Anomalous diffusion driven by the redistribution of internal stresses

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This article explores the mathematical description of anomalous diffusion, driven not by thermal fluctuations but by internal stresses. A continuous time random walk framework is outlined in which the waiting times between displacements (jumps), generated by the dynamics of internal stresses, are described by the generalized Γ distribution. The associated generalized diffusion equation is then identified. The solution to this equation is obtained as an integral over an infinite series of Fox H functions. The probability density function is identified as initially non-Gaussian, while at longer timescales Gaussianity is recovered. Likewise, the second moment displays a transient nature, shifting between subdiffusive and diffusive character. The potential application of this mathematical description to the quaking observed in several soft-matter systems is discussed briefly.

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I. INTRODUCTION

Anomalous diffusion is a transport process observed in a large variety of systems, across many lengths and timescales. As specific examples, it has been observed in charge carrier transport in amorphous semiconductors [1,2], in flow in porous systems [3], in quantum optics [4,5], and in many other systems [6,7]. Mathematically, there are two main avenues for the treatment of anomalous diffusion: either considering the stochastic description afforded by the Langevin equation or studying the deterministic Fokker-Planck equation, where the latter encodes information regarding the evolution in time of the probability density function (PDF). Diffusive behavior is most commonly characterized by the temporal behavior of the second moment of the PDF,

$$\langle x^2(t)\rangle \propto t^{\alpha}. \tag{1}$$

The exponent α is used to distinguish between the characteristic regimes. The behavior is referred to as subdiffusive ($\alpha < 1$), Fickian or normal ($\alpha = 1$), and superdiffusive ($\alpha > 1$). In other cases the exponent may vary in time, known as *transient* diffusion [8]. All of these diffusive behaviors may be found in systems throughout the physical world [9–11]. One of the most natural ways of modeling these systems has been to incorporate fractional derivative operators [12]. These operators can incorporate the influence of nonlocal effects in both time and space as the current state of the system evolves. The application of fractional calculus to diffusion is known as the field of *fractional diffusion*.

The continuous time random walk (CTRW) is considered to be the stochastic framework underpinning fractional diffusion. The discrete time counterpart has also proven to be a valuable tool within anomalous diffusion studies [13,14]; however, in this article we consider the CTRW. By altering the functional forms of the waiting time and jump length distributions in this CTRW framework, new anomalous behaviors may be captured by the corresponding fractional diffusion equations (FDEs). Historically, the connection between these two concepts is that FDEs represent the description of an underlying CTRW in the so-called *fluid limit* of large time and length scales [15]. The connection between fractional diffusion relations and CTRWs has been explored a great deal over the past few decades, perhaps most notably in the review of Metzler and Klafter [8]. The field of fractional diffusion has continued to grow, with a wide range of new connections and applications being produced [16,17]. Fractional diffusion equations owe their success to their capacity to describe a range of non-Gaussian probability density functions and capture nonlocality in time and space as well as incorporate various external forces and boundary conditions [8].

CTRWs are classified in one regard by the relationship between the distribution functions of the spatial and temporal increments. When they depend (in some fashion) on one another, they are referred to as a *coupled* CTRW, while independence of these functions implies *decoupling*. Within the decoupled CTRW framework the waiting time distribution between successive *jump* events is vitally important in determining the Markovian (*memoryless*) or non-Markovian character observed in the corresponding generalized diffusion equation. The classical diffusion equation is connected to a Markovian CTRW model, corresponding to a waiting time distribution that decays exponentially [18]. Waiting time distributions that decay in some nonexponential fashion capture various *non-Markovian* features. Common examples of this assume the waiting time distribution behaves as an inverse power law on large timescales. Such behavior leads to the classic Riemann-Liouville diffusion equation in the hydrodynamic limit [19]. In the natural world variations of waiting time distributions are commonly associated with environmental factors such as trapping events, disorder, and memory [20]. This article considers the influence that a waiting time distribution strongly associated with the timing of

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stress-redistribution events, namely, the generalized Γ distribution, has on the diffusive properties of the CTRW and generalized diffusion equation [21–24]. This article will be structured as follows: Sec. II will cover some preliminary details of fractional derivatives and will outline the decoupled CTRW finishing with the well-known Montroll-Weiss [25] equation; Sec. III will identify the corresponding generalized diffusion equation for the stress-redistribution driven CTRW. Several basic properties of the solution to this equation will also be discussed. Concluding remarks and potential applications will be mentioned in Sec. IV.

II. PRELIMINARIES

A. Fractional calculus

While fractional calculus has existed as long as integer order calculus, the applications of fractional calculus were not immediately apparent. A common difficulty of fractional (or, more accurately, *arbitrary* [26]) order calculus is the lack of a simple geometric interpretation, which is present with its integer order counterpart. Indeed, many have attempted to capture the geometric essence of these operators [27], although a consensus on the matter is, for the most part, unavailable. Perhaps the most widely utilized forms of the fractional operators in the temporal sense are the Riemann-Liouville (RL) and Caputo operators. The RL fractional derivative follows from generalizing Cauchy's repeated integral formula, while the Caputo form follows from generalizing the Laplace transform expression for the derivative. The RL and Caputo derivatives may be defined as follows [27].

Definition 1. The fractional derivative of Riemann and Liouville, for arbitrary order $\alpha \in [0, \infty)$, applied to the function $f(t)$ is defined as

$$ {}^{RL}_{0}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n+1-\alpha)} \left(\frac{d}{dt}\right)^{n+1} \int_0^t (t-t')^{n-\alpha} f(t') dt' \tag{2} $$

for $n < \alpha < n+1$, where $n \in \mathbb{Z}^{0+}$.

