Letter

Bounds on fluctuations for finite-time quantum Otto cycle

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For quantum Otto engine driven quasistatically, we provide exact full statistics of heat and work for a class of working fluids that follow a scale-invariant energy eigenspectra under driving. Equipped with the full statistics we go on to derive a universal expression for the ratio of nth cumulant of output work and input heat in terms of the mean Otto efficiency. Furthermore, for nonadiabatic driving of quantum Otto engine with working fluid consisting of either a (i) qubit or (ii) a harmonic oscillator, we show that the relative fluctuation of output work is always greater than the corresponding relative fluctuation of input heat absorbed from the hot bath. As a result, the ratio between the work fluctuation and the input heat fluctuation receives a lower bound in terms of the square value of the average efficiency of the engine. The saturation of the lower bound is received in the quasistatic limit of the engine.

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I. INTRODUCTION

The quest to build the most efficient and powerful heat engine led Carnot [1] to pioneer the subject what is known today as *Thermodynamics* [2,3]. Although initial development of the subject were motivated by engineering optimization problem, thermodynamics remains as one of the fundamental physical theories in science. In fact its core principles have survived both relativity and quantum revolution. One of the central result of thermodynamics is that efficiency of any engine operating between hot and cold reservoirs with temperatures T_h and T_c, respectively, is upper bounded by Carnot efficiency, $\eta_C = 1 - T_c/T_h$. Traditionally thermodynamics was only concerned with average quantities as fluctuation can be ignored for large systems, for example, steam engines, automobile engines, etc. But with the rapid technological development of miniturization of devices and advancement in accessing very low temperatures one can no longer ignore fluctuations of thermal and/or quantum origin [4–11]. Over the last three decades due to the discovery of fluctuation theorems [12–17] we have taken a big leap in understanding fluctuations of very large class of systems driven out-of-equilibrium.

Very recently, for out-of-equilibrium systems, *thermodynamic uncertainty relations* (TURs) [18–27] provided lower bound on the relative fluctuations of integrated currents (heat, particle, energy, etc.) in terms of the net entropy production. In other words, TUR restricts optimization of relative fluctuations and entropy production in an arbitrary manner by providing a trade-off relation between these quantities. As a consequence of this result, a continuous heat engine operating in a nonequilibrium steady state follows a trade-off relation involving its efficiency, output power, and power fluctuations [19]. For a similar setup operating as an engine, it was recently shown by some of us that, in the linear response regime, relative fluctuation of work current is always lower bounded by the input heat current [28]. In this Letter, we consider a finite-time quantum Otto engine setup and show that a bound similar to Ref. [28] exists. In particular, we show, for two prototypical systems driven arbitrarily, that, the ratio of work fluctuation and input heat fluctuation from the hot bath receives a lower bound which is determined by the square of the average efficiency of the engine. The equality of the bound is received in the quasistatic (QS) limit and can be shown for a class of working fluids following a scale invariant energy eigenspectra.

The plan of this Letter is as follows: We first introduce the quantum Otto cycle along with the projective measurement scheme that allows us to construct the probability distribution function to study fluctuations. We follow a similar scheme, as proposed in Ref. [29], to construct the joint probability distribution of heat and work. However, in our work, we will be primarily focusing on the higher order statistics/cumulants of heat and work instead of the stochastic efficiency which was the main focus of Ref. [29]. Next in the QS limit, we derive a general joint cumulant generating function of heat and work for scale-invariant driven Hamiltonian and show that the ratio of nth cumulant of output work and nth cumulant of input heat is exactly equal to the nth power of the average efficiency and consequently upper bounded by the nth power of the Carnot efficiency. Next we provide two paradigmatic examples [a two-level system (TLS) and a harmonic oscillator (HO)] of nonadiabatic driving of the quantum Otto cycle and show that relative fluctuations of work are always lower bounded by relative fluctuations of heat whenever the Otto cycle works as engine. Finally we summarize our central results.

II. UNIVERSAL QUANTUM OTTO CYCLE

We consider a standard four-stroke quantum Otto cycle [29–41], as illustrated in Fig. 1. The working fluid is initially ($t = 0$) thermalized by placing it in a weak contact with a cold reservoir at inverse temperature $\beta_c = 1/T_c$ (k_B is set to unity). The fluid is then separated from the bath and is sub-

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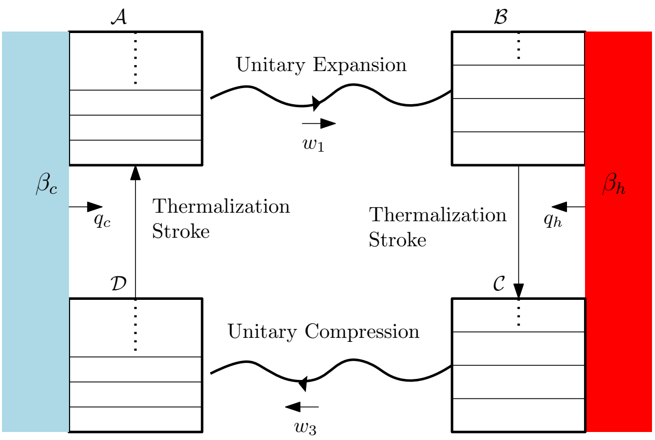

