Optimal non-Gaussian search with stochastic resetting

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In this paper we reveal that each subordinated Brownian process, leading to subdiffusion, under Poissonian resetting has a stationary state with the Laplace distribution. Its location parameter is defined only by the position to which the particle resets, and its scaling parameter is dependent on the Laplace exponent of the random process directing Brownian motion as a parent process. From the analysis of the scaling parameter the probability density function of the stochastic process, subject to reset, can be restored. In this case the mean time for the particle to reach a target is finite and has a minimum, optimal for the resetting rate. If the Brownian process is replaced by the Lévy motion (superdiffusion), then its stationary state obeys the Linnik distribution which belongs to the class of generalized Laplace distributions.

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I. INTRODUCTION

There has been a large amount of research on various stochastic processes under resetting in the last decade [1–5]. The studies on resetting appeared well before the topic of resetting or restart became recently popular (see, e.g., [6]). Any stochastic process evolving under its own natural dynamics is interrupted at random times and brought back (reset) to a fixed state, say, its initial state. The intervals between successive reset events are statistically independent and are drawn from some specified distribution. A particularly simple and illustrative case is the Poissonian resetting with the constant reset rate r [1]. For another popular first-passage resetting, where resetting occurs whenever a diffusing particle reaches a threshold location, see [7]. One of the most optimal search strategies is just based on the stochastic resetting: if it is difficult to find something, it would be better to go back to the beginning to start the search process again [8–10]. This key to success is widely used in nature and human behavior. In the way that animals search for food and relocation [11–13], biomolecules search for proteins on DNA [14,15], people look for a target in a crowd [16], etc. The resetting can lead to a stationary state in natural systems which is off equilibrium [17]. Chemical reactions, producing some product and manifesting a complex stochastic process, may benefit from restarting [18]. All this attracts attention to the study of stochastic processes, subject to reset, again and again [19–22]. Although the systems are different in nature, often they demonstrate similar features. Understanding reasons for their likeness is of particular interest. Many of these works focus on the stationary state, the approach to it, and first-passage properties [7]. In contrast, we study here yet another type of process, namely, the subordinated Brownian process and Levy α-stable motion, undergoing Poissonian resetting. A useful technical tool is the subdiffusive Green's function.

An intuitively simple way [9], leading to such systems, can be represented by using the familiar Gaussian expression (propagator)

$$G_1(x,t|x_0) = \frac{1}{\sqrt{4\pi Dt}}\exp\left[-\frac{(x-x_0)^2}{4Dt}\right], \qquad (1)$$

where D is the diffusion constant, and x_0 the initial position in which the initial condition is expressed in terms of the Dirac delta function, i.e., $G_1(x,0|x_0) = \delta(x-x_0)$. In the presence of resetting the probability density function (PDF) $p_1(x,t|x_0)$ is a sum of two terms [1]. One of them is related to trajectories, where no resetting events have occurred, and another is responsible for summing over trajectories, where the last resetting event occurred at time $t-\tau$. For Poissonian resetting with the rate r the probability of no resetting events up to time t is defined by e^{-rt}, whereas the probability of the last resetting event at time $t-\tau$ (without resetting events after) is $re^{-r\tau}$. The resetting process is independent of the stochastic process which was reset. In this case the PDF $p_1(x,t|x_0)$ is written as

$$p_1(x,t|x_0) = e^{-rt}G_1(x,t|x_0) + r\int_0^t e^{-r\tau}\,G_1(x,\tau|X_r)\,d\tau, \qquad (2)$$

where X_r denotes the position to which the particle was returned after resetting. This case can be also considered as stopping Brownian motion at an exponential time [23]. If $r=0$, this equation manifests the absence of resetting. The stationary state ($t\to\infty$) of Eq. (2) is determined only by the second term which can be calculated exactly [24]. In fact, it

