Letter

Dimension of diffusion-limited aggregates grown on a line

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Diffusion-limited aggregation (DLA) has served for 40 years as a paradigmatic example for the creation of fractal growth patterns. In spite of thousands of references, no exact result for the fractal dimension D of DLA is known. In this Letter we announce an exact result for off-lattice DLA grown on a line embedded in the plane $D = 3/2$. The result relies on representing DLA with iterated conformal maps, allowing one to prove self-affinity, a proper scaling limit, and a well-defined fractal dimension. Mathematical proofs of the main results are available in N. Berger, E. B. Procaccia, and A. Turner, Growth of stationary Hastings-Levitov, arXiv:2008.05792.

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The diffusion-limited aggregation (DLA) model was introduced in 1981 by Witten and Sander [1]. The model has been shown to underlie many pattern forming processes including dielectric breakdown [2], two-fluid flow [3], and electrochemical deposition [4]. The model begins with fixing one particle at the center of coordinates in d dimensions, and follows the creation of a cluster by releasing a random walker from infinity, allowing it to walk around until it hits any particle belonging to the cluster. Once there, the incoming particle is attached to the growing cluster and a new one is released from infinity. The model was studied on and off lattice in several dimensions $d \geqslant 2$; DLA has attracted enormous interest over the years since it is a remarkable example of the spontaneous creation of fractal objects. It is believed that asymptotically (when the number of particles $N \to \infty$) the dimension D of the off-lattice cluster is very close to 1.71 [5,6], although there exists to date no rigorous proof for this fact. In addition, the model has attracted interest since it was among the first [7] to offer a true multifractal measure: The harmonic measure (which determines the probability that a random walker from infinity will hit a point at the boundary) exhibits singularities that are usefully described using the multifractal formalism [8,9]. Nevertheless, DLA still poses more unsolved problems than answers. It is obvious that a new language is needed in order to allow fresh attempts to explain the growth patterns, the fractal dimension, and the multifractal properties of the harmonic measure. In this Letter we announce an exact result on the fractal dimension of DLA grown on a fiber. This model was simulated on the lattice by Meakin in 1983 [10] with the numerical result that a typical tree with N particles reaches a height (radius of gyration) of the order of N^δ with

$$\delta \approx 0.665 \pm 0.03. \tag{1}$$

Rewritten in terms of the fractal dimension of the clusters, this translates to $D = 1.50 \pm 0.01$. Here, we show that the DLA grown on a line off lattice has an exact dimension $D = 3/2$.

The method used to establish this result is based on iterated conformal maps to grow a DLA cluster in a controlled fashion. Introduced by Hastings and Levitov in Ref. [11], the idea is to employ a mapping $\phi_{\lambda,\theta}(\omega)$ that maps the exterior of the unit circle to the exterior of unit circle with and added "bump" or "strike." This addition, whose linear size is λ, is placed on the unit circle at a uniformly distributed angle θ. Iterating this mapping one defines a conformal map $\Phi^n(\omega)$ according to

$$\Phi^n(\omega) \equiv \phi_{\lambda_1,\theta_1} \circ \phi_{\lambda_2,\theta_2} \circ \cdots \circ \phi_{\lambda_n,\theta_n}. \tag{2}$$

A major difficulty associated with the creation of the map for the classical example of DLA in two dimensions has precluded so far the use of this method to determine exactly the fractal dimension of the growing cluster. The first difficulty is that the linear size λ has to be judiciously chosen in each step to grow a *fixed size* addition to the cluster,

$$\lambda_n = \frac{\lambda_0}{|\Phi^{(n-1)'}(e^{i\theta_n})|}. \tag{3}$$

Note the originally in Ref. [11] the size of λ_n was allowed to vary, by taking the denominator in Eq. (3) to the power of $\alpha/2$. Thus the classical DLA model corresponds to $\alpha = 2$. A related difficulty lies in the monotonicity of the logarithmic capacity c_n. As $\omega \to \infty$, the Laurent expansion of Φ^n starts as

$$\Phi^n(\omega) = e^{c_n}\omega + O(1), \tag{4}$$

with $c_n > 0$ and is monotonic increasing in n. In fact, one can show that $c_n = (\log n)/D$ [6,12]. Thus one needs to normalize the size of λ more and more as the cluster grows.

