, \quad (i\geqslant 2), \tag{C20}$$

and

$$\alpha_1 = \tfrac{1}{2}+A, \alpha_2 = \tfrac{1}{2}+B, \quad \alpha_i = \tfrac{1}{2}+\sqrt{n}, \quad (i \geqslant 3). \tag{C21}$$

The rescaled partition function

$$\mathcal{Z}_n(x,t) = n^{tn} Z(tn - x\sqrt{n}/2, tn + x\sqrt{n}/2) \tag{C22}$$

converges as n goes to infinity [41, Claim 4.6] to the solution of Eq. (35) with initial condition given as follows. Let $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ be independent Brownian motions with respective drifts $-(B+\frac{1}{2}), -(A+\frac{1}{2}), -(B+\frac{1}{2})$. Let w_{11} be an inverse Gamma random variable with parameter $A+B+1$ and w_{21} be an inverse Gamma random variable with parameter $2B+1$. For $x\leqslant 0$, the initial condition is given by

$$Z(x,0) = w_{11}w_{21}e^{\mathcal{W}_1(-x)}, \tag{C23}$$

and for $x \geqslant 0$,

$$Z(x,0) = w_{11}w_{21}e^{\mathcal{W}_3(x)} + w_{11}\int_0^x e^{\mathcal{W}_2(y)+\mathcal{W}_3(x)-\mathcal{W}_3(y)}dy. \tag{C24}$$

Let us briefly explain how this initial condition is obtained. Note that under Eqs. (C20) and (C21), the weights w_{11} and w_{22} are independent and inverse Gamma distributed with parameters $A+B+1$ and $2B+1$. We have that for $i \geqslant 3$ $\alpha_i + \beta_1 = B+1+\sqrt{n}$, so that under the scaling given in Eq. (C22), products of weights along the first row converge to $e^{\mathcal{W}_1(-x)}$, where $x<0$ and $\mathcal{W}_1$ has drift $-(B+\frac{1}{2})$ (see Ref. [41, Eq. (4.12)] for details). Since paths need to go through the vertices (1,1) and (2,1) before continuing along the first row until location $(-x\sqrt{n},1)$, this explains Eq. (C23). Along the first columns, we have that for $j\geqslant 2$, $\alpha_1+\beta_j = A+1+\sqrt{n}$, and $\alpha_2+\beta_j = B+1+\sqrt{n}$, so that products of weights along the first row converge to $e^{\mathcal{W}_2(x)}$ and products of weights along the second row converge to $e^{\mathcal{W}_3(x)}$. We need to consider two types of paths: those going through vertices (1,1), (2,1), and collecting a number of weights on the second column until the location $(1, x\sqrt{n})$, hence the first term in Eq. (C24); those going through vertex (1,1), then collecting a number of weights along the first column until a location close to $(1, y\sqrt{n})$ then collecting a number of weights on the second columns between locations $(1, y\sqrt{n})$ and $(1, x\sqrt{n})$, hence the second term in Eq. (C24).

Note that the weight w_{11} is in factor of the IC $Z(x,0)$ for any x, so that the solution of the full-space SHE (35) with such initial condition that we have obtained as a limit of the log-gamma polymer model can be written as $Z(x,t) = w_{11}Z^{(B|A,B)}(x,t)$, and $w_{21} = w$, using the notations in the main text in Sec. V. Assuming the convergence of moments, and taking the limit of the integral Eq. (C19) under the scalings Eqs. (C20), (C21), and (C22), we obtain the following moment formula: For $x_1 \leqslant \cdots \leqslant x_n$,

$$\mathbb{E}\left[\prod_{i=1}^n Z^{(B|A,B)}(x_i,t)\right] = \frac{\Gamma(A+B+1)}{\Gamma(A+B+1-n)}\int_{r_1+\mathbf{i}\mathbb{R}}\frac{dz_1}{2\mathbf{i}\pi}\cdots\int_{r_n+\mathbf{i}\mathbb{R}}\frac{dz_n}{2\mathbf{i}\pi}\prod_{a<b}\frac{z_a-z_b}{z_a-z_b-1}\prod_{j=1}^n\left(\frac{1}{A+z_j}\frac{1}{B^2-z_j^2}e^{tz_j^2+x_jz_j}\right), \tag{C25}$$

where the contours are such that

$$B > r_1 > r_2+1 > \ldots > r_k+k-1, \quad \text{with } r_k > -A, -B, \tag{C26}$$

and we have used that the moments of w_{11} are given by $\mathbb{E}[w_{11}^k] = \frac{\Gamma(A+B-k+1)}{\Gamma(A+B+1)}$. Comparing with Eq. (C6), we have that for $x\geqslant 0$, and any integer $n\geqslant 1$,

$$2^n\mathbb{E}[Z^{(B|A,B)}(-x,t)^n] = \mathbb{E}\big[Z_{A,B}^{\text{shifted}}(x,t)^n\big], \tag{C27}$$

from which we deduce the equality in distribution Eq. (40) (strictly speaking, an equality of moments does not imply an equality in distribution but we will ignore this mathematical subtlety).

4. Degeneration as $B \to +\infty$

In the $B\to+\infty$ limit, we need to multiply both members of Eq. (C27) by B^n before taking the limit. Then, the full space solution in the l.h.s. of Eq. (C27) has half-Brownian IC (in the limit), as can be seen from Eqs. (41) and (42), that is, on $\mathbb{R}_+$ the initial condition is the exponential of a Brownian motion with drift $-(A+\frac{1}{2})$ and on $\mathbb{R}_+$ the initial condition is zero. The half-space solution involved in the r.h.s. of Eq. (C27) has Robin-type boundary condition with parameter A, and delta at 0 IC in the limit (i.e., droplet initial condition). Hence, we obtain the identity in distribution Eq. (38).

The identity in distribution can also be obtained by a comparison of Fredholm determinant formulas. Indeed, the kernel $K_{u,t}$ already appeared in Ref. [15]. This paper was considering the solution $Z^{(A)}(x,t)$ to the full-space SHE (35) with half-Brownian IC $Z^{(A)}(x,0) = e^{\mathcal{B}(x)-(A+\frac{1}{2})x}$ for $x>0$ and $Z^{(A)}(x,0)=0$ for $x<0$. In fact, the solution was obtained there for any x but in the absence of the drift (i.e., for $A=-1/2$); however, it is immediate to extend it to arbitrary drift, using the statistical symmetry (see Appendix D). Comparing Eq. (A9) with Ref. [15, Prop. 2], we obtain the identity in distribution Eq. (36). The correspondence of notations is as follows: one must set $\alpha=1$, $\gamma_t = t^{1/3}$ there, and here $u = e^{-t^{1/3}s}$

APPENDIX D: TILT SYMMETRY

Let us consider the SHE on the full line Eq. (35) with standard space time white noise $\xi(x,t)$. Suppose that $\{Z(x,t)\}_{x\in\mathbb{R},t>0}$ is a solution with IC $Z(x,0)=Z_0(x)$. Consider now for any fixed real a

$$\tilde{Z}(x,t) = e^{ax+a^2t}Z(x+2at,t). \tag{D1}$$

Since $\tilde{\xi}(x,t) = \xi(x+2at,t)$ is also a standard space time white noise, $\tilde{Z}(x,t)$ is also a solution of the SHE (35), with IC $\tilde{Z}(x,0) = \tilde{Z}_0(x) = e^{ax}Z_0(x)$ in another realization of the noise. Hence the (statistical) tilt symmetry (STS) relates the statistics of the solutions of the SHE with "tilted" initial conditions. In the particular case of the droplet IC, $\tilde{Z}_0(x) = Z_0(x) = \delta(x)$ these IC are identical and the statistics of $\{Z(x,t)\}_{x\in\mathbb{R},t>0}$ and $\{\tilde{Z}(x,t)\}_{x\in\mathbb{R},t>0}$ are thus identical (as space time processes).

