First passage under restart for discrete space and time: Application to one-dimensional confined lattice random walks

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First passage under restart has recently emerged as a conceptual framework to study various stochastic processes under restart mechanism. Emanating from the canonical diffusion problem by Evans and Majumdar, restart has been shown to outperform the completion of many first-passage processes which otherwise would take longer time to finish. However, most of the studies so far assumed continuous time underlying first-passage time processes and moreover considered continuous time resetting restricting out restart processes broken up into synchronized time steps. To bridge this gap, in this paper, we study discrete space and time first-passage processes under discrete time resetting in a general setup without specifying their forms. We sketch out the steps to compute the moments and the probability density function which is often intractable in the continuous time restarted process. A criterion that dictates when restart remains beneficial is then derived. We apply our results to a symmetric and a biased random walker in one-dimensional lattice confined within two absorbing boundaries. Numerical simulations are found to be in excellent agreement with the theoretical results. Our method can be useful to understand the effect of restart on the spatiotemporal dynamics of confined lattice random walks in arbitrary dimensions.

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I. INTRODUCTION

There are certain benefits in starting anew. Restart or resetting, a new topic in statistical physics, teaches us that stopping intermittently and starting over again and again can increase the chances of reaching a desired outcome. Although it seems nonintuitive at a first glance, the basic physics is rather simple. Restart works by truncating the tails of long detrimental trajectories thus rendering the large stochastic fluctuations regular. In statistical physics, this mechanism was first observed in the canonical diffusion model with stochastic resetting by Evans and Majumdar [1,2]. Since then, restart has emerged as a very active avenue of research in statistical physics [1–8] and generic stochastic process [9–15] due to its numerous applications spanning across interdisciplinary fields ranging from computer science [16,17], population dynamics [18], queuing theory [19], chemical and biological process [20–23], foraging [24], and search processes with rare events [25–27]. We refer to a recent review [28] (and references therein) for a detailed account of the subject. The subject has also seen advances through single particle experiments using optical tweezers [29,30].

A hallmark of resetting is its ability to reduce the mean and fluctuations of the first-passage time to a target. This observation has led to many first-passage studies under resetting over the years (see Ref. [28] for details). In particular, a framework, namely, first passage under restart, has been quite instrumental to study generic stochastic processes under various restart mechanisms [31]. The power of this approach lies on the fact that it allows one to compute general expressions for important metrics, namely, the moments and the distribution of completion time of restarted process regardless of the specifics of the underlying first-passage process [31–38]. Furthermore, it allows one to discover many universal phenomena such as a criterion for restart to be beneficial [31,35,39,40], a globally dominating restart mechanism [31,41], a general Landau like theory for restart transitions [42] and conditions on other quantiles of first-passage processes [43] that emerge as an effect of restart.

A Brownian walker on a real line subject to a resetting to the origin at certain rate is perhaps the most quintessential example of restart phenomena [1]. In this problem, one is interested in the first-passage time to a target which is located at a given distance and it was shown that restart can expedite the completion. This problem was then studied in higher dimensions [44] and complex geometries [45,46], in the presence of multiple targets [36,47], and also in the presence of generally distributed resetting time density [7,31,35]. It is important to emphasize that resetting occurring at a constant rate essentially implies that the waiting times between resetting events are taken from an exponential distribution. In other words, here, resetting is a continuous time Markov process. Notably, majority of the studies in the field spanned around the continuous time Brownian motion and its variants, e.g., scaled diffusion [48], underdamped [49], and random acceleration process [50] (also see Ref. [28] for other model systems). A general version of random walk (RW), namely, the mesoscopic continuous time random walks with heavy tailed jump distributions (such as Lévy flights) were studied

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in Refs. [51,52]. Continuous-time RWs have also been studied under various resetting strategies (power law, Markovian, etc.) [53–56]. In a recent work [57], a continuous time lattice RW was considered in the presence of resetting conducted at a rate and stationary probability distribution, mean first-passage time were computed for an infinite and semi-infinite chain. Interacting continuous time lattice RWs with exclusion were studied in the presence of stochastic resetting [58,59]. Continuous time RWs on a network in the presence of stochastic resetting were studied in Refs. [60,61]. Arguably, most of these models also use resetting to be a continuous time process for the computations there have neater mathematical structure.

Nonetheless, there are some recent works which shed light on the discrete time lattice RW with discrete resetting. A discrete time RW on a real line (but with jumps drawn from a continuous and symmetric distribution) in the presence of resetting was studied in Ref. [9]. A discrete time unidirectional RW, namely, Sisyphus lattice RW was studied in the presence of the resetting probability which can be random or site-dependent [62]. They analyzed the first-passage time and survival probabilities for the walker to reach a certain threshold in the lattice. Recently, preferential visit models have been introduced to generalize resetting with memory of history [63]. This is most naturally illustrated by a discrete time lattice RW which with some reset probability is returned to its previous position at a randomly selected time from the past. More precisely, the walker relocates to a previously visited site with a probability proportional to the number of past visits to that site. It was shown that the RWs perform slow sub-diffusion due to the dynamics of memory-driven resetting [63,64]. The model above was further studied in the presence of a single defect site where the RW stays with a finite probability [65,66]. Another discrete time RW on a lattice was considered in Ref. [67] where resetting with a probability relocated the walker to the previous maximum. Very recently, discrete time random walks on arbitrary complex networks was studied in the presence of stochastic resetting [68]. The authors studied the stationary probability distribution as well as the mean and global first-passage times. Unlike the continuous time and space first-passage processes under continuous time resetting, the discrete counterparts are only handful. Also it is not apparent how the nonstochastic restarts can ramify the first-passage processes. To this end, we build a general framework for discrete first-passage processes under discrete time resetting. Crucially, we need not specify the details of the underlying first-passage process or the resetting time. The derivation follows the steps from Ref. [31] developed by one of the authors of the current paper. Importantly, this approach holds even when both the first-passage and restart processes are not necessarily Markovian as long as the memory from the past is erased after each resetting. Pertaining to this property, renewal framework has been used extensively in the above-mentioned fields along with in stochastic thermodynamics [69], quantum mechanics [70] and nonlinear dynamics [71]. We provide working formulas for the moments and the probability density function for the first-passage time under restart. We then derive a condition which asserts when restart is going to expedite the completion. We employ our results to the application of 1D lattice RWs in confined geometry under two different restart strategies.

Lattice RWs are a special class of Markov processes which were popularized following Pólya's seminal work on the dimensionality dependence of the recurrence probability, that is, the probability that a random walker on an *infinite space d*-dimensional lattice eventually returns to its starting point [72–77]. Needless to say that the subject is now a text book material and the applications are myriad. Somewhat surprisingly, and in contrast to Brownian walks, the space-time dependence of the *confined lattice walk* probability and first-passage time quantities has been accessible mainly via computational techniques due to combinatorial hindrances and only a few exact results exist [78–80]. Although for 1D confined domains, the time-dependent propagator for absorbing boundaries [81] and periodic domains [78] were known, no analytical results were known for mixed or reflective boundary conditions in 1D and in high dimensions for all the above-mentioned boundary conditions. Only recently, Giuggioli has derived exact results for the space and time dependence of the occupation probability, first-passage time probability for confined Pólya's walks in arbitrary dimensions with reflective, periodic, absorbing, and mixed (reflective and absorbing) boundary conditions along each direction [82]. Further generalizations were made by Sarvaharman and Giuggioli for the biased RWs with different boundary conditions [83]. Our derivation shows that the solution for the restarted process can be given in terms of the solution for the underlying process in accordance with many other previous works (see, e.g., Refs. [6–10,31]). This often reduces the overall complexity since one may not need to solve the master equations for position density function or the backward Fokker-Planck equations for the survival probability with resetting terms. Instead, one can directly use some of the existing solutions for the underlying process in the renewal relations. Taking advantage of this, we show here, how these very recent results by Giuggioli and coauthors naturally set the perfect stage for us to employ them directly when restart is involved.

