Letter

Counting statistics for noninteracting fermions in a d-dimensional potential

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We develop a first-principles approach to compute the counting statistics in the ground state of N noninteracting spinless fermions in a general potential in arbitrary dimensions d (central for $d > 1$). In a confining potential, the Fermi gas is supported over a bounded domain. In $d = 1$, for specific potentials, this system is related to standard random matrix ensembles. We study the quantum fluctuations of the number of fermions $\mathcal{N}_\mathcal{D}$ in a domain $\mathcal{D}$ of macroscopic size in the bulk of the support. We show that the variance of $\mathcal{N}_\mathcal{D}$ grows as $N^{(d-1)/d}(A_d \log N + B_d)$ for large N, and obtain the explicit dependence of A_d, B_d on the potential and on the size of $\mathcal{D}$ (for a spherical domain in $d > 1$). This generalizes the free-fermion results for microscopic domains, given in $d = 1$ by the Dyson-Mehta asymptotics from random matrix theory. This leads us to conjecture similar asymptotics for the entanglement entropy of the subsystem $\mathcal{D}$, in any dimension, supported by exact results for $d = 1$.

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An important concept to study quantum noise and correlations in many-body fermionic systems is the counting statistics (CS), which characterizes the fluctuations of the number of particles $\mathcal{N}_\mathcal{D}$ inside a domain $\mathcal{D}$. Applications include shot noise [1], quantum transport [2,3], quantum dots [4,5], spin and fermionic chains [6–9], and trapped fermions [10,11]. In the related context of random matrix theory (RMT), the statistics of the number of eigenvalues in an interval also generated a lot of interest [6,12–23]. The CS is particularly important for noninteracting fermions because of its connection [24–27] to the bipartite entanglement entropy (EE) of the subsystem $\mathcal{D}$ with its complement $\overline{\mathcal{D}}$. The EE is a highly nonlocal quantity, much studied in the context of quantum information [28,29], conformal field theory [30–32], topological phases [33], quantum phase transitions [34,35], or quantum spin chains [36,37]. Both the CS and the EE are difficult to compute analytically, in particular in the presence of an external potential. There exist, however, standard results for free fermions, in the absence of an external potential. In this case, at zero temperature, both the variance of $\mathcal{N}_\mathcal{D}$ and the EE grow as $\sim R^{d-1} \log R$ with the typical size R of the domain $\mathcal{D}$ [38–45].

In cold Fermi gases [46], the quantum microscopes [47–49] allow one to take an instantaneous "picture" and measure the counting statistics. In experiments the fermions are in a trapping potential, of tunable shape and interaction [46,50]. It is thus important to calculate both the CS and the EE in an inhomogeneous background, for which very few analytical results exist even for noninteracting fermions, apart from the $d = 1$ harmonic oscillator [32,51,52], and the rotating harmonic trap in $d = 2$ [53].

There has been recent progress to describe noninteracting spinless fermions in traps in d dimensions [11]. In $d = 1$, for a single-particle Hamiltonian $\hat{H} = \frac{p^2}{2} + V(x)$ (in units $\hbar = m = 1$), there is a useful connection with random matrices for a few specific potentials $V(x)$. The many-body ground-state wave function Ψ_0 of N fermions is a Slater determinant with all energy levels of $\hat{H}$ occupied up to the Fermi energy μ, a function of N. The quantum joint probability $|\Psi_0|^2$ of the positions $\{x_j\}$ of the N fermions, maps onto the joint probability for the eigenvalues $\{\lambda_j\}$ of random matrices of size $N \times N$. For the harmonic oscillator (HO), $V(x) = \frac{x^2}{2}$, the random matrix is Hermitian from the Gaussian unitary ensemble (GUE). At large N, the mean fermion density, i.e., the quantum average $\rho(x) = \langle \sum_i \delta(x - x_i) \rangle$, has support $[x^-, x^+]$, with $x^\pm \simeq \pm\sqrt{2N}$. In the bulk, i.e., away from the edges $x^\pm$, it takes the semicircle form $\rho(x) \simeq \rho^{\text{bulk}}(x) = k_F(x)/\pi$, where $k_F(x) = \sqrt{2\mu - x^2}$ is the local Fermi momentum, and in this case $\mu \simeq N$. There are two natural length scales, the microscopic one of order the interparticle distance $\sim 1/k_F(x)$, and the macroscopic one of order $x^+ - x^-$. For an interval $\mathcal{D} = [a, b]$ of microscopic size, it is well known from standard results of RMT [54,55] that for $\sqrt{N}|b - a| = O(1) \gg 1$ the variance behaves as [6,12–14,17,20–22]