Growing a DLA cluster on a fiber removes these difficulties altogether. One can map the upper half plane to the upper half plane with a strike of size 1 above x using the map $\phi_x(\omega)$,

$$\phi_x(\omega) = x + \sqrt{(\omega - x)^2 - 1}. \tag{5}$$

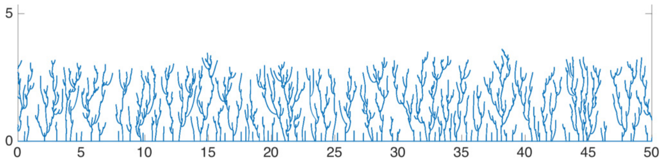

FIG. 1. Computer simulation of cluster growth in a window of the real axis. This simulation employed 1000 particles of size $\lambda = 0.3$, accreted on the base of a cylinder of size 100, and see below the discussion on cylinder growth.

To represent the growth of a cluster on a fiber, one considers the whole upper half plane and orders the arrival times of particles according to a homogeneous Poisson point process of intensity 1. Focus on a window of the real axis of length R and mark the arrivals to particles into positions $x_1, x_2, \dots, x_k \dots$ in this window, at times $0 < t_1 < t_2 < \cdots < t_k \cdots$. Define

$$F_t^{(R)}(\omega) = \begin{cases} \omega & \text{for } 0 \leqslant t < t_1, \\ \phi_{x_1} \circ \cdots \phi_{x_k}(z) & \text{for } t_k \leqslant t < t_{k+1}. \end{cases} \tag{6}$$

The wanted process is finally defined by taking the limit $F_t(\omega) = \lim_{R\to\infty} F_t^{(R)}(\omega)$. This process was proven to exist in Ref. [13], was denoted "stationary Hastings-Levitov(0)," and shown to define a conformal map. In addition, this process is invariant to horizontal shifts. See Fig. 1 for a computer simulation of the process. Note that this limit must be employed to define the process, as a uniform distribution on the infinite line does not exist. In other words, as the window increases in size, so does the number of arrivals.

The relative simplicity of the resulting process is demonstrated by the Laurent expansion of $F_t(\omega)$. By a tedious but straightforward calculation one shows that in the limit $\omega \to \infty$,

$$F_t(\omega) = \omega + \frac{i\pi}{2}t + O(1/\omega). \tag{7}$$

This is important since it implies that no repeated normalization of the strike sizes is necessary in this process. Thus employing the power $\alpha = 0$ is sufficient to generate a cluster growth in which the added particles to the physical domain remain of fixed size. Moreover, it implies that redefining the map by inverting the order of iterations in Eq. (6) results, in a fixed time t, in an inverted growth where the last particles grown normally appear first, and further particles push them up in the half plane to end up with a cluster sharing the same distribution as the original one.

Another immediate consequence of Eq. (7) is that the average height h_c of the growth sites (known as the half-place capacity) can be determined from the second term in the Laurent expansion [14],

$$h_c = -i \lim_{\omega\to\infty} [F_t(\omega) - \omega] = \frac{\pi}{2}t. \tag{8}$$

In other words, the positions in which random walkers coming from infinity meet the growing cluster increase on the average linearly with time.

Having control on the average height of the arrivals along the imaginary axis, we next focus on the fluctuations of these arrivals along the real axis. To this aim we first consider the complex integral

$$\int_{-\infty}^{\infty} dx |\phi_x(\omega) - \omega|^2 \leqslant \frac{C}{1 + \operatorname{Im}\omega} \leqslant C, \tag{9}$$

where C is a constant independent of ω and the last inequality stems from the fact that we work in the upper half plane. Using this we can immediately derive a sharp estimate for the expectation of the fluctuations of the real part of the arrival points $\mathbf{E}(|F_t(\omega) - \omega - \frac{i\pi}{2}t|^2)$:

$$\mathbf{E}\left(\left|F_t(\omega) - \omega - \frac{i\pi}{2}t\right|^2\right) \equiv \mathbf{E}\left(\int_0^t ds \int_{-\infty}^{\infty} dx\, |\phi_x(\omega)F_s(\omega) - F_s(\omega)|^2\right) \leqslant Ct. \tag{10}$$

We can deduce from Eq. (7) that the fluctuations in the real position of the arrival of new particles grow as $\sqrt{t}$. One should know, however, that this is an estimate of the global fluctuation over the whole real axis rather than on a single growing tree. To achieve a statement about the fractal dimension requires a local result on the fluctuations.