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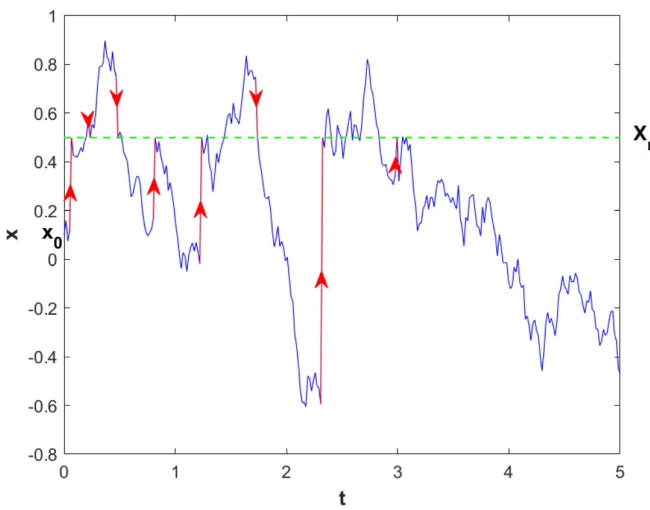

FIG. 1. Illustration of the Brownian motion with resetting process: the particle starts at initial position x_0 and resets to position X_r with rate r.

takes the form of the Laplace distribution [25], namely,

$$p_1(x,\infty|x_0) = \frac{c_1}{2} e^{-c_1|x-X_r|}, \tag{3}$$

where $c_1 = \sqrt{r/D}$ is the constant. The result is based on the consideration of Brownian motion as a stochastic process under resetting and was obtained in Ref. [9]. This example is illustrated in Fig. 1. Note that the renewal equation (2) holds for more general stochastic processes with their propagators, which can be different from this case having $G_1(x,t|x_0)$. Below we will show that the Laplace distribution is a stationary state not only for Brownian motion under Poissonian resetting, but also it is typical for many other stochastic processes with the same resetting. In addition, this allows one to explain why the Laplace distribution shows up so often in the statistics of off-equilibrium non-Brownian systems. Recall that the sum of random variables with the Laplace distribution does not belong to the same distribution as in the case of Gaussian variables. Thus, the well-known method of limit theorems from the theory of probability cannot be used for interpretations of experimental data with the Laplace distribution. Nevertheless, Laplace distributions are popular enough in different studies [26–28]. Moreover, this analysis also brings us to another interesting feature induced by resetting: the mean first-passage time is minimized for an optimal choice of the resetting rate.

II. SUBDIFFUSION WITH RESETTING

Consider the subdiffusion instead of Brownian motion as a stochastic process under Poissonian resetting with the constant rate r. The subdiffusive Green's function (also known as the propagator for the subdiffusion equation) is easy to represent with the help of the subordination integral:

$$G_\alpha(x,t|x_0) = \int_0^\infty G_1(x,\xi|x_0)\, g_\alpha(\xi,t)\, d\xi, \tag{4}$$

for which the PDF $G_1(x,\xi|x_0)$ describes the parent process, whereas the PDF $g_\alpha(\xi,t)$ is related to the directing process being inverse α stable [29]. The latter has a simple Laplace transform

$$\bar{g}_\alpha(\xi,u) = u^{\alpha-1} e^{-\xi u^\alpha}, \quad 0<\alpha<1, \tag{5}$$

which we use for the subsequent analysis. The PDF is expressed in terms of the M-Wright function [30], for which the Laplace transform with respect to ξ leads to the Mittag-Leffler function. Then the corresponding time-dependent equation, accounting for Poissonian resetting, yields

$$p_\alpha(x,t|x_0) = e^{-rt} G_\alpha(x,t|x_0) + r\int_0^t e^{-r\tau}\, G_\alpha(x,\tau|X_r)\, d\tau, \tag{6}$$

similar to Eq. (2), but with another propagator corresponding to the given case. For $\alpha = 1$ the PDF $g_\alpha(\xi,t)$ becomes the Dirac δ function (no subordination), and Eq. (6) is transformed into Eq. (2) for Brownian motion. This explains why the propagator index equal to 1 is used for Eq. (2). In the stationary state Eq. (6) is determined, as before, by the second term:

$$p_\alpha(x,\infty|x_0) = r\int_0^\infty e^{-r\tau}\, G_\alpha(x,\tau|X_r)\, d\tau. \tag{7}$$

Using Eqs. (4) and (5) and integrating over ξ, we find the subdiffusive Green's function as the inverse Laplace transform:

$$G_\alpha(x,t|X_r) = \frac{1}{2\pi i}\int_{\mathrm{Br}} e^{ut - \frac{|x-X_r|}{\sqrt{D}} u^{\alpha/2}} \frac{du}{2\sqrt{D}u^{1-\alpha/2}}. \tag{8}$$

