Letter

Presence and absence of delocalization-localization transition in coherently perturbed disordered lattices

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A new type of delocalization induced by coherent harmonic perturbations in one-dimensional Anderson-localized disordered systems is investigated. With only a few M frequencies a normal diffusion is realized, but the transition to a localized state always occurs as the perturbation strength is weakened below a critical value. The nature of the transition qualitatively follows the Anderson transition (AT) if the number of degrees of freedom $M+1$ is regarded as the spatial dimension d. However, the critical dimension is found to be $d = M + 1 = 3$ and is not $d = M + 1 = 2$, which should naturally be expected by the one-parameter scaling hypothesis.

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Introduction. Since the proposal of Anderson, the localization of electrons in disordered lattices has been one of the most fundamental problems associated with the essence of the electron conduction process [1–3]. No matter how high the spatial dimension may be, the Anderson localized state exists prior to the delocalized conducting state, and a transition from the localized state to the delocalized state, the so called Anderson transition (AT), occurs as the relative strength of disorder decreases [4–8]. Theoretical predictions have been obtained by using several theoretical tools such as the one-parameter scaling hypothesis, the self-consistent theory, and so on [9,10].

On the other hand, in the study of chaotic systems the ergodic transition of quantum maps is equivalent to the AT of a disordered lattice [11–13]. Based on this equivalence, the dynamical AT was first experimentally observed for quantum map systems which is implemented on an optical lattice [14,15]. In this case the number of dynamical degrees of freedom corresponds to the number of spatial dimensions of the disordered lattices, and so the features of AT in high-dimensional lattices can be explored by quantum maps.

The dynamical interaction among the degrees of freedom thus enables the delocalization transition. Then the following question immediately follows: Can the Anderson localization in the disordered lattices be destroyed as it is perturbed by dynamical degrees of freedom such as phonon modes? The perturbation by infinitely many phonon modes can be modeled by a stochastic perturbation, and it is well known that the stochastic perturbation destroys the localization and induces a normal diffusion [16–19]. However, the effect of dynamical perturbation composed of a *finite number* of coherent modes has still not been determined. In previous papers, we investigated the effect of finite-number harmonic perturbations on a one-dimensional disordered lattice (ODDL), and showed that the ODDL exhibits a normal diffusion at least on a finite timescale [20–22]. On the other hand, numerical and mathematical studies show that the localization is persistent for finite-number harmonic perturbations [23,24], and whether localization and delocalization dominates is still open to question. It is quite interesting whether or not a coherent dynamical perturbation composed of finite number of harmonic modes can dynamically destroy the localization. In this Letter, we present results answering the question.

Model. We consider a tightly binding ODDL perturbed by coherent periodic perturbations with different incommensurate frequencies. It is given by

$$i\hbar\frac{\partial \Psi_n(t)}{\partial t} = \Psi_{n-1}(t) + \Psi_{n+1}(t) + V_L(n,t)\Psi_n(t), \quad (1)$$

where $V_L(n,t) = V(n)[1 + f(t)]$. The coherently time-dependent part $f(t)$ is given as

$$f(t) = \frac{\epsilon}{\sqrt{M}}\sum_i^M \cos(\omega_i t), \quad (2)$$

where M and ϵ are the number of frequency components and the strength of the perturbation, respectively. Note that the long-time average of the total power of the perturbation is normalized to $\overline{f(t)^2} = \epsilon^2/2$. The frequencies $\{\omega_i\}$ $(i = 1,\ldots,M)$ are taken as mutually incommensurate numbers of order $O(1)$. The static on-site disorder potential takes random value $V(n)$ uniformly distributed over the range $[-W/2, W/2]$, where W denotes the disorder strength.

It is important to note that the harmonic source can be interpreted as the quantum linear oscillator of the Hamiltonian $\sum_i^M \omega J_i$ interacting with the irregular lattice with the quantum amplitude $\frac{\epsilon}{\sqrt{M}}\sum_i^M \cos\phi_i$ instead of the classical force $f(t)$, where $(J_i,\phi_i) = (-i\partial/\partial\phi_i, \phi_i)$ are the action-angle operators of the ith oscillator. Each quantum oscillator has the action eigenstates $|n_i\rangle$ with the action eigenvalue $J_i = n_i\hbar$ (n_i integer) and the energy $n_i \hbar\omega_i$. Thus the system (1) is regarded as a quantum autonomous system of $(M+1)$ degrees of freedom spanned by the quantum states $|n\rangle \prod_{i=1}^{M} |n_i\rangle$ [23].