In order to find the number of particles added to a given tree in the cluster we consider the harmonic measure of an interval on the real axis. Since particles are being added according to a homogeneous Poisson process, the harmonic measure must be proportional to the length of the interval. Denote the harmonic measure of the interval $[a, b]$ at time t as $\mathcal{H}_{[a,b]}(t)$,

$$\mathcal{H}_{[a,b]}(t) \equiv F_t^{-1}(b) - F_t^{-1}(a). \tag{11}$$

We will demonstrate now that for any chosen a and b the preimages $F_t^{-1}(a)$ and $F_t^{-1}(b)$ are diffusion processes for times of the order $t < (b-a)^2$. At time of the order of $(b-a)^2$ these diffusion processes collide, at which point in time the harmonic measure $\mathcal{H}_{[a,b]}(t)$ becomes so small that in later times no particles can reach this interval [15]. Physically this means that all the trees that grow out of the interval $[a, b]$ become shadowed by higher and broader trees and eventually no new particle can ever reach these trees. We note in passing that this result means that any set of trees that starts to grow from any finite-size interval will eventually get shadowed and stop growing. This is a warning that simulating on a fiber with periodic boundary conditions is different, and will result in a single tree occupying all of the harmonic measure. It is remarkable that Meakin [10] had the intuition to terminate his simulation at the "right" time to get the correct result for this growth process. Note that this distinguishes the present process from standard DLA growth in which there is a part of every tree that keeps growing forever.

The way that the Poisson process is defined it is clear that the number of particles arriving into any given area in the upper half plane is proportional to that area. We know now that if we choose trees that start growing from an interval $[a, b]$ of the order of unity, and the condition on the harmonic measure not vanishing before or at time t, their typical height will be of the order of t. Moreover, tracing the area bounded between the curves defined by $F_s^{-1}(a)$ and $F_s^{-1}(b)$ for $s \in [0, t]$, we know that this area scales as $t \times \sqrt{t}$, and therefore the number of incoming particles belonging to the trees that survives until

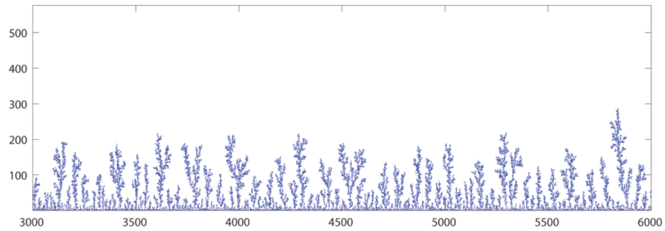

FIG. 2. Computer simulation of cluster growth on lattice in a cylinder of circumference 8000 at $t = 30$ (i.e., 240 000 particles).

time t is indeed proportional to $t^{3/2}$. This provides the desired result that the height scales as $N^{2/3}$, or

$$\boxed{\delta = 2/3}\,. \qquad (12)$$

To discuss the dimension of the cluster we stress that the growing trees are not self-similar but rather self-affine. The Hausdorff dimension therefore requires covering the set with different rescaling in the real and the imaginary directions. The scaling is the natural one of a random walk, i.e., rescaling by t in the imaginary direction and by $\sqrt{t}$ in the real direction. The result then, in the limit $t \to \infty$, under the above affine scaling, is that $D = 3/2$. A rigorous proof of this result is Theorem 7.6 in Ref. [13].

All the results presented above pertain to growth in all the upper half plane, and the relation to growth on a finite fiber as executed in Ref. [10] must be discussed. Moreover, the simulations presented in Ref. [10] were done *on lattice* whereas the considerations above were all for random walks *off lattice*. Consider then a cylinder of circumference of length N (in units of the lattice constant) and infinite height. The process then involves sending off $t \times N$ random walkers from infinity (the random walkers are periodic with period N in the direction of the x axis). To proceed we invoke the rigorous proof (cf. Refs. [16,17]) that in the limit $N \to \infty$ the cluster that includes $t \times R$ particles grown over any finite interval of length $R \ll N$ is equivalent in all properties to a cluster grown over an interval of length R belonging to the infinite real axis. Accordingly, Meakin's simulation can be considered relevant for DLA on-lattice growth on an infinite line. Since for the off-lattice growth we could show that trees that contain t particles are of height of the order of $t^{2/3}$, we now elaborate on Meakin's simulations and show an equivalent result for the on-lattice simulation for large enough trees.