In fact, Eq. (7) manifests the Laplace transform of the function $G_\alpha(x,t|X_r)$, whereas Eq. (8) presents the inverse Laplace transform giving the same function. Thus, the stationary solution of Eq. (6) can be found from the integrand of Eq. (8), calculating the following integral:

$$p_\alpha(x,\infty|x_0) = \frac{r}{2\sqrt{D}}\int_0^\infty \delta(s-r)\, e^{-\frac{|x-X_r|}{\sqrt{D}} s^{\alpha/2}} \frac{ds}{s^{1-\alpha/2}} = \frac{c_\alpha}{2} e^{-c_\alpha|x-X_r|}, \tag{9}$$

where $c_\alpha = \sqrt{r^\alpha/D}$ is the constant. In this case we again obtain the Laplace distribution as a stationary state after Poissonian resetting. The evolution of $p_\alpha(x,t|x_0)$ in time is shown in Fig. 2. Recall that in the logarithmic scale along the y axis the Laplace distribution has typically a triangular form.

III. INVERSE INFINITELY DIVISIBLE SUBORDINATOR

Next, we will move on to a more general case, when the subordinator is described by an inverse infinitely divisible distribution [31]. Such a distribution has the following Laplace transform:

$$\bar{g}_\Psi(\xi,u) = \frac{\bar{\Psi}(u)}{u} e^{-\xi\bar{\Psi}(u)}, \tag{10}$$

where $\bar{\Psi}(u)$ is the Laplace exponent expressed in terms of Bernstein functions [32]. Then the propagator is

$$G_\Psi(x,t|x_0) = \int_0^\infty G_1(x,\xi|x_0)\, g_\Psi(\xi,t)\, d\xi. \tag{11}$$

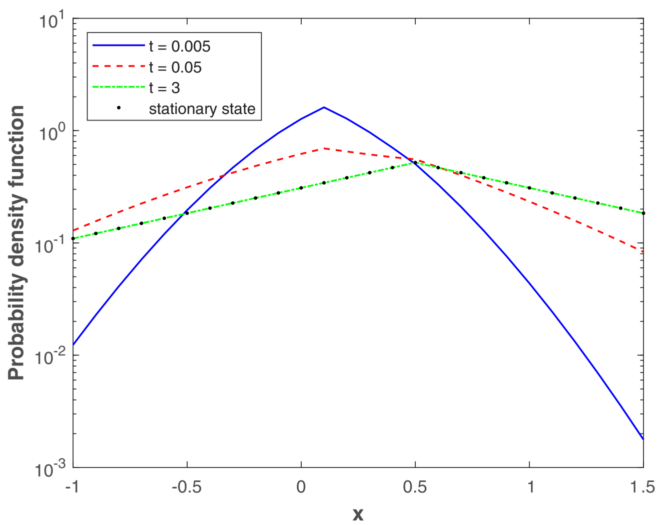

FIG. 2. Propagator $p_\alpha(x,t|x_0)$ of subdiffusion with resetting for $\alpha = 0.7$, $r = 3$, $D = 2$, $x_0 = 0.1$, and $X_r = 0.5$, drawn for several instances of time, starting with the Dirac δ function at x_0 and passing to the subdiffusive PDF, which for $t \to \infty$ becomes the Laplace distribution, shown by the black dotted line on the panel, the maximum of which is located at X_r.

Based on Eq. (10), it is not difficult to find the function $G_\Psi(x,t|x_0)$ as the inverse Laplace transform, namely,

$$G_\Psi(x,t|x_0) = \frac{1}{2\pi i}\int_{\mathrm{Br}} e^{ut-\frac{|x-x_0|}{\sqrt{D}}\sqrt{\bar\Psi(u)}}\,\frac{\sqrt{\bar\Psi(u)}}{2\sqrt{D}u}\,du. \quad (12)$$

The corresponding Green's function $p_\Psi(x,t|x_0)$ of this stochastic process with resetting is written as Eq. (2) expressed in terms of the propagator $G_\Psi(x,t|x_0)$. It is clear that the second term determines a stationary state of this equation, and this term is nothing but the Laplace transform for $t \to \infty$. Consequently, the PDF of the stationary state takes the form

$$p_\Psi(x,\infty|x_0) = \frac{c_\Psi}{2}e^{-c_\Psi|x-X_r|}, \quad (13)$$

where $c_\Psi = \sqrt{\bar\Psi(r)/D}$ is the constant. Consequently, the brief analysis of the anomalous diffusion under resetting leads to the Laplace distribution, if $t \to \infty$. It is interesting to note that the Laplace distribution also appears, as a stationary state, from the conjugate property of Bernstein functions, connecting the tempered stable subdiffusion with the confinement [33].