1. Half-Brownian IC

Let us denote now $Z_v(x,t)$ the solution with the half Brownian IC with drift v, i.e., $Z_v(x,0) = e^{B(x)+vx}\theta(x)$. The solution $Z_{v+a}(x,t)$ with a half Brownian IC with drift $v+a$, i.e., $Z_{v+a}(x,0) = e^{B(x)+(v+a)x}\theta(x)$ can thus be constructed using the STS, i.e., one has in law

$$Z_{v+a}(x,t) = e^{ax+a^2t} Z_v(x+2at,t). \tag{D2}$$

This is an identity between space time processes. We want now to focus only on the distribution at a single fixed space-time point (x,t). Then we can choose $a = -x/(2t)$ and $v = w - a$ and obtain the equality in law

$$Z_w(x,t) = e^{-\frac{x^2}{4t}} Z_{w+\frac{x}{2t}}(0,t). \tag{D3}$$

Setting $w = -(A+\frac{1}{2})$ and $x = -X$ we obtain the identity in law $Z^{(A)}(-X,t) = e^{-\frac{X^2}{4t}} Z^{(A+\frac{X}{2t})}(0,t)$ given in the main text.

2. Droplet IC

Consider now the solution $Z(x,t) = Z(x,t|0,0)$ of the full line SHE with the droplet IC. Let us define $G(\mathsf{f},t) = \int_{-\infty}^{+\infty} dx e^{\mathsf{f}x} Z(x,t)$. It is the partition sum of a directed polymer with one fixed endpoint at (0,0) and one free endpoint (x,t) but with an applied force f on that endpoint. From the STS property we have that $G(\mathsf{f},t)$ has the same distribution as

$$\int_{-\infty}^{+\infty} dx e^{\mathsf{f}x} e^{ax+a^2t} Z(x+2at,t) = G(\mathsf{f}+a,t) e^{-(a^2+2a\mathsf{f})t}. \tag{D4}$$

Hence, choosing $a = -\mathsf{f}$

$$\mathbb{E} \log G(\mathsf{f},t) = \mathbb{E} \log G(0,t) + \mathsf{f}^2 t. \tag{D5}$$

It follows, by differentiation, that the averaged thermal cumulants of the free endpoint (x,t) in the absence of the force, i.e., for $\mathsf{f} = 0$, are simply $\mathbb{E}\langle x^p \rangle_c = \partial_{\mathsf{f}}^p \mathbb{E} \log G(\mathsf{f},t) = 2t\delta_{p,2}$ on the full line, as mentioned in the main text. Similar remarkable identities for thermal fluctuations occur in a larger class of disordered models [76]. While it is valid for any t, for large t this result is usually interpreted within the droplet picture [51,77]. The typical Gibbs measure of the endpoint is localized, i.e., the thermal fluctuations of the endpoint are typically $\delta x = O(1)$. However, with probability $p(t) \sim T/t^{1/3}$ (where the temperature is $T = 1$ here) there exists two distant states, almost degenerate in energy (within $O(T)$): the Gibbs measure is splitted between them and that leads to a much larger $\delta x \sim t^{2/3}$. Putting these factors together leads to $\mathbb{E}\langle x^2 \rangle_c \sim T t^{-1/3} t^{4/3} \sim Tt$

Note that by the tilt symmetry the polymer configurations are mapped into each others. In the case of the half-space the STS maps a problem with a vertical wall to a problem with a tilted wall, so a priori one cannot readily use it. In the main text we have found the curious relation Eq. (38) and used on the r.h.s. the STS for the half-Brownian in full space (shown above) to deduce the equality in law $Z_A^{\text{shifted}}(X,t) = e^{-\frac{X^2}{4t}} Z_{A+\frac{X}{2t}}^{\text{shifted}}(0,t)$. Although it has a flavor of STS in half space, it is not, and in fact there is no simple correspondence between the polymer trajectories on both sides of this relation.

A similar puzzle occurs upon applying a force f to the endpoint in the half space. A tilt transformation which removes the force would also tilt the wall, so no obvious consequence can be obtained. Nevertheless, as shown in the main text, the force induces an additional drift in the drifted Brownian stationary measure leading to the simple shift $A \to A + \mathsf{f}$. There also, it does not seem to exist any simple picture in terms of tilted polymer paths.

APPENDIX E: REPLICA BETHE ANSATZ APPROACH

In this Appendix we explore the interplay between the energy spectrum of the replica delta Bose gas in the half-space and the stationary measure of increments of partition function that is used in the main text in Sec. IV to study endpoint distributions of polymers.

1. Moments of partition sum

The replica Bethe ansatz method (RBA) allows to write the multipoint equal time moments of $Z(x,t)$, solution of the SHE equation (in full or half space), as a quantum mechanical expectation [denoting $\vec{x} = (x_1,\dots,x_n)$]

$$\mathbb{E}[Z(x_1,t)\cdots Z(x_n,t)] = \langle \vec{x}|e^{-tH_n}|\Psi(t=0)\rangle = \sum_\mu \Psi_\mu(x_1,\dots,x_n)\langle\Psi_\mu|\Psi(t=0)\rangle \frac{e^{-tE_\mu}}{||\Psi_\mu||^2}, \tag{E1}$$

i.e., a sum over the unnormalized eigenfunctions Ψ_μ (of norm denoted $||\Psi_\mu||$) of the n-body Lieb-Liniger (LL) Hamiltonian

$$H_n = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} - 2 \sum_{1\leqslant i<j\leqslant n} \delta(x_i - x_j), \tag{E2}$$

with eigenenergies E_μ. In Eq. (E1), we have denoted $\Psi_\mu(x_1,\ldots,x_n)=\langle\vec{x}|\Psi_\mu\rangle$ and the initial state $|\Psi(t=0)\rangle$ encodes the initial condition of the SHE, with $\langle\vec{x}|\Psi(t=0)\rangle=\prod_{i=1}^n Z(x_i,0)$ for a deterministic IC and $\langle\vec{x}|\Psi(t=0)\rangle=\mathbb{E}[\prod_{i=1}^n Z(x_i,0)]$ for a random IC. These initial conditions being symmetric in $(x_1,\cdots,x_n)$ only the symmetric, i.e., bosonic, eigenstates contribute to the sum in Eq. (E1). The representation Eq. (E1) is valid in full space and half space. The sum Eq. (E1) is weighted by the overlaps $\langle\Psi_\mu|\Psi(t=0)\rangle=\int\prod_i dx_i\Psi_\mu^*(\vec{x})\langle\vec{x}|\Psi(t=0)\rangle$.

In the half space case H_n acts on wave-functions which satisfy the boundary conditions $\partial_{x_i}\Psi(\vec{x})|_{x_i=0}=A\Psi(\vec{x})|_{x_i=0}, i=1\ldots,n$ (hence the eigenstates Ψ_μ satisfy this condition). The n body spectrum of the half-space problem is complicated and was obtained from the Bethe ansatz in Ref. [42]. We also refer to Ref. [42] for a more detailed presentation and for the references to the literature on the Bethe ansatz for the half-space LL model. In addition to the usual bulk string bound states which exist in the full space problem and have an arbitrary center of mass momentum, there are also n body boundary bound states which are localized at the boundary.