FIG. 1. Schematic of a four-stroke quantum Otto cycle. For a detailed description about the cycle please refer to the text. As per our convention, energy flowing towards the working fluid is considered as positive. The cycle operates as a heat engine when $\langle w\rangle = \langle w_1\rangle + \langle w_3\rangle < 0$ and $\langle q_h\rangle > 0$.

jected to four strokes. (i) *Unitary expansion stroke* ($A \to B$). In this stroke, the working fluid expands unitarily under a time-dependent driving that takes the initial Hamiltonian H_0 to a final Hamiltonian H_τ in a time duration τ. The working fluid, in this step, consumes an amount of work w_1 which is not a fixed number but rather a stochastic quantity due to the random thermal initial condition and possible quantum fluctuations during the unitary evolution. (ii) *Isochoric heating stroke* ($B \to C$). During this step, the working medium is put in weak contact with a hot bath at inverse temperature β_h to achieve full thermalization. The Hamiltonian for the working fluid therefore remains the same while the fluid absorbs an amount of heat q_h. Here, we assume that the interaction time with the bath is long enough to achieve equilibration. (iii) *Unitary compression stroke* ($C \to D$). In the next stroke, the system is detached from the hot bath and unitarily compressed via driving the working fluid back to the initial Hamiltonian H_0 starting from H_τ while the fluid consuming an amount of work w_3. For simplicity, we assume that the time duration for this stroke is the same as the expansion stroke. In this study, we are going to assume that the compression protocol is a time-reversed version of the corresponding expansion protocol. (iv) *Isochoric cooling stroke* ($D \to A$). In the final stroke, the fluid is put in contact with a cold bath at inverse temperature β_c to reach equilibrium and thereby closing the cycle. It is important to note that, as per our convention (see Fig. 1), energy flowing into the fluid is always considered to be positive. From here onwards, we denote $w = w_1 + w_3$ as the net work performed on the working fluid.

III. JOINT PROBABILITY DISTRIBUTION FOR WORK AND INPUT HEAT IN QUANTUM OTTO ENGINE

In the quantum regime, a thermodynamically consistent way of studying fluctuations for nonequilibrium systems is via the two-point projective measurement scheme. Such a measurement scheme is also consistent with the quantum fluctuation relations [12–17]. In fact, very recently, following this scheme, an expression for efficiency statistics for the Otto cycle with arbitrary working fluid was obtained [29]. We follow a similar procedure here. Since in this work we are interested only in the heat engine regime, we construct the joint probability distribution $p(w_1, q_h, w_3)$ by performing projective measurements of the respective Hamiltonians involving in the first three strokes ($A \to B \to C \to D$). We then receive

$$p(w_1, q_h, w_3) = \sum_{nmkl} \delta\big(w_1 - (\epsilon_m^\tau - \epsilon_n^0)\big)\ \delta\big(q_h - (\epsilon_k^\tau - \epsilon_m^\tau)\big) \times \delta\big(w_3 - (\epsilon_l^0 - \epsilon_k^\tau)\big)\, T^\tau_{n\to m}\ T^\tau_{k\to l} \frac{e^{-\beta_c \epsilon_n^0}}{\mathcal{Z}_0} \times \frac{e^{-\beta_h \epsilon_k^\tau}}{\mathcal{Z}_\tau}, \tag{1}$$

where $\epsilon_n^0(\epsilon_n^\tau)$ are the energy eigenvalues of initial (final) Hamiltonian during the unitary expansion stroke $A \to B$. Here $\mathcal{Z}_0 = \sum_n \exp(-\beta_c \epsilon_n^0)$ and $\mathcal{Z}_\tau = \sum_n \exp(-\beta_h \epsilon_n^\tau)$ are the partition functions. $T^\tau_{n\to m} = |\langle m_\tau | U_{\rm exp} | n_0\rangle|^2$ ($T^\tau_{k\to l} = |\langle l_0 | U_{\rm com} | k_\tau\rangle|^2$) is the transition probability between the eigenstates of H_0 and H_τ during the unitary expansion (compression) stoke. From the above distribution function, the joint distribution for net work $w = w_1 + w_3$ and input heat q_h can also be obtained easily. As mentioned before, we focus in the engine regime, (i.e., as per our convention, $\langle w\rangle < 0$ and $\langle q_h\rangle > 0$) and correspondingly investigate the bound for the ratio for the output work fluctuation to the input heat fluctuation by defining our central quantity

$$\eta^{(2)} = \frac{\langle\langle w^2\rangle\rangle}{\langle\langle q_h^2\rangle\rangle}. \tag{2}$$