Definition 2. The fractional derivative of Caputo, for arbitrary order $\alpha \in [0, \infty)$, applied to the function $f(t)$ is defined as

$$ {}^{C}_{0}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(t')}{(t-t')^{1+\alpha-n}} dt' \tag{3} $$

for $n-1 < \alpha < n$, where $n \in \mathbb{Z}^{0+}$.

There is an intimate connection between these two formalisms which can be made apparent through the following manipulation. If $\alpha = 1-\beta$ and $0 < \beta < 1$,

$$ {}^{RL}_{0}D^{1-\beta}_{t} f(t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^{1-\beta}} dt'. \tag{4} $$

Due to the convolution theorem, after Laplace transformation of Eq. (4), it appears as

$$ \mathcal{L}\left[\frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^{1-\beta}} dt'\right] = \frac{1}{\Gamma(\beta)} u^{1+\beta} F(u). \tag{5} $$

Applying the Laplace inversion yields

$$ \mathcal{L}^{-1}\left[\frac{1}{\Gamma(\beta)} u^{\beta} (uF(u))\right] = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f^{(1)}(t') + f(0)}{(t-t')^{1-\beta}} dt', \tag{6} $$

which is nothing more than the Caputo fractional derivative plus a power law decaying contribution from the initial conditions [28]. Inspection of the form of the Caputo fractional derivative in Eq. (6) reveals that it generates an infinite sum of weighted infinitesimal contributions of the process over $[0, t]$. This integral represents the *memory* of the system and is of great importance to non-Markovian processes.

B. Continuous time random walks

The continuous time random walk theory considers the motion of a *walker* as it progresses through time continuously. The framework was first described by Weiss and Montroll and has been utilized to describe an enormous variety of stochastic processes [25]. In a CTRW the walker, or object of interest, progresses by taking steps of size x, after which it waits a time t prior to taking the next step. Both variables x and t are distributed according to a probability density function $\Psi(x, t)$. If the sizes of the steps and waiting times are uncorrelated, the following expressions hold:

$$ \lambda(x) = \int_0^{\infty} \Psi(x, t) dt, \tag{7} $$

$$ \omega(t) = \int_{-\infty}^{\infty} \Psi(x, t) dx, \tag{8} $$

where $\lambda(x)$ and $\omega(t)$ are the step-length and waiting time probability density functions, respectively. Equally, in the decoupled framework $\Psi(x, t)$ factors into the independent distributions $\lambda(x)$ and $\omega(t)$. From these quantities one may construct an arrival probability density $\eta(x, t)$ describing the probability density of arriving at various positions x in time t, defined as

$$ \eta(x, t) = \int_{-\infty}^{\infty} \int_0^t \eta(x', t') \lambda(x-x') \omega(t-t') dt' dx' + \delta(x)\delta(t). \tag{9} $$

This expression contains two terms; the first describes the probability associated with a walker at x' at time t' having made a jump of length $x - x'$ in the remaining time $t - t'$, summed over all x and causally relevant t. The second term represents the initial conditions here that at time $t = 0$ the walker is localized at a location defined by $\delta(x)$.

The position PDF $P(x, t)$ is then defined as the probability density of arriving and remaining at a position x at time t, defined as

$$ P(x, t) = \int_0^t \eta(x, t') \Phi(t-t') dt', \tag{10} $$

where $\Phi(t)$ is referred to as the survival PDF, which provides the probability density for a waiting time longer than t, defined as

$$ \Phi(t) = 1 - \int_0^t \omega(t') dt'. \tag{11} $$

Thus, Eq. (10) represents the probability density associated with a walker remaining at position x for the at least the time $t - t'$. At this point it is easier to manipulate the series of equations by passing into the Fourier $[\mathcal{F} : \lambda(x) \to \hat{\lambda}(k)]$–Laplace $[\mathcal{L} : \omega(t) \to \hat{\omega}(u)]$ space, by virtue of the convolution theorems for these transforms [29]. Transforming Eqs. (9) and (11) and then substituting into the Fourier-Laplace equivalent of Eq. (10) gives the following form for the probability density [30]:

$$P(k,u) = \frac{1-\hat{\omega}(u)}{u}\frac{1}{1-\hat{\lambda}(k)\hat{\omega}(u)}. \tag{12}$$

III. RESULTS

A. Stress redistribution diffusion

Equation (12) represents a common starting point when constructing anomalous diffusion models. By altering the functional forms of the distributions, $\hat{\omega}(u)$ and $\hat{\lambda}(k)$, nonlocal features, of both time and space, can be incorporated in the corresponding generalized diffusion equation. A variety of models ranging in complexity have been described in the literature that attempt to capture statistical aspects of *stress redistribution* processes, akin to those of plate tectonics and more broadly in stick-slip systems [23,24]. A common feature of both simulated and real-world statistical data is that the waiting time distribution between successive events appears to be contained within the functional form of a *generalized* Γ *distribution* [31,32]. The generalized Γ distribution is defined as

$$\omega(t) = \frac{t^{\gamma-1}}{\tau^{\gamma}\Gamma(\gamma)}\exp\left(\frac{-t^{\alpha}}{\tau^{\alpha}}\right). \tag{13}$$

In this article the influence of waiting times of this form on diffusive processes is explored, with $\alpha = 1$ and $\gamma \in [0, 1]$. Thus, the events exhibit random behavior in the long time regime. The motivation for exploring this influence is the observation that a large number of soft-matter systems appear to restructure internally over time. The development of internal stresses and these restructuring events are expected to contribute to the dynamics of the system. The Laplace transform of Eq. (13) is

$$\hat{\omega}(u) = \frac{1}{\tau^{\gamma}\left(\frac{1}{\tau}+u\right)^{\gamma}}. \tag{14}$$