$$\text{Var}\, \mathcal{N}_{[a,b]} \simeq \frac{1}{\pi^2}[\log(\sqrt{2N - a^2}|b - a|) + c_2], \tag{1}$$

with $c_2 = \gamma_E + 1 + \log 2$, where γ_E is Euler's constant. The fermions (or eigenvalues) correlations can be expressed as determinants of a central object called the kernel, which depends on $V(x)$ (see below). At microscopic scales, the kernel takes a universal scaling form, called the sine kernel, independent of the (smooth) potential, which leads to (1). However, except for free fermions on the infinite line, it does not apply when both a, b are well separated in the bulk. For the HO, some results in that regime were obtained in Refs. [20,21] using a Coulomb gas method, and for the GUE in the math literature [56–59].

Despite recent advances a general framework is still lacking for computing the counting statistics and entanglement entropy for noninteracting fermions in a general potential and arbitrary dimension. In this Letter we provide a first-principles approach to compute these quantities in $d = 1$ for a general potential $V(x)$, and in $d > 1$ for a general central potential. Our method recovers the existing results in various special cases (see below).

Let us summarize our main results. For a confining potential in $d = 1$, such that the bulk density $k_F(x)/\pi$, $k_F(x) = \sqrt{2[\mu - V(x)]}$, has a single support $[x^-, x^+]$, we obtain an explicit formula for $\mathrm{Var}\,\mathcal{N}_{[a,b]}$, with a, b well separated in the bulk, $|a - b| \gg 1/k_F(a)$. In the limit $N \gg 1$ (i.e., $\mu \gg 1$) where $N \simeq \int_{x^-}^{x^+} \frac{dx}{\pi} k_F(x)$,

$$(2\pi^2)\mathrm{Var}\,\mathcal{N}_{[a,b]} = 2\log\left(2k_F(a)k_F(b)\int_{x^-}^{x^+}\frac{dz}{\pi k_F(z)}\right) + \log\left(\frac{\sin^2\frac{\theta_a-\theta_b}{2}}{\sin^2\frac{\theta_a+\theta_b}{2}}|\sin\theta_a\sin\theta_b|\right) + 2c_2 + o(1), \tag{2}$$

where

$$\theta_x = \pi\frac{\int_{x^-}^{x} dz/k_F(z)}{\int_{x^-}^{x^+} dz/k_F(z)}, \qquad \begin{cases}\theta_{x^-} = 0,\\ \theta_{x^+} = \pi.\end{cases} \tag{3}$$

We then consider noninteracting fermions in a general central potential in d dimension, with single-particle Hamiltonian $\hat{H} = \frac{\mathbf{p}^2}{2} + V(r)$, where $r = |\mathbf{x}|$. We obtain the variance $\mathrm{Var}\,\mathcal{N}_{\mathcal{D}}$ for any rotationally invariant domain $\mathcal{D}$. For instance, for the HO, $V(r) = \frac{1}{2}r^2$, the support of the density is the ball of radius $\sqrt{2\mu}$, and for a sphere of macroscopic radius $R = \tilde{R}\sqrt{2\mu}$ we obtain for large μ, with fixed $\tilde{R} \in [0, 1[$,

$$\mathrm{Var}\,\mathcal{N}_{\mathcal{D}} = \mu^{d-1}[A_d(\tilde{R})\log\mu + B_d(\tilde{R}) + o(1)], \tag{4}$$

$$A_d(\tilde{R}) = \frac{1}{\pi^2\Gamma(d)}(2\tilde{R}\sqrt{1-\tilde{R}^2})^{d-1}, \tag{5}$$

and $B_d(\tilde{R})$ is given below for $d = 2$ in (26) and for $d = 3$ in (27). As seen from the comparison to simulations in Fig. 1 (see Ref. [60] for details on the simulations), the prediction in (4) for a disk in $d = 2$ is already excellent for $\mu = 100$ [it is crucial to include the subleading term $B_d(\tilde{R})$]. In the microscopic limit $\tilde{R} \to 0$ we obtain

