

Nonhydrodynamic initial conditions are not soon forgottenT. R. Kirkpatrick,¹ D. Belitz^{2,3} and J. R. Dorfman¹¹*Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742, USA*²*Department of Physics and Institute for Fundamental Science, University of Oregon, Eugene, Oregon 97403, USA*³*Materials Science Institute, University of Oregon, Eugene, Oregon 97403, USA*

(Received 25 February 2021; accepted 28 July 2021; published 10 August 2021)

Solutions to hydrodynamic equations, which are used for a vast variety of physical problems, are assumed to be specified by boundary conditions and initial conditions on the hydrodynamic variables only. Initial values of other variables are assumed to be irrelevant for a hydrodynamic description. We show that this assumption is not correct because of the existence of long-time-tail effects that are ubiquitous in systems governed by hydrodynamic equations. We illustrate this breakdown of a hydrodynamic description by means of the simple example of diffusion in a disordered electron system.

DOI: [10.1103/PhysRevE.104.024111](https://doi.org/10.1103/PhysRevE.104.024111)**I. INTRODUCTION**

Hydrodynamic descriptions of matter are abundant in virtually all subfields of physics. The most obvious example is provided by classical fluids [1]; others include plasmas [2,3], superfluids, spin transport in magnets, and excitations in liquid crystals and in solids [4,5]. More recently, hydrodynamic descriptions have been used to study active matter [6], general relativity [7], supersymmetric field theories and quantum gravity [8], quark-gluon plasmas [9], and the unusual properties of Weyl and Dirac metals and semimetals [10].

The basic assumption of any hydrodynamic theory is that the behavior of a macroscopic system on “hydrodynamic” length and time scales, i.e., scales large compared to the microscopic ones, can be described in terms of a small number of “hydrodynamic” variables or modes, while all other degrees of freedom are effectively integrated out and just renormalize coefficients. The feasibility of such a reduction from a huge (on the order of 10^{23}) number of degrees of freedom to a few variables that are essential for the problem at hand is a crucial assumption underlying much of statistical mechanics. An essential feature of any such reduced description is that it is closed in the sense that the initial values of the hydrodynamic variables completely determine the time evolution of the system. In this paper we will show that this is not true, for the following reasons. Let h be a set of hydrodynamic variables (we will consider explicit examples below), and let $\delta h(t=0)$ be an initial perturbation. Kinetic theory shows that in general the relaxation of h towards its equilibrium value on hydrodynamic time scales is also governed by a set of initial values of “nonhydrodynamic” variables h_{\perp} that are orthogonal to h with respect to an appropriate scalar product in the space of modes. While this violates the closure assumption of the hydrodynamic description, by itself it does not completely invalidate the underlying concept: If $\delta h(t=0)$ and $\delta h_{\perp}(t=0)$ multiply the same relaxational dynamics, then one can adjust the hydrodynamic boundary conditions to still obtain

a complete description. This is analogous to the concept of a ‘slip length’ used to describe the flow of a fluid near a surface, i.e., the distance between the actual surface and the fictitious one where the hydrodynamic boundary conditions can be used [11,12]. However, if the different sets of initial conditions multiply *different* dynamics, then such an adjustment is not possible and the very concept of a closed hydrodynamic description breaks down. As we will show, this is the generic situation whenever perturbations decay algebraically in time, rather than exponentially, a phenomenon known as long-time tails (LTTs) [13,14]. LTTs are not described by the Boltzmann equation, and are generically present in all systems for which hydrodynamic descriptions are used [6,8,11]. As a consequence of their ubiquitous existence, all hydrodynamic descriptions are inherently incomplete.

II. A SIMPLE EXAMPLE: DIFFUSION

To illustrate these ideas, let us consider the arguably simplest hydrodynamic process, diffusion, which has applications in physics, chemistry, biology, and engineering [15–17]. It is phenomenologically described by Fick’s law, which states that the current density \mathbf{j} associated with a conserved density n is proportional to the gradient of n , $\mathbf{j}(\mathbf{x}, t) = -D \nabla n(\mathbf{x}, t)$, with D the diffusion coefficient. Together with the continuity equation for n , $\partial_t n(\mathbf{x}, t) = -\nabla \cdot \mathbf{j}(\mathbf{x}, t)$, this constitutive equation leads to the diffusion equation