The ground state (lowest energy state) was found by Kardar in Ref. [19], and this was confirmed in Ref. [42]. One has

(1) For $n\leqslant 1+2A$ the ground state is a state made of a single bulk string with a vanishing momentum. Far from the boundary its wave-function behaves like the ground state of the full-space problem

$$\Psi_0(\vec{x})\sim e^{-\frac{1}{2}\sum_{1\leqslant i<j\leqslant n}|x_i-x_j|}=e^{\frac{1}{2}\sum_{j=1}^n(n+1-2j)x_j},\quad \text{for large } x_1\leqslant\cdots\leqslant x_n. \tag{E3}$$

The ground-state energy is $E_0(n)=-\frac{1}{12}n(n^2-1)$.

(2) For $n>1+2A$ the ground state is made of a single boundary string. The ground state energy is $E_0(n)=-n[A+\frac{1}{2}(1-n)]^2-\frac{1}{12}n(n^2-1)$. It has the form

$$\Psi_0(\vec{x})\propto e^{\sum_{j=1}^n(A-j+1)x_j},\quad \text{for any } 0\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_n, \tag{E4}$$

and its norm was computed in Ref. [42]. Note that shifting all $x_i\to x_i+\bar{x}$ the wave-function decays as $e^{-\frac{\bar{x}}{2}n[n-(1+2A)]}$.

Exactly at the transition for $n=1+2A$, the two states are identical as the r.h.s. of Eqs. (E3) and (E4) match. This state should be considered as a bulk string ground state as its center of mass is delocalized in the full volume. The ground-state energy is continuous across the transition.

In the limit $t\to+\infty$, for any fixed positive integer n, the sum over states is dominated by the ground state $\Psi_0(\vec{x})$. It is thus tempting to follow the following two steps:

(i) write

$$\mathbb{E}[Z_A(x_1,t)\cdots Z_A(x_n,t)]\simeq\frac{\Psi_0(\vec{x})}{||\Psi_0||^2}\langle\Psi_0|\psi(t=0)\rangle e^{-E_0(n)t}, \tag{E5}$$

where $Z_A(x,t)$ solves the half-space SHE (2),

(ii) postulate that the form of the expression found for positive integer n can be extended to real $n>0$ in the limit $n\to 0$ to calculate moments.

This is what was done by Kardar in Ref. [19] to predict $\mathbb{E}(\langle x\rangle)$ in the bound phase. In the following it will be convenient to rewrite Eq. (E5) as

$$\mathbb{E}[Z_A(x_1,t)\cdots Z_A(x_n,t)]\simeq c_n(t)\tilde{\Psi}_0(\vec{x}),\quad c_n(t)=\mathbb{E}[Z_A(0,t)^n],\quad \tilde{\Psi}_0(\vec{x})|_{x_1\leqslant\cdots\leqslant x_n}=e^{\sum_{j=1}^n(A-j+1)x_j}. \tag{E6}$$

In that form it is clear that the continuation of $c_n(t)$ to $n=0$ is simply unity.

It is well known that in the full space problem (and we can expect the same in the unbound phase for the half-space problem), in step (i) the amplitude in Eq. (E5) is not correct since the spectrum is gap-less and one must further integrate over the low lying center of mass excitations, but this integration can be performed for a given initial condition. In step (ii), more severely, the limit $t\to+\infty$ and $n\to 0$ do not commute (i.e., one would need to perform the limit $n\to 0$ on the full sum and then take the limit $t\to+\infty$). However, once these two issues are addressed, this program enables to obtain the right tails of the free-energy $\log Z$ [78,79].

In the bound phase however, for $A<-1/2$, step (i) is more reasonable as there are no center of mass excitations and there is a finite gap between the ground state and the excited states [42], so Eq. (E5) should give the correct asymptotics. In step (ii) it is quite likely that the limits $t\to+\infty$ and $n\to 0$ commute in that case: for $A<-1/2$ the ground state holds for any $n>0$, and it is a system of effectively finite size. Indeed our results below confirm that.

2. Thermal cumulants of the endpoint position via the RBA

Let us recall the definition of the endpoint distribution (in a given noise realization) and of the thermal averages

$$\mathcal{P}(x,t)=\frac{Z_A(x,t)}{Z},\quad Z=\int_0^{+\infty}dyZ_A(y,t),\quad \langle O(x)\rangle=\int_0^{+\infty}dxO(x)\mathcal{P}(x,t), \tag{E7}$$

where the time dependence of the averages is implicit. Note that we have not specified the initial condition, so it can be fixed endpoint at $t=0$ in some position, or more general. Let us first calculate the thermal cumulants.

The generating function $f(v)$ of the averaged thermal cumulants can be written as

$$f(v) = \mathbb{E}\left[\log \int_0^{+\infty} dx e^{vx} Z_A(x,t)\right] = \partial_n \mathbb{E}\left[\left(\int_0^{+\infty} dx e^{vx} Z_A(x,t)\right)^n\right]\Big|_{n\to 0} \tag{E8}$$

$$= \partial_n \int_0^{+\infty} d\vec{x} e^{v\sum_{i=1}^n x_i} \mathbb{E}[Z_A(x_1,t)\cdots Z_A(x_n,t)]\Big|_{n\to 0}. \tag{E9}$$

Thus, in the large time limit (under the above assumptions) it becomes

$$\mathbb{E}[\langle x^p\rangle_c] = \partial_v^p f(v)\big|_{v=0}, \quad f(v) = \partial_n (c_n(t) I_n(v))\big|_{n=0}, \quad I_n(v) := \int d\vec{x} e^{v\sum_{i=1}^n x_i} \tilde{\Psi}_0(\vec{x}). \tag{E10}$$

Let us calculate the integral

$$I_n(v) = n! \int_{0<x_1<\cdots<x_n} e^{\sum_{j=1}^n (A+v-j+1)x_j} = n! \prod_{j=1}^n \frac{-2}{j(1+2A+2v+j-2n)} = \frac{2^n \Gamma(-2A+n-2v-1)}{\Gamma(-2A+2n-2v-1)}, \tag{E11}$$

where we have used the identity

$$\int_{0<y_1<\cdots<y_p} e^{\sum_{j=1}^p z_j y_j} = \prod_{j=1}^p \frac{-1}{z_p+\cdots+z_{p-j+1}}. \tag{E12}$$

We see that $I_0(v) = 1$ as expected, and recall that $c_0(t) = 1$, so that

$$f(v) = \partial_n c_n(t) + \partial_n I_n(v)\big|_{n=0} = \mathbb{E}[\log Z_A(0,t)] + \log(2) - \psi(-2A-2v-1). \tag{E13}$$

Note that this is exactly compatible with what was obtained in the main text around (28) since

$$\mathbb{E}\left[\log \int_0^{+\infty} dx e^{vx} Z_A(x,t)\right] - \mathbb{E}[\log Z_A(0,t)] = \mathbb{E}\left[\log \int_0^{+\infty} dx e^{vx} \frac{Z_A(x,t)}{Z_A(0,t)}\right], \tag{E14}$$

where the limit $\frac{Z_A(x,t)}{Z_A(0,t)}$ was shown to converge to the exponential of the Brownian with drift $A+\frac{1}{2}$. So the RBA method reproduces exactly the result Eq. (28) for the general averaged thermal cumulant. In the case $p=1$ this is the result obtained by Kardar [19]. This coincidence between the results of the RBA and of the method used in the main text appears to extend to all moments as we now discuss.