Note that the double angular bracket refers to the cumulants. It is important to note that this definition for the ratio of fluctuations or $\eta^{(2)}$ is different than what follows from the stochastic efficiency definition $\tilde{\eta}^2 = \langle\langle \frac{w^2}{q_h^2}\rangle\rangle$, which was recently investigated in Ref. [29]. In what follows, we first present universal result for $\eta^{(2)}$ for quasi-static Otto cycle with working fluid satisfying a scaling relation and then extend our study to the nonadiabatic regime for two paradigmatic models, consisting of (i) a TLS and (ii) a HO.

IV. QUASISTATIC LIMIT

Before discussing the most-general situation, we first focus on the QS driving limit for the unitary strokes for an Otto cycle. In this limit, one receives universal results for $\eta^{(2)}$. As per the *quantum adiabatic theorem*, in the slow-driving limit, the occupation probabilities between the instantaneous energy eignestates do not change with time which imply for the transition probabilities in Eq. (1) $p_{n\to m} = \delta_{nm}$ and $p_{k\to l} = \delta_{kl}$. As a result, the joint distribution of input heat (q_h) and the net work $w = w_1 + w_3$ simplifies to

$$p_{\rm QS}(w, q_h) = \sum_{n,k} \delta\big(w - \big[(\epsilon_n^\tau - \epsilon_n^0) + (\epsilon_k^0 - \epsilon_k^\tau)\big]\big) \times \delta\big(q_h - (\epsilon_k^\tau - \epsilon_n^\tau)\big) \frac{e^{-\beta_c\epsilon_n^0}}{\mathcal{Z}_0} \frac{e^{-\beta_h\epsilon_k^\tau}}{\mathcal{Z}_\tau}. \tag{3}$$

Instead of looking at a most general eigenpectra for the driving Hamiltonians, we consider a scale-invariant energy eigenspectra under driving, given as $\epsilon_n^\tau = \epsilon_n^0/\lambda_\tau^2$ where λ_τ is the scaling factor. Such a scaling can be realized for driving Hamiltonians of the form $H_t = \mathbf{p}^2/2m + V(\mathbf{x}, \lambda_t)$ with the interaction following a scaling property $V(\mathbf{x}, \lambda_t) = V_0(\mathbf{x}/\lambda_t)/\lambda_t^2$. Such Hamiltonians represent a broad class of single particle and many-body systems [42–46]. Under these conditions, the corresponding characteristic function (CF) $\chi_{\rm QS}(\alpha_1, \alpha_2)$ with α_1 and α_2 being the counting parameters for w and q_h, respectively, simplifies to

$$\chi_{\rm QS}(\alpha_1, \alpha_2) = \sum_{n,k} e^{i(\epsilon_n^0-\epsilon_k^0)\left[\frac{1}{\lambda_\tau^2}(\alpha_1-\alpha_2)-\alpha_1\right]}\, \frac{e^{-\beta_c\epsilon_n^0}}{\mathcal{Z}_0}\, \frac{e^{-\beta_h\epsilon_k^\tau}}{\mathcal{Z}_\tau}. \quad (4)$$

A relation between work and heat cumulants immediately follows from it as

$$\langle\langle w^n\rangle\rangle = (-1)^n\big(1-\lambda_\tau^2\big)^n\langle\langle q_h^n\rangle\rangle. \quad (5)$$

Consequently the nth order ratio for net work and input heat from the hot bath is given as

$$\eta_{\rm QS}^{(n)} = (-1)^n\frac{\langle\langle w^n\rangle\rangle}{\langle\langle q_h^n\rangle\rangle} = \big(1-\lambda_\tau^2\big)^n = \langle\eta\rangle_{\rm QS}^n \leqslant \eta_C^n, \quad (6)$$

where $\langle\eta\rangle = \langle -w\rangle/\langle q_h\rangle$ is the standard thermodynamic efficiency which for an Otto engine in the QS limit reduces to $\langle\eta\rangle_{\rm QS} = (1-\lambda_\tau^2)$. Note that the upper bound can be simply obtained by demanding the positivity of the net entropy production for the Otto cycle. This is our first central result. We note that a similar result was obtained for central moments instead of cumulants in Ref. [47] for a classical Carnot engine where one receives an equality for the upper bound instead of the inequality in Eq. (6). Interestingly, the results in Eq. (5) also holds true for the central moments.