We take a lattice-site eigenstate as the initial state $|\Psi(t=0)\rangle$, i.e., $\langle n|\Psi(t=0)\rangle = \delta_{n,n_0}$, and numerically observe the spread of the wave packet measured by the mean square displacement (MSD), $m_2(t) = \sum_n (n-n_0)^2 \langle |\Psi(n,t)|^2 \rangle$.

First, we consider the limit $M \to \infty$. In this case $f(t)$ can be identified with the delta-correlated stochastic force $\langle f(t) f(t') \rangle = \Gamma \delta(t-t')$, where $\Gamma \propto \epsilon^2$ is a noise strength. The localization is surely destroyed and the normal diffusion $m_2(t) = Dt$ with the diffusion constant D is recovered for $t \to \infty$ [21,22], as was first pointed out by Haken and his co-workers [16,17]. They predicted analytically, for Gaussian white noise,

$$D = \lim_{t\to\infty} \frac{m_2(t)}{t} \propto \frac{\Gamma}{\Gamma^2 + W^2/12}, \tag{3}$$

for weak enough ϵ. If $W \gg \Gamma$, $D \propto W^{-2}$, but recently it was shown that $D \propto W^{-4}$ for the strong disorder region $W \gg 1$ [18]. The noise-induced diffusion has been extended for a random lattice driven by colored noise, including the fluctuation of the off-diagonal terms [17–19].

However, for finite M, $f(t)$ can no longer be replaced by random noise, and it play the role of a coherent dynamical perturbation, and the system is a quantum dynamical system with $(M+1)$ degrees of freedom. The main purpose of this study is to investigate how the nature of the quantum dynamics of the irregular lattice changes as the number M decreases from ∞ to 0.

Delocalized states and normal diffusion. We show typical examples of time evolution of MSD for $M=7$ and $M=3$ in Figs. 1(a) and 1(b), respectively. If ϵ is large enough, it is evident that MSD follows asymptotically the normal diffusion $m_2 = Dt$, which means that only a finite number of coherent periodic modes plays the same role as the stochastic perturbation in the disordered lattice. The W and ϵ dependencies of the diffusion constant D depicted in Fig. 1 follow the main feature of the stochastically induced diffusion constants: as shown in Fig. 1(c) the W dependence changes from $D \propto W^{-2}$ for weak W in Eq. (3) to $D \propto W^{-4}$ for $W \gg 1$, following the theoretical prediction of stochastic perturbations [18]. Moreover, as depicted in Fig. 1(d), even for $M=3$ the ϵ dependence reproduces the characteristic behavior of the stochastically induced D, which first increases but finally decreases with ϵ after reaching a maximum value. It is a remarkable feature of ODDLs that a normal diffusion, which mimics the one induced by a stochastic force composed of an infinite number of frequencies, is self-generated by a coherent perturbation composed of only three incommensurate frequencies.

On the other hand, a coherently perturbed ODDL always undergoes a definite phase transition from the diffusing state to a localized state as ϵ decreases, crossing over a critical value ϵ_c. The transition is quite similar to the AT of a high-dimensional disordered lattice. As shown in Fig. 2, at $\epsilon = \epsilon_c$, the MSD exhibits a subdiffusion $m_2 \sim t^\alpha$ with a critical diffusion index α $(0 < \alpha < 1)$. Close to ϵ_c, typical critical transient phenomena are observed. To show them we define the function $\Lambda(t)$ as the scaled MSD:

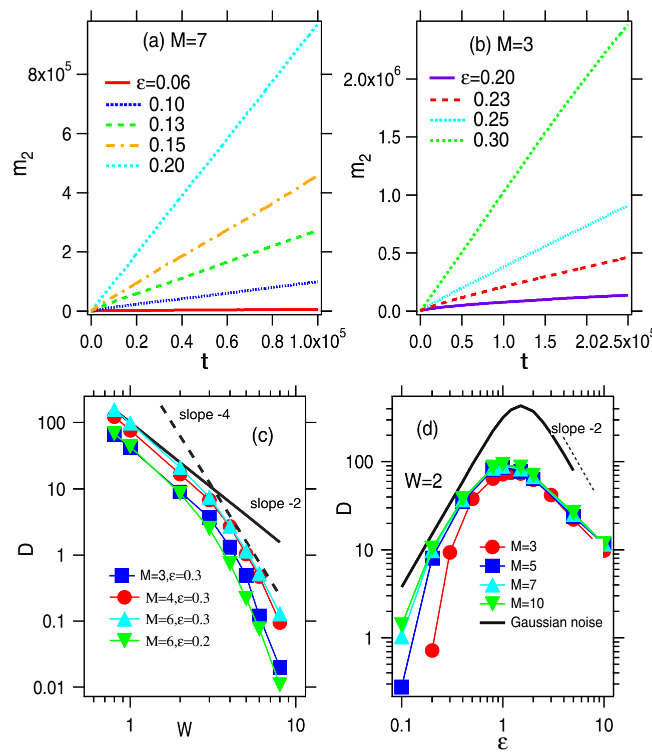

FIG. 1. $m_2(t)$ as a function of time in the ODDLs of (a) $M=7$ and (b) $M=3$ with $W=2$ for some values of the perturbation strength ϵ increasing from $\epsilon = 0.06$ (bottom) to $\epsilon = 0.2$ (top) for $M=7$ and from $\epsilon = 0.2$ (bottom) to $\epsilon = 0.3$ (top) for $M=3$, respectively. Note that the axes are in the real scale. (c) The diffusion coefficient D as a function of W and (d) D as a function of ϵ for several M, determined by least-square fit of $m_2(t)$ for $t \gg 1$. The system and ensemble sizes are $N = 2^{14}$–2^{15} and 10–40, respectively, throughout this paper. We used a second-order symplectic integrator with time step size $\Delta t = 0.02$–0.05, and we take $\hbar = 0.125$ as the Planck constant.

$$\Lambda(t) \equiv \frac{m_2(t)}{t^\alpha}, \tag{4}$$

divided by the subdiffusion. In the inset of Fig. 2 the $\Lambda(t)$ at various ϵ close to ϵ_c are displayed for $M=7$; they form a characteristic fan pattern spreading outward.

As demonstrated in Fig. 3(a), the index of the critical subdiffusion decreases with M, following the result of the one-parameter scaling hypothesis

$$\alpha = \frac{2}{d} = \frac{2}{M+1} \tag{5}$$

for the d-dimensional disordered lattice, if we regard d as the total number of degrees of freedom of our system, i.e., $d = M+1$, which seems to be quite reasonable.

The localization in the side of $\epsilon < \epsilon_c$ is characterized by the localization length ξ_M, which diverges at ϵ_c as $\xi_M(\epsilon) \sim (\epsilon - \epsilon_c)^{-\nu}$ with the critical exponent ν (>0). A remarkable feature of the critical state is that the fan pattern of $\Lambda(t)$ are represented by two unified curves depending on whether $\epsilon > \epsilon_c$ or $\epsilon < \epsilon_c$ by using the scaling variable $x = \xi_M(\epsilon) t^{\alpha/2\nu}$, as demonstrated by Fig. 3(b) for $M=3$. The $d = M+1$

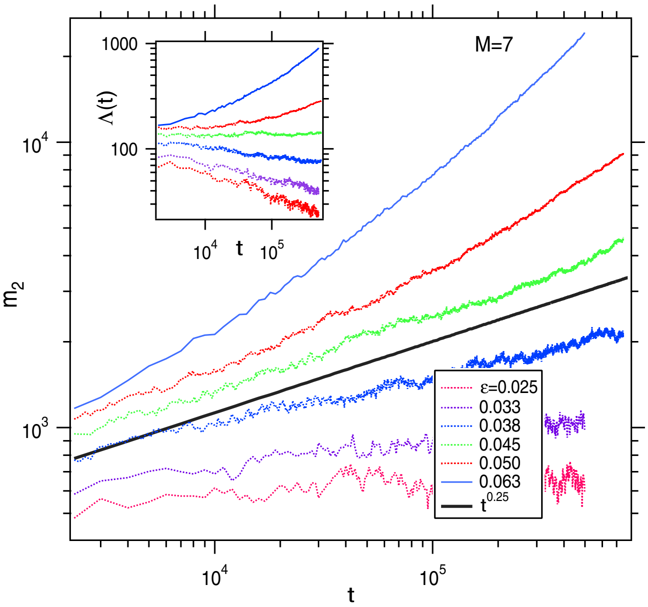

FIG. 2. The double-logarithmic plots of $m_2(t)$as a function of time for some values of the perturbation strength ϵ increasing from $\epsilon = 0.025$ (bottom) to $\epsilon = 0.063$ (top), where the diffusion exponent α is determined by the least-square fit of $m_2(t)$ in the critical case, in the perturbed ODDL of $M = 7$ with $W = 2$. The data near the critical value ϵ_c are shown by bold black lines. $\epsilon_c \simeq 0.045$, $\alpha \simeq 2/8 = 0.25$. Note that the axes are in logarithmic scale. The inset shows the scaled MSD $\Lambda(t)$.