3. Comparison of the two methods and general moments

Let us put side by side the results of the two methods. In the method based on stationary measures of increments described in the main text in Sec. IV, one states that

$$\mathbb{E}\left[\frac{Z_A(x_1,t)}{Z_A(0,t)}\cdots\frac{Z_A(x_n,t)}{Z_A(0,t)}\right] \xrightarrow[t\to+\infty]{} \Phi_0(\vec{x}) := \mathbb{E}[e^{\sum_{i=1}^n \mathcal{B}(x_i)+(A+\frac{1}{2})x_i}],\ \Phi_0(\vec{x})|_{x_1\leqslant\cdots\leqslant x_n} = e^{\sum_{j=1}^n (n-j+\frac{1}{2})x_j + (A+\frac{1}{2})x_j}, \tag{E15}$$

while in the RBA method, one obtains

$$\mathbb{E}[Z(x_1,t)\cdots Z(x_n,t)] \simeq_{t\to+\infty} \mathbb{E}[Z(0,t)^n]\, \tilde{\Psi}_0(\vec{x}), \quad \tilde{\Psi}_0(\vec{x})|_{x_1\leqslant\cdots\leqslant x_n} = e^{\sum_{j=1}^n (A-j+1)x_j}. \tag{E16}$$

The functions $\Phi_0(\vec{x})$ and $\tilde{\Psi}_0(\vec{x})$ are both fully symmetric in their arguments and very similar, although different.

4. General moments

The multipoint average of the endpoint distribution can be written as

$$\mathbb{E}[\mathcal{P}(x_1,t)\cdots\mathcal{P}(x_p,t)] = \mathbb{E}\left[\frac{1}{Z^p} Z_A(x_1,t)\cdots Z_A(x_p,t)\right] = \lim_{n\to 0} \mathbb{E}\left[\int_0^{+\infty} dx_{p+1}\cdots \int_0^{+\infty} dx_n Z_A(x_1,t)\cdots Z_A(x_n,t)\right], \tag{E17}$$

where we used that $\frac{1}{Z^p} = \lim_{n\to 0} Z^{n-p}$, and we recall that $Z = \int_0^{+\infty} dy Z_A(y,t)$.

Using the RBA result Eq. (E16), in the large time limit it thus becomes

$$\mathbb{E}[\mathcal{P}(x_1,t)\cdots\mathcal{P}(x_p,t)] \simeq \lim_{n\to 0} \int_0^{+\infty} dx_{p+1}\cdots \int_0^{+\infty} dx_n\, \tilde{\Psi}_0(\vec{x}). \tag{E18}$$

Upon multiplicision by $O_1(x_1)\ldots O_p(x_p)$ and integration, it implies in particular for the most general type of moment

$$\mathbb{E}[\langle O_1(x)\rangle\cdots\langle O_p(x)\rangle] \simeq \lim_{n\to 0} \int_0^{+\infty} dx_1\cdots \int_0^{+\infty} dx_n O_1(x_1)\ldots O_p(x_p)\, \tilde{\Psi}_0(\vec{x}). \tag{E19}$$

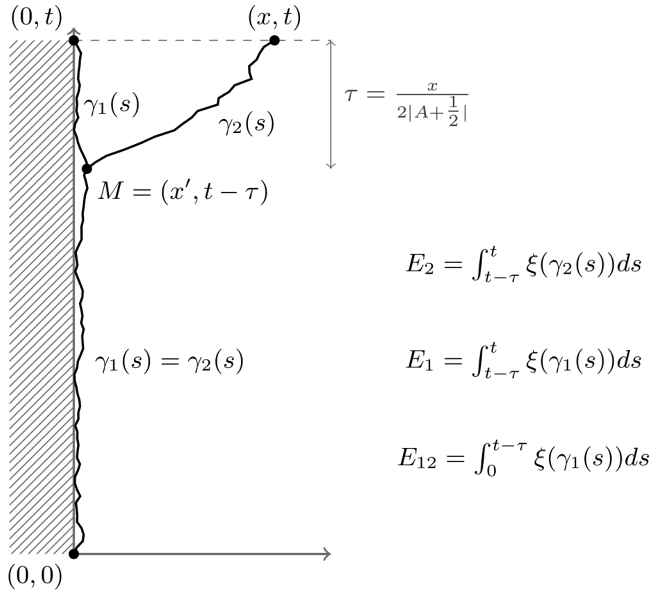

FIG. 5. Illustration of the geodesics $\gamma_1 = [\gamma_1(s), s]_{0\leqslant s\leqslant t}$ and $\gamma_2 = [\gamma_2(s), s]_{0\leqslant s\leqslant t}$.

Within the method based on the stationary measure, one has from Eq. (27), $\lim_{t\to\infty} \mathcal{P}_A(x,t) = p_A(x)$ with $p_A(x) = \frac{1}{\mathsf{Z}} e^{\mathcal{B}(x)+(A+\frac{1}{2})x}$ and $\mathsf{Z} = \int_0^{+\infty} dy e^{\mathcal{B}(y)+(A+\frac{1}{2})y}$. Using the same steps as in Eq. (E17) with $\frac{1}{\mathsf{Z}^p} = \lim_{n\to 0} \mathsf{Z}^{n-p}$ one obtains using Eq. (E15)

$$\mathbb{E}[p_A(x_1)\cdots p_A(x_p)] = \lim_{n\to 0} \int_0^{+\infty} dx_{p+1}\cdots \int_0^{+\infty} dx_n \Phi_0(\vec{x}), \tag{E20}$$

which provides a starting formula for the evaluations of the many point correlations of the stationary measure via the replica method.

We can now compare Eqs. (E20) and (E18) and use that $\Phi_0(\vec{x})$ and $\tilde{\Psi}_0(\vec{x})$ are very similar. More precisely, $\Phi_0(\vec{x}) = e^{n\sum_{i=1}^n x_i}\tilde{\Psi}_0(\vec{x})$, i.e., they differ by a term which contains n explicitly and vanishes at $n = 0$. This indicates that the results of the two methods for the endpoint probability correlations, and thus for all the moments, are the same.

5. Correlation between endpoint positions and free energy

The RBA ground state $\tilde{\Psi}_0(\vec{x})$ thus seems to contain the information about the stationary measure of the partition sum ratios. However, it should contain more, and encode also for some information about the correlations between $Z(0,t)$ and these ratios, or with the endpoint distribution, which remains to be explored. Indeed putting together Eqs. (E15) and (E16) we obtain that at large time

$$\mathbb{E}[Z(0,t)^n Z_t(x_1)\dots Z_t(x_n)] \simeq e^{-n\sum_{i=1}^n x_i}\mathbb{E}[Z(0,t)^n]\,\mathbb{E}[Z_t(x_1)\dots Z_t(x_n)], \tag{E21}$$

where we denoted the ratios as $Z_t(x) = Z(x,t)/Z(0,t)$. Taking $x_i = x$ for $i = 1,\dots,p$ and $x_i = 0$ for $i = p+1,\dots,n$, it gives $\mathbb{E}[Z(0,t)^n Z_t(x)^p] = e^{-npx}\mathbb{E}[Z(0,t)^n]\,\mathbb{E}[Z_t(x)^p]$ for all positive integers $n \leqslant p$. This suggest that at large time the only non zero joint cumulant between $\log Z(0,t)$ and $\log Z_t(x)$ is the two-point covariance