A similar exercise can be carried out in the refrigerator regime as well, following the strokes ($C \to D \to A \to B$) and the corresponding nth order ratio for input heat from cold bath and net work is given as

$$\varepsilon^{(n)} = \frac{\langle\langle q_c^n\rangle\rangle}{\langle\langle w^n\rangle\rangle} = \left(\frac{\lambda_\tau^2}{1-\lambda_\tau^2}\right)^n = \langle\varepsilon\rangle_{QS}^n \leqslant \left(\frac{1-\eta_C}{\eta_C}\right)^n, \quad (7)$$

where $\varepsilon^{(1)} = \langle q_c\rangle/\langle w\rangle$ is the average coefficient of performance of an Otto refrigerator which in the QS limit reduces to $\langle\varepsilon\rangle_{QS} = (\frac{\lambda_\tau^2}{1-\lambda_\tau^2})$.

V. BEYOND QUASISTATIC LIMIT - NON ADIABATIC DRIVING

Beyond the quasistatic limit, it is non-trivial to derive universal bound for arbitrary working fluid. We therefore focus on two paradigmatic model systems to understand the nonadiabatic driving situation. We consider the working fluid for Otto engine consisting of (i) a TLS, and (ii) a simple HO. Interestingly, a TLS as a working fluid was recently implemented and studied from the perspective of an Otto heat engine [8].

A. Working fluid consisting of a two-level system

We first consider a TLS with a Hamiltonian evolving unitarily from $H_A = \frac{1}{2}\omega_0\sigma_x$ to $H_B = \frac{1}{2}\omega_\tau\sigma_y$ during the expansion ($A \to B$) stroke and back to H_A during the compression stroke ($B \to A$) ($\hbar$ is set to unity). For the compression stroke we consider here the reverse protocol of the expansion stroke. Here $\sigma_{x,y,z}$ are the standard Pauli matrices, $\omega_{0,\tau}$ denote the angular frequencies with $\omega_\tau > \omega_0$ corresponding to the energy gap expansion. The evolution of the density operator during the expansion (compression) protocol is governed by a unitary operator $U_{\rm exp}(U_{\rm com} = U_{\rm exp}^\dagger)$. It is not necessary for our case to specify this operator explicitly, as the below results are valid for arbitrary time-dependent protocol under the above-mentioned initial and final TLS Hamiltonians.

One can obtain the the joint CF for the net work w and q_h [48] and compute the first- and second-order cumulants of heat and work. We present here the expressions

$$\langle w\rangle \equiv \langle w_1\rangle + \langle w_3\rangle = \left[\frac{\omega_0}{2} + \frac{\omega_\tau}{2}(1-2u)\right]\tanh\left(\frac{\beta_c\omega_0}{2}\right) + \left[\frac{\omega_\tau}{2} + \frac{\omega_0}{2}(1-2u)\right]\tanh\left(\frac{\beta_h\omega_\tau}{2}\right), \quad (8)$$

$$\langle q_h\rangle = -\frac{\omega_\tau}{2}\left[\tanh\left(\frac{\beta_h\omega_\tau}{2}\right) + \tanh\left(\frac{\beta_c\omega_0}{2}\right)(1-2u)\right], \quad (9)$$

$$\langle\langle w^2\rangle\rangle = \frac{1}{2}(\omega_\tau+\omega_0)^2 - 2\,u\,\omega_\tau\omega_0 - \langle w_1\rangle^2 - \langle w_3\rangle^2, \quad (10)$$

$$\langle\langle q_h^2\rangle\rangle = \frac{\omega_\tau^2}{4}\left[2 - \tanh^2\left(\frac{\beta_h\omega_\tau}{2}\right) - (1-2u)^2\tanh^2\left(\frac{\beta_c\omega_0}{2}\right)\right], \quad (11)$$

where u represents the probability of no transition between the final and the initial eigenstates during the unitary strokes. The QS limit therefore corresponds to $u = 1$. It is easy to check from the above expressions that, in the QS limit, one receives $\eta_{\rm QS}^{(2)} = \langle\eta\rangle_{\rm QS}^2 = (1-\omega_0/\omega_\tau)^2$ which matches with the result obtained in the previous section with a proper scaling factor $\lambda_\tau^2 = \omega_0/\omega_\tau$.

Beyond the QS limit, we provide a rigorous proof in Ref. [48] that while the TLS medium working as a heat engine i.e., under the conditions $\langle w\rangle < 0$ and $\langle q_h\rangle > 0$, the following quantity:

$$\mathcal{A} \equiv \langle\langle w^2\rangle\rangle\langle q_h\rangle^2 - \langle\langle q_h^2\rangle\rangle\langle w\rangle^2 \geqslant 0 \quad (12)$$

is always non-negative with the equality sign achieved in the QS limit ($u = 1$). Therefore, for the TLS Otto cycle, while operating as an engine, we receive

$$\eta^{(2)} \equiv \frac{\langle\langle w^2\rangle\rangle}{\langle\langle q_h^2\rangle\rangle} \geqslant \frac{\langle w\rangle^2}{\langle q_h\rangle^2} = \langle\eta\rangle^2. \quad (13)$$