$$\partial_t n(\mathbf{x}, t) - D \nabla^2 n(\mathbf{x}, t) = 0. \quad (1)$$

This is expected to be valid for times $t \gg t_0$ and wave numbers $q \ll q_0$, with t_0 and $1/q_0$ the microscopic time and length scales that depend on the nature of the diffusive process. In this description there is only one hydrodynamic variable, viz., the density. With n_0 the constant equilibrium density, Eq. (1) describes the relaxation of a density perturbation $\delta n(\mathbf{x}, t) = n(\mathbf{x}, t) - n_0$. It is solved by a spatial Fourier transform and

a temporal Laplace transform [4,18]. The result is the well known diffusion pole in the complex-frequency (z) plane,

$$\delta n(\mathbf{q}, z) = \delta n(\mathbf{q}, t = 0) i/[z + iDq^2 s_z], \quad (2a)$$

with \mathbf{q} the wave vector, and $s_z = \text{sgn Im}(z)$. In the time domain, this corresponds to exponential decay with a relaxation time that diverges as $1/q^2$ for small \mathbf{q} ,

$$\delta n(\mathbf{q}, t) = \delta n(\mathbf{q}, t = 0) e^{-Dq^2 t}. \quad (2b)$$

According to Eq. (2b), the initial value $\delta n(\mathbf{q}, t = 0)$ completely determines the relaxation for $t \gg t_0$ and $q \ll q_0$.

A derivation of the diffusion pole from kinetic theory (see below) shows that $\delta n(\mathbf{q}, t)$ also depends on initial conditions in higher angular-momentum channels; e.g., an initial current. Within diffusion theory, these initial conditions are “nonhydrodynamic” since they involve quantities other than the density. As we will show in Sec. III B, solving the standard Boltzmann equation for the scattering of electrons by static pointlike impurities (s -wave scattering only) yields

$$\delta n(\mathbf{q}, t \gg t_0) = [\delta n(\mathbf{q}, t = 0) + \delta n_{\perp}(\mathbf{q}, t = 0)] e^{-Dq^2 t}. \quad (3a)$$

Here the diffusion coefficient is $D = v_F^2 \tau / 3$, with v_F the Fermi velocity and τ the elastic mean-free time. In this case, the microscopic time and wave number are $t_0 = \tau$ and $q_0 = 1/v_F \tau$, respectively. $\delta n_{\perp}(\mathbf{q}, t = 0)$ is a set of nonhydrodynamic initial conditions that are orthogonal in angular-momentum (l) space to the density, with the longitudinal part of the current density \mathbf{j} ($l = 1$) the leading contribution,

$$\delta n_{\perp}(\mathbf{q}, t = 0) \approx -i\tau \mathbf{q} \cdot \mathbf{j}(\mathbf{q}, t = 0). \quad (3b)$$

Despite this incompleteness of a purely hydrodynamic description, the density relaxation can still be described by the solution of the diffusion equation, Eq. (2b), if we replace $\delta n(t = 0)$ by δn taken at an adjusted initial time t_s defined by

$$\begin{aligned} \delta n(\mathbf{q}, t_s) &\equiv \delta n(\mathbf{q}, t = 0) e^{-Dq^2 t_s} \\ &= \delta n(\mathbf{q}, t = 0) + \delta n_{\perp}(\mathbf{q}, t = 0). \end{aligned} \quad (3c)$$

The “slip time”

$$t_s = \frac{-1}{Dq^2} [1 + \delta n_{\perp}(\mathbf{q}, t = 0)/\delta n(\mathbf{q}, t = 0)] \quad (3d)$$

is the temporal analog of the “slip length” mentioned above. Its value depends on the initial conditions [19].

What allows for this concept of a slip time is that the two orthogonal initial conditions in Eq. (3a) multiply the same exponential relaxation. Within the framework of the Boltzmann equation, an effective hydrodynamic description therefore is still possible. However, the exponential decay predicted by Eq. (3a) is not the true asymptotic long-time behavior: Correlation, or mode-mode-coupling, effects lead to a nonanalytic frequency dependence of the diffusion coefficient. As a function of time, this corresponds to a power-law decay of δn , which is an example of a LTT in the context of diffusion. This effect is not contained in the standard Boltzmann equation.