The paper is organized as follows. In Sec. II, we build up the framework of first-passage step under restart, derive the working formulas for the statistical moments, and the probability mass function for the completion time. In the consecutive subsections, we discuss two different restart strategies, namely, the geometric and sharp protocols. We herein derive the sufficient criterion for geometric restart to be beneficial for any first-passage process. In Sec. III, we apply the framework to one-dimensional lattice RWs. In Sec. II C, we discuss the simple random walk, while in Sec. III B we present the biased random walk in the presence of the above-mentioned restart strategies. Our conclusions are summarized in Sec. IV. Some of the derivations from the main text and additional discussions are reserved for the Appendix. Before we proceed, it will be useful to introduce the notations used throughout the paper. We will use $P_X(x)$, $\langle X\rangle$, σ_X^2, and $G_X(z) \equiv \langle z^X\rangle$ to denote, respectively, the probability mass/density function (PMF/PDF), mean/expectation, variance, and the probability generating function (PGF) of a discrete random variable X taking values in the nonnegative integers.

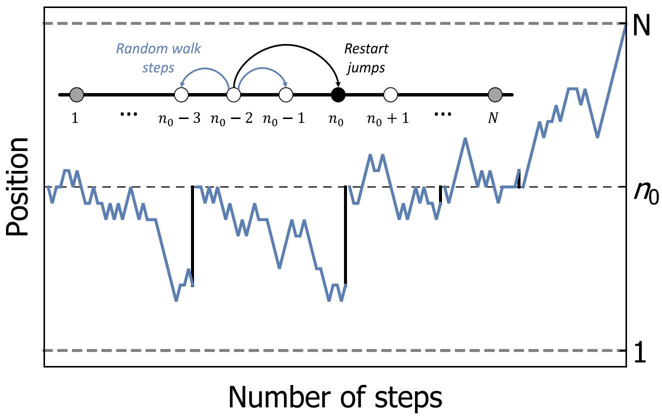

FIG. 1. Schematic of a lattice random walker in a 1D confined geometry under discrete restart. First passage occurs as soon as the walker reaches one of the boundaries located at 1 and N. The restart coordinate is n_0, same as the initial condition.

II. FIRST-PASSAGE STEP UNDER RESTART

Consider a generic discrete step first-passage process that starts at the origin and, if allowed to take place without interruptions, ends after a random number of steps N (see Fig. 1). The process is, however, restarted after some random number of steps R. Thus, if the process is completed prior, or at the same time as the restart, then we mark a completion of the event. Otherwise, the process will start from scratch and begin completely anew. This procedure repeats itself until the process reaches completion. Denoting the random completion number of steps of the restarted process by N_R, it can be seen that

$$N_R = \begin{cases} N & N \leqslant R, \\ R + N_R' & N > R, \end{cases} \tag{1}$$

where N_R' is an independent and identically distributed copy of N_R. Equation 1 is the central renewal equation for first-passage step under restart and assumes that after each restart, the memory is erased from the previous trial. To obtain the mean number of steps for the restarted process, we note that Eq. (1) can be written as $N_R = \min(N, R) + I\{N > R\}N_R'$, where $I\{N > R\}$ is an indicator random variable that takes the value 1 when $N > R$ and zero otherwise. Taking expectations on the both sides and using that N and R are independent of each other, we find the *mean completion time* under restart to be

$$\langle N_R \rangle = \frac{\langle \min(N, R) \rangle}{\Pr(N \leqslant R)}. \tag{2}$$

The numerator can be computed by noting that the probability $\Pr(\min(N, R) > n) = \Pr(N > n)\Pr(R > n)$ and thus

$$\begin{aligned} \langle \min(N, R) \rangle &= \sum_{n=0}^{\infty} \Pr[\min(N, R) > n] \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} P_N(k) \right) \left(\sum_{m=n+1}^{\infty} P_R(m) \right), \end{aligned} \tag{3}$$

where recall that $P_N(n)$ and $P_R(n)$ are the probability density functions for the first-passage and restart process, respectively. The denominator in Eq. (2) can also be computed easily:

$$\Pr(N \leqslant R) = \sum_{n=0}^{\infty} P_N(n) \sum_{m=n}^{\infty} P_R(m). \tag{4}$$

We now turn our attention to derive the generating function for the restarted process. The probability generating function of the discrete random variable N_R taking values in the non-negative integers $0, 1, \ldots, n$, is defined as

$$G_{N_R}(z) \equiv \langle z^{N_R} \rangle = \sum_{n=0}^{\infty} P_{N_R}(n) z^n, \tag{5}$$

where $P_{N_R}(n)$ is the probability mass function of N_R. It will now prove useful to introduce the following conditional random variables

$$N_{\min} \equiv \{N | N = \min(N, R)\} = \{N | N \leqslant R\}, \tag{6}$$

$$R_{\min} \equiv \{R | R = \min(N, R)\} = \{R | N > R\}, \tag{7}$$

with their respective densities

$$P_{N_{\min}}(n) = P_N(n) \frac{\sum_{m=n}^{\infty} P_R(m)}{\Pr(N \leqslant R)}, \tag{8}$$

$$P_{R_{\min}}(n) = P_R(n) \frac{\sum_{m=n+1}^{\infty} P_N(m)}{\Pr(N > R)}, \tag{9}$$

where $\Pr(N > R) = 1 - \Pr(N \leqslant R)$. Using the renewal Eq. (1) and the new random variables in Eqs. (6)–(7), we can write (see Appendix A)

$$G_{N_R}(z) = \Pr(N \leqslant R) \langle z^{N_{\min}} \rangle + \Pr(N > R) \langle z^{R_{\min} + N_R'} \rangle. \tag{10}$$

Now using the fact that N_R' is an independent and identically distributed copy of N_R in above, we arrive at the following expression for the generating function of the restarted process,

$$G_{N_R}(z) = \frac{(1 - \Pr(N > R)) G_{N_{\min}}(z)}{1 - \Pr(N > R) G_{R_{\min}}(z)}, \tag{11}$$

where $G_{X_{\min}}(z)$ is the generating function for the random variable $X_{\min}$. Equation (11) is extremely useful since it allows one to compute *all the moments*,

$$\langle N_R^k \rangle = \left(z \frac{\partial}{\partial z} \right)^k G_{N_R}(z)|_{z=1^-}, \tag{12}$$

and, importantly, also the *probability density function* of N_R,

$$P_{N_R}(n) = \Pr(N_R = n) = \frac{G_{N_R}^{(n)}(0)}{n!}, \tag{13}$$

where $\langle N_R^k \rangle$ is the kth moment and $G_X^{(n)}(z)$ is the nth derivative of $G_X(z)$ with respect to z, and $n = 0, 1, 2....$ It is important to emphasize that in continuous time setups, deriving the first-passage time density for the restarted process requires Laplace inversions, and thus it remains intractable in most of the cases (except some asymptotic limits [1]). In stark contrast, in discrete setup, computation of the first-passage time density requires only derivatives of the generating function as seen in Eq. (13), and thus is more accessible. It is easy to see that the expression for the mean in Eq. (2) can be easily recovered using the generating function given in Eq. (11) and noting

$$\langle N_R\rangle = z\frac{\partial G_{N_R}(z)}{\partial z}\Big|_{z=1^-}$$
$$= \frac{z(1-\Pr(N>R))(1-\Pr(N>R)G_{R_{\min}}(z))G'_{N_{\min}}(z)}{(1-\Pr(N>R)G_{R_{\min}}(z))^2}\Bigg|_{z=1^-}$$
$$-\frac{\Pr(N>R)G_{N_{\min}}(z)G'_{R_{\min}}(z)}{(1-\Pr(N>R)G_{R_{\min}}(z))^2}\Bigg|_{z=1^-}. \quad (14)$$

Substituting $z\to 1^-$, and utilizing the relations $G_{N_{\min}}(1^-) = G_{R_{\min}}(1^-) = 1$, $G'_{N_{\min}}(1^-) = \langle N_{\min}\rangle$, $G'_{R_{\min}}(1^-) = \langle R_{\min}\rangle$, obtained from Eqs. (8) and (9), we get

$$\langle N_R\rangle = \frac{\Pr(N\leqslant R)\langle N_{\min}\rangle + \Pr(N>R)\langle R_{\min}\rangle}{\Pr(N\leqslant R)}$$
$$= \frac{\langle \min(N,R)\rangle}{\Pr(N\leqslant R)}, \quad (15)$$

which is indeed Eq. (2). Similarly, the *second moment* can be computed using the following relation:

$$\langle N_R^2\rangle = \left(zG'_{N_R}(z) + z^2G''_{N_R}(z)\right)\big|_{z=1^-}. \quad (16)$$

Higher-order moments can also be computed in a similar way. So far, we have kept our formalism extremely general without specifying the forms of the restart time density. In what follows, we will use two different distributions for the restart time, namely, the geometric and the sharp distribution.