$$\mathrm{Cov}[\log Z(0,t), \log Z_t(x)] = -x. \tag{E22}$$

It is possible to obtain some understanding of how this relation (which is a conjecture at this stage) could come about. At large t and large x we may approximate the polymer partition functions by the exponential of the energy collected along geodesics (i.e., paths with maximal energy). Consider the geodesics $\gamma_1(t)$ from (0,0) to $(0,t)$ and $\gamma_2(t)$ from (0,0) to (x,t). They first coincide and then split at some point M of coordinate $(x', t-\tau)$; see Fig. 5. In the bound phase $A < -1/2$, $\gamma_1(t)$ remains close to the wall, and x' remains bounded. Let us call E_{12} the energy of the common segment (that is $E_{12} = \int_0^{t-\tau} \xi(\gamma_1(s))ds$), E_1 the one of the segment from $M = (x', t-\tau)$ to $(0,t)$, and E_2 from M to (x,t). At large t, one has $\log Z(0,t) \simeq E_1 + E_{12}$ and $\log Z(x,t) \simeq E_2 + E_{12}$. Conditionally on the position of M, E_{12} is independent from E_1 and E_2, and for large t and x, E_1 and E_2 are asymptotically independent. Hence the left hand side of Eq. (E22) is thus

$$\mathrm{Cov}(E_{12} + E_1, E_2 - E_1) = \mathrm{Cov}(E_1, E_2 - E_1) \simeq -\mathrm{Var}(E_1). \tag{E23}$$

We have obtained in the main text, see also Ref. [42], that for large τ, we have $\mathrm{Var}E_1 \simeq 2\tau|A + \frac{1}{2}|$. Further one expects that for large τ and x, the length τ is proportional to x. If one equates the elastic energy $x^2/(4\tau)$ with $(A + \frac{1}{2})^2\tau$, then one obtains

$\tau \simeq x/(2|A+\frac{1}{2}|)$, which makes Eq. (E23) consistent with Eq. (E22) in the limit of large x. A similar understanding when x is not going to infinity seems more difficult.

Remark: Midpoint. The partition function with two fixed endpoints in $x=0$ at times 0 and $2t$ and a given midpoint position $x(t)=x$ is given by $Z(0,2t|x,t)Z(x,t|0,0)$, which has the same law as two independent copies of $Z(x,t|0,0)$. The moments of this partition function will thus be the square of the moments of $Z(x,t|0,0)$. Within the RBA it will thus amount to the same formula as above, replacing $\tilde{\Psi}_0(x) \to \tilde{\Psi}_0(x)^2$. From (E16) it is an exponential linear in the x_i, hence this replacement amounts to change $x_i \to 2x_i$. This agrees with the result of the main text.

Remark: Basin of attraction. We expect that the property of "ground-state dominance" Eq. (E5) in the RBA will hold in the bound phase upon some condition on the initial condition (i.e., on the behavior of the overlaps). The condition for the convergence to the stationary measure for the ratios was discussed in the main text in Sec. IV. It would be interesting to see whether it can also be obtained with the RBA.

APPENDIX F: CORRELATIONS OF $p_A(x)$ VIA LIOUVILLE QUANTUM MECHANICS

The stationary measure for the endpoint position at large time in the bound phase is $p_A(x)$ given in Eq. (27). It is possible to compute its m-point correlations (and therefore all the moments) using methods developped in Refs. [56,58] and in Ref. [55] based on stochastic processes, replica, and most notably on Liouville quantum mechanics. Other works addressed similar questions in various contexts [80–82], often motivated by multifractal properties of eigenfunctions of random Dirac type operators. Although it is a simple extension of these works (which often focus on periodic boundary conditions) the formula for the case of the Brownian (i.e., with free boundary conditions) and in presence of a drift have not been given, so we display them here (for details of the method we refer to Refs. [55] and [56]).

1. Moments of $p_A(x)$

Let us denote $w=-(A+\frac{1}{2})>0$. As in these works we consider a finite L truncation, denoting $Z_L^w=\int_0^L dx e^{\mathcal{B}(x)-wx}$, and we take $L\to+\infty$ at the end. A simple but useful observation [55,58] is that $p_A(x)$ in Eq. (27) can be rewritten as

$$p_A(x)=\lim_{L\to\infty}\frac{1}{\int_0^L dy e^{\mathcal{B}(y)-\mathcal{B}(x)-w(y-x)}}=\lim_{L\to\infty}\frac{1}{Z_x^{-w}+\tilde{Z}_{L-x}^w}=\frac{1}{Z_x^{-w}+\tilde{Z}_\infty^w}, \tag{F1}$$

where $\tilde{Z}$ contains an independent realisation of the Brownian. This is obtained splitting the integral over y on the two interval $[0,x]$ and $[x,L]$ and performing the change of variable $y=x-z$ on the first and $y=x+z$ on the second, with z positive. In the last equation we used the property that Z_L^w converges for $L\to+\infty$ to an inverse Gamma random variable $\tilde{Z}_\infty^w=\Gamma(2w,\frac{1}{2})$ where here $1/2$ is the scale parameter, i.e., $z=\tilde{Z}_\infty^w$ has PDF $p(z)=\frac{1}{2\Gamma(2w)}(\frac{2}{z})^{1+2w}e^{-2/z}$. Thus, the moments of $p_A(x)$ are given by

$$\mathbb{E}[p_A(x)^n]=\int_0^{+\infty}dp\frac{p^{n-1}}{\Gamma(n)}\mathbb{E}[e^{-pZ_x^{-w}}]\,\mathbb{E}[e^{-pZ_\infty^w}]=\frac{2}{\Gamma(2w)}\int_0^{+\infty}dp\frac{p^{n-1}}{\Gamma(n)}\phi^{\mu=-2w}(p,x)(2p)^w K_{2w}(2\sqrt{2p}), \tag{F2}$$

where we inserted the exact result for Z_∞^w, and the function $\phi^\mu(p,x)$ was obtained in Ref. [56, Eq. (3.6)] (setting $\beta=\sigma=1$, $\alpha=1/2$ there),

$$\phi^{\mu=-2w}(p,x)=\mathbb{E}[e^{-pZ_x^{-w}}]=\frac{(2p)^{-w}}{4\pi^2}\int_{-\infty}^{+\infty}dq q\sinh(\pi q)\left|\Gamma\left(w+\frac{\mathbf{i}q}{2}\right)\right|^2 K_{\mathbf{i}q}(2\sqrt{2p})e^{-\frac{x}{8}(q^2+4w^2)}. \tag{F3}$$

Inserting into Eq. (F2) the factors $(2p)^w$ and $(2p)^{-w}$ cancel. Let us specialize now to $n=1$. For $w<1$ we can interchange the integrals and use that

$$\int_0^{+\infty}dpK_{2w}(2\sqrt{2p})K_{\mathbf{i}q}(2\sqrt{2p})=\int_0^{+\infty}\frac{du}{4}uK_{2w}(u)K_{\mathbf{i}q}(u)=\frac{\pi^2(q^2+4w^2)}{16[\cosh(\pi q)-\cos(2\pi w)]}, \tag{F4}$$

leading to our first result, valid for $0<w\leqslant 1$, $w=-(A+\frac{1}{2})$,

$$\mathbb{E}[p_A(x)]=\frac{1}{32\Gamma(2w)}\int_{-\infty}^{+\infty}dq e^{-\frac{1}{8}(q^2+4w^2)x}q\sinh(\pi q)\left|\Gamma\left(w+\frac{\mathbf{i}q}{2}\right)\right|^2\frac{q^2+4w^2}{\cosh(\pi q)-\cos(2\pi w)}. \tag{F5}$$

Under the change of variables $\mathbf{i}q=2z$, and after using some trigonometric identities and Euler's reflection formula, Eq. (F5) can be rewritten as

$$\mathbb{E}[p_A(x)]=\frac{1}{4\Gamma(2w)}\int_{-\mathbf{i}\infty}^{\mathbf{i}\infty}\frac{dz}{2\mathbf{i}\pi}\frac{(w^2-z^2)e^{\frac{-x}{2}(w^2-z^2)}\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(1-w-z)\Gamma(1-w+z)}{\Gamma(2z)\Gamma(-2z)}. \tag{F6}$$

We may now analytically continue this formula for all $w>0$ by subtracting and adding the necessary residues when $w>1$ (see Fig. 6).