Substituting Eq. (14) into Eq. (12) under natural boundary conditions [$P(x,0) = \delta(x)$ and $P(\pm\infty, t) = 0$] yields

$$P(k,u) = \frac{1-\frac{1}{\tau^{\gamma}\left(\frac{1}{\tau}+u\right)^{\gamma}}}{u}\frac{1}{1-\hat{\lambda}(k)\frac{1}{\tau^{\gamma}\left(\frac{1}{\tau}+u\right)^{\gamma}}}. \tag{15}$$

1. Gaussian jump distribution

After inserting this new functional form of $\hat{\omega}(u)$, the effects of a Gaussian jump distribution $\lambda(x)$ are considered. This distribution may be represented in Fourier space by the following approximation for small k:

$$\hat{\lambda}(k) \sim 1 - \sigma^2 k^2, \tag{16}$$

where σ^2 represents the variance. With some rearranging of Eq. (15),

$$P(k,u) = \frac{1}{u}\frac{1}{1+\frac{\sigma^2 k^2}{\tau^{\gamma}\left[\left(\frac{1}{\tau}+u\right)^{\gamma}-\frac{1}{\tau^{\gamma}}\right]}}. \tag{17}$$

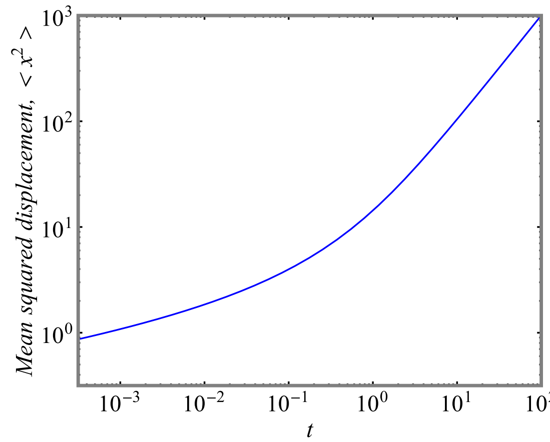

FIG. 1. Second moment behavior for $P(x,t)$, found in Eq. (22), with $\tau = \sigma = 1$ and $\gamma = 1/10$.

Defining the ratio σ^2/τ^{γ} to be the generalized diffusion coefficient D_{γ} (with units m^2/s^{γ}, where $\gamma \to 1$ recovers the standard diffusion coefficient),

$$P(k,u) = \frac{1}{u}\frac{1}{1+\frac{D_{\gamma}k^2}{\left[\left(\frac{1}{\tau}+u\right)^{\gamma}-\frac{1}{\tau^{\gamma}}\right]}}. \tag{18}$$

2. Second moment behavior

The transient nature of the process driving the temporal evolution of the system is apparent in the second moment of the PDF. The second moment $\mu_2(t)$ is related to the mean squared displacement (MSD) in the following way:

$$\mu_2(t) = \langle x^2\rangle(t) + \langle x\rangle^2(t). \tag{19}$$

Thus, the MSD and second moment are equivalent in the present work, where $\langle x\rangle(t) = 0$, by virtue of beginning with a bias free CTRW. The second moment of the PDF in Eq. (18) may be obtained from the Laplace inversion of the relation

$$\langle x^2(u)\rangle = -\frac{\partial^2}{\partial k^2}P(k,u)|_{k=0}. \tag{20}$$

After Laplace inversion of Eq. (20)

$$\langle x^2(t)\rangle = D_{\gamma}\int_0^t \exp\left(-\frac{t'}{\tau}\right)t'^{\gamma-1}E_{\gamma,\gamma}\left(\frac{t'^{\gamma}}{\tau^{\gamma}}\right)dt', \tag{21}$$

where $E_{a,b}(t)$ is the generalized Mittag-Leffler function [33]. Figure 1 displays the behavior of this expression. The processes evolves subdiffusively $\langle x^2(t)\rangle \propto t^{\gamma}$ on short timescales ($t \ll \tau$) prior to transitioning to Fickian behavior, $\langle x^2(t)\rangle \propto t$, as the random nature of long timescale events begins to manifest ($t \gg \tau$).

B. Diffusion equation

Applying the inverse Laplace and Fourier transforms to Eq. (18), the following generalized diffusion equation is

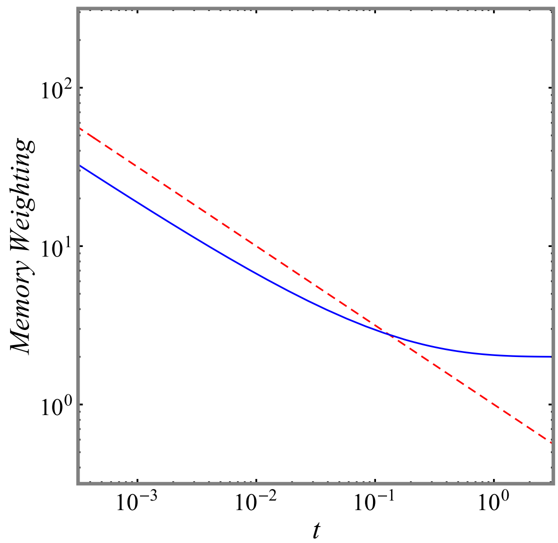

FIG. 2. Comparison of the Riemann-Liouville memory kernel (red dashed line) and memory kernel of Eq. (22) (blue solid line). Here $\tau = \sigma = 1$, and $\gamma = \frac{1}{2}$.