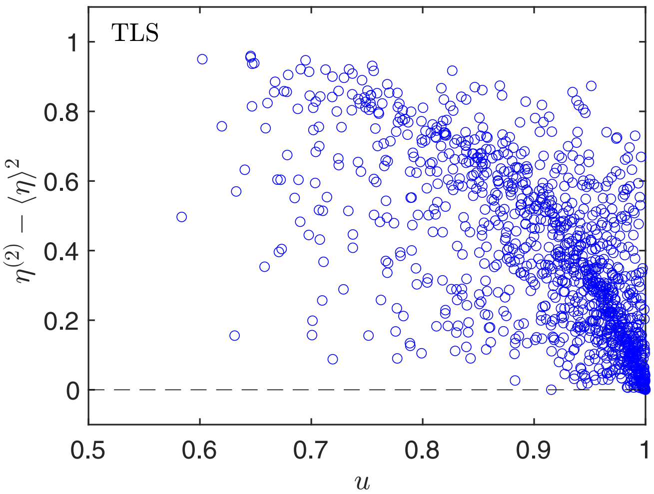

FIG. 2. Scatter plot of $\eta^{(2)} - \langle\eta\rangle^2$ for the TLS Otto cycle while operating as a heat engine. All parameters except u here are chosen randomly in the interval between [0, 10]. u is varied randomly between [0, 1]. Simulation is done for one million random points.

In other words, in the engine regime, $\eta^{(2)}$ is always lower bounded by the square value of the average efficiency. Another way to interpret Eq. (12) is that the relative fluctuation of output work is always greater than relative fluctuations of input heat in the engine regime i.e.,

$$\frac{\langle\langle w^2\rangle\rangle}{\langle w\rangle^2} \geqslant \frac{\langle\langle q_h^2\rangle\rangle}{\langle q_h\rangle^2}. \tag{14}$$

This is the second central result of this Letter. Note that, we did not find such a result or proof in the refrigerator regime. In Fig. 2 we present a scatter plot for $\eta^{(2)} - \langle\eta\rangle^2$ for the TLS, while operating as a heat engine, by choosing all the parameters randomly. As the Hamiltonian here is scale invariant under the driving protocol, we expect that $\eta^{(2)} - \langle\eta\rangle^2$ approaches zero in the quasistatic driving limit which in this case corresponds to $u = 1$. Thus, in Fig. 2, close to $u = 1$, large number of points cluster around the zero value.

B. Working fluid consisting of a harmonic oscillator (HO).

We next consider another paradigmatic example with working fluid consisting of a single harmonic oscillator. The time-dependent Hamiltonian for the unitary strokes is given as $H(t) = p^2/2m + \frac{1}{2}m\omega^2(t)x^2$ where in this case, the trapping frequency $\omega(t)$ is modulated as a function of time from ω_0 at $t = 0$ to ω_τ at $t = \tau$ during the stroke $A \to B$. For the unitary compression stroke $C \to D$ a reverse protocol is considered which can be obtained from the expansion stroke by replacing t by $\tau - t$.

The CF for this case can be obtained exactly [48]. We write down the expressions for the average and the noise for both absorbed heat and net work in the nonadiabatic limit [49,50]

$$\langle w\rangle = \langle w_1\rangle + \langle w_3\rangle = \frac{1}{2}\bigg[\big(\mathcal{Q}\,\omega_\tau - \omega_0\big)\coth\left(\frac{\beta_c\omega_0}{2}\right) + \big(\mathcal{Q}\,\omega_0 - \omega_\tau\big)\coth\left(\frac{\beta_h\omega_\tau}{2}\right)\bigg], \tag{15}$$

$$\langle q_h\rangle = \frac{\omega_\tau}{2}\bigg[\coth\left(\frac{\beta_h\omega_\tau}{2}\right) - \mathcal{Q}\coth\left(\frac{\beta_c\omega_0}{2}\right)\bigg], \tag{16}$$

$$\langle\langle w^2\rangle\rangle = \langle w_1\rangle^2 + \langle w_3\rangle^2 - \frac{1}{2}(\omega_0 - \omega_\tau)^2 + (\mathcal{Q} - 1)\,\omega_0\omega_\tau + \frac{1}{4}(\mathcal{Q}^2 - 1)\bigg[\omega_\tau^2\coth^2\left(\frac{\beta_c\omega_0}{2}\right) + \omega_0^2\coth^2\left(\frac{\beta_h\omega_\tau}{2}\right)\bigg], \tag{17}$$

$$\langle\langle q_h^2\rangle\rangle = -\frac{\omega_\tau^2}{4}\bigg[2 - \coth^2\left(\frac{\beta_h\omega_\tau}{2}\right) - (2\,\mathcal{Q}^2 - 1)\coth^2\left(\frac{\beta_c\omega_0}{2}\right)\bigg]. \tag{18}$$