A. Geometrically distributed restart

Consider a resetting number step taken from a geometric distribution with parameter $p\,(0<p<1)$,

$$P_R(n) = (1-p)^n p,\ n\geqslant 0. \quad (17)$$

In other words, restart would take place with a probability p after n unsuccessful trials. Notably, this distribution is the discrete analog of the exponential distribution, being one discrete distribution possessing the memory-less property [84]. For this distribution, following Appendix B, we have

$$\langle \min(N,R)\rangle = \frac{1-p}{p}[1-G_N(1-p)], \quad (18)$$
$$\Pr(N\leqslant R) = G_N(1-p). \quad (19)$$

Substituting Eqs. (18) and (19) into the formula for the mean of the restarted process given in Eq. (2) yields

$$\langle N_R\rangle = \frac{1-G_N(1-p)}{G_N(1-p)}\frac{1-p}{p}. \quad (20)$$

The above expression can be understood intuitively from the knowledge of the mean number of restart events till the first passage and the mean number of steps taken between any two restart events [here, we have assumed that restart can also occur at $n=0$ with probability p – see Eq. (17)]. At the limit $p\to 0^+$, namely, when restart is rare, Eq. (20) reduces to (following L'Hospital's rule) $\lim_{p\to 0^+}\langle N_R\rangle = G'_N(1^-) = \langle N\rangle$.

We now turn to the derivation of the PGF of the restarted process under geometric restart. We first compute the PGFs for the conditional random variables $N_{\min}$ and $R_{\min}$. Following Appendix B, we find

$$G_{N_{\min}}(z) = \frac{G_N[z(1-p)]}{G_N(1-p)}, \quad (21)$$
$$G_{R_{\min}}(z) = \frac{p\{G_N[z(1-p)]-1\}}{[(1-p)z-1][1-G_N(1-p)]}. \quad (22)$$

Substituting Eqs. (21)–(22) into Eq. (11) we find

$$G_{N_R}(z) = \frac{[1-(1-p)z]G_N[(1-p)z]}{(1-p)(1-z)+pG_N[(1-p)z]}, \quad (23)$$

from which one can derive the probability mass function of the restarted process by taking the derivatives of the generating function using Eq. (13). Another equivalent way is to derive the moments from PGF of the survival function $Q_{N_R}(n) = \Pr[N_R>n]$, i.e., probability that the process has not ended by the time step n with restart steps. This is given by (Appendix C)

$$G_{Q_{N_R}}(z) \equiv \sum_{n=0}^{\infty} z^n Q_{N_R}(n) = \frac{(1-p)G_{Q_N}[(1-p)z]}{1-pG_{Q_N}[(1-p)z]}. \quad (24)$$

We also refer to Eq. (C2) which derives a relation between the PGFs of the survival and first passage function in discrete case. A similar relation like in Eq. (24) was first derived in Ref. [9] (only difference is that our process can also start with a restart event in zero step). This difference becomes more apparent in the mean first passage of the restarted process, namely, in Eq. (20), which was also obtained in Ref. [9]. We mention that the mean can also be obtained from the survival PGF by noting $\langle N_R\rangle = G_{Q_{N_R}}(z\to 1^-)$, as was done in Ref. [9]. We now compute the second moment using Eq. (23) in Eq. (16)

$$\langle N_R^2\rangle = \frac{(1-p)G_N(1-p)[3p-2-pG_N(1-p)]}{p^2G_N^2(1-p)}$$
$$-\frac{2(1-p)^2(pG'_N(1-p)-1)}{p^2G_N^2(1-p)}. \quad (25)$$

One of the key properties of the restart is its ability to lower the underlying mean first-passage time and this often leads to an optimal value of the restart rate at which the mean time reaches a global minimum. To elucidate this in the discrete setup, we now study whether there exists a sufficient enough criterion under which restart is always beneficial.

B. A criterion for geometric restart to be beneficial

To derive the criterion, we first observe a first-passage time process and turn on an infinitesimal restart probability $p\to 0^+$. If restart has to lower the mean time, then it is sufficient enough to check whether $d\langle N_R\rangle/dp|_{p\to 0}<0$, where $\langle N_R\rangle$ is given by Eq. (20). A small p expansion of $\langle N_R\rangle$ gives

$$\langle N_R\rangle \approx G'_N(1) + \tfrac{1}{2}p[2G'_N(1)^2 - 2G'_N(1) - G''_N(1)]. \quad (26)$$

Now noting that $G'_N(1) = \langle N\rangle$, $G''_N(1) + G'_N(1) - G'_N(1)^2 = \mathrm{Var}(N)$ and substituting into Eq. (26), the criterion can be recast as

$$\mathrm{CV}^2 > 1 - \frac{1}{\langle N\rangle}, \quad (27)$$

where $\mathrm{CV}^2 = \frac{\mathrm{Var}(N)}{\langle N\rangle^2}$ is the squared coefficient of variation of the underlying first-passage process. This essentially means that whether restart would favour a completion depends on the underlying first-passage time process. Moreover, this criterion is also not sensitive to the entire density, but only to the first two moments of the underlying process. We refer to a similar criterion that was derived for the continuous stochastic resetting case in Ref. [31].

It is, however, clear that repeated restart will only prolong the completion since the process is almost "frozen" to the resetting configuration, and thus the average completion time will be exceedingly large. Taking this fact along with the criterion Eq. (27) simply implies that the mean completion time must be having at-least one minimum as a function of p. The optimal probability, p^*, which minimizes this mean first-passage time [Eq. (20)], can be determined from the following root equation:

$$G_N^2(1-p^*) - G_N(1-p^*) + (1-p^*)p^*G_N'(1-p^*) = 0. \tag{28}$$

Clearly, the valid solution of the above equation provides us with an optimal probability p^* for which the mean completion time attains a minimum and thus adheres to the criterion Eq. (27).

C. Sharp restart

We also consider a strategy when restart events always take place after a fixed number of steps. This is often known as sharp or deterministic restart protocol (see Refs. [7,31,85] for different properties of this strategy in the continuous setup). Since the resetting takes place always after a fixed period, the density can be written as

$$P_R(n) = \delta_{n,r} = \begin{cases} 0, & n \neq r \\ 1, & n = r \end{cases}, \tag{29}$$

where $\delta_{n,r}$ is the Kronecker δ. So, we will refer to this as sharp distribution with restart step or period r. For sharp restart, we have $\Pr(N \leqslant R) = \sum_{n=0}^r P_N(n) = \Pr(N \leqslant r)$. Furthermore, we find (Appendix D)

$$G_{N_{\min}}(z) = \frac{1}{\Pr(N \leqslant r)} \sum_{n=0}^{r} P_N(n) z^n, \tag{30}$$

$$G_{R_{\min}}(z) = z^r. \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (11) we get

$$G_{N_R}(z) = \frac{\sum_{n=0}^r P_N(n) z^n}{1 - \Pr(N > r) z^r}. \tag{32}$$

Again, using Eq. (13) one can derive the full probability mass function of the restarted process by taking the derivatives of the PGF given in Eq. (32). The moments can be computed using Eq. (12) e.g., the first and second moment take the following forms:

$$\langle N_r\rangle = \frac{\sum_{n=0}^r nP_N(n)}{\Pr(N \leqslant r)} + \frac{[1 - \Pr(N \leqslant r)]r}{\Pr(N \leqslant r)}, \tag{33}$$