IV. NON-LAPLACE STATIONARY STATE

Is it possible to get away from the Laplace distribution as a stationary state in a stochastic process with Poissonian resetting? The answer to this question is affirmative. For this purpose the process must have an infinitely divisible distribution [34,35]. The characteristic exponent $\Xi(|k|)$ of the distribution is a Bernstein function. As for the β-stable Lévy motion, the characteristic exponent is equal to $|k|^\beta$ with $\beta \in (0, 2)$. This is only one of the families of non-Gaussian infinitely divisible distributions. There are others: (i) $(|k|^2 + m^{\beta/2})^{2/\beta} - m$, $\beta \in (0,2)$; (ii) $\log(1+|k|^\beta)$, $\beta \in (0,2]$; (iii) $b|k|^2 + |k|^\beta$; (iv) $\log((1+|k|^2)+\sqrt{(1+|k|^2)^2-1})$; and so on [36]. Some of them have special names: (i) relativistic, (ii) gamma. On this basis we can describe any non-Brownian motion with $p(x,0) = \delta(x)$ by the characteristic function in the form

$$\hat h(k,t) = e^{-tD^*\,\Xi(|k|)}, \quad (14)$$

where D^* is a generalized diffusive constant. Next, it is convenient to consider the characteristic function as the Fourier transformation of the Green's function in this resetting process, i.e.,

$$\hat p(k,t|x_0) = \int_{-\infty}^{\infty} p(x,t|x_0)\,e^{ikx}\,dx. \quad (15)$$

The resetting leads to the expected equation [Eq. (2)] with the propagator $G(x,t|x_0)$, which is easy to study in the Fourier space, passing from x to k. The stationary state of the characteristic function $\hat p(k,t|x_0)$ is described by the Laplace transform integral type,

$$\hat p(k,\infty|X_r) = r\int_0^\infty e^{-r\tau}\,\hat G(k,\tau|X_r)\,d\tau = \frac{e^{ikX_r}}{1+D^*\Xi(|k|)/r}. \quad (16)$$

Note that $1/[1+A\,\Xi(|k|)]$ with $A = D^*/r > 0$ is an even function, and thus its Fourier transform is equivalent to the cosine transform. The term e^{ikX_r} indicates that the PDF maximum is located at X_r. Considering the β-stable Lévy motion with the characteristic exponent $\Xi(|k|) = |k|^\beta$ under $\beta \in (0, 2)$, we obtain the Linnik distribution [37] as a stationary PDF. Since $\Xi(|k|)$ is a Bernstein function (or otherwise the function having a complete monotone derivative), the characteristic function $1/[1+A\,\Xi(|k|)]$ is typical for geometrical infinitely divisible PDFs [38]. In any case the PDF form is symmetric and unimodal. Depending on the type of function $\Xi(|k|)$ it has a finite or infinite maximum. This feature is explained by the integral $\int_0^\infty 1/[1+A\,\Xi(k)]\,dk$ having a single improper point, namely, $k \to \infty$, where the integral is either convergent or divergent. Nevertheless, $p(x,t|x_0)$ evolves from a bimodal form to unimodal. The case of the Linnik distribution as a stationary state is shown in Fig. 3.

V. MEAN TIME TO REACH A TARGET

The key question for stochastic processes with resetting concerns the mean time for the particle (or searcher) to reach a target. For Poissonian resetting a convenient way is based on the renewed equation for survival probabilities,

$$Q_r(x_0,t) = e^{-rt}Q_0(x_0,t) + r\int_0^t e^{-r\tau}\,Q_0(X_r,\tau)\,Q_r(x_0,t-\tau)\,d\tau, \quad (17)$$

in which $Q_r(x_0,t)$ is the shorthand notation of $Q_r(x_0,t|X_r)$, and similarly for Q_0, accepted for convenience [1]. The first term of Eq. (17) is related to trajectories without resetting, whereas the second term represents trajectories under resetting. After the Laplace transform and when the initial position