obtained:

$$\frac{\partial}{\partial t}P(x,t) = D_\gamma \frac{\partial}{\partial t}\int_0^t \exp\left(-\frac{(t-t')}{\tau}\right)(t-t')^{\gamma-1} \times E_{\gamma,\gamma}\left(\frac{(t-t')^\gamma}{\tau^\gamma}\right)\frac{\partial^2}{\partial x^2}P(x,t')dt'. \tag{22}$$

Generalized diffusion equations were investigated recently by Tateishi *et al.* [34] and were shown to describe a range of different diffusive behaviors. Within this work Tateishi *et al.* also discussed the concept of a *memory kernel* $\mathcal{M}(t)$, which represents the function being convolved with the standard diffusive component $\frac{\partial^2}{\partial x^2}P(x,t)$. The form of the memory kernel relevant to this work appears as

$$\mathcal{M}(t) = \exp(-t/\tau)t^{\gamma-1}E_{\gamma,\gamma}(t^\gamma/\tau^\gamma). \tag{23}$$

The memory kernel describes the influence past diffusive behavior has on the present dynamics. Figure 2 highlights the comparison of this memory kernel with that found in the Riemann-Liouville fraction diffusion equation. For small values of t the functions behave identically; thus, they account for the *recent memory* in the same way. However, as the value of t grows, the two kernels deviate in behavior, with the kernel of interest to this research decaying to a *constant* value. This decay reflects the transition back to a Markovian process on these timescales and the return to Fickian dynamics.

C. Probability density function

Obtaining the PDF of interest in x-t space requires the Fourier-Laplace inversion of the following equation:

$$P(k,u) = \frac{1}{u}\frac{1}{1+\frac{\sigma^2k^2}{\tau^\gamma\left[\left(\frac{1}{\tau}+u\right)^\gamma-\frac{1}{\tau^\gamma}\right]}}, \tag{24}$$

which has been developed through a CTRW framework, with a Gaussian jump distribution and Γ waiting time distribution. The derivation of this equation also employed the small k limit. This equation has been discussed once before [35]. Here the application to physical systems, the Fox H function solution, the short and long timescale asymptotic behaviors, and the means by which the Gaussian solution may be recovered upon $\gamma \to 1$ are described. It is worth noting that by employing Monte Carlo methods, as discussed in [36,37], the PDF may be numerically identified. However, this article considers an analytic approach. First, the Fourier inverse transform of the expression is performed, readily identified by a common Fourier inversion result [38],

$$P(x,u) = \mathcal{F}^{-1}[P(k,u)] = \mathcal{F}^{-1}\left[\frac{1}{u}\frac{1}{1+\frac{\sigma^2k^2}{\tau^\gamma\left(\left(\frac{1}{\tau}+u\right)^\gamma-\frac{1}{\tau^\gamma}\right)}}\right]. \tag{25}$$

Evaluating the Fourier transform gives us the following expression in $x - u$ space:

$$P(x,u) = \frac{\tau^{\frac{\gamma}{2}}}{2u\sigma}\sqrt{\left(\frac{1}{\tau}+u\right)^\gamma - \frac{1}{\tau^\gamma}}\exp\left[-\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma}\sqrt{\left(\frac{1}{\tau}+u\right)^\gamma-\frac{1}{\tau^\gamma}}\right]. \tag{26}$$

Taking the Laplace transform yields

$$P(x,t) = \mathcal{L}^{-1}\left[\frac{\tau^{\frac{\gamma}{2}}}{2u\sigma}\sum_{n=0}^{\infty}\left(-\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma}\right)^n\left[\left(\frac{1}{\tau}+u\right)^\gamma\right]^{\frac{1}{2}+\frac{n}{2}}\left(1-\frac{1}{\tau^\gamma\left(\frac{1}{\tau}+u\right)^\gamma}\right)^{\frac{1}{2}+\frac{n}{2}}\frac{1}{\Gamma(n+1)}\right](t). \tag{27}$$

Taking out the $1/u$ factor as the t space definite integral over $[0, t]$,

$$P(x,t) = \int_0^t \mathcal{L}^{-1}\left\{\frac{\tau^{\frac{\gamma}{2}}}{2\sigma}\sum_{n=0}^{\infty}\left(-\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma}\right)^n\left[\left(\frac{1}{\tau}+u\right)^\gamma\right]^{\frac{1}{2}+\frac{n}{2}}\left(1-\frac{1}{\tau^\gamma\left(\frac{1}{\tau}+u\right)^\gamma}\right)^{\frac{1}{2}+\frac{n}{2}}\frac{1}{\Gamma(n+1)}\right\}(t')dt'. \tag{28}$$

Invoking the *shift theorem* of the Laplace transform and utilizing the binomial theorem leaves us with the expression

$$P(x,t) = \int_0^t \exp\left(-\frac{t'}{\tau}\right)\mathcal{L}^{-1}\left[\frac{\tau^{\frac{\gamma}{2}}}{2\sigma}\sum_{n=0}^{\infty}\left(-\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma}\right)^n(u^\gamma)^{\frac{1}{2}+\frac{n}{2}}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{3}{2}+\frac{n}{2}\right)}{\Gamma(m+1)\Gamma\left(\frac{3}{2}+\frac{n}{2}-m\right)}\left(-\frac{1}{\tau^\gamma u^\gamma}\right)^m\frac{1}{\Gamma(n+1)}\right](t')dt'. \tag{29}$$