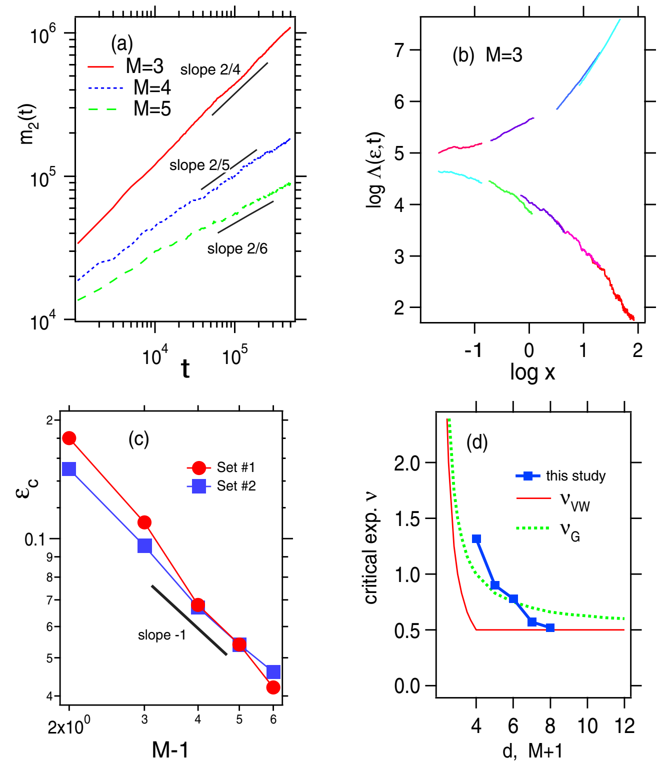

FIG. 3. (a) The double-logarithmic plots of $m_2(t)$ as a function of time near the critical pints ϵ_c in a polychromatically perturbed ODDL with $W = 2$ ($M = 3, 4, 5$ from top). (b) The scaled variable $\log_{10} \Lambda(\epsilon, t)$ as a function of $\log_{10} x$ where $x = \xi_M(\epsilon)t^{\alpha/2\nu}$. The delocalized (localized) regime is the upper (lower) branch. (c) The critical perturbation strength ϵ_c as a function of $(M - 1)$. The results for different frequency sets $\{\omega_i\}$ are shown. Note that the axes are in logarithmic scale. The line with slope -1 is shown as a reference. (d) The dimensionality $(M + 1) = d$ dependence of the critical exponent ν which characterizes the critical dynamics. The red solid line and green dashed line are the results of the analytical prediction by ν_{VW} and ν_G, respectively.

dependence of the critical index ν is shown in Fig. 3(d). More details of the finite-time scaling analysis for the numerical data are given in Refs. [15] and [25].

Such a remarkable critical subdiffusion exists at ϵ_c for an arbitrary M, but the critical value ϵ_c decreases with M:

$$\epsilon_c \propto \frac{1}{(M-1)^{\delta}}, \qquad \delta \simeq 1, \tag{6}$$

which does not depend upon W as shown in Fig. 3(c). Thus the ODDL is always localized if ϵ is small enough, but no matter how small ϵ may be, normal diffusion mimicking stochastically induced diffusion is realized if M is taken large enough.

The mathematical research of [24] using a model very similar to ours asserts that the a localized phase exists for small enough ϵ as long as M is finite. In particular, the persistence of the localization for $M = 2$ was numerically confirmed up to a value of ϵ beyond the perturbation regime [23]. On the other hand, in the large limit of M, the perturbation can be well approximated by white noise, which makes the system delocalize for any $\epsilon \neq 0$ [16–19]. To be compatible with the above observations, a delocalization-localization transition (DLT) should exists for arbitrarily large finite M, and it should disappear in a limit $M \to \infty$, which is just the background supporting our result of Eq. (6). An important fact is that the change to the delocalized state is not a crossover process but a quantum phase transition.