The above expressions are valid for arbitrary protocol of $\omega(t)$. Here $\mathcal{Q} \in [1, \infty]$ is the so-called adiabaticity parameter which characterizes the degree of adiabaticity. The QS limit corresponds to $\mathcal{Q} = 1$ and it is easy to check that $\eta^{(2)}$ saturates the lower bound, i.e., $\eta^{(2)}_{\rm QS} = \langle\eta\rangle^2_{\rm QS} = (1 - \omega_0/\omega_\tau)^2$ which is expected as the energy eigenspectra follow the scaling relation. We perform extensive numerical simulation by choosing the parameters randomly and notice that, for this model as well, in the engine regime, the lower bound is always respected. In Fig. 3 we present a scatter plot for $\eta^{(2)} - \langle\eta\rangle^2$ for the HO working fluid in the engine regime with the parameters chosen randomly over a broad range. It is clear that the lower bound is always respected for this model with the difference disappearing in the QS limit i.e., for $\mathcal{Q} = 1$.

VI. CONNECTION OF THE BOUND $\eta^{(2)} > \langle\eta\rangle^2$ WITH THE TUR.

At this point, it is important to make a connection with the TUR studies [18–27] which provide independent bounds on relative fluctuations of individual observables (work, heat) in terms of total entropy production. In Ref. [23] it was shown that if a nonequilibrium process satisfying the exchange fluctuation theorem, which for our case is $\chi(\alpha_1, \alpha_2) = \chi(-\alpha_1 + i\beta_c, -\alpha_2 + i(\beta_c - \beta_h))$ [48], then the relative fluctuations of individual integrated currents are lower bounded by a function which solely depends on the total entropy production $\langle\Sigma\rangle$. More precisely,

$$\frac{\langle\langle w^2\rangle\rangle}{\langle w\rangle^2} \geqslant f(\langle\Sigma\rangle), \quad \frac{\langle\langle q_h^2\rangle\rangle}{\langle q_h\rangle^2} \geqslant f(\langle\Sigma\rangle), \tag{19}$$

where $\langle\Sigma\rangle = \beta_c\langle w\rangle + (\beta_c - \beta_h)\langle q_h\rangle$ is the net entropy production and $f(x) = \mathrm{csch}^2(g(x/2))$ where $g(x)$ is the inverse function of $x\tanh(x)$. For the two paradigmatic examples studied in this letter and satisfying the exchange fluctuation theorem, we have shown that in the heat engine regime the relative fluctuation of total work is always greater than the relative fluctuation of heat absorbed from the hot bath. This implies that these relative fluctuations are not independent

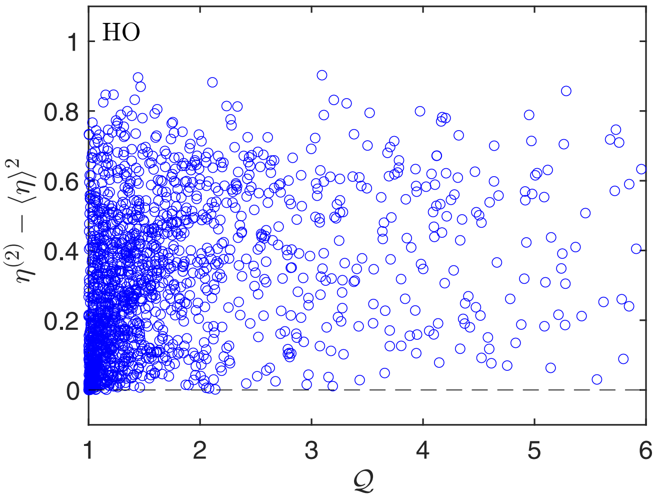

FIG. 3. Scatter plot of $\eta^{(2)} - \langle\eta\rangle^2$ for the harmonic oscillator (HO) Otto cycle while operating as a heat engine. All parameters except $\mathcal{Q}$ are chosen randomly in the interval between [0, 10]. $\mathcal{Q}$ is varied randomly between [1, 6]. Simulation is done for one million random points.

of each other but rather follows the following sequence of bounds in the engine regime:

$$\frac{\langle\langle w^2\rangle\rangle}{\langle w\rangle^2} \geqslant \frac{\langle\langle q_h^2\rangle\rangle}{\langle q_h\rangle^2} \geqslant f(\langle\Sigma\rangle). \tag{20}$$

VII. SUMMARY

In summary, we have investigated bounds on the ratio of nonequilibrium fluctuation for output work and input heat for a finite-time Otto cycle operating as an engine. We provide a universal result for the ratio $\eta^{(n)}$ in the quasistatic limit which is exactly equal to the nth power of the corresponding average efficiency $\langle\eta\rangle$. In the nonadiabatic limit we show for two paradigmatic models that $\eta^{(2)}$ receives a lower bound determined by $\langle\eta\rangle^2$. Importantly, this result further connects to the TUR study where as a consequence of the lower bound, the relative fluctuation of work, always surpass the corresponding relative fluctuation of heat absorbed from the hot bath. Future work will be directed towards providing a general proof for the lower bound for arbitrary working fluid operating as an engine in the nonadiabatic regime. Also, it will be interesting to analyze the validity of such bounds in presence of both driving force and external magnetic field [51].