This expression can be cast as a Fox H function [39] by virtue of its series expansion [40]. Recognizing the H function form allows for more straightforward manipulations with regard to the inverse transforms and reduction to known results. For these reasons and others the H function sees regular employment within the realm of fractional and generalized diffusion equations [41]:

$$P(x,t)=\int_0^t \exp\left(-\frac{t'}{\tau}\right)\mathcal{L}^{-1}\left\{\frac{\tau^{\frac{\gamma}{2}}}{2\sigma}\sum_{m=0}^{\infty}\frac{1}{\Gamma(m+1)}\left(-\frac{1}{\tau^{\gamma}u^{\gamma}}\right)^m u^{\frac{\gamma}{2}}H_{1,2}^{1,1}\left[\frac{|x|\tau^{\frac{\gamma}{2}}u^{\frac{\gamma}{2}}}{\sigma}\middle|_{(0,1)(m-\frac{1}{2},\frac{1}{2})}^{(-\frac{1}{2},\frac{1}{2})}\right]\right\}(t')dt'. \tag{30}$$

The Laplace inversion of the Fox H function, followed by the use of identity 3.5 described by Skibiński in 1970 [42], is now employed. Due to the flexibility of the coefficients a_i and α_i (by virtue of the value of p in the H function) it can be cast in the following form:

$$P(x,t)=\frac{1}{2}\int_0^t \exp\left(-\frac{t'}{\tau}\right)\sum_{m=0}^{\infty}\frac{1}{\Gamma(m+1)}\left(\frac{t'^{\gamma}}{\tau^{\gamma}}\right)^m\frac{1}{|x|}\frac{1}{t'}H_{2,2}^{2,0}\left[\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\middle|_{(m,\frac{1}{2}),(1,1)}^{(0,\frac{1}{2}),(\gamma m,\frac{\gamma}{2})}\right]dt'. \tag{31}$$

The behavior of this solution across a range of timescales and γ values is portrayed in Fig. 3. This solution describes a system exhibiting short timescale subdiffusive behavior which transitions on longer timescales to normal diffusion and thus may capture many real-world systems with this phenomenology.

D. Recovery of the Gaussian PDF

We can test the behavior of this result by first establishing the return to the Gaussian PDF upon setting $\gamma \to 1$ in Eq. (31). This change immediately allows for the reduction of the H function by way of identity 3.2 of Skibiński [42]. After the reduction, there is no longer any occurrence of the index m within the H function structure; thus, all that remains is the series form of the exponential function. These steps yield the following:

$$P(x,t)=\int_0^t \exp\left(-\frac{t'}{\tau}\right)\exp\left(\frac{t'}{\tau}\right)\frac{1}{2|x|}\frac{1}{t'}H_{1,1}^{1,0}\left[\frac{|x|\tau^{\frac{1}{2}}}{\sigma t'^{\frac{1}{2}}}\middle|_{(1,1)}^{(0,\frac{1}{2})}\right]dt', \tag{32}$$

where the exponential functions cancel one another, leaving

$$P(x,t)=\int_0^t \frac{1}{2|x|}\frac{1}{t'}H_{1,1}^{1,0}\left[\frac{|x|\tau^{\frac{1}{2}}}{\sigma t'^{\frac{1}{2}}}\middle|_{(1,1)}^{(0,\frac{1}{2})}\right]dt'. \tag{33}$$

We now utilize identity 3.4 of Skibiński and use it in conjunction with Legendre's duplication formula for the Γ function. These steps produce the following result:

$$P(x,t)=\frac{1}{\sqrt{4\pi}}\int_0^t \frac{1}{|x|}\frac{1}{t'}H_{1,2}^{2,0}\left[\frac{x^2\tau}{4\sigma^2 t'}\middle|_{(\frac{1}{2},1),(1,1)}^{(0,1)}\right]dt'. \tag{34}$$

Next, identity 3.6 of Skibiński is invoked, in combination with taking the partial derivative with respect to t of both sides of the equation, thus producing

$$\frac{\partial}{\partial t}P(x,t)=\frac{1}{\sqrt{4\pi}}\frac{-1}{|x|}\frac{1}{t}H_{1,2}^{1,1}\left[\frac{x^2\tau}{4\sigma^2 t}\middle|_{(1,1),(\frac{1}{2},1)}^{(0,1)}\right]. \tag{35}$$

By virtue of the known differential results for the H function covered by Skibiński as well as later texts on the subject, the following result is identified:

$$\frac{\partial}{\partial t}P(x,t)=\frac{1}{\sqrt{4\pi}}\sqrt{\frac{\tau}{\sigma^2}}\frac{\partial}{\partial t}\frac{1}{\sqrt{t}}H_{0,1}^{1,0}\left[\frac{x^2\tau}{4\sigma^2 t}\middle|_{(0,1)}\right]. \tag{36}$$

Taking the antiderivative of both sides and utilizing the initial conditions discussed earlier, namely, the initial Dirac δ form over x, provide

$$P(x,t)=\frac{1}{\sqrt{4\pi}}\sqrt{\frac{\tau}{\sigma^2}}\frac{1}{\sqrt{t}}H_{0,1}^{1,0}\left[\frac{x^2\tau}{\sigma^2 t}\middle|_{(0,1)}\right], \tag{37}$$

which is the H function form of the Gaussian PDF, as expected.