It is quite interesting that the dependencies of both α and ϵ_c upon M are the same as those of the AT observed for quantum maps simulating a high-dimensional disordered lattice [25–27]. If we are allowed to extrapolate the above results for smaller M, ϵ_c diverges at $M = 1$, at which the critical diffusion index becomes $\alpha = 1$. This fact implies that for $M = 1$ the critical subdiffusion is realized at $\epsilon = \epsilon_c = \infty$ as normal diffusion; namely, that $M = 1$ is the critical dimension of the DLT, which has been established for quantum maps and high-dimensional disordered lattices. However, our numerical results reject the above conjecture.

Number of critical modes ($M = 2$). If the above conjecture is correct, $M = 2$ ($d = 3$) should exhibit the critical phenomenon. In Fig. 4(a) log-log plots of MSD curves for $M = 2$ are shown for various values of ϵ. Clearly, the $m_2(t)$ with $\epsilon = 0.6$ follows the expected critical subdiffusion of the exponent $\alpha = 2/3$ in the initial stage, which is roughly predicted from the interplolation of the numerical data for $M \geqslant 3$, but it drops from the straight line as time elapses.

To get an overview of the features of the MSD curves, it is instructive to show the time evolution of the diffusion exponent, defined as the instantaneous slope of the log-log

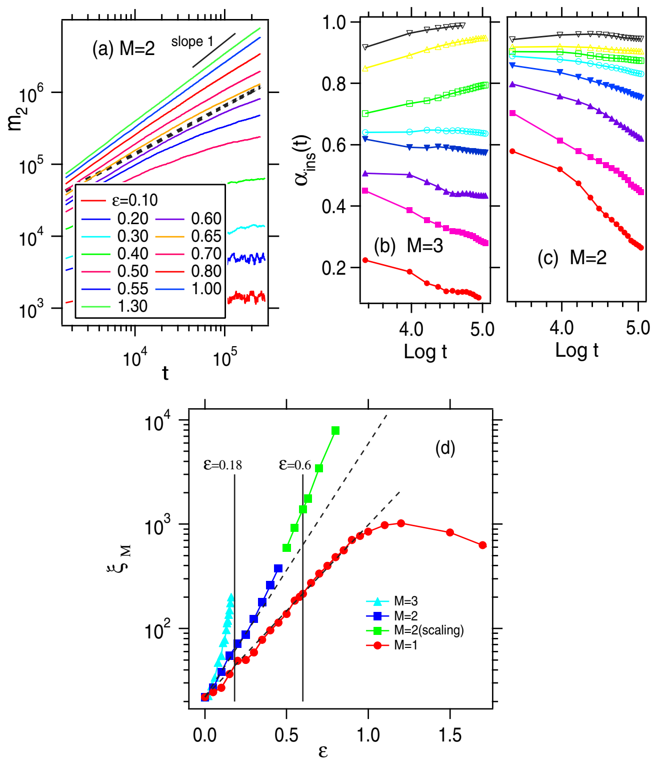

FIG. 4. (a) Double-logarithmic plots of $m_2(t)$ as a function of time for some values of the perturbation strength ϵ increasing from bottom to top in a trichromatically perturbed ODDL of $M=2$. The panels (b) and (c) show the diffusion exponent $\alpha_{\rm ins}(t)$ as a function of $\log_{10} t$ in the cases $M=3$ and $M=2$, respectively. (d) Localization length (LL) as a function of ϵ for $M=1,2,3$ with $W=2$. Some LLs of $M=2$ are obtained by scaling relation $m_2(t) \sim \xi_M(\epsilon)^2 F(t/\xi_M(\epsilon)^2)$ for $\epsilon \geqslant 0.5$. Note that the horizontal axis is in logarithmic scale. The dashed lines show $\xi_M \propto e^{5.5\epsilon}$ and $\xi_M \propto e^{3.8\epsilon}$, respectively. The lines $\epsilon=0.18$ and $\epsilon=0.6$ are shown as a reference, and $\xi_M(\epsilon=0) \simeq 20$ for $W=2$.

plot of MSD,

$$\alpha_{\rm ins}(t) = \frac{d \log m_2(t)}{d \log t}. \tag{7}$$

If DLT happens at a finite $\epsilon = \epsilon_c$, then $\alpha_{\rm ins}(t)$ should keep a constant value $\alpha(\epsilon_c) < 1$. Above ϵ_c, as time passes, $\alpha_{\rm ins}(t)$ increases up to the exponent 1, indicating normal diffusion, while below ϵ_c it decreases to zero, indicating localization. Indeed, the expected feature is evident for the $\alpha_{\rm ins}(t)$ plot of $M=3$ shown in Fig. 4(b). The same feature is observed also for $M \geqslant 4$.

