Reply to "Comment on 'Non-Markovian harmonic oscillator across a magnetic field and time-dependent force fields' "

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Recently in a paper by Hidalgo-Gonzalez and Jiménez-Aquino [Phys. Rev. E **100**, 062102 (2019)], the generalized Fokker-Planck equation (GFPE) for a Brownian harmonic oscillator in a constant magnetic field and under the action of time-dependent force fields, has been explicitly calculated using the characteristic function method. Although the problem is linear it is not easy to solve, however, the method of the characteristic function is effective and allows to obtain an exact and precise solution of the problem. Our theoretical result has been compared with the one reported by Das *et al.* in a recently submitted paper [arXiv:2011.09771] using another solution method. The proposed method consists in constructing the GFPE and then calculating each time-dependent coefficient associated with this equation. However, in a more complicated case, one cannot know *a priori* the exact number of terms that this equation must contain. The precise number is further provided by the characteristic function method.

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I. INTRODUCTION

In 1976 a clever method to derive the generalized Fokker-Planck equation for a free particle and for a particle bounded by a harmonic potential for simple non-Markovian systems was reported by Adelman [1]. Twenty years later, the study of the statistical properties of linear oscillators driven by both internal and external Gaussian colored noise was reported by Wang and Masoliver [2]. By means of the explicit solution of the generalized Langevin equation (GLE) with a general friction memory kernel, the GFPE for the harmonic oscillator could also be derived using the characteristic function method. It was shown that the GFPE is exactly the same as the one derived by Adelman. In 2016, the method of the characteristic function was used to obtain the GFPE for an electrically charged Brownian particle in the presence of a constant magnetic field and under the action of time-dependent force fields [3]. According to the obtained results, it was shown that the characteristic function method is exact and provides the precise analytical expressions for each GFPE reported in Ref. [3].

A year after the paper reported in Ref. [3], the GFPE for the charged Brownian harmonic oscillator in the presence of a constant magnetic field was reported in Ref. [4] using another solution method. The method proposed in Ref. [4] consists in constructing the GFPE based on the structure of the Markovian Fokker-Planck equation plus additional contributions. It is worth commenting that in the Markovian Fokker-Planck equation all the coefficients are constant. According to the authors, the GFPE is constructed depending on the structure of a time-dependent matrix $\mathcal{A}(t)$ contained in the Gaussian conditional probability density. If the matrix is diagonal, it seems to be easy to construct the GFPE for a Brownian particle. However, in a more complicated case the matrix is not already diagonal, and, therefore, it is not clear in a general way to know *a priori* the exact number and the structure of additional contributions the GFPE should have.

In 2019 the characteristic function method was again applied to calculate the GFPE for a harmonic oscillator across a constant magnetic field and time-dependent force fields [5]. In this case, the exact number and explicit structure of each contribution have been explicitly obtained. Our theoretical results show that the characteristic function method is exact and precisely provides all the contributions the GFPE should have. Our result has been compared with the one obtained reported in a submitted paper by Das *et al.* [6].

II. GFPE FOR A HARMONIC OSCILLATOR ACROSS A MAGNETIC FIELD AND TIME-DEPENDENT FORCE FIELDS

The problem a non-Markovian harmonic oscillator across a magnetic field and under the action of time-dependent force fields has been studied and solved in a paper by Hidalgo and Jiménez-Aquino [5]. In this case the associated GLE can be written as

$$\ddot{x} - \Omega\dot{y} + \omega^2 x + \int_0^t \gamma(t-t')\dot{x}(t')\,dt' - a_x(t) = f_x(t), \quad (1)$$

$$\ddot{y} + \Omega\dot{x} + \omega^2 y + \int_0^t \gamma(t-t')\dot{y}(t')\,dt' - a_y(t) = f_y(t), \quad (2)$$

$$\ddot{z} + \omega^2 z + \int_0^t \gamma(t-t')\dot{z}(t')\,dt' - a_z(t) = f_z(t), \quad (3)$$

where $\gamma(t)$ is the friction memory kernel, $a_i(t)$'s are the components of the time-dependent force field per unit mass $\mathbf{a}(t)$, and $f_i(t)$'s are the components of the fluctuating force per unit mass $\mathbf{f}(t)$ which satisfies the fluctuation-dissipation relation of the second kind given by

$$\langle f_i(t) f_j(t')\rangle = k_B T\, \delta_{ij}\, \gamma(t-t'), \quad (4)$$

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being k_B as Boltzmann's constant and T as the temperature of a thermal bath. Along the z axis and in the absence of time-dependent external fields, the problem was solved first by Adelman [1] and later by Wang and Masoliver [2]. The alternative solution reported by Wang and Masoliver was given using the characteristic function method, which has also been used in Refs. [3,5]. The GFPE for the harmonic oscillator across a magnetic field and under the action of time-dependent force fields is explicitly calculated in Ref. [5], and its exact analytical expression on the xy plane is given by