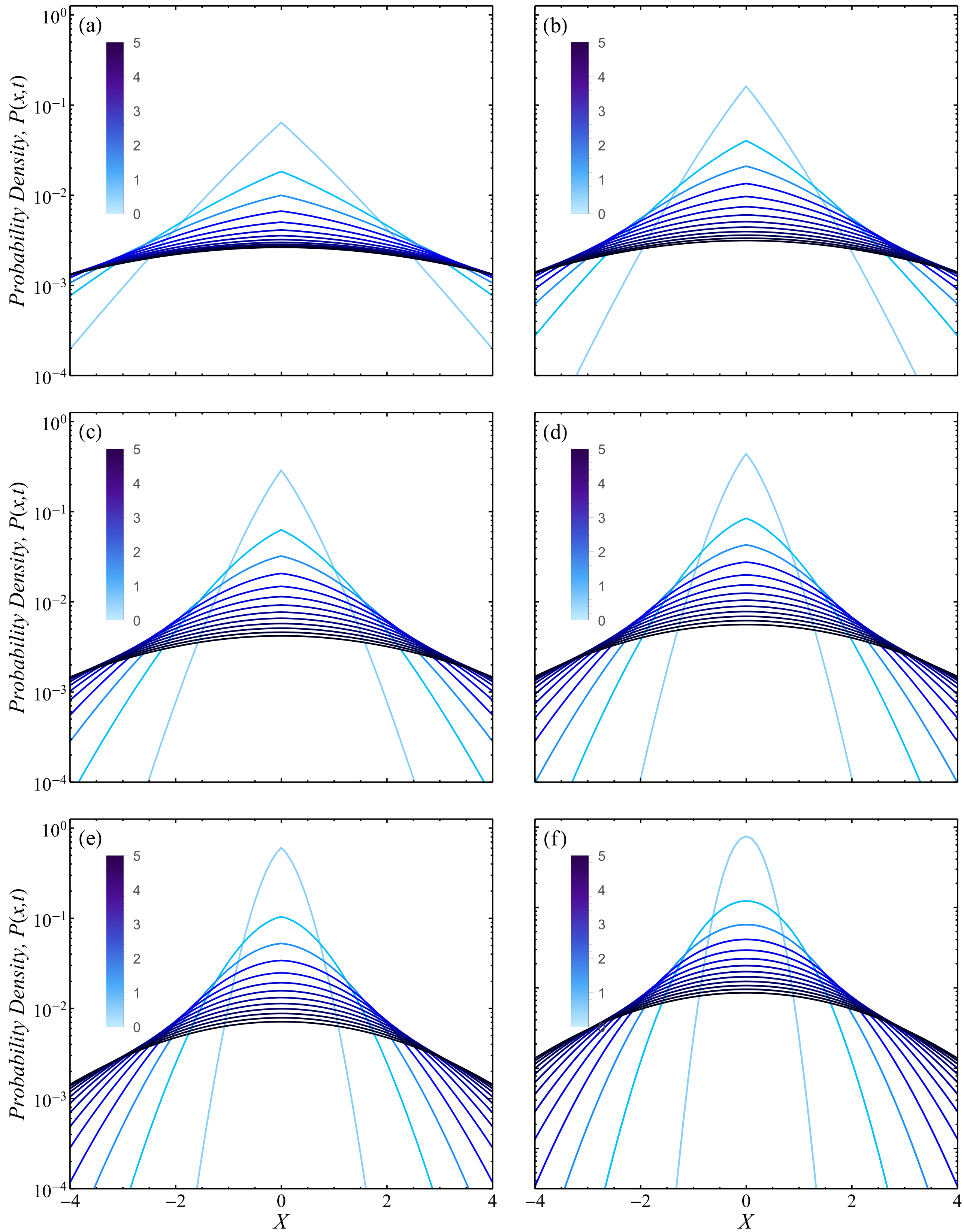

FIG. 3. PDFs corresponding to increasing values of γ in Eq. (31), where X is dimensionless, $X = x/\sigma$. The value of γ ranges from $\gamma = 1/6$ in (a) to $\gamma = 1$ in (f), in increments of 1/6. The color scale correspond to units of time, t/τ, within the open range (0,5) [light blue (light gray) to black]. These plots use values of τ and σ set to be $\tau = \sigma = 1$.

E. Long timescale asymptotics

The H function, present in Eq. (31), in its series form as defined in the article by Sandev *et al.* [40] is

$$H_{2,2}^{2,0}\left[\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t^{\frac{\gamma}{2}}}\middle|\begin{matrix}(0,\frac{1}{2}),(\gamma m,\frac{\gamma}{2})\\(m,\frac{1}{2}),(1,1)\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{\Gamma(1-2m-2n)}{\Gamma(\gamma m-\gamma(m+n))}\frac{(-1)^n}{\Gamma(-m-n)\Gamma(n+1)\Gamma\left(\frac{3}{2}\right)}\left(\frac{x^2\tau^{\gamma}}{\sigma^2t^{\gamma}}\right)^{m+n}+\sum_{n=0}^{\infty}\frac{\Gamma\left(m-\frac{1}{2}(1+n)\right)}{\Gamma(\gamma m-\gamma(1+n))\Gamma\left(-\frac{1}{2}-\frac{n}{2}\right)}\frac{(-1)^n}{\Gamma(n+1)}\left(\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t^{\frac{\gamma}{2}}}\right)^{1+n};\tag{38}$$

because $\frac{1}{\Gamma(-m-n)}=0$ for $m,n\in\mathbb{Z}^{0+}$, only the second term contributes. Inserting this result back into Eq. (31) yields

$$P(x,t)=\frac{1}{2}\int_0^t\exp\left(-\frac{t'}{\tau}\right)\sum_{m=0}^{\infty}\frac{1}{\Gamma(m+1)}\left(\frac{t'^{\gamma}}{\tau^{\gamma}}\right)^m\frac{1}{|x|}\frac{1}{t'}\sum_{n=0}^{\infty}\frac{\Gamma\left(m-\frac{1}{2}(1+n)\right)}{\Gamma[\gamma m-\gamma(1+n)]\Gamma\left(-\frac{1}{2}-\frac{n}{2}\right)\Gamma(n+1)}(-1)^n\left(\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\right)^{(1+n)}dt'.\tag{39}$$