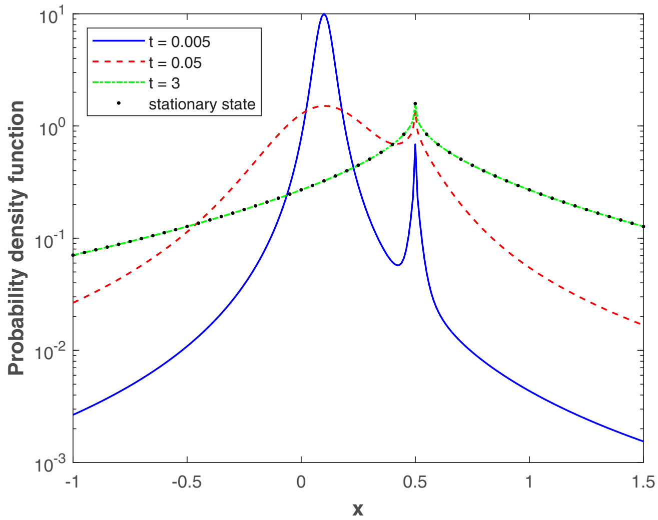

FIG. 3. Propagator $p(x, t|x_0)$ of Lévy motion with resetting for $\beta = 1.3$, $r = 3$, $D^* = 2$, $x_0 = 0.1$, and $X_r = 0.5$, drawn for several instances of time, starting with the Dirac δ function at x_0 and passing to the Lévy PDF, which for $t \to \infty$ becomes the Linnik distribution, shown by the black dotted line on the panel, the maximum of which is located at X_r.

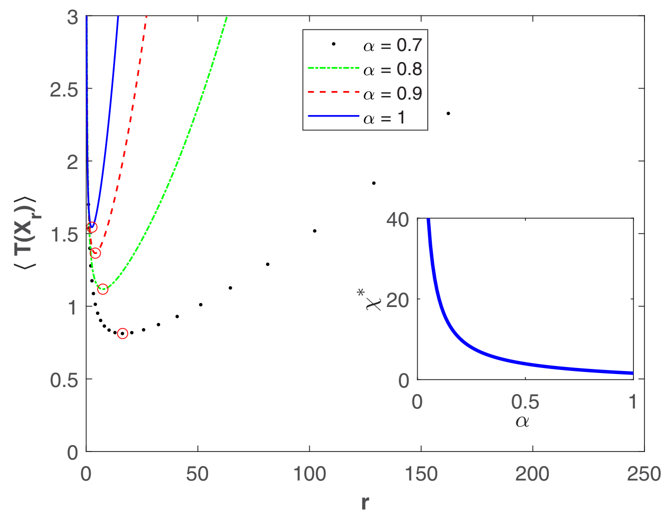

FIG. 4. Mean time of subdiffusion with Poissonian resetting as a function of r ($D = 1$ and $X_r = 1$), drawn for several values of α. The red circles indicate minimums of $\langle T(X_r)\rangle$ with different α. Inset shows evolution of the minimum of χ dependent on $0 < \alpha \leqslant 1$.

and resetting position coincide, we have

$$\bar{Q}_r(X_r, s) = \frac{\bar{Q}_0(X_r, r+s)}{1 - r\bar{Q}_0(X_r, r+s)}. \tag{18}$$

Then the mean time for the particle to reach a target is $\langle T(X_r)\rangle = \bar{Q}_r(X_r, 0)$, expressed in terms of the survival probability $Q_0(X_r, t)$ in the absence of resetting. The general equation for the propagator from x_0 to x, as an integral over the first time to reach x, reads

$$G_\Psi(x, t|x_0) = \int_0^t \phi_0(x, \tau|x_0)\, G_\Psi(x, t-\tau|x)\, d\tau, \tag{19}$$

where $\phi_0(x, t|x_0)$ is the probability density of reaching x for the first time at t. As $\phi_0(0, t|x_0) = -\partial Q_0(x_0, t)/\partial t$, the Laplace transform of $Q_0(x_0, t)$ can be presented by using the Laplace transform of $G_\Psi(x, t|x_0)$ from Eq. (12). In this case we obtain

$$\bar{Q}_0(x_0, s) = \frac{1 - e^{-(\bar{\Psi}(s)/D)^{1/2}x_0}}{s}. \tag{20}$$

This gives us the following expression:

$$\bar{Q}_r(X_r, s) = \frac{1 - \exp(-c_\Psi X_r)}{s + r\exp(-c_\Psi X_r)}, \tag{21}$$