The result of computer simulations on a cylinder of circumference 8000 at $t = 30$ (i.e., 240 000 particles) is shown in Fig. 2. The figure shows the cluster growth in the interval [3000,6000]. Contrary to Meakin who considered the radius of gyration of the whole cluster, we compute the height versus the number of particles belonging to individual trees. To accomplish this we identify each tree by the location of its root, paying attention to the particles added to the same tree starting from this root. A log-log plot of the heights versus the logarithm of the number of particles belonging to individual trees is shown in Fig. 3. The expected slope of 2/3 is obtained asymptotically for large trees. Thus we can conclude from the present simulation that the on-lattice model has the same Brownian fluctuations for the width of the growing trees. A tree will arrive to a given height having a width that

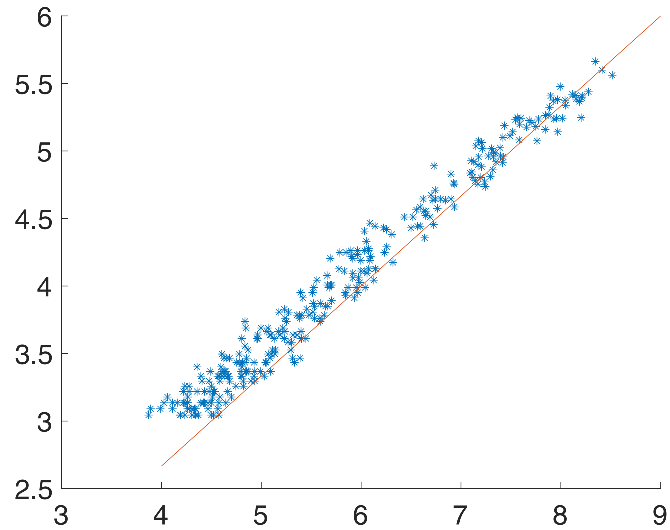

FIG. 3. Measurement of the logarithm of the heights h vs the logarithm of the number of particles belonging to individual trees n, using the data from Fig. 2. The line is a guide to the eye showing the convergence to slope 2/3 for large trees.

is determined by the distance between two Brownian paths conditioned on nonintersection.

Finally, we should note that the pure Brownian scaling will fail in a finite cylinder when the simulation time gets too long. When a given tree reaches the height of $\sqrt{N}$ then particles that might typically attach to this tree will already feel the periodic boundary conditions. One expects that such a tree will occupy eventually the entire harmonic measure and all the other trees will not be able to increase their width in subsequent times. Similar caution should be exercised for a growth on a finite-sized fiber (without boundary conditions). There the edges of the fiber will act as singular attracting points, and the growth far away from the edges will exhibit Brownian scaling only for a finite time. An example of a simulation of growth on a finite fiber is shown in Fig. 4.

In summary, the DLA process over the real axis provides a relatively transparent example for the employment of iterated conformal maps to represent the cluster growth. The reason for the relative ease is that the size of the strike does not depend on the order of iteration, in contrast to the classical off-lattice DLA in two dimensions where the strike size changes in every iteration to conform with the addition of a fixed size particle in the physical domain. As a consequence, one can derive in the present case an exact result for the

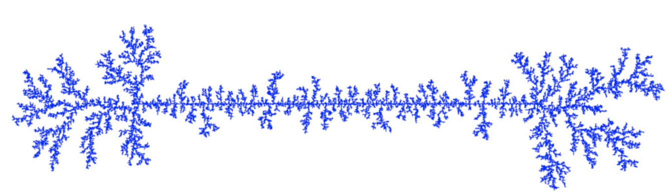

FIG. 4. A typical cluster grown on a finite fiber embedded in a two-dimensional lattice. Here, 1000 particles accreted to a line of length 300. Only far away from the edges the growing trees exhibit Brownian scaling.

growth rate and fractal dimension of the whole cluster or of individual trees. We note in passing that the dimension $3/2$ was offered by Kesten as a rigorous lower bound to the dimension of DLA grown on the square lattice in two dimensions [18,19]. It is known that DLA grown on the square lattice in two dimensions looks asymptotically as a cross with four long arms [20–22]. While the fractal dimension of the whole cluster appears to exceed $3/2$, it is not impossible that further analysis might lead to the possibility that the Brownian scaling is appropriate for individual trees growing far away from the tips, for reasons akin to the discussion offered above.

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