Expressing the series over m as a Fox H function,

$$\begin{aligned}P(x,t)&=\frac{1}{2}\int_0^t\exp\left(-\frac{t'}{\tau}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(-\frac{1}{2}-\frac{n}{2})\Gamma(n+1)}\left(\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\right)^{1+n}\frac{1}{|x|}\frac{1}{t'}H_{1,2}^{1,1}\left[-\frac{t^{\gamma}}{\tau^{\gamma}}\middle|\begin{matrix}(\frac{3}{2}+\frac{n}{2},1)\\(0,1),(1+\frac{\gamma}{2}+\frac{\gamma n}{2},\gamma)\end{matrix}\right]dt'\\&=\frac{1}{2}\int_0^t\exp\left(-\frac{t'}{\tau}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(-\frac{1}{2}-\frac{n}{2}\right)\Gamma(n+1)}\left(\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\right)^{1+n}\frac{1}{|x|}\frac{1}{t'}\,{}_1\Psi_1\left[\frac{t^{\gamma}}{\tau^{\gamma}}\middle|\begin{matrix}(-\frac{1}{2}-\frac{n}{2},1)\\(-\frac{\gamma}{2}-\gamma\frac{n}{2},\gamma)\end{matrix}\right]dt',\end{aligned}\tag{40}$$

where ${}_p\Psi_q$ is the Wright function [43]. The asymptotic behavior of this function is described in the article by Wright published in 1940 [44]. For the Wright function of interest, these parameters take on the following values, which are defined in the work of Wright [44] as well as briefly in the Appendix:

$$\begin{aligned}&\kappa=\gamma,\quad h=\gamma^{-\gamma},\quad \theta=-\frac{1}{2}-\frac{n}{2}+\frac{\gamma}{2}+\frac{\gamma n}{2},\quad Z=\gamma\left(\gamma^{-\gamma}\frac{t^{\gamma}}{\tau^{\gamma}}\right)^{\frac{1}{\gamma}}=\frac{t}{\tau},\quad A_0=\gamma^{\frac{n}{2}-\frac{\gamma}{2}-\frac{\gamma n}{2}}\gamma^{\frac{1}{2}+\frac{\gamma}{2}+\frac{\gamma n}{2}}=\gamma^{\frac{1}{2}+\frac{n}{2}},\\&I\left(\frac{t}{\tau}\right)=\left(\frac{t}{\tau}\right)^{-\frac{1}{2}-\frac{n}{2}+\frac{\gamma}{2}+\frac{\gamma n}{2}}\exp\left(\frac{t}{\tau}\right)\left[\sum_{m=0}^{M-1}A_m\left(\frac{t}{\tau}^{-m}\right)+O\left(\frac{t}{\tau}^{-M}\right)\right].\end{aligned}\tag{41}$$

In the article by Wright, M is free to be chosen as long as $M\in\mathbb{Z}$. Therefore, $M=1$ is chosen, but the terms $O(\frac{t}{\tau}^{-1})$ are neglected, given the fact that $t\to\infty$. Thus,

$$I\left(\frac{t}{\tau}\right)=\left(\frac{t}{\tau}\right)^{-\frac{1}{2}-\frac{n}{2}+\frac{\gamma}{2}+\frac{\gamma n}{2}}\exp\left(\frac{t}{\tau}\right)\left(\gamma^{\frac{1}{2}+\frac{n}{2}}\right).\tag{42}$$

Thus, $P(x,t)$ appears in this regime as

$$\begin{aligned}P(x,t)&=\frac{1}{2}\int_0^t\exp\left(-\frac{t'}{\tau}\right)\frac{1}{|x|}\frac{1}{t'}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(-\frac{1}{2}-\frac{n}{2})}\frac{1}{\Gamma(n+1)}\left(\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\right)^{1+n}I\left(\frac{t'}{\tau}\right)dt'\\&=\frac{1}{2}\int_0^t\exp\left(-\frac{t'}{\tau}\right)\frac{1}{|x|}\frac{1}{t'}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(-\frac{1}{2}-\frac{n}{2}\right)\Gamma(n+1)}\left(\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\right)^{1+n}\frac{t'}{\tau}^{-\frac{1}{2}-\frac{n}{2}+\frac{\gamma}{2}+\frac{\gamma n}{2}}\exp\left(\frac{t'}{\tau}\right)\left(\gamma^{\frac{1}{2}+\frac{n}{2}}\right)dt'\\&=\frac{1}{2}\int_0^t\frac{1}{|x|}\frac{1}{t'}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(-\frac{1}{2}-\frac{n}{2}\right)}\frac{1}{\Gamma[n+1]}\left(\frac{\gamma^{\frac{1}{2}}|x|\tau^{\frac{1}{2}}}{\sigma t'^{\frac{1}{2}}}\right)^{1+n}dt',\end{aligned}\tag{43}$$

where, after expressing the series in its Fox H function form,

$$P(x,t)=\frac{1}{2}\int_0^t\frac{1}{|x|}\frac{1}{t'}H_{1,1}^{1,0}\left[\frac{\gamma^{\frac{1}{2}}|x|\tau^{\frac{1}{2}}}{\sigma t'^{\frac{1}{2}}}\middle|\begin{matrix}(0,\frac{1}{2})\\(1,1)\end{matrix}\right]dt',\tag{44}$$

which arrives at a Gaussian PDF in the same manner as demonstrated in Eq. (33). Interestingly, the remnants of the prior non-Gaussian behavior are visible in the occurrence of γ in the standard deviation of the resulting Gaussian.