The combination of LTTs and nonhydrodynamic initial conditions yields a long-time behavior of the density relaxation that cannot be reconciled with the diffusion equation by adjusting the initial condition as in Eq. (3c), even with a

frequency-dependent diffusivity that yields the correct LTT. The reason is that the initial values of δn and δn_{\perp} multiply different LTTs, as we will now demonstrate by means of a more elaborate example.

III. DENSITY RELAXATION OF DISORDERED ELECTRONS

As an example, we consider the density relaxation of noninteracting conduction electrons in a three-dimensional (3D) system with weak quenched disorder. The latter leads to LTTs known as weak-localization (WL) effects [20,21], but in 3D systems it does not lead to Anderson localization; the system remains metallic. Let ϵ_F and k_F be the Fermi energy and wave number, respectively, and m the effective mass. The density of states per spin at the Fermi surface is $N_F = k_F m / 2\pi^2$, and the mean-free path is $\ell = v_F \tau$. For weak disorder ($k_F \ell \gg 1$, or $\epsilon_F \tau \gg 1$), and $t \gg 1/Dq^2$, we find

$$\begin{aligned} \delta n(\mathbf{q}, t) &= \delta n(\mathbf{q}, t = 0) \left[\frac{3}{4\sqrt{\pi}} \frac{1}{k_F \ell} \frac{q/k_F}{(Dq^2 t)^{3/2}} + O\left(\frac{1}{t^{5/2}}\right) \right] \\ &+ \delta n_{\perp}(\mathbf{q}, t = 0) \frac{9}{8\sqrt{\pi}} \frac{1}{k_F \ell} \frac{q/k_F}{(Dq^2 t)^{5/2}}. \end{aligned} \quad (4a)$$

The different LTTs in the two terms make it impossible to account for the nonhydrodynamic initial conditions represented by $\delta n_{\perp}(t = 0)$ by means of a slip time as in Eqs. (3). A diffusion pole, Eq. (2a), generalized to allow for the LTT, thus fails to describe the density relaxation at long times. This remains true for intermediate times $\tau \ll t \ll 1/Dq^2$, where the leading time dependence (ignoring an $O(1/k_F \ell)$ contribution to the constant part of the δn_{\perp} term) is

$$\begin{aligned} \delta n(\mathbf{q}, t) &= \delta n(\mathbf{q}, t = 0) [1 + O(Dq^2 t)] \\ &+ \delta n_{\perp}(\mathbf{q}, t = 0) \left[1 - \frac{3}{2\sqrt{\pi}} \frac{1}{k_F \ell} \frac{q/k_F}{(Dq^2 t)^{1/2}} \right]. \end{aligned} \quad (4b)$$

Here the difference between the hydrodynamic and nonhydrodynamic terms is even more striking than in Eq. (4a): The former is a constant, as described by the diffusion equation, whereas the latter has a $1/t^{1/2}$ long-time tail. Again, the concept of a slip time breaks down.

Equations (4), and the related comments, represent our main result which we will now derive from kinetic theory. The dependence on nonhydrodynamic initial conditions arises already at the level of the Boltzmann equation, while the weak-localization LTTs require a more sophisticated treatment of collision processes.

A. Kinetic equation for the single-particle distribution

Consider the single-electron distribution function $f(\mathbf{p}, \mathbf{x}, t)$ as a function of the electron momentum \mathbf{p} , real-space position \mathbf{x} , and time t . In the absence of external forces, the kinetic equation that governs f reads

$$(\partial_t + \mathbf{p} \cdot \nabla_{\mathbf{x}}/m) f(\mathbf{p}, \mathbf{x}, t) = (\partial f / \partial t)_{\text{coll}}. \quad (5)$$