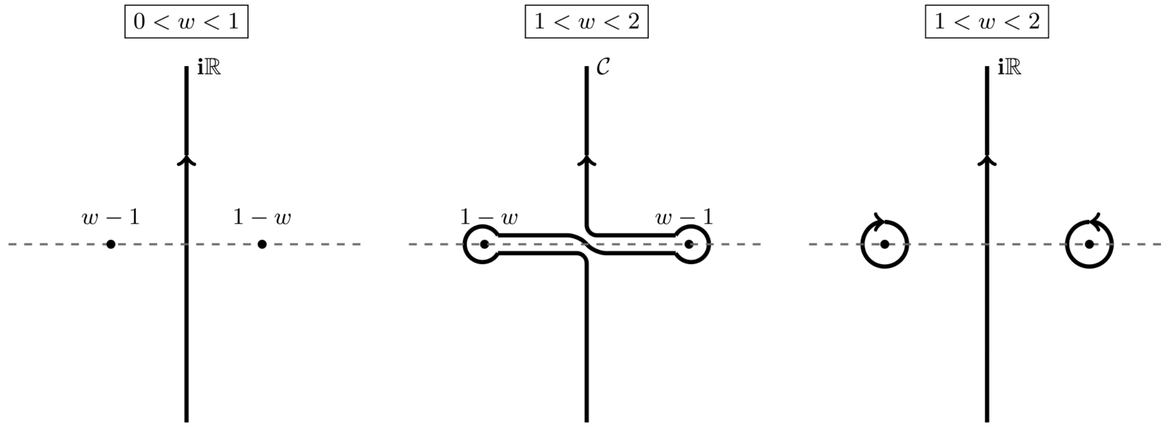

FIG. 6. We consider a function $I(w)$ such that for $0<w<1$, we have $I(w)=\int_{\mathrm{i}\mathbb{R}} f(z,w)$, and $f(z,w)$ is analytic in both z and w except at some isolated poles. In particular it has poles at $z=w-1, w-2, w-3,\dots$ and $z=1-w, 2-w, 3-w,\dots$. Then, its analytic continuation to w such that $n<w<n+1$ is $\int_{\mathcal{C}} f(z,w)$ where $\mathcal{C}$ is the contour shown above (in the case $n=1$). This contour is such that $1-w,\dots,n-w$ still lie on the right of the contour (as when $0<w<1$), and the poles at $w-1,\dots,w-n$ still lie on the left of the contour (as when $0<w<1$). Since the poles do not cross the contour, the formula remains analytic in w. This contour can be then deformed to the union of a vertical line and small circles around simple poles whose contribution can be computed by the residue Theorem.

The analytic continuation of the r.h.s. of Eq. (F6) to $w>1$ is

$$\frac{1}{4\Gamma(2w)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dz}{2\mathrm{i}\pi}\frac{(w^2-z^2)e^{\frac{-x}{2}(w^2-z^2)}\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(1-w-z)\Gamma(1-w+z)}{\Gamma(2z)\Gamma(-2z)}+\frac{1}{4\Gamma(2w)}\sum_{1\leqslant n<w}(R_{w-n}-R_{n-w}), \tag{F7}$$

where R_{w-n} and R_{n-w} are residues of the integrand at $z=w-n$ and $z=n-w$, respectively. We have

$$R_{w-n}=-R_{n-w}=2(-1)^{n-1}n!e^{\frac{-x}{2}(2wn-n^2)}(w-n)\frac{\Gamma(2w)\Gamma(1-2w)}{\Gamma(n-2w)}, \tag{F8}$$

so that, for $w=-(A+\frac{1}{2})>0$,

$$\begin{aligned}\mathbb{E}[p_A(x)]=&\frac{1}{4\Gamma(2w)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dz}{2\mathrm{i}\pi}\frac{(w^2-z^2)e^{\frac{-x}{2}(w^2-z^2)}\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(1-w-z)\Gamma(1-w+z)}{\Gamma(2z)\Gamma(-2z)}\\&+\sum_{1\leqslant n<w}n!(-1)^{n-1}e^{\frac{-x}{2}(2wn-n^2)}(w-n)\frac{\Gamma(1-2w)}{\Gamma(n-2w)}.\end{aligned} \tag{F9}$$

Note that for large w the finite series dominates and the integral can be neglected for most averages.

Remark: Normalization. Let us first check that Eq. (F9) obeys the normalization condition $\int_0^{+\infty}dx\mathbb{E}[p_A(x)]=1$. By analyticity, it suffices to check it for $0<w<1$. Then, using Eq. (F6), this is equivalent to the identity

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dz}{2\mathrm{i}\pi}\frac{\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(1-w-z)\Gamma(1-w+z)}{\Gamma(2z)\Gamma(-2z)}=2\Gamma(2w), \tag{F10}$$

which is a particular case of the known identity [83, Eq. (3.6.1)]

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dz}{2\mathrm{i}\pi}\frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)}{\Gamma(2z)\Gamma(-2z)}=2\Gamma(a+b)\Gamma(a+c)\Gamma(b+c), \tag{F11}$$

valid for a,b,c with positive real part.

Remark: First moment. We may also check that Eq. (F6) is consistent with the result Eq. (29) for the first thermal cumulant given in Sec. IV, i.e., $\mathbb{E}\langle x\rangle=\int_0^{+\infty}x\mathbb{E}[p_A(x)]dx=2\psi'(2w)$. Indeed, from Eq. (F6), we have

$$\int_0^{+\infty}x\mathbb{E}[p_A(x)]dx=\frac{1}{\Gamma(2w)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dz}{2\mathrm{i}\pi}\frac{\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(-w-z)\Gamma(-w+z)}{\Gamma(2z)\Gamma(-2z)}. \tag{F12}$$

The integral in Eq. (F12) cannot be simplified directly using Eq. (F11), but we may use that

$$\text{Eq. }(F12)=\lim_{\epsilon\to0}\frac{1}{\Gamma(2w)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dz}{2\mathrm{i}\pi}\frac{\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(\epsilon-w-z)\Gamma(\epsilon-w+z)}{\Gamma(2z)\Gamma(-2z)}. \tag{F13}$$

When $0 < w < 1$, this integral above is analytic in $\epsilon \in (0, +\infty)$, and the expression for $0 < \epsilon < w$ can be obtained by an analytic continuation from the expression when $\epsilon > w$, which is given by Eq. (F11). More precisely, for $0 < \epsilon < w$,

$$\frac{1}{\Gamma(2w)}\int_{-\mathbf{i}\infty}^{\mathbf{i}\infty}\frac{dz}{2\mathbf{i}\pi}\frac{\Gamma(w+z)^2\Gamma(w-z)^2\Gamma(\epsilon-w-z)\Gamma(\epsilon-w+z)}{\Gamma(2z)\Gamma(-2z)} = \frac{1}{\Gamma(2w)}[2\Gamma(2w)\Gamma(\epsilon)^2 + R_{\epsilon-w} - R_{-\epsilon+w}], \tag{F14}$$

where $R_{\pm\epsilon\mp w}$ are residues of the integrand at $z = \pm\epsilon \mp w$, which can be computed as

$$R_{\epsilon-w} = -R_{w-\epsilon} = -\frac{\Gamma(2w-\epsilon)^2\Gamma(\epsilon)^2}{\Gamma(2w-2\epsilon)}. \tag{F15}$$