F. Short timescale behavior

It can be shown that, in the regime where $\exp(-\frac{t}{\tau}) \approx 1$ (or $t \ll \tau$), taking the $m = 0$ term of the summation ($m = 0$ is the dominant term in the series for small t) yields the solution to the RL fractional diffusion equation. Under these conditions Eq. (31) is approximated by

$$P(x,t) = \frac{1}{2}\int_0^t \frac{1}{|x|}\frac{1}{t'} H_{1,1}^{1,0}\left[\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t'^{\frac{\gamma}{2}}}\middle|_{(1,1)}^{(0,\frac{\gamma}{2})}\right]dt'. \tag{45}$$

Making use of the properties of the H function [42], the following form of Eq. (45) can be found:

$$\frac{\partial}{\partial t}P(x,t) = \frac{1}{2}\frac{1}{|x|}\frac{1}{t}(-1)H_{2,2}^{1,1}\left[\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t^{\frac{\gamma}{2}}}\middle|_{(1,1),(1,\frac{\gamma}{2})}^{(0,\frac{\gamma}{2}),(1,\frac{\gamma}{2})}\right]dt'. \tag{46}$$

The right-hand side can be identified as the partial derivative of the RL solution [42] with respect to time. Given the initial condition for $P(x,t)$, the RL solution appears after integration of both sides:

$$P(x,t) = \frac{1}{2}\frac{1}{|x|}H_{1,1}^{1,0}\left[\frac{|x|\tau^{\frac{\gamma}{2}}}{\sigma t^{\frac{\gamma}{2}}}\middle|_{(1,1)}^{(1,\frac{\gamma}{2})}\right]dt'. \tag{47}$$

IV. DISCUSSION AND CONCLUSION

This article began by framing the basics of fractional diffusion and the CTRW, where the former is connected to the latter in the so-called *hydrodynamic* limit [45]. A functional form of the waiting time distribution, widely associated with the timing of stress driven displacements, is inserted into the CTRW. The corresponding generalized diffusion equation was then obtained by making use of the small k approximation. The *memory kernel* of this equation was plotted in Fig. 2, where it was compared to the power law decay of the Riemann-Liouville kernel. The memory kernel represents the functional form of the *weighting* of past diffusive contributions. The memory kernel described within this article exhibits power law decay which tails off to a constant value: this represents the transition from subdiffusive behavior towards standard diffusion. The MSD was obtained as well, plotted in Fig. 1; the behavior is transient in nature, moving between subdiffusive and diffusive regimes. The early subdiffusive behavior can be attributed to the power law decay observed on short timescales for the underlying waiting time distribution. This power law decay is indicative of a correlation of events on shorter timescales. The recovery of Fickian behavior is a direct consequence of the exponential decay on longer timescales for the waiting time distribution. The solution to the generalized diffusion equation was obtained as an integral over an infinite series of Fox H functions. The solution was arrived at through a combination of Fourier and Laplace transforms used in conjunction with the known properties of the H function. It was further shown that the solution reduced to known and expected results across the relevant regimes of short and long timescales, as well as in the instance that the anomalous parameter γ took on the Fickian value of 1.

This article explored the implications of *stress-redistribution* type timing for diffusion behavior. Restructuring events, which may be described in this manner, have been observed among a range of soft materials such as dense colloids, foams, gels, and granular fluids [46–50]. It is therefore suggested that the diffusive contribution outlined in this article may be found across these systems.

A more specific example may be seen in studies on pectin gel systems, where, through more recent experimental techniques, the diffusive contribution of restructuring events has been identified [46]. By constructing a coupling of the Brownian contribution and the stress redistribution portion discussed here, it is suggested the full transient diffusive behavior of such a pectin gel system may be modeled. In the applications of this mathematical treatment to physical systems it is expected that the origins of the parameters may be connected back to physical parameters of the gel, thus providing insight into the potential avenues for tuning the system towards desirable features for the various applications.

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APPENDIX: WRIGHT FUNCTION ASYMPTOTIC BEHAVIOR

If $\kappa > 0$ and $|\arg(z)| \leqslant \frac{1}{2}\pi\min(\kappa, 2) - \epsilon$, then for $z \to \infty$ [44]

$${}_p\Psi_q\left[z\middle|_{(\mu_r,\rho_r)}^{(\beta_r,\alpha_r)}\right] = I(Z), \tag{A1}$$

where

$$I(Z) = Z^\theta \exp(Z)\left(\sum_{m=0}^{M-1} A_m Z^{-m} + O(Z^{-M})\right) \tag{A2}$$

and

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)}\kappa^{-\frac{1}{2}-\theta}\prod_{r=1}^{p}\rho_r^{\frac{1}{2}-\mu_r}\prod_{r=1}^{q}\alpha_r^{\beta_r-\frac{1}{2}}, \tag{A3}$$

with

$$Z = \kappa(h|z|)^{\frac{1}{\kappa}}\exp\left(\frac{i\arg(z)}{\kappa}\right), \tag{A4}$$

$$\kappa = 1 + \sum_{r=1}^{q}\rho_r - \sum_{r=1}^{p}\alpha_r, \tag{A5}$$

$$h = \left(\prod_{r=1}^{p}\alpha_r^{\alpha_r}\right)\left(\prod_{r=0}^{q}\rho_r^{-\rho_r}\right), \tag{A6}$$

and

$$\theta = \sum_{r=1}^{p}\beta_r - \sum_{r=1}^{q}\mu_r + \frac{1}{2}(q-p). \tag{A7}$$

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Correction: Equation (11) contained an error and has been fixed.