This is completely general: The total time derivative of f on the left-hand side equals the collision integral, i.e., the temporal change of f due to collision, on the right-hand side. The

equilibrium distribution function is given by the Fermi-Dirac distribution $f_0(\mathbf{p}) = 1/(e^{(\epsilon_p - \mu)/T} + 1)$, with $\epsilon_p = \mathbf{p}^2/2m$ the single-electron energy, μ the chemical potential, and T the temperature. We parametrize the deviation from equilibrium, $\delta f = f - f_0$, by

$$\delta f(\mathbf{p}, \mathbf{x}, t) = w(\epsilon_p) \phi(\mathbf{p}, \mathbf{x}, t), \quad (6)$$

where $w(\epsilon_p) = -\partial f_0/\partial \epsilon_p$ is a weight function. For later reference, the number density n and the number current density \mathbf{j} are

$$n(\mathbf{x}, t) = \frac{1}{V} \sum_{\mathbf{p}} f(\mathbf{p}, \mathbf{x}, t), \quad (7a)$$

$$\mathbf{j}(\mathbf{x}, t) = \frac{1}{V} \sum_{\mathbf{p}} \mathbf{p} f(\mathbf{p}, \mathbf{x}, t), \quad (7b)$$

where V is the system volume.

B. Boltzmann equation

The Boltzmann collision integral for elastic scattering by impurities is [22,23]

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}^{\text{B}} = \frac{-1}{N_{\text{F}}V} \sum_{\mathbf{p}'} \delta(\epsilon_p - \epsilon_{p'}) \frac{1}{\tau} W(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \times [\delta f(\mathbf{p}, \mathbf{x}, t) - \delta f(\mathbf{p}', \mathbf{x}, t)]. \quad (8a)$$

The δ function reflects the elastic nature of the collisions, and W is a form factor. For pointlike scatterers, the latter is a constant equal to one and we obtain

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}^{\text{B}} = \frac{-1}{\tau} [\delta f(\mathbf{p}, \mathbf{x}, t) - \delta \bar{f}(\epsilon_p, \mathbf{x}, t)], \quad (8b)$$

where

$$\delta \bar{f}(\epsilon_p, \mathbf{x}, t) = \frac{1}{V N_{\text{F}}} \sum_{\mathbf{p}'} \delta(\epsilon_p - \epsilon_{p'}) \delta f(\epsilon_{p'}, \mathbf{x}, t) \quad (8c)$$

is δf averaged over the ϵ_p -energy shell. Integrating over the momentum yields $\sum_{\mathbf{p}} \delta \bar{f}(\epsilon_p, \mathbf{x}, t) = \sum_{\mathbf{p}} \delta f(\mathbf{p}, \mathbf{x}, t)$, so the number density, Eq. (7a), is conserved. After a Fourier-Laplace transform as in Eq. (2a), the linearized Boltzmann equation reads

$$\left(-iz + \frac{i}{m} \mathbf{p} \cdot \mathbf{q} + \frac{1}{\tau}\right) \phi(\mathbf{p}, \mathbf{q}, z) = \phi(\mathbf{p}, \mathbf{q}, t=0) + \frac{1}{\tau} \bar{\phi}(\epsilon_p, \mathbf{q}, z), \quad (9)$$

with ϕ from Eq. (6) and $\bar{\phi}$ defined in analogy to $\delta \bar{f}$. Now consider $\delta n = n - n_0$, the deviation of the particle-number density from its equilibrium value n_0 . From Eq. (9) we find

$$\delta n(\mathbf{q}, z) = J_0(\mathbf{q}, z) + \frac{i}{\tau} \frac{1}{V} \sum_{\mathbf{p}} \frac{w(\epsilon_p) \bar{\phi}(\epsilon_p, \mathbf{q}, z)}{z - \mathbf{p} \cdot \mathbf{q}/m + i/\tau}, \quad (10a)$$

where

$$J_0(\mathbf{q}, z) = \frac{i}{V} \sum_{\mathbf{p}} \frac{\delta f(\mathbf{p}, \mathbf{q}, t=0)}{z - \mathbf{p} \cdot \mathbf{q}/m + i/\tau}. \quad (10b)$$

For $T \ll \epsilon_{\text{F}}$ we can replace $\mathbf{p} \cdot \mathbf{q}/m$ in Eq. (10a) by $v_{\text{F}} \hat{\mathbf{p}} \cdot \mathbf{q}$, which neglects corrections of $O(T^2)$ to the \mathbf{q} -dependence of δn . The resulting closed equation for δn yields

$$\delta n(\mathbf{q}, z) = J_0(\mathbf{q}, z)/(1 - iJ(\mathbf{q}, z)/\tau), \quad (11a)$$

where

$$J(\mathbf{q}, z) = \frac{1}{4\pi} \int d\Omega_{\mathbf{p}} \frac{1}{z - v_{\text{F}} \hat{\mathbf{p}} \cdot \mathbf{q} + i/\tau}, \quad (11b)$$