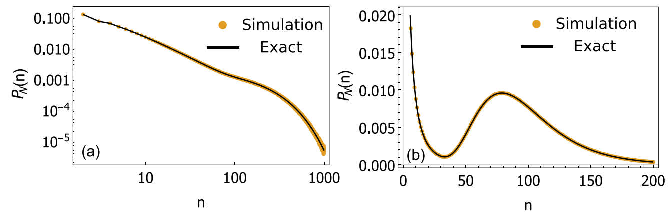

FIG. 2. First-passage time densities $P_N(n)$ for the underlying process. Analytical results are taken from Eqs. (36) and (42) and plotted against numerical simulations. (a) Symmetric RW with diffusivity parameter $q = 0.7$. (b) Biased RW with $q = 0.7$ and bias $g = 0.3$. Other parameters are kept fixed for both the simulations: left boundary = 1, right boundary = 25, and initial position $n_0 = 3$.

and

$$\langle N_r^2\rangle = \frac{[\Pr(N > r)(2r-1)+1]\sum_{n=0}^r nP_N(n)}{\Pr(N \leqslant r)^2} + \frac{\Pr(N \leqslant r)\sum_{n=0}^r (n-1)nP_N(n)}{\Pr(N \leqslant r)^2} + \frac{\Pr(N > r)[1+\Pr(N > r)]r^2}{\Pr(N \leqslant r)^2}. \tag{34}$$

Intuitively, for small r, restart events are quite frequent and thus the completion times can be large since the particle is almost localized near the resetting coordinate. However, for $r \gg 1$, restart rarely occurs and we reach to the limit of the underlying process, e.g., $\langle N_r\rangle \to \langle N\rangle$, etc.

III. APPLICATIONS TO RANDOM WALKS

So far, we have built up a general framework for first passage under restart in discrete space and time. We have provided working formulas to compute the mean, higher moments, and even the probability mass function for the completion time without specifying any details of the underlying process. To see how they work in practice, we apply the formalism to a symmetric and biased random walk in a 1D lattice in the presence of two absorbing boundaries. The dynamics is further subjected to restart which after a random step brings the walker back to its initial state. We start with the symmetric unbiased random walk.

A. Symmetric random walk in 1D

Let us consider the so-called symmetric lazy random walker [82,86], in 1D bounded domain. The dynamics of this walker are governed by the following evolution equation for the site occupation probability, $\mathbb{P}(m, n)$,

$$\mathbb{P}(m, n+1) = (1-q)\mathbb{P}(m, n) + \frac{q}{2}\mathbb{P}(m-1, n) + \frac{q}{2}\mathbb{P}(m+1, n), \tag{35}$$

with m representing a lattice site on the line and n being time step. The q parameter for this kind of random walk represents the tendency of the walker to move, with $q = 0$ representing a walker that does not move, and $q = 1$ a walker that moves at each time step. The walk is symmetric since the probability to go either to the left or to the right is $q/2$. The walker sets

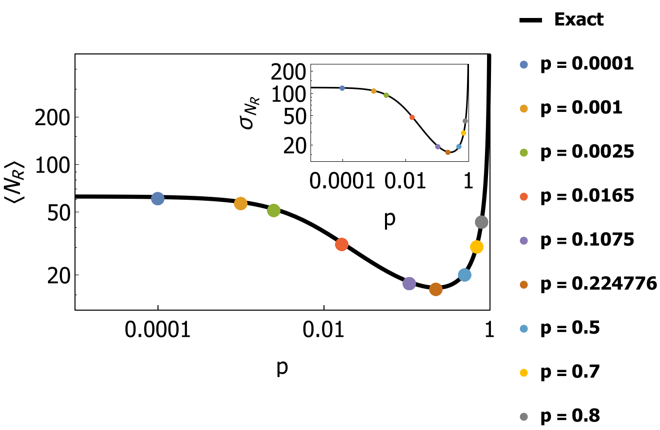

FIG. 3. Mean first-passage time $\langle N_R\rangle$ and fluctuations σ_{N_R} (inset) of a symmetric RW in confinement as a function of restart probability p. Simulations are indicated by the markers while the continuous line represents the exact result. For $p\to 0^+$, these statistical quantities saturate to their underlying values. As can be seen, for a range of restart probabilities, mean time could be significantly reduced. Parameters set for the simulations: left boundary =1, right boundary =25, initial position $n_0=3$, and diffusivity parameter $q=0.7$.

off from $m=n_0$ and stays inside the bounded interval $(1,N)$, with absorbing boundaries located at $m=1$ and $m=N$ so that $\mathbb{P}(1,n)=\mathbb{P}(N,n)=0$ for all $n\geqslant 0$. The first-passage time probability density i.e., the time for the walker to reach either of the two boundaries, $P_N(n)$, is given by [82]

$$P_N(n)=\frac{q}{N-1}\sum_{k=1}^{N-2}[1-(-1)^k]\sin\left[\frac{n_0-1}{N-1}\pi k\right]\times\sin\left(\frac{\pi k}{N-1}\right)\left[1-q+q\cos\left(\frac{\pi k}{N-1}\right)\right]^{n-1},\quad(36)$$

also see Fig. 2(a) for a numerical verification. The generating function of the first-passage time density can be obtained by noting $G_N(z)=\sum_{n=0}^{\infty}z^nP_N(n)$, where $P_N(n)$ is given by Eq. (36). Substituting the resulting expression for $G_N(z)$ into Eq. (20) with $z\to 1-p$ yields the mean first-passage time under geometric restart. This is demonstrated in Fig. 3. Moreover, from the generating function Eq. (20) we obtain the second moment using Eq. (25), and then plot the fluctuations σ_{N_R} of first-passage time as a function of p as shown in the inset of Fig. 3. It is clear from Fig. 3 that restart aids in the favour of completion. In other words, this must satisfy the restart criterion given in Eq. (27). The criterion is illustrated in Fig. 4(a) where we have plotted the left-hand side (LHS) and right-hand side (RHS) of the inequality Eq. (27) keeping n_0, the initial position/restart location, as the controlling parameter. We pick $n_0=3$ for which $\mathrm{CV}^2>1-\frac{1}{\langle N\rangle}$, and thus the criterion is naturally satisfied (also see Appendix E for counter cases). Note from Fig. 3 and the inset of Fig. 4(b) that $\langle N_R\rangle$ has a minimum at the optimal probability p^*. This value can furthermore be conferred from the root Eq. (28). Finally, the probability mass function of the first-passage time, namely, $P_{N_R}(n)$ is obtained analytically from Eqs. (13) and (23) by using $G_N(z)$ and plotted in Fig. 4(b) against the data obtained from the numerical simulations. The inset of Fig. 4(b) corroborates with the parameter $n_0=3$ [taken from Fig. 4(a)], which guarantees a reduction in $\langle N_R\rangle$ in the presence of restart. Here, we comment on the asymptotic large n-form of $P_{N_R}(n)$. It is intuitive to understand that a successful first-passage event would occur after a few restart events that brings back the walker to its initial position. Thus, if Y denotes the number of restart events until a first passage then it simply conforms to a geometric distribution. Moreover, if $\tilde{Y}$ is the time until the first passage under Y, then the average of this random time is given by $\langle N_R\rangle$. It is then trivial to show that $\mathrm{Prob}(\tilde{Y}>t)\to e^{-t/\langle N_R\rangle}$ as $t\to\infty$ [84], which essentially implies that $P_{N_R}(n)$ is asymptotically exponential in n. We demonstrate this asymptotic form in Fig. 5.