$$\mathrm{Var}\,\mathcal{N}_{\mathcal{D}} \simeq \frac{1}{\pi^2\Gamma(d)}(k_F R)^{d-1}[\log(k_F R) + b_d], \tag{6}$$

where $k_F = \sqrt{2\mu}$. The leading term reproduces the free-fermion result [38–43] for a sphere and we further obtain the subleading term

$$b_d = 2\log 2 - \frac{\gamma_E}{2} + 1 - \frac{3}{2}\psi^{(0)}\left(\frac{d+1}{2}\right), \tag{7}$$

$\psi^{(0)}(x)$ being the digamma function. These results lead us to the conjecture (34) for the entanglement entropy of the subsystem $\mathcal{D}$ in any dimension for arbitrary smooth central potential, corroborated by exact results in $d = 1$.

Let us start with fermions on the infinite line in $d = 1$. It is useful to introduce the height field $h(x)$ [61], also called the

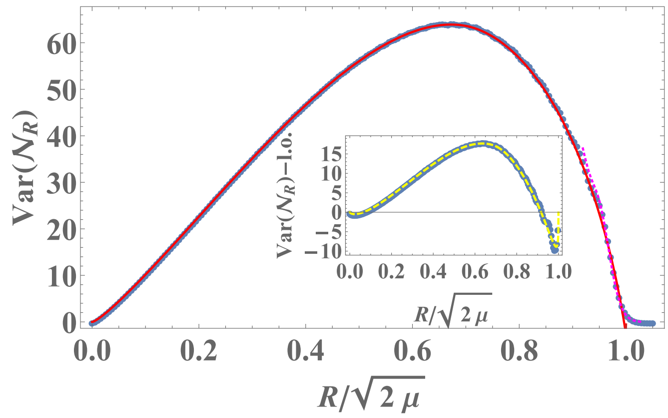

FIG. 1. Variance of $\mathcal{N}_{\mathcal{D}} = \mathcal{N}_R$ for a disk of radius R in $d = 2$, plotted vs $\tilde{R} = R/\sqrt{2\mu}$ for $\mu = 100$ corresponding to $N = \mu(\mu+1)/2 = 5050$. The simulations (symbols) [60] show excellent agreement with our predictions: In the bulk, with (4) (solid line), where $A_2(\tilde{R})$ is given in (5) and $B_2(\tilde{R})$ in (26), and near the edge $\tilde{R} = 1$, with the scaling form (28) (dotted line). Inset: The subleading term $B_2(\tilde{R})$ plotted vs $\tilde{R}$ (dashed line), compared to the simulations (symbols), the leading term $A_2(\tilde{R})\mu\log\mu$ being subtracted from the variance.

"index" in RMT [15,16,18,19], and its two-point covariance function $H(x, y)$, from which the variance of $\mathcal{N}_{\mathcal{D}}$ for any interval $\mathcal{D} = [a, b]$ is obtained as

$$h(x) = \mathcal{N}_{]-\infty,x]}, \quad H(x, y) = \mathrm{Cov}[h(x), h(y)], \tag{8}$$

$$\mathrm{Var}\,\mathcal{N}_{[a,b]} = H(a, a) + H(b, b) - 2H(a, b), \tag{9}$$

with $\mathrm{Var}\,\mathcal{N}_{]-\infty,a]} = \mathrm{Var}\,\mathcal{N}_{[a,+\infty[} = H(a, a)$, for a semi-infinite interval [62].

For N noninteracting fermions the correlation functions are obtained from the kernel

$$K_\mu(x, y) = \sum_{k=1}^{N}\psi_k^*(x)\psi_k(y), \tag{10}$$

where the $\psi_k(x)$ are the eigenstates of $\hat{H} = \frac{p^2}{2} + V(x)$. We will denote $\{\epsilon_k\}_{k=1,2,\ldots}$ the eigenenergies in increasing order. The mean density is $\rho(x) = K_\mu(x, x)$, and the n-point correlation is given by $\det_{n\times n} K_\mu(x_i, x_j)$ (see, e.g., Ref. [11]). This leads to the exact relation [60]

$$K_\mu(x, y)^2 = -\partial_x\partial_y H(x, y) + \delta(x - y)\rho(x), \tag{11}$$

from which we determine the height field covariance (8).