with $\Omega_{\mathbf{p}}$ the solid angle associated with \mathbf{p} . An expansion in the limit of small wave numbers ($q\ell \ll 1$) and frequencies ($z\tau \ll 1$) yields

$$J_0(\mathbf{q}, z) = \frac{i\delta n(\mathbf{q}, t=0)}{z + i/\tau} + \frac{i\mathbf{q} \cdot \mathbf{j}(\mathbf{q}, t=0)}{(z + i/\tau)^2} + O(q^2), \quad (12a)$$

$$1 - \frac{i}{\tau} J(\mathbf{q}, z) = -i\tau(z + iD_0q^2) + O(z^2, q^4), \quad (12b)$$

where $D_0 = v_{\text{F}}^2\tau/3$ is the Boltzmann diffusion coefficient. Inserting these results into Eq. (11a), and performing a Laplace back transform, yields Eqs. (3a) and (3b).

Equations (10b) and (12a) show that the solution depends, in addition to the initial density perturbation, on the initial current density, as well as higher modes. At the level of the Boltzmann equation, after a few mean-free times the various initial-condition terms all multiply the same time dependence and therefore can be incorporated into a ‘‘slip time’’ as in Eq. (3c). This is no longer the case if one uses a more sophisticated collision integral that accounts for LTTs.

C. Weak-localization effects

The LTTs associated with weak-localization effects arise from two-particle correlations that are not included in the Boltzmann collision integral and lead to a frequency-dependent diffusivity that is nonanalytic at zero frequency [24,25]. Reference [26] showed that these effects can be incorporated into the kinetic equation for the single-particle distribution by adding an additional term to the collision integral. This additional term can still be written in the form of Eq. (8a) [27], with two modifications: First, the scattering rate $1/\tau$ must be replaced by a time-dependent memory function α (to be specified below) that encodes correlations. The weak-localization contribution to the collision integral then has the form

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}^{\text{WL}} = - \int_0^t dt' \alpha(t-t') \frac{1}{N_{\text{F}}V} \sum_{\mathbf{p}'} \delta(\epsilon_p - \epsilon_{p'}) \times W^{\text{WL}}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') [\delta f(\mathbf{p}, \mathbf{x}, t') - \delta f(\mathbf{p}', \mathbf{x}, t')]. \quad (13)$$

Second, the form factor is strongly angle dependent, which reflects the fact that the weak-localization effects are due to backscattering events [28]:

$$W^{\text{WL}}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') = \frac{1}{4\pi} \delta(\hat{\mathbf{p}} + \hat{\mathbf{p}}') - 1. \quad (14a)$$

The additional contribution to the collision integral does not change the total scattering rate [27]:

$$\int d\Omega_{p'} W^{\text{WL}}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') = 0. \quad (14b)$$

The weak-localization contribution to the collision integral becomes

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}^{\text{WL}} = \int_0^t dt' \alpha(t-t') [\delta f(-\mathbf{p}, \mathbf{x}, t') - \delta \bar{f}(\epsilon_p, \mathbf{x}, t')]. \quad (15a)$$

The function $\alpha(t)$ and its Laplace transform are [29,30]

$$\alpha(t) = \frac{1}{\pi N_F \tau} \frac{1}{V} \sum_k e^{-D_0 k^2 t} \quad (t > 0). \quad (15b)$$

$$\alpha(z) = \frac{i}{\pi N_F \tau} \frac{1}{V} \sum_k \frac{1}{z + iD_0 k^2 s_z} \quad (15c)$$

$$= s_z \frac{1}{\tau} \frac{3}{\pi} \frac{1}{k_F \ell} \left[c - \frac{\pi}{2} \zeta^{1/2} + O(\zeta) \right], \quad (15d)$$

where $\zeta = z s_z / iD_0 k_F^2$, c is the nonuniversal zero-frequency contribution [29], and Eq. (15d) is valid for $d = 3$. Note the nonanalytic frequency dependence that results from integrating over the diffusion pole. Only this contribution to α matters for the arguments that follow.