Finally, taking the limit $\epsilon \to 0$, we obtain

$$\int_0^{+\infty} x\mathbb{E}[p_A(x)]dx = \lim_{\epsilon\to 0}\frac{1}{\Gamma(2w)}\left(2\Gamma(2w)\Gamma(\epsilon)^2 - 2\frac{\Gamma(2w-\epsilon)^2\Gamma(\epsilon)^2}{\Gamma(2w-2\epsilon)}\right) = 2\psi'(2w). \tag{F16}$$

Remark: Value of $\mathbb{E}[p_A(0)]$. Letting $x = 0$ in Eq. (F6) and using the identity (F11) yields $\mathbb{E}[p_A(0)] = w$. This result has a simple origin: when $x = 0$ $p_A(0) = (\int_0^\infty e^{\mathcal{B}(x)-wx}dx)^{-1} \sim \Gamma(2w, 1/2)$ and $\mathbb{E}[\Gamma(2w, 1/2)] = w$.

Remark: Decay for large x. Let us start with $0 < w < 1$. Saddle point analysis and rescaling in formula (F6) gives that at large x the decay is exponential with a $-3/2$ power law prefactor, $\mathbb{E}[p_A(x)] \simeq c_w x^{-3/2}e^{-w^2x/2}$, with $c_w = \frac{\pi^{3/2}\csc^2(\pi w)\Gamma(w+1)^2}{\sqrt{2}\Gamma(2w)}$. For $w > 1$ however the decay at large x is dominated by the term $n = 1$ in the discrete series in Eq. (F9) and $\mathbb{E}[p_A(x)] \sim e^{\frac{-x}{2}(2w-1)}$ for $w > 1$, i.e., a much slower decay than $e^{\frac{-x}{2}w^2}$.

Remark: Limit $w = -(A + \frac{1}{2}) \to 0$. In the limit $w \to 0$, it is easy to check, upon rescaling $q \to wq$ and $x \to y/w^2$ in Eq. (F5) that as $w \to 0^+$ one has $\mathbb{E}[p_A(x)]dx \to P(y)dy$ where $P(y)$ is the probability given in the main text (of moments given by (31)). In that limit the large x tail obtained above matches the tail of $P(y)$ at large y, since $c_w \sim \sqrt{\frac{2}{\pi}}/w$ for $w \to 0$. The $-3/2$ exponent, ubiquitous in this types of problems [55,57], is known to originate from quasi-degenerate extrema of the Brownian [84].

2. *m*-point correlations

To compute the m point correlations with $m \geqslant 2$ one uses the Liouville quantum mechanics. One introduces the Liouville Hamiltonian H_p on the real axis $U \in \mathbb{R}$, and its eigenfunctions $\psi_k(U)$ which are real and indexed by $k \geqslant 0$

$$H_p = -\frac{1}{2}\frac{d^2}{dU^2} + pe^U, \quad H_p\psi_k(U) = \frac{k^2}{8}\psi_k(U), \quad \psi_k(U) = \frac{1}{\pi}\sqrt{k\sinh(\pi k)}K_{ik}(2\sqrt{2p}e^{U/2}). \tag{F17}$$

These eigenfunctions form a continuum orthonormal basis (we use the conventions in Ref. [55] with $\beta = \sigma = 1$ and $\alpha = p$). It allows to compute our observables of interest. The first one is expressed as follows, using the path integral representation for the Brownian motion with drift, $U(x) = \mathcal{B}(x) - wx$, with $U_0 = U(0) = 0$ and free $U(L) = U_L$, followed by the Feynman-Kac formula

$$\begin{aligned}\phi^{\mu=2w}(p, L) = \mathbb{E}[e^{-pZ_L^w}] &= e^{-\frac{w^2L}{2}}\int_{-\infty}^{+\infty} dU_L e^{-wU_L}\langle U_L|e^{-LH_p}|U_0 = 0\rangle \\ &= \int_0^{+\infty} dk\int_{-\infty}^{+\infty} dU_L\psi_k(U_L)\psi_k^*(0)e^{-wU_L - \frac{L}{8}(k^2+4w^2)},\end{aligned} \tag{F18}$$

where the dependence in the drift $-w$ is made explicit through a trivial shift. In the last equation we have used the spectral decomposition of H_p in terms of its eigenvectors, $\langle U|k\rangle = \psi_k(U)$ introduced above. For $w < 0$ one can use the identity

$$\int_{-\infty}^{+\infty} dU e^{-wU}K_{\mathbf{i}k}(2\sqrt{2p}e^{U/2}) = \frac{(2p)^w}{2}\left|\Gamma\left(-w + \frac{\mathbf{i}k}{2}\right)\right|^2 \tag{F19}$$

and one checks that Eq. (F18) yields Eq. (F3) after the change $w \to -w$.

The m-point correlation can be written following closely [55] upon adding the drift w. Upon exponentiation of the denominators $\frac{1}{Z^m} = \int_0^{+\infty} dq\frac{q^{m-1}}{\Gamma(m)}e^{-qZ}$ and using the same path integral representation, one obtains for $L \geqslant x_1 \geqslant \dots x_m \geqslant 0$,

$$\begin{aligned}\mathbb{E}[p_A(x_1)\dots p_A(x_m)] &= \int_0^{+\infty} dq\frac{q^{m-1}}{\Gamma(m)}e^{-\frac{w^2L}{2}}\int_{-\infty}^{+\infty} dU_L e^{-wU_L}\langle U_L|e^{-H_q(L-x_1)}e^{\hat{U}}e^{-H_q(x_1-x_2)}e^{\hat{U}}\dots e^{-H_q x_m}|U_0 = 0\rangle \\ &= \frac{e^{-\frac{w^2L}{2}}p^m}{\Gamma(m)}\int_{-\infty}^{+\infty} dU_0\int_{-\infty}^{+\infty} dU_L e^{-w(U_L-U_0)}\langle U_L|e^{-H_p(L-x_1)}e^{\hat{U}}e^{-H_p(x_1-x_2)}e^{\hat{U}}\dots e^{-H_p x_m}|U_0\rangle,\end{aligned} \tag{F20}$$

where $e^{\hat{U}} = \int_{-\infty}^{+\infty} dU|U\rangle e^U\langle U|$. The second line is obtained after the standard trick in Liouville theory, i.e., the change of variable $q = pe^{U_0}$ followed by the shift $U(x) \to U(x) - U_0$. Introducing the eigenbasis of H_p, and choosing $p = 1/2$ for convenience, one obtains (here $w = -(A + \frac{1}{2}) > 0$)

$$\mathbb{E}[p_A(x_1)\dots p_A(x_m)] = \frac{1}{2^m\Gamma(m)} e^{-\frac{w^2 L}{2}} \int_{-\infty}^{+\infty} dU_0 \int_{-\infty}^{+\infty} dU_L e^{-w(U_L-U_0)} \int_0^{+\infty} dk \psi_k(U_L) \tag{F21}$$

$$\times \prod_{j=1}^m \int_0^{+\infty} dk_j F(k,k_1)F(k_1,k_2)\dots F(k_{m-1},k_m)\psi^*_{k_m}(U_0) e^{-\frac{k^2}{8}(L-x_1)-\frac{k_1^2}{8}(x_1-x_2)-\dots-\frac{k_m^2}{8}x_m}, \tag{F22}$$

where we have defined the matrix elements [55]

$$F(k,k') = \langle k|e^{\hat{U}}|k'\rangle = \int_{-\infty}^{+\infty} dU \psi^*_k(U) e^U \psi_{k'}(U) = \frac{1}{8}\sqrt{kk'\sinh(\pi k)\sinh(\pi k')}\frac{k^2-(k')^2}{\cosh(\pi k)-\cosh(\pi k')}. \tag{F23}$$