We now obtain an estimate of $K_\mu(x, y)^2$, and of $H(x, y)$, valid anywhere in the bulk in the large N limit. In this regime, the sum over k in (10) is dominated by $k \gg 1$ [63]. One can thus use the WKB asymptotics [64,65]

$$\psi_k(x) \simeq \frac{C_k}{\{2[\epsilon_k - V(x)]\}^{1/4}}\sin\left(\phi_k(x) + \frac{\pi}{4}\right), \tag{12}$$

where $\phi_k(x) = \int_{x^-}^{x} dz\sqrt{2[\epsilon_k - V(z)]}$ and $C_k^2 = \frac{2}{\pi}\frac{d\epsilon_k}{dk}$ is a normalization [66,67]. Inserting (12) in (10), we relabel $k = N - m$ around the Fermi energy $\mu = \epsilon_N$. Noting that the phase $\phi_N(x)$ at large N is also very large, we can expand $\phi_{N-m}(x) = \phi_N(x) - m\frac{d\phi_N(x)}{dN} + o(1) = \phi_N(x) - m\theta_x + o(1)$, where θ_x is given in (3), using $\frac{dN}{d\mu} \simeq \int_{x^-}^{x^+} \frac{dx}{\pi k_F(x)}$. Performing the geometric sum over m we obtain

$$K_\mu(x,y) \simeq \frac{d\mu/dN}{2\pi\sqrt{k_F(x)k_F(y)}} \sum_{\sigma=\pm 1} \frac{\sin[\tilde{\phi}_N(x) - \sigma\tilde{\phi}_N(y)]}{\sin[(\theta_x - \sigma\theta_y)/2]}, \quad (13)$$

with $\tilde{\phi}_N(x) = \phi_N(x) + O(1)$. In Eq. (13) the sine terms oscillate on microscopic scales. For $|x-y| \sim 1/k_F(x)$ the term $\sigma = 1$ dominates [68]. Using $\tilde{\phi}_N'(x) \simeq k_F(x)$ and $\frac{d\theta_x}{dx} = \frac{d\mu}{dN}\frac{1}{k_F(x)}$, one recovers the sine kernel

$$K_\mu(x,y) \simeq \frac{\sin(k_F(x)|x-y|)}{\pi|x-y|}, \quad (14)$$

valid on microscopic scales. On the other hand, for x, y well separated on macroscopic scales in the bulk $]x^-, x^+[$, taking the square of (13), one can neglect the cross term and replace the $\sin^2$ by 1/2, leading to

$$K_\mu(x,y)^2 \simeq \frac{(d\mu/dN)^2}{2\pi^2 k_F(x)k_F(y)} \frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}, \quad (15)$$

up to fast oscillating terms averaging to zero on scales larger than microscopic. Note that Eq. (15) is valid for any smooth potential: For the HO we also derived these estimates using the Plancherel-Rotach asymptotics for the Hermite polynomials [60]. Having obtained $K_\mu(x,y)^2$ in the two regimes, we use (11) to compute the height correlator.

(i) For x, y well separated in the bulk, i.e., $|x-y| \gg 1/k_F(x)$, the two-point height covariance is given by

$$H(x,y) \simeq \frac{1}{2\pi^2}\left(\log\left|\sin\frac{\theta_x+\theta_y}{2}\right| - \log\left|\sin\frac{\theta_x-\theta_y}{2}\right|\right), \quad (16)$$

up to $o(1)$ terms at large μ. One checks that (16) is consistent with (15) and (11) (in this regime the δ function does not contribute). Using (3), the right-hand side (rhs) in (16) vanishes when x is in the bulk and y reaches an edge $y = x^\pm$, and for $y \notin]x^-, x^+[$, $H(x,y) \simeq o(1)$ [60]. The rhs in (16) coincides with the correlator of the two-dimensional (2D) Gaussian free field (GFF) in the upper-half plane (with Dirichlet boundary conditions) along part of a circle $z = e^{i\theta_x}$, thus extending the result of Ref. [57] for the GUE-HO [69]. Similar connections to the GFF also emerge in recent approaches using inhomogeneous bosonization [32,72–74].