The kinetic equation that generalizes Eq. (9) now reads

$$\begin{aligned} & \left(-iz + \frac{i}{m} \mathbf{p} \cdot \mathbf{q} + \frac{1}{\tau}\right) \phi(\mathbf{p}, \mathbf{q}, z) - \alpha(z) \phi(-\mathbf{p}, \mathbf{q}, z) \\ &= \phi(\mathbf{p}, \mathbf{q}, t=0) + \left(\frac{1}{\tau} - \alpha(z)\right) \bar{\phi}(\epsilon_p, \mathbf{q}, z). \end{aligned} \quad (16)$$

By letting $\mathbf{p} \rightarrow -\mathbf{p}$ we obtain two coupled equations for $\phi(\mathbf{p})$ and $\phi(-\mathbf{p})$ that can be solved exactly. The result for the density relaxation generalizes Eq. (11a) to

$$\delta n(\mathbf{q}, z) = \frac{\tilde{J}_0(\mathbf{q}, z) + i\alpha(z)K_0(\mathbf{q}, z)}{1 - [i/\tau - i\alpha(z)][\tilde{J}(\mathbf{q}, z) + i\alpha(z)K(\mathbf{q}, z)]}. \quad (17a)$$

In terms of a denominator

$$N(\mathbf{p}, \mathbf{q}, z) = z - \mathbf{p} \cdot \mathbf{q}/m + i/\tau + \frac{(\alpha(z))^2}{z + \mathbf{p} \cdot \mathbf{q}/m + i/\tau}, \quad (17b)$$

we find

$$\tilde{J}_0(\mathbf{q}, z) = \frac{i}{V} \sum_p \delta f(\mathbf{p}, \mathbf{q}, t=0)/N(\mathbf{p}, \mathbf{q}, z), \quad (17c)$$

$$\tilde{J}(\mathbf{q}, z) = \frac{1}{4\pi} \int d\Omega_p \frac{1}{N(k_F \hat{\mathbf{p}}, \mathbf{q}, z)}, \quad (17d)$$

$$K_0(\mathbf{q}, z) = \frac{i}{V} \sum_p \frac{\delta f(-\mathbf{p}, \mathbf{q}, t=0)}{N(\mathbf{p}, \mathbf{q}, z)(z + \mathbf{p} \cdot \mathbf{q}/m + i/\tau)}, \quad (17e)$$

$$K(\mathbf{q}, z) = \int \frac{d\Omega_p}{4\pi} \frac{1}{N(k_F \hat{\mathbf{p}}, \mathbf{q}, z)(z + v_F \hat{\mathbf{p}} \cdot \mathbf{q} + i/\tau)}. \quad (17f)$$

To determine the long-wavelength behavior it suffices to expand the numerator in Eq. (17a) to $O(q)$ and the

denominator to $O(q^2)$. Neglecting corrections to the diffusion pole that are analytic in z , we find

$$\delta n(\mathbf{q}, z) = \frac{i\delta n(\mathbf{q}, t=0)}{z + iD(z)q^2} + \frac{i\delta n_{\perp}(\mathbf{q}, t=0)}{[z + iD(z)q^2][1 + s_z\alpha(z)]} \quad (18a)$$

with δn_{\perp} from Eq. (3b), and a renormalized diffusivity

$$D(z) = D_0 s_z / [1 + s_z\alpha(z)\tau]. \quad (18b)$$

The nonanalytic frequency dependence of the diffusivity is a well-known manifestation of the existence of LTTs [20]. However, Eq. (18a) demonstrates another striking consequence of their existence, which has not been discussed before. To see this, we determine the leading nonanalytic z -dependence of δn , which follows from using Eq. (15d) in Eqs. (18). In the asymptotic low-frequency regime $z \ll D_0 q^2$, the latter is $z^{1/2}$ in the first term on the right-hand side of Eq. (18a), and $z^{3/2}$ in the second one. For $z \gg D_0 q^2$, the first term has no nonanalytic z -dependence, while the second term goes as $z^{-1/2}$. The corresponding behavior in the time domain can be obtained by using the concept of generalized functions and their Fourier transforms [31,32]. By making use of the identities

$$(-izs_z)^{1/2} = \frac{1}{\sqrt{2}} [z^{1/2} + (-z)^{1/2}], \quad (19a)$$

$$s_z z^{1/2} = i(-z)^{1/2}, \quad (19b)$$

all of the relevant nonanalyticities can be expressed in terms of

$$H_{\nu}(z) = -i[z^{\nu} - (-z)^{\nu}]. \quad (20a)$$

The Laplace back transform for noninteger ν is [31]

$$H_{\nu}(t) = \frac{-2}{\pi} \sin(\pi\nu) \sin(\pi\nu/2) \Gamma(1+\nu) |t|^{1+\nu}, \quad (20b)$$

with Γ the Gamma function. $H_{-1}(t)$ is a constant. Applying these results to Eqs. (18) we obtain Eqs. (4).