Examination of the calculations in Ref. [56] [Sec. 5, in particular Eq. (5.6)] for a simpler quantity, indicates that the limit $L \to +\infty$ is controlled by setting $k = -2\mathbf{i}w$ (the integral $\int_{-\infty}^{+\infty} dU_L e^{-wU_L}\psi_k(U_L)$ is divergent for $w > 0$, but one can extract the residue of its analytic continuation). This amount to use that as $L \to +\infty$

$$e^{-\frac{w^2 L}{2}} \int_{-\infty}^{+\infty} dU_L e^{-wU_L} \int_0^{+\infty} dk \psi_k(U_L) F(k,k_1) e^{-\frac{k^2}{8}L} \underset{L\to\infty}{\longrightarrow} \frac{2\pi}{\Gamma(2w)} \frac{F(k,k_1)}{\sqrt{k\sinh(\pi k)}}\bigg|_{k=-2iw} (2p)^w \tag{F24}$$

valid for p arbitrary, and recalling our choice $p = 1/2$. Using further Eq. (F19) (with $w \to -w$) to integrate over U_0 this leads to the final result, for $0 < w \leqslant 1$ and $x_1 \geqslant x_2 \dots \geqslant x_m \geqslant 0$

$$\mathbb{E}[p_A(x_1)\dots p_A(x_m)] = \frac{1}{2^m\Gamma(2w)\Gamma(m)} \prod_{j=1}^m \int_0^{+\infty} \frac{dk_j}{8} k_j \sinh(\pi k_j)$$
$$\times \prod_{j=1}^{m-1} \frac{k_j^2 - k_{j+1}^2}{\cosh(\pi k_j)-\cosh(\pi k_{j+1})} \frac{k_1^2+4w^2}{\cosh(\pi k_1)-\cos(2\pi w)} \left|\Gamma\left(w+\frac{ik_m}{2}\right)\right|^2 e^{-\frac{w^2}{2}x_1 - \frac{k_1^2}{8}(x_1-x_2)-\dots-\frac{k_m^2}{8}x_m}, \tag{F25}$$

which for $m = 1$ agrees with Eq. (F5) obtained by a different method. The normalization is checked below. For $m = 2$, we have checked numerically using Mathematica, using also Eq. (F5), that $\int_0^{+\infty} dx x^2 \mathbb{E}[p_A(x)] - \int_0^{+\infty} dx_1 \int_0^{+\infty} dx_2 x_1 x_2 \mathbb{E}[p_A(x_1)p_A(x_2)]$ agrees numerically with the result Eq. (30) for the thermal cumulant $\mathbb{E}[\langle x^2\rangle_c]$.

Using similar manipulations as around Eq. (F6), we may write (with the convention $x_{m+1} = 0$)

$$\mathbb{E}[p_A(x_1)\dots p_A(x_m)] = \frac{1}{2^{2m}\Gamma(2w)\Gamma(m)} \int_{\mathbf{i}\mathbb{R}} \frac{dz_1}{2\mathbf{i}\pi} \dots \int_{\mathbf{i}\mathbb{R}} \frac{dz_m}{2\mathbf{i}\pi} \prod_{j=1}^m \frac{e^{\left(\frac{z_j^2}{2}-\frac{w^2}{2}\right)(x_j-x_{j+1})}}{\Gamma(2z_j)\Gamma(-2z_j)}$$
$$\times \Gamma(w+z_m)\Gamma(w-z_m)\big(w^2-z_1^2\big)\Gamma(w+z_1)\Gamma(w-z_1)\Gamma(1-w+z_1)\Gamma(1-w-z_1)$$
$$\times \prod_{j=1}^{m-1} \Gamma(1+z_{j+1}-z_j)\Gamma(1+z_{j+1}+z_j)\Gamma(1-z_{j+1}+z_j)\Gamma(1-z_{j+1}-z_j). \tag{F26}$$

A similar analytic continuation as in Eq. (F9) can be performed on Eq. (F26) to obtain a formula when $w > 1$.

3. Verification of the normalization

Let us compute

$$C_m = m! \int_{x_1\geqslant x_2 \dots \geqslant x_m \geqslant 0} \mathbb{E}[p_A(x_1)\dots p_A(x_m)] dx_1 \dots dx_m \tag{F27}$$

and check that $C_m = 1$. We use the change of variables $y_j = x_j - x_{j+1}$ and compute the integrals over $y_1, z_1, y_2, z_2 \dots$ sequentially. We will need the identity [83, Th. 3.6.2]

$$\int_{\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)}$$
$$= \frac{2\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}, \tag{F28}$$

valid for a, b, c, d with positive real part. Performing the integration over y_1 and z_1 using Eq. (F28) with $\{a, b, c, d\} = \{w, 1 - w, 1 + z_2, 1 - z_2\}$, we obtain

$$C_m = \frac{\Gamma(m+1)}{2^{2(m-1)}\Gamma(2w)\Gamma(m)} \int_{\mathbf{i}\mathbb{R}} \frac{dz_2}{2\mathbf{i}\pi} \cdots \int_{\mathbf{i}\mathbb{R}} \frac{dz_m}{2\mathbf{i}\pi} \prod_{j=2}^{m} \frac{e^{\left(\frac{z_j^2}{2} - \frac{w^2}{2}\right)(x_j - x_{j+1})}}{\Gamma(2z_j)\Gamma(-2z_j)}$$
$$\times \Gamma(w + z_m)\Gamma(w - z_m)\left(w^2 - z_2^2\right)\Gamma(w + z_2)\Gamma(w - z_2)\Gamma(2 - w + z_2)\Gamma(2 - w - z_2)\frac{\Gamma(1)\Gamma(2)}{\Gamma(3)}$$
$$\times \prod_{j=2}^{m-1} \Gamma(1 + z_{j+1} - z_j)\Gamma(1 + z_{j+1} + z_j)\Gamma(1 - z_{j+1} + z_j)\Gamma(1 - z_{j+1} - z_j). \tag{F29}$$

Then we integrate over y_2 and z_2 using Eq. (F28) with $\{a, b, c, d\} = \{w, 2 - w, 1 + z_3, 1 - z_3\}$, we integrate over y_3 and z_3 using Eq. (F28) with $\{a, b, c, d\} = \{w, 3 - w, 1 + z_4, 1 - z_4\}$, and we continue until we are left with variables y_m, z_m, where we use Eq. (F11) with $\{a, b, c\} = \{w, w, m - w\}$. Keeping track of all the Gamma factors involved at each step, we find that

$$C_m = \frac{\Gamma(m+1)}{\Gamma(m)\Gamma(2w)} \times \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)} \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} \cdots \frac{\Gamma(2)\Gamma(m-1)}{\Gamma(m+1)} \times \Gamma(2w)\Gamma(m)^2 = 1. \tag{F30}$$

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