where c_Ψ is the same as in Eq. (13). Thus, the mean time of one-dimensional subordinated diffusion with resetting is written as

$$\langle T(X_r)\rangle = \frac{1}{r}(e^{c_\Psi X_r} - 1). \tag{22}$$

As an example, we consider the ordinary subdiffusion having $\bar{\Psi}(s) = s^\alpha$ with $0 < \alpha \leqslant 1$. In this case it is convenient to introduce the dimensionless parameter $\chi = c_\alpha X_r$ which is the ratio of two length parameters: X_r is the distance from the resetting position to the target and $1/c_\alpha$ is the typical diffusion length between resets. To minimize Eq. (22), the equation reduces in terms of χ to the transcendental equation

$$\frac{\alpha\chi}{2} = 1 - e^{-\chi}, \tag{23}$$

which has a unique nonzero solution χ^* dependent on α. This result is represented in Fig. 4. Note that for $\alpha = 1$ Eq. (23) recovers the transcendental equation, characteristic for the ordinary diffusion with resetting [1,9], having the root equal to ≈1.5936. Thus the minimal mean time to locate the target is achieved when the ratio of the distance X_r to the target to the typical diffusion length between resets is χ^*. The value $\langle T(X_r)\rangle$ diverges as $r \to 0$ because of $\langle T(X_r)\rangle \sim r^{\alpha/2-1}$, which leads to the well-known result that the mean time for a subdiffusive particle to reach the origin (in the absence of resetting) is infinite. Also $\langle T(X_r)\rangle$ diverges for $r \to \infty$, which has a clear explanation: if the reset rate increases, then the subdiffusing particle has less time between resets to reach the origin. If $\alpha \to 0$, χ^* tends to infinity, which is caused by the particle entering the confinement. The presence of temporal traps in the subdiffusion provides an increase in the value χ^* with decreasing α.

In the most general case of Eq. (22) the transcendental equation for χ cannot be represented in the simple form like Eq. (23), but there is another useful relation,

$$1 - e^{-\zeta} = r\frac{d\zeta}{dr}, \quad \zeta = c_\Psi X_r = X_r\sqrt{\frac{\bar{\Psi}(r)}{D}}, \tag{24}$$

dependent on r and permitting to find the minimum position of $\langle T(X_r)\rangle$ along the axis r. Then we calculate ζ^* by using the second expression in Eqs. (24). It is not difficult to notice also that the function $1 - e^{-\zeta} \leqslant 1$ is bounded, whereas $r\,d\zeta/dr$ is a monotonically increasing function. Both functions start from scratch at $r = 0$. The first of them grows faster, but then slows down so that the second catches up and overtakes it. Thus, between two divergences of $\langle T(X_r)\rangle$ for $r \to 0$ and $r \to \infty$ there is a single minimum determining the optimal resetting.

VI. CONCLUSIONS

Finally, it should be emphasized that the Laplace distribution has a crucial role to play in Poissonian resetting of a stochastic process. The stationary state after such resetting obeys the Laplace PDF that was well known for ordinary Brownian motion. Nevertheless, this feature persists, if the resetting acts on subdiffusion which arises from Brownian motion subordinated by an inverse infinitely divisible random process. Each Laplace PDF is characterized by two parameters: X_r is a location parameter and $1/c$ is a scale parameter. The first of them is the same for any subdiffusive process under resetting, whereas the second depends on r and D. The contribution of r is determined by the Laplace exponent of the stochastic process subject to resetting, but D is not. By changing the rate r (reset protocol) at constant value D and finding the scaling parameter of the Laplace distribution, one can restore the Laplace exponent of its directing random process. The second effect of resetting in our case concerns the mean time to search or capture a target. We have shown that there is an optimal resetting rate at which the mean capture time is minimal that makes the search process efficient. The Lévy processes with resetting manifest also a stationary state, but its PDF is described by the Linnik distribution which belongs to the class of geometrical stable distributions which forms a subclass of generalized Laplace distributions. Geometric stable distributions are heavy tailed and stable with respect to geometric summation (see [25,38]). However, differently from the stable ones, their densities are more "peaked" and "blow up" at zero. Moreover, since geometric stable distributions approximate random sums, they naturally arise in a variety of applied problems and are particularly appropriate in modeling heavy-tailed phenomena, when the variable of interest may be thought of as the result of a random number of independent innovations (heterogeneous environment).

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