(ii) On microscopic scales, $|x-y| \sim 1/k_F(x)$, one uses the sine kernel (14) in the left-hand side of (11). The integration constants are fixed so that $H(x,y)$ for $|x-y| \gg 1/k_F(x)$ matches with the limit $y \to x$ in (16), leading to

$$H(x,y) \simeq \frac{1}{2\pi^2}\left[U(k_F(x)|x-y|) + \log\frac{2k_F(x)\sin\theta_x}{d\theta_x/dx}\right], \quad (17)$$

where

$$U(z) = \mathrm{Ci}(2z) + 2z\,\mathrm{Si}(2z) - \log z + 1 - 2\sin^2(z) - \pi z, \quad (18)$$

with $U(z \gg 1) = -\log z + o(1)$ and $U(z \ll 1) = 1 + \gamma_E + \log 2 - \pi z + z^2 + o(z^2)$. One checks, using $U''(z) = 2\sin^2 z/z^2$, that (17) is consistent with (11) (including the delta function) and K_μ given by the sine kernel (14), since $k_F(x) \simeq k_F(y)$ on microscopic scales. Using (9), it leads to the Dyson-Mehta behavior

$$\pi^2 \,\mathrm{Var}\,\mathcal{N}_{[a,b]} \simeq U(0) - U(k_F(a)|a-b|) \simeq \log k_F(a)|a-b| + c_2. \quad (19)$$

From (16), (17), and (9), we obtain our result (2) as well as for any a in the bulk,

$$H(a,a) = \mathrm{Var}\mathcal{N}_{[a,+\infty[} \simeq \frac{1}{2\pi^2}\left(\log\frac{2k_F(a)^2\sin\theta_a}{d\mu/dN} + c_2\right). \quad (20)$$

Expanding (20) for $a \to x^+$, $a < x^+$, one obtains [60] $H(a,a) \simeq \frac{1}{2\pi^2}[\frac{3}{2}\log(-\hat{a}) + c_2 + 2\log 2]$ for $-\hat{a} \gg 1$. Here, the edge scaling variable is $\hat{a} = (a - x^+)/w_N$, and $w_N = [2V'(x^+)]^{-1/3}$, the width of the edge regime [11], appears naturally. Inside the edge regime, i.e., for $\hat{a} = O(1)$, $H(a,a) \simeq \frac{1}{2}\mathcal{V}_2(\hat{a})$, where the scaling function $\mathcal{V}_2$ was defined in Refs. [20,21] for the HO, but is universal for a smooth potential [60]. The matching with the bulk for $\hat{a} \to -\infty$ obtained above agrees with known results for the HO-GUE [20,75,77].

For the HO, $x^\pm = \pm\sqrt{2\mu}$, $\theta_x = \arccos(-x/\sqrt{2\mu})$, and (16) agrees with the rigorous results for the GUE [57]. In this case (2) gives a general result [60] which agrees with known results in special cases [15,16,20,21,58].

Another important example is the inverse square well $V(x) = \frac{x^2}{2} + \frac{\alpha(\alpha-1)}{2x^2}$ for $x > 0$ and $\alpha \geqslant 1/2$. It corresponds [60] to the Wishart-Laguerre unitary ensemble (LUE) of random matrices [78] with the correspondence between fermion positions x_j and eigenvalues $\lambda_j \sim x_j^2$ [79–81]. One has $\mu = 2N + \alpha + 1/2$, hence $d\mu/dN \simeq 2$ and $\cos\theta_x = \frac{\mu - x^2}{\sqrt{\mu^2 - \alpha(\alpha-1)}}$. We focus on the interval $[0,a]$ and scale both $a = O(\sqrt{\mu})$ and $\alpha = O(\mu)$ in the large μ limit. This scaling, used below for d-dimensional central potentials, is also the standard large-N limit for Wishart matrices. Setting $\tilde{a} = a/\sqrt{2\mu}$ and $\lambda = \alpha/\mu$, one obtains from (20) in the bulk $|2\tilde{a}^2 - 1| < \sqrt{1-\lambda^2}$,

$$2\pi^2\,\mathrm{Var}\,\mathcal{N}_{[0,a]}^{\mathrm{LUE}} \simeq \log(\mu) + \log\left(4\tilde{a}\frac{\left(1-\tilde{a}^2-\frac{\lambda^2}{4\tilde{a}^2}\right)^{3/2}}{(1-\lambda^2)^{1/2}}\right) + c_2, \quad (21)$$

with the superscript LUE added for later convenience. A similar result was recently reported in the mathematics literature [82,83]. The result (16) also agrees with rigorous GFF results for the LUE [85,86]. We have extended these results to other cases related to RMT [60].