It is only imperative to revisit the simple random walker (with $q=1$ and starting at the origin) in the presence of only one absorbing boundary located at x. In this case, the generating function for the underlying process can be easily obtained following steps from Ref. [76] and this reads

$$G_N(z)=\left(\frac{1-\sqrt{1-z^2}}{z}\right)^{|x|},\quad x\neq 0.\quad(37)$$

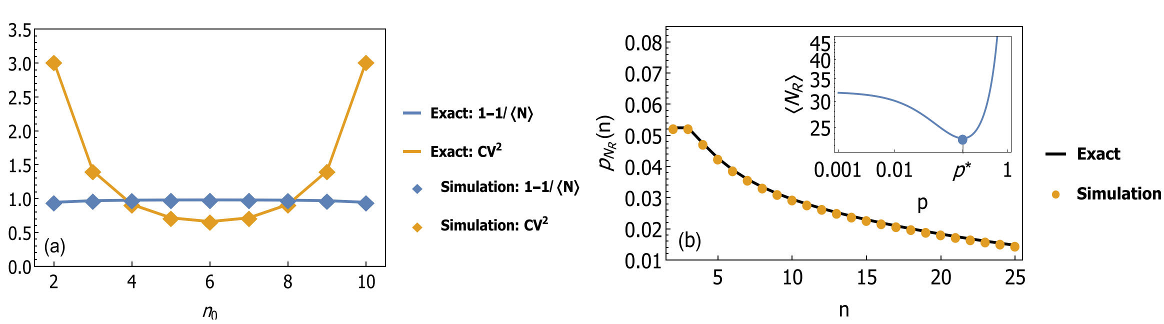

FIG. 4. (a) Demonstration of the restart criterion Eq. (27) for the symmetric RW in confinement. We plotted the LHS and RHS of the criterion as a function of the initial position n_0. (b) We choose $n_0=3$ from panel (a) for which restart is beneficial, and then plot the probability mass function $P_{N_R}(n)$ for this given parameter value. In the inset, we plot $\langle N_R\rangle$ to demonstrate that indeed restart is able to reduce the mean completion time. The optimal probability $p^*=0.1610\ldots$ is found to match exactly with the theoretical prediction. Parameters fixed for the simulations: left boundary $=1$, right boundary $=11$, and $q=0.5$. The probability mass function in panel (b) is obtained using the property of the PGF, given in Eq. (13), by taking the first n-derivatives of $G_{N_R}(z)$.

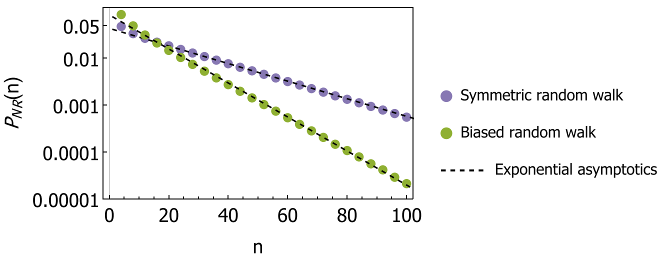

FIG. 5. Comparison between $P_{N_R}(n)$ and numerical data. The asymptotic forms of $P_{N_R}(n) \sim e^{-n/\langle N_R \rangle}$ are taken along with their respective $\langle N_R \rangle$ for the symmetric and biased RW in confinement. Parameters fixed for the symmetric RW: left boundary =1, right boundary =11, initial position $n_0 = 3$, $q = 0.5$ and $p = 0.1610..$. Parameters fixed for the biased RW: left boundary =1, right boundary =11, initial position $n_0 = 9$, $q = 0.5$, $g = 0.3$, and $p = 0.0813...$

In fact, it is known that for large n, the first-passage time density has a power law tail $n^{-3/2}$ [72–75], which is similar to the Lévy-Smirnov distribution for the first-passage time of a Brownian walker in one dimension. This power law trivially leads to a diverging mean first-passage time for a RW. Naturally, it is only expected that restart will always expedite the first-passage time as was shown in the classic diffusion problem with stochastic resetting [1]. For completeness, in this case, we compute the mean $\langle N_R \rangle$ and the fluctuations σ_{N_R} as a function of the restart probability p from the generating function using Eqs. (20) and (25). The theoretical results are in excellent agreement with the numerical simulations of the walker in the semi-infinite geometry (see Fig. 6). We note that the result for the mean first-passage time of a random walker on this geometry, namely, $\langle N_R \rangle = [(\frac{1+\sqrt{2p-p^2}}{1-p})^{|x|} - 1]\frac{1-p}{p}$ [which can be obtained by substituting Eq. (37) into Eq. (20)] was previously obtained by Riascos *et al.* in Ref. [68]. Thus, our results are consistent with theirs. We end this discussion by referring the readers to Fig. 7 where we have plotted mean first-passage time under sharp restart ($\langle N_r \rangle$) as a function of the restart period r. Clearly, the limit $r \gg 1$ describes the scenario when there is hardly any resetting event (i.e., convergence to the original process itself)

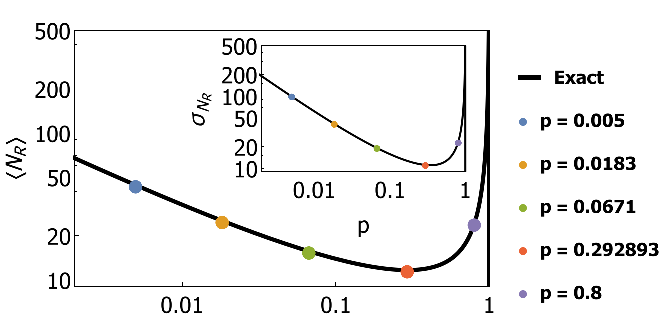

FIG. 6. Mean and standard deviation for the simple RW on the semi-infinite line with resetting. Parameters: left boundary $= -2$, initial position $n_0 = 0$, and $q = 1$. The right boundary is set to infinity.

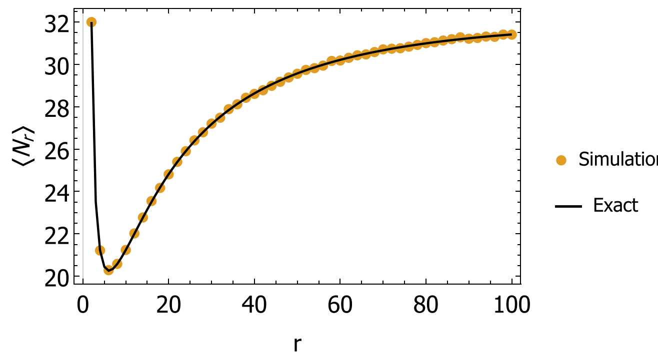

FIG. 7. Mean first-passage time $\langle N_r \rangle$ for symmetric RW as a function of the restart step r. Parameters: left boundary =1, right boundary =11, and $n_0 = 3$, $q = 0.5$.

and thus the mean times saturate to their underlying values as can be seen from the figure.

B. Biased random walks in 1D

We now consider a biased random walker in the presence of restart. The underlying problem was recently studied by Sarvaharman and Giuggioli [83] generalizing the results obtained in Ref. [82]. Biased lattice RWs are quite important since their applications include cell migration due to concentration gradients in biology (chemotaxis) [87,88], drifting bacteria by light modulation (phototaxis) [89], or upwards movement of single-celled algae in response to gravity (gravitaxis) [90]. Moreover, the model has been employed to study wireless sensor networks [91] and model-driven tracer particles [92]. Although seemingly there is a lot of interests, an attempt to derive exact expressions for first-passage statistics for biased lattice RW in confined space has been extremely limited. The work by Sarvaharman and Giuggioli [83] develops a general framework that allows to derive analytically various transport quantities in arbitrary dimensions and arbitrary boundary conditions for biased lattice random walk. Since the problem with restart can be elegantly mapped to the problem without restart, we use some of the results obtained in Ref. [83]. We first recall the model for brevity.