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[1] S. Carnot, Réflexions sur la Puissance Motrice du feu et sur les Machines Propres ‘a Développer Cette Puissance; Annales Scientifiques de l’Ecole Normale 1872, 1, 393–457. (In French)

[2] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed., (Wiley, New York, 1985).

[3] E. H. Lieb and J. Yngvason, The physics and mathematics of the second law of thermodynamics, Phys. Rep **310**, 1 (1999).

[4] J. Klatzow *et al.*, Experimental Demonstration of Quantum Effects in the operation of Microscopic Heat Engines, Phys. Rev. Lett. **122**, 110601 (2019).

[5] H. E. D. Scovil and E. O. Schulz-DuBois, Three-Level Masers as Heat Engines, Phys. Rev. Lett. **2**, 262 (1959).

[6] R. Uzdin, A. Levy, and R. Kosloff, Equivalence of Quantum Heat Machines, and Quantum-Thermodynamic Signatures, Phys. Rev. X **5**, 031044 (2015).

[7] S. Krishnamurthy, S. Ghosh, D. Chatterji, R. Ganapathy, and A. K. Sood, A micrometre-sized heat engine operating between bacterial reservoirs, Nat. Phys. **12**, 1134 (2016).

[8] J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental Characterization of a Spin Quantum Heat Engine, Phys. Rev. Lett. **123**, 240601 (2019).

[9] D. von Lindenfels, O. Gräb, C. T. Schmiegelow, V. Kaushal, J. Schulz, M. T. Mitchison, J. Goold, F. Schmidt-Kaler, and U. G. Poschinger, Spin Heat Engine Coupled to a Harmonic-Oscillator Flywheel, Phys. Rev. Lett. **123**, 080602 (2019).

[10] G. Maslennikov, S. Ding, R. Hablutzel, J. Gan, A. Roulet, S. Nimmrichter, J. Dai, V. Scarani, and D. Matsukevich, Quantum absorption refrigerator with trapped ions, Nat. Commun. **10**, 202 (2019).

[11] Q. Bouton, J. Nettersheim, S. Burgardt, D. Adam, E. Lutz, and A. Widera, A quantum heat engine driven by atomic collisions, Nat. Commun. **12**, 2063 (2021).

[12] U. Seifert, Stochastic thermodynamics, fluctuation theorems, and molecular machines, Rep. Prog. Phys. **75**, 126001 (2012).

[13] U. Seifert, Stochastic thermodynamics: Principles and perspective, Eur. Phys. J. B **64**, 423 (2008).

[14] S. Vinjanampathy and J. Anders, Quantum thermodynamics, Contemp. Phys. **57**, 545 (2016).

[15] M. Esposito, U. Harbola, and S. Mukamel, Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems, Rev. Mod. Phys. **81**, 1665 (2009).

[16] M. Campisi, P. Hänggi, and P. Talkner, Quantum fluctuation relations: Foundations and applications, Rev. Mod. Phys. **83**, 771 (2011).

[17] C. Jarzynski, Equalities and inequalities: Irreversibility and the second law of thermodynamics at the nanoscale, Ann. Rev. Condens. Matter Phys. **2**, 329 (2011).

[18] A. C. Barato and U. Seifert, Thermodynamic Uncertainty Relation for Biomolecular Processes, Phys. Rev. Lett. **114**, 158101 (2015).

[19] P. Pietzonka and Udo Seifert, Universal Trade-Off between Power, Efficiency, and Constancy in Steady-State Heat Engines, Phys. Rev. Lett. **120**, 190602 (2018).

[20] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Dissipation Bounds all Steady State Current Fluctuations, Phys. Rev. Lett. **116**, 120601 (2016).

[21] G. Falasco, M. Esposito, and J.-C. Delvenne, Unifying thermodynamic uncertainty relations, New J. Phys. **22**, 053046 (2020).

[22] K. Macieszczak, K. Brandner, and J. P. Garrahan, Unified Thermodynamic Uncertainty Relations in Linear Response, Phys. Rev. Lett. **121**, 130601 (2018).

[23] A. M. Timpanaro, G. Guarnieri, J. Goold, and G. T. Landi, Thermodynamic Uncertainty Relations from Exchange Fluctuation Theorems, Phys. Rev. Lett. **123**, 090604 (2019).