We now address a central potential $V(r)$ in $d > 1$ and focus on the number of fermions $\mathcal{N}_R$ in a spherical domain $\mathcal{D}$ of radius R centered at the origin. The single-particle Hamiltonian $\hat{H}$ commutes with the angular momentum $\hat{\boldsymbol{L}}$, and with $\hat{\boldsymbol{L}}^2$ of eigenvalues $\ell(\ell+d-2)$, $\ell = 0, 1, \ldots$, defining the sector of angular momentum ℓ. The eigenstates of $\hat{H}$ are obtained from those of a collection of 1D radial problems $\hat{H}_\ell = -\frac{1}{2}\partial_r^2 + V_\ell(r)$, $r \geqslant 0$, with potentials [80,87]

$$V_\ell(r) = V(r) + \frac{\left(\ell + \frac{d-3}{2}\right)\left(\ell + \frac{d-1}{2}\right)}{2r^2} \quad (22)$$

and eigenenergies $\epsilon_{n,\ell}$, each with degeneracy $g_d(\ell)$, which behaves as $g_d(\ell \gg 1) \simeq \frac{2\ell^{d-2}}{\Gamma(d-1)}$. We consider the N fermion ground state where all levels with $\epsilon_{n,\ell} \leqslant \mu$ are filled. In each sector ℓ, the levels $n = 1, \ldots, m_\ell$ are occupied, with $N = \sum_\ell g_d(\ell) m_\ell$, with $m_\ell = 0$ for $\ell > \ell_{\max}(\mu)$. Remarkably, we show [60] that the quantum joint probability of the radial positions $\{r_i\}_{i=1,\ldots,N}$ of the fermions decouples into a symmetrized product over the angular sectors. As a consequence, the cumulants $\langle \mathcal{N}_R^p \rangle^c$ for $p \geqslant 1$ are simply sums over the angular sectors as

$$\langle \mathcal{N}_R^p \rangle^c = \sum_{\ell=0}^{\ell_{\max}(\mu)} g_d(\ell) \langle \mathcal{N}_{[0,R]}^p \rangle_\ell^c, \tag{23}$$

where $\langle \mathcal{N}_{[0,R]}^p \rangle_\ell^c$ are the cumulants of $\mathcal{N}_{[0,R]}$ for the 1D potential $V_\ell(r)$ in (22) with m_ℓ fermions. In the large μ limit, the sum in (23) is dominated by large values of ℓ and m_ℓ, and, for $p > 1$, is effectively cut off at $\ell_c(\mu, R) \simeq R k_F(R) \leqslant \ell_{\max}$, where $k_F(r) = \sqrt{2[\mu - V(r)]}$. This allows us to use our results in 1D and to obtain the variance of $\mathcal{N}_R$ for a general central potential (see Ref. [60]).

We discuss here the HO $V(r) = r^2/2$, for which the density has a spherical support, with $\rho^{\rm bulk}(r) \sim (2\mu - r^2)^{d/2}$, and an edge at $r = r_e = \sqrt{2\mu}$ [88]. In this case $V_\ell(r)$ in (22) is the inverse square well studied above with $\alpha = \ell + \frac{d-1}{2}$. For large μ, the occupation numbers m_ℓ are determined by $\epsilon_{m_\ell,\ell} \simeq 2m_\ell + \ell \simeq \mu$. Hence, defining $\lambda = \ell/\mu$, one has $m_\ell \simeq \frac{\mu}{2}(1 - \lambda)$ for $\lambda < 1$ and $m_\ell = 0$ for $\lambda > 1$. The total number of fermions is thus

$$N \simeq \frac{\mu^d}{\Gamma(d-1)} \int_0^1 d\lambda (1-\lambda)\lambda^{d-2} = \frac{\mu^d}{\Gamma(d+1)}. \tag{24}$$