We start by considering the dynamics of a random walker with bias on a 1D confined lattice with two absorbing points at the end. Here, the strength and direction of the bias is described by the parameter g. Thus, we assume that the probability of jumping to the neighboring site on the left is given by $\frac{q}{2}(1-g)$, and the probability of jumping to right is then given by $\frac{q}{2}(1+g)$, where recall that q is the diffusivity parameter. Thus, probability of staying in the same site is given by $1-q$. When $g = 0$, the movement is diffusive, whereas the cases $g = 1$ and $g = -1$ are, respectively, the ballistic limit to the right and left sites. For this RW, the dynamics is governed by the following evolution equation for the occupation probability

$$\mathbb{P}(m, n+1) = (1-q)\mathbb{P}(m, n) + \frac{q}{2}(1-g)\mathbb{P}(m-1, n) + \frac{q}{2}(1+g)\mathbb{P}(m+1, n). \tag{38}$$

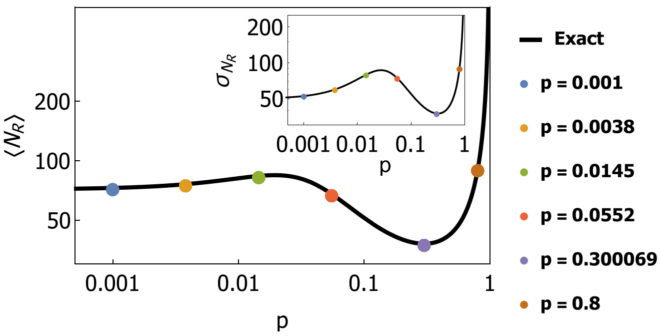

FIG. 8. Comparison between exact and simulation results for mean and standard deviation for a biased RW in confinement subject to restart conducted at a probability p. Parameters for the current setup: left boundary =1, right boundary =25, initial position $n_0 = 3$, diffusive parameter $q = 0.7$, and bias $g = 0.3$.

For a walker performing the RW described in Eq. (38), in a bounded interval $(1, N)$ and starting at $1 < n_0 < N$, where $m = 1$ and $m = N$ are absorbing boundaries (with $\mathbb{P}(1, n) = \mathbb{P}(N, n) = 0$), the propagator is given by [83]

$$\mathbb{P}(m, n) = \sum_{k=1}^{N-2} h_k(m, n_0)[1 + s_k]^n, \tag{39}$$

where s_k and h_k are given by [83]

$$s_k = \frac{q}{\eta} \cos\left(\frac{k\pi}{N-1}\right) - q,$$
$$h_k(m, n_0) = \frac{2f^{\frac{m-n_0}{2}} \sin\left[\left(\frac{m-1}{N-1}\right)k\pi\right] \sin\left[\left(\frac{n_0-1}{N-1}\right)k\pi\right]}{N-1}, \tag{40}$$

and

$$f = \frac{1+g}{1-g}, \quad \eta = \frac{1+f}{2\sqrt{f}}. \tag{41}$$

Again, here we need only the generating function $G_N(z)$ for the underlying first-passage time density $P_N(n)$ to make use of our renewal formulas. To compute this, we note that, by definition,

$$P_N(n) = \mathcal{S}_{n_0}(n-1) - \mathcal{S}_{n_0}(n), \tag{42}$$

where $\mathcal{S}_{n_0}(n)$ is the survival probability i.e., the probability that the walker survives upto the time step n starting from n_0 (without restart) and is given by $\mathcal{S}_{n_0}(n) = \sum_{m=1}^{N} \mathbb{P}(m, n)$, where $\mathbb{P}(m, n)$ is given by Eq. (39) [72]. Replacing $\mathcal{S}_{n_0}(n)$ in Eq. (42) gives an exact expression for $P_N(n)$ [also see Fig. 2(b) for a numerical verification]. Next, to obtain the generating function $G_N(z)$, we multiply z^n on the both sides of Eq. (42) and sum over n. From the resulting expression for $G_N(z)$, the mean $\langle N_R \rangle$ and the fluctuations σ_{N_R} are computed using Eqs. (20) and (25) simply by replacing $z \to 1 - p$. We have plotted these expressions as a function of restart probability p in Fig. 8. Similar to the symmetric RW, we now investigate the criterion Eq. (27) in Fig. 9(a). We plot CV^2 and $1 - \frac{1}{\langle N \rangle}$ as a function of the initial position n_0. We choose an n_0 for which restart is beneficial and corroborate with Fig. 9(b) to plot the probability mass function for the completion time of the restarted process. From the inset, it is clear that restart lowered the completion time and thus implying the existence of an optimal restart probability p^*. We compare this value with that obtained from the theory [Eq. (28)] to get an exact match. The large n asymptotic form of $P_{N_R}(n)$ for the biased RW case has also been verified in Fig. 5. Finally, we refer to Fig. 10 for the results in the presence of sharp restart.

IV. CONCLUSIONS

In summary, we have studied first passage under restart when both underlying and restart processes are spatially and temporally discrete. Although there exists a plethora of works dedicated on Brownian walkers subject to restart, results on a discrete time and space random walker are only limited. Taking advantage of the renewal properties, we have derived working formulas for the mean and higher-order moments of the completion time under restart. Importantly, the formulation is quite general and can be used as a platform to extend our findings to a wide range of other discrete stochastic motions and restart time distributions. In particular, we have shown that when restart is geometrically distributed, there is a sufficient criterion which determines when restart is beneficial. We apply the theory in two paradigmatic setups, namely, the canonical 1D lattice random walker (symmetric and biased) in the presence of absorbing boundaries at two given end points. Using them as underlying first-passage time processes, we analyze the effect of restart in full details. We stress the fact that it is also possible to compute the density function of the first-passage time under restart at any time unlike the continuous cases where only large time

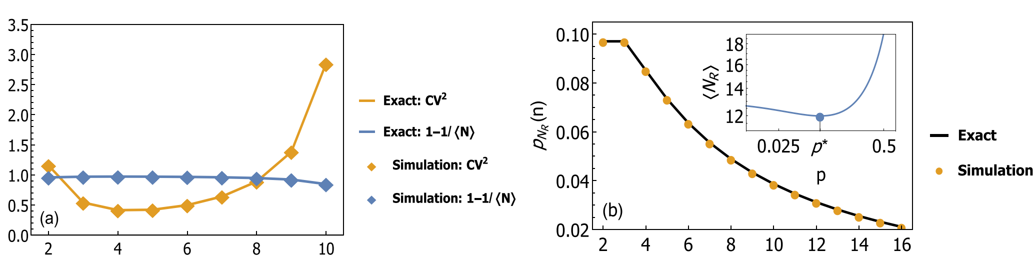

FIG. 9. (a) Demonstration of the restart criterion Eq. (27) for the biased RW in confinement. We plotted the LHS and RHS of the criterion as a function of the initial position n_0. (b) We choose $n_0 = 9$ from panel (a) and then plot $P_{N_R}(n)$ for this given parameter value. In the inset, we plot $\langle N_R \rangle$ to demonstrate that indeed restart is able to reduce the mean completion time. The optimal probability $p^* = 0.0813..$ is found to match exactly with the theoretical prediction. Parameters fixed for the setup: left boundary =1, right boundary =11, and $q = 0.5$, $g = 0.3$.

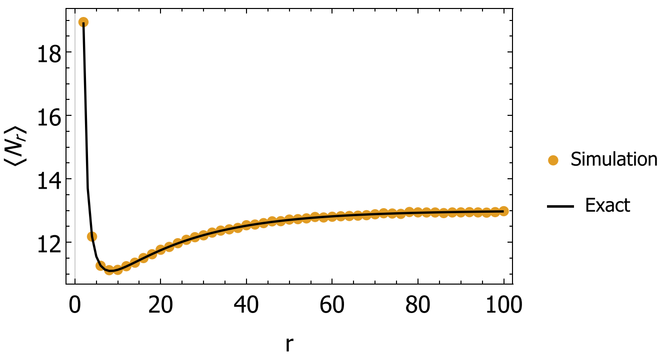

FIG. 10. Mean first-passage time $\langle N_r\rangle$ under sharp restart for the biased RW as a function of the restart step r. Parameters: left boundary =1, right boundary =11, and $n_0 = 3,\ q = 0.5,\ g = -0.3$. For $r \gg 1$, the mean saturates to its original value.

asymptotics are tractable. Our analytical results provide an excellent agreement with all the simulation results. There are many extensions possible within this current setup. Our method can be employed directly to understand the effects of restart on the search or first-passage time dynamics of confined lattice RWs in arbitrary dimensions. It will be interesting to study the effects of sticky or mixed boundaries for RW models in the presence of restart with an overhead time [24,93–95] or a refractory period [96]. Random walks in a spatially nonhomogeneous condition under restart is also another challenging direction. Consider a simple RW on a lattice and start diluting the lattice. By this we mean that some fraction of the lattice sites is removed, i.e., is declared inaccessible for the walker. This is often known as random walks on a percolation structure [76]. It would be interesting to study how the probability of finding a finite cluster among all finite clusters scales in the presence of restart.