However, as shown in Fig. 4(c) the $\alpha_{\rm ins}(t)$ plots of $M=2$ show a quite different feature. No curves follow the critical behavior $\alpha_{\rm ins}(t) = $ const < 1, and all the curves tends to decrease from the initial values, which approaches 1 as ϵ increases. As $\alpha_{\rm ins}(t)$ comes close to 1, the tim scale beyond which $\alpha_{\rm ins}(t)$ begins to decrease becomes longer. Certainly it seems as if the normal diffusion $\alpha_{\rm ins}(t) = 1$, which would be realized in the limit $\epsilon \to \infty$, were the critical diffusion. These facts indicate that the DLT does not exists for $M=2$, in contradiction with the prediction of Eqs. (5) and (6), and that $M=2 (d=3)$ is the critical dimension.

TABLE I. Dimensionality of the DLT. For $4 \leqslant M < \infty$ the result is the same as the case of $M=3$. The lower lines are the results of d-dimensional disordered systems. Loc: exponential localization; Diff: normal diffusion.

$d(=M+1)$	1	2	3	4	5	...	∞
this study	Loc	Loc	Loc	DLT	DLT	...	Diff
quantum maps [25]	Loc	Loc	DLT	DLT	DLT	...	Diff
Anderson model	Loc	Loc	DLT	DLT	DLT	...	DLT

Comparison by localization length. Localization, of course, occurs with $M=1$. Then what is the difference of the localizations between the case of $M=1$ $(d=2)$ and the case of $M=2$ $(d=3)$? In both cases $M=d-1=1$ and $M=d-1=2$, the localization length grow exponentially when ϵ is small enough ($\epsilon < 0.8$ for $M=1$ and $\epsilon < 0.3$ for $M=2$), which coincides with the case of the $d=2$ ordinary disordered lattice.

However, with a further increase of ϵ, ξ_M begins to decrease steeply for $M=1$. Such a behavior is a direct result of the intersite transfer being suppressed by the random potential enhanced with the increasing perturbation strength ϵ. Let us remember that, as shown in Fig. 1(d), even the recovered diffusion constant of the the system of $d=M+1 \gg 1$ in general decreases steeply with ϵ $(> \epsilon_c)$. This is the reason why, unlike the ordinary d-dimensional irregular lattice, $d=2$ cannot be the critical dimension of our system. The growth of ξ_M with ϵ takes place only by increasing the dimension from $d=2$ to 3.

Indeed, for $d=M+1=3$ the localization still remains, but ξ_M increases with ϵ exponentially. The exponential growth rate is further enhanced and a superexponential growth occurs as ϵ increases beyond $O(1)$, as depicted in Fig. 4(d). And it is for $M=3$ that the divergence of ξ_M is first observed at a finite ϵ_c.

Summary and discussion. In the present paper, we investigated the delocalized and the localized motion in a one-dimensional irregular lattice coherently perturbed by harmonic modes. In order to induce a delocalized motion the stochastic perturbation composed of an infinite number of harmonic mode is not necessary: the diffusive motion is always induced only by a few harmonic modes if the perturbation strength is strong enough. The critical perturbation strength (ϵ_c) and the critical subdiffusion exponent (α) decrease with the number of modes M, and their dependencies upon M are almost same as those of the Anderson transitions numerically established for multidimensional quantum maps, which can be considered as modified versions of the many-dimensional Anderson model [25]. However, the critical number of the degrees of freedom is not $d=M+1=2$ but $d=3$ in our system. Thus our system provides with an example demonstrating that the critical dimension of the DLT may be larger than $d=2$ and depend upon the nature of recovered diffusion, as summarized in the Table I.

The Anderson-like transition discussed in the present paper affords an example of a quantum phase transition in which the coherent localized state changes to a decoherent diffusive state. Existence of such a quantum transition has been known in some quantum chaos systems which exhibit chaotic diffusion [12,14]. More generally, it will play a crucial role when quantum systems with a small number of degrees of freedom get ergodic properties [28]. We expect that investigations of the transition process to the decoherent and delocalized states in quantum dynamical systems with small degrees of freedom would contribute much to the fundamental study of quantum statistical physics.

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