IV. CONCLUSION

We conclude with some general remarks.

(1) We have illustrated our ideas by means of a specific example, which has the advantage that the kinetic equation is exactly soluble, which in general is not the case. However, the conclusions are general: Kinetic theory generically yields a dependence on nonhydrodynamic initial conditions, and LTTs are known to be present generically. As a result, all hydrodynamic descriptions are necessarily incomplete. One cannot simply integrate out the nonhydrodynamic variables and still correctly describe transport or other nonequilibrium effects. We note that, due to the ubiquity of LTTs, this conclusion is much more general than other mechanisms that lead to the failure of hydrodynamic descriptions, such as many-body localization, where strong disorder in certain quantum system leads to a failure of the system to equilibrate [33].

(2) In our example the only hydrodynamic variable is the density. Accordingly, the “nonhydrodynamic” initial conditions include all higher moments of the single-particle distribution function with respect to the momentum, taken at

the initial time. In general, the set of hydrodynamic variables is larger. For instance, in the case of the Navier-Stokes theory of a classical fluid, it consists of the particle number density, momentum density, and energy or entropy density. The leading nonhydrodynamic initial conditions in an angular-momentum expansion are the initial values of the stress tensor and the energy current density.

(3) More generally, even a description in terms of the initial value of the entire single-particle distribution function f , as we have used in this paper, is not complete, and this observation implies that there is no simple remedy for this incompleteness. To obtain a closed description in terms of f alone, with the solution determined by the initial condition $f(\mathbf{p}, \mathbf{x}, t = 0)$, one has to effectively integrate out higher order distribution functions, such as the pair correlation function $G^{(2)} = f^{(2)} - ff$, with $f^{(2)}$ the two-particle distribution function. A complete description in terms of f therefore depends on $G^{(2)}(t = 0)$, which in general is nonzero and also multiplies a LTT contribution [11]. The problem is thus

not confined to hydrodynamic theories. Rather, we conclude that, due to LTT effects, no reduced description can be truly complete.

To summarize: Long-time-tail effects have been investigated in a host of physical systems including classical fluids, superfluids, granular matter, active matter, electrons in solids, quark-gluon plasmas, and quantum gravity. The main message of this paper is that a purely hydrodynamic description that is routinely used in all these fields is incomplete because nonhydrodynamic initial-condition effects cannot be ignored in general.

ACKNOWLEDGMENTS

We thank Alex Levchenko for a discussion. This work was performed in part at the Aspen Center for Physics, which is supported by National Science Foundation Grant No. PHY-1607611.