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APPENDIX A: DERIVATION FOR THE GENERATING FUNCTION EQ. (11)

In this section, we present the derivation of the PGF of the first-passage process under generic restart. To do so, we note that

$$G_{N_R}(z) = \Pr(N \leqslant R)\langle Z^{N_R}|N \leqslant R\rangle + \Pr(N > R)\langle Z^{N_R}|N > R\rangle, \qquad (A1)$$

which gives

$$G_{N_R}(z) = \Pr(N \leqslant R)\langle Z^{\{N_R|N\leqslant R\}}\rangle + \Pr(N > R)\langle Z^{\{N_R|N>R\}}\rangle. \qquad (A2)$$

Recall the random variables defined in the main text

$$N_{\min} \equiv \{N|N = \min(N,R)\} = \{N|N \leqslant R\}, \qquad (A3)$$

$$R_{\min} \equiv \{R|R = \min(N,R)\} = \{R|N > R\}, \qquad (A4)$$

where $\min(R,N)$ is the minimum of N and R. Thus $\{N_R|N \leqslant R\} = \{N|N \leqslant R\} = N_{\min}$ and

$$\{N_R|N > R\} = \{R + N_R'|N > R\} = \{R|R = \min(R,N)\} + N_R' = R_{\min} + N_R', \qquad (A5)$$

where in the second transition in Eq. (A5) we have further used the fact that N_R' is an independent and identically distributed copy of N_R and hence independent of both R and N. We thus have

$$G_{N_R}(z) = \Pr(N \leqslant R)\langle Z^{N_{\min}}\rangle + \Pr(N > R)\langle Z^{R_{\min}+T_R'}\rangle = \Pr(N \leqslant R)G_{N_{\min}}(z) + \Pr(N > R)G_{R_{\min}}(z)G_{N_R}(z), \qquad (A6)$$

where in the last step we have again used the fact that N_R' is an independent and identically distributed copy of N_R. Rearranging Eq. (A6) we have

$$G_{N_R}(z) = \frac{\Pr(N \leqslant R)G_{N_{\min}}(z)}{1 - \Pr(N > R)G_{R_{\min}}(z)} = \frac{(1 - \Pr(N > R))G_{N_{\min}}(z)}{1 - \Pr(N > R)G_{R_{\min}}(z)}, \qquad (A7)$$

which is Eq. (11) in the main text.

APPENDIX B: DERIVATIONS FOR FIRST-PASSAGE STEP UNDER GEOMETRIC RESTART

In this section, we present the derivations for Eqs. (20) and (23). We start the derivation by noting that under geometrically distributed resetting, $\Pr(N \leqslant R)$ is given by

$$\Pr(N \leqslant R) = \sum_{n=0}^{\infty} P_N(n) \sum_{m=n}^{\infty} P_R(m) = \sum_{n=0}^{\infty} P_N(n) \sum_{m=n}^{\infty} (1-p)^m p = \sum_{n=0}^{\infty} P_N(n)(1-p)^n = G_N(1-p), \qquad (B1)$$

where in the last step we utilized the definition of the PGF of random variable X, given in Eq. (5). Using Eq. (3), one can also compute $\langle \min(N,R)\rangle$ for the case of geometrically distributed restart

$$\begin{aligned}\langle \min(N,R)\rangle &= \sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty}P_N(k)\right)\left(\sum_{m=n+1}^{\infty}P_R(m)\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty}P_N(k)\right)\sum_{m=n+1}^{\infty}(1-p)^m p\\ &= \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}P_N(k)(1-p)^{n+1}\\ &= \sum_{k=1}^{\infty}\sum_{n=0}^{k-1}P_N(k)(1-p)^{n+1}\\ &= \sum_{k=1}^{\infty}P_N(k)\sum_{n=0}^{k-1}(1-p)^{n+1}\\ &= \sum_{k=1}^{\infty}P_N(k)\left(\frac{1-p}{p}+\frac{(p-1)(1-p)^k}{p}\right)\\ &= \sum_{k=0}^{\infty}P_N(k)\left(\frac{1-p}{p}-\frac{(1-p)(1-p)^k}{p}\right)\\ &= \frac{1-p}{p}\left(1-\sum_{k=0}^{\infty}P_N(k)(1-p)^k\right)\\ &= \frac{1-p}{p}[1-G_N(1-p)],\end{aligned} \tag{B2}$$

where in the second to last step we once again utilize the definition of PGF. Equation (20) is obtained by substituting Eqs. (B1) and (B2) into Eq. (2).

We now turn to the derivation of the PGF of the restarted first-passage process N_R. We start by deriving $G_{N_{\min}}(z)$ and $G_{N_R}(z)$. We substitute the PMF of the geometric distribution into Eqs. (8) and (9) to find

$$\begin{aligned}G_{N_{\min}}(z) &= \sum_{n=0}^{\infty}P_N(n)\frac{\sum_{m=n}^{\infty}P_R(m)}{\Pr(N\leqslant R)}z^n\\ &= \sum_{n=0}^{\infty}P_N(n)\frac{\sum_{m=n}^{\infty}(1-p)^m p}{G_N(1-p)}z^n\\ &= \sum_{n=0}^{\infty}P_N(n)\frac{(1-p)^n}{G_N(1-p)}z^n\\ &= \frac{1}{G_N(1-p)}\sum_{n=0}^{\infty}P_N(n)(1-p)^n z^n\\ &= \frac{G_N[z(1-p)]}{G_N(1-p)},\end{aligned} \tag{B3}$$

and

$$\begin{aligned}G_{R_{\min}}(z) &= \sum_{n=0}^{\infty}P_R(n)\frac{\sum_{m=n+1}^{\infty}P_N(m)}{\Pr(N>R)}z^n\\ &= \sum_{n=0}^{\infty}(1-p)^n p\frac{\sum_{m=n+1}^{\infty}P_N(m)}{1-\Pr(N\leqslant R)}z^n\\ &= \frac{1}{1-G_N(1-p)}\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}(1-p)^n pz^n P_N(m)\\ &= \frac{1}{1-G_N(1-p)}\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}(1-p)^n pz^n P_N(m)\\ &= \frac{1}{1-G_N(1-p)}\sum_{m=1}^{\infty}P_N(m)\sum_{n=0}^{m-1}(1-p)^n pz^n\\ &= \frac{1}{1-G_N(1-p)}\sum_{m=1}^{\infty}P_N(m)\left(\frac{p\{[(1-p)z]^m-1\}}{(1-p)z-1}\right)\\ &= \frac{p\{\sum_{m=1}^{\infty}P_N(m)[(1-p)z]^m-\sum_{m=1}^{\infty}P_N(m)\}}{[(1-p)z-1][1-G_N(1-p)]}\\ &= \frac{p\{\sum_{m=0}^{\infty}P_N(m)[(1-p)z]^m-\sum_{m=0}^{\infty}P_N(m)\}}{[(1-p)z-1][1-G_N(1-p)]}\\ &= \frac{p\{G_N[z(1-p)]-1\}}{[(1-p)z-1][1-G_N(1-p)]}.\end{aligned} \tag{B4}$$

Equation (23) is obtained by substituting Eqs. (B1), (B3), and (B4) into Eq. (11).