Substituting the result (21) with $a = R$, i.e., $\tilde{a} = \tilde{R} = R/\sqrt{2\mu}$, into (23) with $p = 2$, and approximating the sum by an integral, one obtains, using $\ell_c(\mu, R)/\mu = 2\tilde{R}\sqrt{1 - \tilde{R}^2}$,

$$\mathrm{Var}\,\mathcal{N}_R \simeq \frac{2\mu^{d-1}}{\Gamma(d-1)} \int_0^{2\tilde{R}\sqrt{1-\tilde{R}^2}} d\lambda\, \lambda^{d-2}\, \mathrm{Var}\,\mathcal{N}_{[0,R]}^{\rm LUE}. \tag{25}$$

Performing the integral over λ yields the result in (4) and (5) for the HO in the large μ limit. The coefficient $A_d(\tilde{R})$ has a maximum at $\tilde{R} = 1/\sqrt{2}$ for any $d > 1$, and vanishes at the edge as $A_d(\tilde{R}) \sim (1 - \tilde{R})^{(d-1)/2}$. The $O(\mu^{d-1})$ term B_d is obtained in Ref. [60] for general d. For $d = 2$ and $d = 3$ it reads

$$\begin{aligned} 2\pi^2 B_2(x) = {} & \log\left(\frac{|1 - 2x\sqrt{1-x^2}|}{1 + 2x\sqrt{1-x^2}}\right) \\ & + 2x\sqrt{1-x^2}\left\{\log\left[\left(\frac{64x}{1-2x^2}\right)^2 (1-x^2)^3\right]\right. \\ & \left. + 2\gamma_E - 2\right\} \end{aligned} \tag{26}$$

and

$$\begin{aligned} 2\pi^2 B_3(x) = {} & (1-2x^2)^2 \log|1 - 2x^2| \\ & + 4x^2(1-x^2)\{\log[8x(1-x^2)^{3/2}] + \gamma_E\}, \end{aligned} \tag{27}$$

respectively. $B_d(x)$ has a singularity $(1-x)^{\frac{d-1}{2}} \log(1-x)$ near the edge at $x = 1$. As in $d = 1$, there is an edge region of width $w_N = [2V'(r_e)]^{-1/3}$ where the variance becomes a universal function of $\hat{R} = (R - r_e)/w_N$,

$$\mathrm{Var}\,\mathcal{N}_R \simeq \left(\frac{r_e}{w_N}\right)^{d-1} \int_0^\infty \frac{d\xi\, \xi^{\frac{d-3}{2}}}{2\Gamma(d-1)} \mathcal{V}_2(\hat{R} + \xi). \tag{28}$$

Here, $(r_e/w_N)^{d-1}$ is the typical number of fermions in the edge region [11] and $\mathcal{V}_2$ is the above scaling function for $d = 1$, defined in Refs. [20,21]. For the HO, Eq. (28) matches, for $\hat{R} \to -\infty$, the behavior of $B_d(x)$ for $x \to 1^-$ [60]. Finally, the small R limit corresponding to free fermions, given in the introduction, can also be obtained directly [60] using the sine-kernel analog in d dimensions [11,42].

One can ask about higher cumulants of $\mathcal{N}_\mathcal{D}$. In $d = 1$, for potentials related to RMT they can be extracted from known Fisher-Hartwig asymptotics of Hankel and Toeplitz determinants [17,58,82,89]. In all cases we find for $n \geqslant 2$ [90],

$$\langle \mathcal{N}_{[a,b]}^{2n} \rangle^c = \kappa_{2n} + o(1), \quad \kappa_{2n} = (-1)^{n+1}(2n)!\frac{2\zeta(2n-1)}{n(2\pi)^{2n}}, \tag{29}$$

and $\langle \mathcal{N}_{[a,b]}^{2n+1} \rangle^c = o(1)$, where $\zeta(x)$ is the Riemann zeta function. This leads to two important observations. First, from very recent results [84], Eq. (29) also holds for the potential $V_\ell(r) \simeq \frac{\ell^2}{2r^2}$ even when $\ell \sim \mu$. Using our Eq. (23) we obtain [60] the cumulants of $\mathcal{N}_R$ for free fermions in dimension $d > 1$, with $k_F R \gg 1$,

$$\langle \mathcal{N}_R^{2n} \rangle^c = \frac{(k_F R)^{d-1}}{\Gamma(d)}[\kappa_{2n} + o(1)], \quad n \geqslant 2. \tag{30}$$