-
- [1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
- [2] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Butterworth-Heinemann, Oxford, 1981).
- [3] G. Belmont, L. Rezeau, C. Riconda, and A. Zaslavsky, *Introduction to Plasma Physics* (Elsevier, Amsterdam, 2019).
- [4] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, MA, 1975).
- [5] P. C. Martin, O. Parodi, and P. S. Pershan, *Phys. Rev. A* **6**, 2401 (1972).
- [6] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, *Rev. Mod. Phys.* **85**, 1143 (2013).
- [7] S. Minwalla, V. E. Hubeny, and M. Rangamani, in *Proceedings of the 2010 Theoretical Advanced Study Institute in Elementary Particle Physics*, edited by M. Dine, T. Banks, and S. Sachdev (World Scientific, Singapore, 2011), p. 817.
- [8] P. Kovtun and L. G. Yaffe, *Phys. Rev. D* **68**, 025007 (2003).
- [9] A. Mazeliauskas and J. Berges, *Phys. Rev. Lett.* **122**, 122301 (2019).
- [10] J. Zaanen, *Science* **351**, 1026 (2016).
- [11] J. R. Dorfman, H. van Beijeren, and T. R. Kirkpatrick, *Contemporary Kinetic Theory of Matter* (Cambridge University Press, Cambridge, UK, 2021).
- [12] J. C. Nieuwoudt, T. R. Kirkpatrick, and J. R. Dorfman, *J. Stat. Phys.* **34**, 203 (1984).
- [13] J. R. Dorfman and E. G. D. Cohen, *Phys. Rev. Lett.* **25**, 1257 (1970).
- [14] M. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, *Phys. Rev. Lett.* **25**, 1254 (1970).
- [15] H. Mehrer, *Diffusion in Solids* (Springer, Berlin, 2007).
- [16] E. L. Cussler, *Diffusion. Mass Transfer in Fluid Systems* (Cambridge University Press, Cambridge, UK, 1997), 2nd ed.
- [17] W. D. Stein, *Transport and Diffusion across Cell Membranes* (Academic Press, Orlando, FL, 1986).
- [18] Following Refs. [4,34], we define the Laplace transform $f(z)$ of a time-dependent function $f(t)$ by $f(z) = \pm \int_{-\infty}^{\infty} dt \Theta(\pm t) e^{izt} f(t)$, with \pm for $\text{sgn Im } z \gtrless 0$.
- [19] If the initial conditions are such that δn and δn_{\perp} are related by Fick's law, $\delta n_{\perp}(\mathbf{q}, t = 0) \approx -Dq^2 \tau \delta n(\mathbf{q}, t = 0)$, then $|t_s| \approx \tau$, but generically this will not be the case.
- [20] P. A. Lee and T. V. Ramakrishnan, *Rev. Mod. Phys.* **57**, 287 (1985).
- [21] The same type of LTT arises by a different mechanism in interacting electron systems via an interplay between the interaction and quenched disorder, see Refs. [35–37].
- [22] W. Kohn and J. M. Luttinger, *Phys. Rev.* **108**, 590 (1957).
- [23] D. A. Greenwood, *Proc. Phys. Soc.* **71**, 585 (1958).
- [24] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, *Phys. Rev. Lett.* **42**, 673 (1979).
- [25] F. Wegner, *Z. Phys. B* **35**, 207 (1979).
- [26] S. Hershfield and V. Ambegaokar, *Phys. Rev. B* **34**, 2147 (1986).
- [27] G. Strinati, C. Castellani, and C. Di Castro, *Phys. Rev. B* **40**, 12237 (1989).
- [28] L. P. Gorkov, A. Larkin, and D. E. Khmel'nitskii, *Pis'ma Zh. Eksp. Teor. Fiz.* **30**, 248 (1979) [*JETP Lett.* **30**, 228 (1979)].
- [29] This is an integral over the density relaxation in Boltzmann approximation. The integral extends over all wave numbers, but the integrand is diffusive only for small wave numbers. This observation accounts for the nonuniversal constant in Eq. (15d) that depends on the density response in the nonhydrodynamic regime.
- [30] The object $M(z) = i/\tau + i\alpha(z)$ is identical with the memory kernel that appears in the mode-mode coupling theory of weak localization [38,39].
- [31] M. J. Lighthill, *Introduction to Fourier analysis and generalised functions* (Cambridge University Press, Cambridge, UK, 1958).
- [32] The algebraic decay at long times is entirely determined by the singularities at $z = 0$ and the associated branch cuts [31]. In addition, there are poles in the complex-frequency plane that lead to exponential decay with a nonanalytic wave-number dependence of the diffusivity.
- [33] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, *Rev. Mod. Phys.* **91**, 021001 (2019).

- [34] D. N. Zubarev, *Usp. Fiz. Nauk* **71**, 71 (1960) [*Sov. Phys. Usp.* **3**, 320 (1960)].
- [35] B. L. Altshuler and A. G. Aronov, *Solid State Commun.* **30**, 115 (1979).
- [36] B. L. Altshuler, A. G. Aronov, and P. A. Lee, *Phys. Rev. Lett.* **44**, 1288 (1980).
- [37] D. Belitz and T. R. Kirkpatrick, *Rev. Mod. Phys.* **66**, 261 (1994).
- [38] D. Vollhardt and P. Wölfle, *Phys. Rev. B* **22**, 4666 (1980).
- [39] D. Belitz, A. Gold, and W. Götze, *Z. Phys. B* **44**, 273 (1981).