Taking the derivative of Eq. (23), $z\to 1^-$, one can recover the mean first-passage time under geometric restart given in Eq. (20),

$$\begin{aligned}\langle N_R\rangle &= z\frac{\partial G_{N_R}(z)}{\partial z}\Big|_{z=1^-} = z\frac{(1-p)((1-p)(z-1)((1-p)z-1)G_N'((1-p)z))}{((1-p)(z-1)-pG_N((1-p)z))^2}\Big|_{z=1^-}\\ &\quad + z\frac{(1-p)(-pG_N((1-p)z)^2+pG_N((1-p)z))}{((1-p)(z-1)-pG_N((1-p)z))^2}\Big|_{z=1^-}\\ &= \frac{1-p}{p}\frac{1-G_N(1-p)}{G_N(1-p)},\end{aligned} \tag{B5}$$

which is identical to the result obtained in Eq. (20).

APPENDIX C: DERIVATION OF Eq. (24) IN THE MAIN TEXT

In this section, we present the derivation of Eq. (24). We start with the definition of the survival function for the underlying process

$$Q_N(n) = \Pr[N>n] = \sum_{k=n+1}^{\infty}P_N(k), \tag{C1}$$

where $P_N(n)$ is the probability mass function of N. We now calculate the PGF of the survival function in the following:

$$\begin{aligned} G_{Q_N}(z) &= \sum_{n=0}^{\infty} z^n Q_N(n) = \sum_{n=0}^{\infty} z^n \Pr[N > n] \\ &= \sum_{n=0}^{\infty} z^n \sum_{k=n+1}^{\infty} P_N(k) \\ &= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} z^n P_N(k) \\ &= \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} z^n P_N(k) \\ &= \sum_{k=1}^{\infty} P_N(k) \sum_{n=0}^{k-1} z^n \\ &= \sum_{k=1}^{\infty} P_N(k) \frac{1-z^k}{1-z} \\ &= \frac{1}{1-z} \sum_{k=1}^{\infty} P_N(k) - \frac{1}{1-z} \sum_{k=1}^{\infty} P_N(k) z^k \\ &= \frac{1-P_N(0)}{1-z} - \frac{G_N(z)-P_N(0)}{1-z} \\ &= \frac{1-G_N(z)}{1-z}, \end{aligned} \tag{C2}$$

where recall again $G_N(z)$ is the PGF of the first-passage time density $P_N(n)$. Similarly, with restarts, we have

$$G_{Q_{N_R}}(z) \equiv \sum_{n=0}^{\infty} z^n Q_{N_R}(n) = \frac{1-G_{N_R}(z)}{1-z}, \tag{C3}$$

where $G_{N_R}(z)$ is given by Eq. (5). Thus, the relations Eqs. (C2) and (C3) connect the survival function and the first-passage time density in z-space. We now write $G_{N_R}(z)$ in terms of $G_N(z)$ using Eq. (23). Next, we replace $G_N(z)$ by $G_{Q_N}(z)$ using Eq. (C2). Finally, we substitute this resulting expression into Eq. (C3) to arrive at the desired relation

$$G_{Q_{N_R}}(z) = \frac{(1-p)G_{Q_N}[(1-p)z]}{1-pG_{Q_N}[(1-p)z]}, \tag{C4}$$

which is Eq. (24) in the main text.

APPENDIX D: DERIVATIONS FOR FIRST-PASSAGE STEP UNDER SHARP RESTART

In this section, we sketch the steps to derive Eq. (32). As for the geometric resetting case, we start the derivation by obtaining $G_{N_{\min}}(z)$ and $G_{N_R}(z)$ using Eqs. (8) and (9):

$$\begin{aligned} G_{N_{\min}}(z) &= \sum_{n=0}^{\infty} P_N(n) \frac{\sum_{m=n}^{\infty} P_R(m)}{\Pr(N \leqslant R)} z^n \\ &= \sum_{n=0}^{\infty} P_N(n) \frac{\sum_{m=n}^{\infty} \delta_{m,r}}{\Pr(N \leqslant r)} z^n = \frac{1}{\Pr(N \leqslant r)} \sum_{n=0}^{r} P_N(n) z^n. \end{aligned} \tag{D1}$$

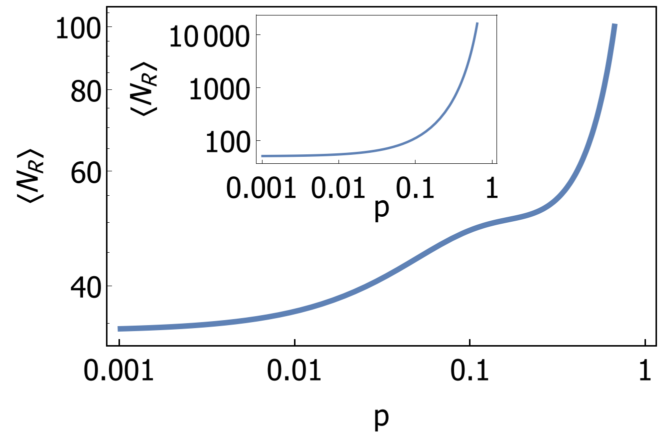

FIG. 11. Mean first-passage time under geometric restart for the case when restart is detrimental. We refer to Figs. 4(a) and 9(a) for the simple and biased RWs, respectively. In particular, we choose those values of n_0 for which the criterion Eq. (27) is violated. It can be seen that adding restart only increases $\langle N_R \rangle$. Parameters set for the biased RW: left boundary =1, right boundary =11, and $n_0 = 3$, $q = 0.5$, $g = 0.3$ and for the symmetric RW (inset): left boundary = 1, right boundary = 11, and $n_0 = 6$, $q = 0.5$.

Moreover, a similar exercise gives

$$\begin{aligned} G_{R_{\min}}(z) &= \sum_{n=0}^{\infty} P_R(n) \frac{\sum_{m=n+1}^{\infty} P_N(m)}{\Pr(N > R)} z^n \\ &= \sum_{n=0}^{\infty} \delta_{n,r} \frac{\sum_{m=n+1}^{\infty} P_N(m)}{\Pr(N > r)} z^n \\ &= \frac{\sum_{m=r+1}^{\infty} P_N(m)}{\Pr(N > r)} z^r \\ &= \frac{\Pr(N > r)}{\Pr(N > r)} z^r = z^r. \end{aligned} \tag{D2}$$

Equation (32) is then obtained by substituting Eqs. (D1) and (D2) into Eq. (11), i.e.,

$$\begin{aligned} G_{N_R}(z) &= \frac{\Pr(N \leqslant R) G_{N_{\min}}(z)}{1 - \Pr(N > R) G_{R_{\min}}(z)} \\ &= \frac{\Pr(N \leqslant r) \frac{1}{\Pr(N \leqslant r)} \sum_{n=0}^{r} P_N(n) z^n}{1 - \Pr(N > r) z^r} \\ &= \frac{\sum_{n=0}^{r} P_N(n) z^n}{1 - \Pr(N > r) z^r}. \end{aligned} \tag{D3}$$

APPENDIX E: CASES WHEN RESTART IS DETRIMENTAL

In Sec. II B, we derived a sufficient criterion for restart to be beneficial [see Eq. (27)]. We presented examples of RW with restart to demonstrate the criterion. Here, we present the same examples, but show that restart can be detrimental on the violation of the criterion Eq. (27). Recall that for a biased RW, Fig. 9(a) pictorially depicts the criterion as a function of n_0. Thus, we take a value of n_0 (say $n_0 = 3$) for which the criterion is not satisfied. Restart is expected only to prolong the completion in such case. This is depicted in Fig. 11. A similar analysis is also made for the symmetric RW following

Fig. 4(a) and the resulting plot for the mean completion time as a function of restart probability p is shown in the inset of Fig. 11. As expected, mean time increases as p varies showcasing another example of restart being detrimental.

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