Second, since (29) coincides with the results from the sine kernel (and the Circular Unitary Ensemble) [6,17,22,27], it is natural to conjecture that these higher cumulants arise solely from fluctuations on microscopic scales and that (29) actually holds in $d = 1$ for any smooth potential $V(x)$ [91]. For $d > 1$, using $\ell_c(\mu, R) \simeq R k_F(R)$ in Eq. (23), our conjecture leads to $\langle \mathcal{N}_R^{2n} \rangle^c = \frac{(k_F(R)R)^{d-1}}{\Gamma(d)}[\kappa_{2n} + o(1)]$, a natural extension of our result for free fermions (30), where $k_F(R)$ now depends on R. In fact, for $V(r) = \frac{1}{2}r^2$ the argument is already close to being rigorous [92].

We now apply our results to the calculation of the bipartite Rényi entanglement entropy of a d-dimensional domain $\mathcal{D}$ with its complement $\overline{\mathcal{D}}$. It is defined for $q \geqslant 1$ as

$$S_q(\mathcal{D}) = \frac{1}{1-q} \ln \mathrm{Tr}[\rho_\mathcal{D}^q], \tag{31}$$

where $\rho_\mathcal{D} = \mathrm{Tr}_{\overline{\mathcal{D}}}[\rho]$ is obtained by tracing out the density matrix ρ of the system over $\overline{\mathcal{D}}$. For noninteracting fermions $S_q(\mathcal{D})$ can be expressed as a series,

$$S_q(\mathcal{D}) = \sum_{n \geqslant 1} s_n^{(q)} \langle \mathcal{N}_\mathcal{D}^{2n} \rangle^c, \tag{32}$$

in the cumulants of $\mathcal{N}_\mathcal{D}$, where $s_n^{(q)}$ are given in Ref. [27] and $s_1^{(q)} = \frac{\pi^2}{6}(1 + \frac{1}{q})$. In $d = 1$ this relation leads to the well-known result for the entropy of free fermions,

$$S_q^{\rm ff}([a,b]) \simeq \frac{q+1}{6q} \log(2k_F|a-b|) + E_q, \tag{33}$$

where E_q is given in Eq. (11) in Ref. [94] (see also Ref. [36]). Our conjecture for the higher cumulants in an arbitrary potential (central for $d > 1$) leads to

$$S_q(\mathcal{D}) = \frac{\pi^2}{6}\frac{q+1}{q}\mathrm{Var}\,\mathcal{N}_{\mathcal{D}} + \frac{[k_F(R)R]^{d-1}}{\Gamma(d)}[\tilde{E}_q + o(1)], \tag{34}$$

with $\tilde{E}_q = E_q - \frac{q+1}{6q}(1+\gamma_E)$. It holds in $d > 1$ for the sphere centered at the origin, and in $d = 1$ for any interval $\mathcal{D} = [a, b]$ with both a, b in the bulk [95]. In (34) the simple form of the second term arises from the common R dependence of the cumulants of order 4 and higher. This conjecture is corroborated by the rigorous results leading to (29) in $d = 1$ for the HO, the inverse square well, and the hard box [60]. It also agrees with existing results for $d = 1$ [32,52,94,96]. Owing to (30), Eq. (34) is exact for free fermions in $d > 1$. The leading term $S_q(\mathcal{D}) \propto R^{d-1}\log R$ at large R is consistent with the result obtained using the Widom conjecture applied to a spherical domain [38–41,43], and also with the rigorous proof in Ref. [97]. Here, in addition, we obtain the first correction $O(R^{d-1})$.

In conclusion, we obtained analytically the counting statistics and the entanglement entropy for $N \gg 1$ noninteracting fermions at temperature $T = 0$ in a general potential in $d = 1$, and a central potential in $d > 1$. They depend nontrivially on the shape of the potential, already at leading order in $d > 1$, e.g., in (4). These results can be extended to finite T [98] and it would be interesting to extend them to interacting particles, as was done recently for bosons [99].

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