[24] K. Brandner, T. Hanazato, and K. Saito, Thermodynamic Bounds on Precision in Ballistic Multiterminal Transport, Phys. Rev. Lett. **120**, 090601 (2018).

[25] J. Liu and D. Segal, Thermodynamic uncertainty relation in quantum thermoelectric junctions, Phys. Rev. E **99**, 062141 (2019).

[26] S. Saryal, O. Sadekar, and B. K. Agarwalla, Thermodynamic uncertainty relation for energy transport in a transient regime: A model study, Phys. Rev. E **103**, 022141 (2021).

[27] B. K. Agarwalla and D. Segal, Assessing the validity of the thermodynamic uncertainty relation in quantum systems, Phys. Rev. B **98**, 155438 (2018).

[28] S. Saryal, M. Gerry, I. Khait, D. Segal, and B. K. Agarwalla, Universal Bounds on Fluctuations in Continuous Thermal Machines, arXiv:2103.13513.

[29] T. Denzler and E. Lutz, Efficiency fluctuations of a quantum heat engine, Phys. Rev. Res. **2**, 032062(R) (2020).

[30] R. Kosloff, Quantum thermodynamics and open-systems modeling, J. Chem. Phys. **150**, 204105 (2019).

[31] E. Geva and R. Kosloff, A quantum-mechanical heat engine operating in finite time. A model consisting of spin-1/2 systems as the working fluid, J. Chem. Phys. **96**, 3054 (1992).

[32] T. Feldmann and R. Kosloff, Performance of discrete heat engines and heat pumps in finite time, Phys. Rev. E **61**, 4774 (2000).

[33] T. D. Kieu, The Second Law, Maxwell's Demon, and Work Derivable from Quantum Heat Engines, Phys. Rev. Lett. **93**, 140403 (2004).

[34] M. J. Henrich, F. Rempp, and G. Mahler, Quantum thermodynamic Otto machines: A spin-system approach, Eur. Phys. J.: Spec. Top. **151**, 157 (2007).

[35] M. O. Scully, Quantum Afterburner: Improving the Efficiency of an Ideal Heat Engine, Phys. Rev. Lett. **88**, 050602 (2002).

[36] G. Watanabe, B. P. Venkatesh, P. Talkner, and A. del Campo, Quantum Performance of Thermal Machines Over Many Cycles, Phys. Rev. Lett. **118**, 050601 (2017).

[37] Y. Rezek and R. Kosloff, Irreversible performance of a quantum harmonic heat engine, New J. Phys. **8**, 83 (2006).

[38] B. Lin and J. Chen, Performance analysis of an irreversible quantum heat engine working with harmonic oscillators, Phys. Rev. E **67**, 046105 (2003).

[39] O. Abah, J. Roßnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz, Single Ion Heat Engine with Maximum Efficiency at Maximum Power, Phys. Rev. Lett. **109**, 203006 (2012).

[40] J. M. Park, S. Lee, H.-M. Chun, and J. D. Noh, Quantum mechanical bound for efficiency of quantum Otto heat engine, Phys. Rev. E **100**, 012148 (2019).

[41] S. Lee, M. Ha, and H. Jeong, Quantumness and thermodynamic uncertainty relation of the finite-time Otto cycle, Phys. Rev. E **103**, 022136 (2021).

[42] V. Gritsev, P. Barmettler, and E. Demler, Scaling approach to quantum non-equilibrium dynamics of many-body systems, New J. Phys. **12**, 113005 (2010).

[43] C. Jarzynski, Generating shortcuts to adiabaticity in quantum and classical dynamics, Phys. Rev. A **88**, 040101(R) (2013).

[44] A. del Campo, Shortcuts to Adiabaticity by Counterdiabatic Driving, Phys. Rev. Lett. **111**, 100502 (2013).

[45] S. Deffner, C. Jarzynski, and A. del Campo, Classical and Quantum Shortcuts to Adiabaticity for Scale-Invariant Driving, Phys. Rev. X **4**, 021013 (2014).

[46] M. Beau, J. Jaramillo, and A. del Campo, Scaling-up quantum heat engines efficiently via shortcuts to adiabaticity, Entropy **18**, 168 (2016).

[47] K. Ito, C. Jiang and G. Watanabe, Universal Bounds for Fluctuations in Small Heat Engines, arXiv:1910.08096.

[48] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.103.L060103 for details about the characteristic functions for the TLS and HO and a proof for the bound for TLS.

[49] T. Denzler, Fluctuations and correlations of quantum heat engines, Ph.D. thesis, University of Stuttgart (2020).

[50] S. Deffner and E. Lutz, Nonequilibrium work distribution of a quantum harmonic oscillator, Phys. Rev. E **77**, 021128 (2008).

[51] Y. Tang, Free energy amplification by magnetic flux for driven quantum systems, Commun. Phys. **4**, 9 (2021).