$$\begin{aligned}
&\frac{\partial P}{\partial t}+\left(\dot p_x\frac{\partial P}{\partial v_x}+\dot p_y\frac{\partial P}{\partial v_y}\right)+\mathcal{P}_1(t)\left(q_x\frac{\partial P}{\partial y}-q_y\frac{\partial P}{\partial x}\right)-\mathcal{Q}_1(t)\left(q_x\frac{\partial P}{\partial x}+q_y\frac{\partial P}{\partial y}\right)\\
&\quad+\mathcal{P}_2(t)\left(p_x\frac{\partial P}{\partial y}-p_y\frac{\partial P}{\partial x}\right)+\mathcal{Q}_2(t)\left(q_x\frac{\partial P}{\partial v_y}-q_y\frac{\partial P}{\partial v_x}\right)-\mathcal{Q}_3(t)\left(p_x\frac{\partial P}{\partial v_x}+p_y\frac{\partial P}{\partial v_y}\right)+\mathcal{Q}_4(t)\left(p_x\frac{\partial P}{\partial v_y}-p_y\frac{\partial P}{\partial v_x}\right)\\
&\quad+\left(v_x\frac{\partial P}{\partial x}+v_y\frac{\partial P}{\partial y}\right)+\mathcal{Q}_1(t)\left(x\frac{\partial P}{\partial v_x}+y\frac{\partial P}{\partial v_y}\right)+\mathcal{P}_2(t)I\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)\\
&=\mathcal{P}_1(t)\left(x\frac{\partial P}{\partial y}-y\frac{\partial P}{\partial x}\right)+\mathcal{P}_2(t)\left(v_x\frac{\partial P}{\partial y}-v_y\frac{\partial P}{\partial x}\right)-\mathcal{R}_1(t)\left(x\frac{\partial P}{\partial v_y}-y\frac{\partial P}{\partial v_x}\right)-\mathcal{R}_3(t)\left(v_x\frac{\partial P}{\partial v_y}-v_y\frac{\partial P}{\partial v_x}\right)\\
&\quad-\mathcal{Q}_3(t)\left(\frac{\partial v_xP}{\partial v_x}+\frac{\partial v_yP}{\partial v_y}\right)-\mathcal{S}_1(t)\left(\frac{\partial^2P}{\partial v_x^2}+\frac{\partial^2P}{\partial v_y^2}\right)-\mathcal{S}_2(t)\left(\frac{\partial^2P}{\partial x\,\partial v_x}+\frac{\partial^2P}{\partial y\,\partial v_y}\right)+\mathcal{S}_3(t)\left(\frac{\partial^2P}{\partial x\,\partial v_y}-\frac{\partial^2P}{\partial y\,\partial v_x}\right). \qquad (5)
\end{aligned}$$

In the first comment by Das and Bag [7], it was pointed out that the time-dependent coefficient $\mathcal{P}_1(t)=\mathcal{P}_2(t)=0$, a fact which was verified to be true. In such a case the GFPE, thus, becomes

$$\begin{aligned}
&\frac{\partial P}{\partial t}+\left(\dot p_x\frac{\partial P}{\partial v_x}+\dot p_y\frac{\partial P}{\partial v_y}\right)-\mathcal{Q}_1(t)\left(q_x\frac{\partial P}{\partial x}+q_y\frac{\partial P}{\partial y}\right)+\mathcal{Q}_2(t)\left(q_x\frac{\partial P}{\partial v_y}-q_y\frac{\partial P}{\partial v_x}\right)\\
&\quad-\mathcal{Q}_3(t)\left(p_x\frac{\partial P}{\partial v_x}+p_y\frac{\partial P}{\partial v_y}\right)+\mathcal{Q}_4(t)\left(p_x\frac{\partial P}{\partial v_y}-p_y\frac{\partial P}{\partial v_x}\right)+\left(v_x\frac{\partial P}{\partial x}+v_y\frac{\partial P}{\partial y}\right)+\mathcal{Q}_1(t)\left(x\frac{\partial P}{\partial v_x}+y\frac{\partial P}{\partial v_y}\right)\\
&=-\mathcal{R}_1(t)\left(x\frac{\partial P}{\partial v_y}-y\frac{\partial P}{\partial v_x}\right)-\mathcal{R}_3(t)\left(v_x\frac{\partial P}{\partial v_y}-v_y\frac{\partial P}{\partial v_x}\right)-\mathcal{Q}_3(t)\left(\frac{\partial v_xP}{\partial v_x}+\frac{\partial v_yP}{\partial v_y}\right)-\mathcal{S}_1(t)\left(\frac{\partial^2P}{\partial v_x^2}+\frac{\partial^2P}{\partial v_y^2}\right)\\
&\quad-\mathcal{S}_2(t)\left(\frac{\partial^2P}{\partial x\,\partial v_x}+\frac{\partial^2P}{\partial y\,\partial v_y}\right)+\mathcal{S}_3(t)\left(\frac{\partial^2P}{\partial x\,\partial v_y}-\frac{\partial^2P}{\partial y\,\partial v_x}\right). \qquad (6)
\end{aligned}$$

As can be seen, the precise number of nonzero terms that arise in a natural way using the characteristic function method is then 13. In a short notation we have

$$\begin{aligned}
&\frac{\partial P}{\partial t}-\mathcal{Q}_1(t)\,\mathbf{q}\cdot\nabla_{\mathbf{x}}P+\mathcal{Q}_2(t)\,[\mathbf{q}\times\nabla_{\mathbf{u}}P]_z\;+[\dot{\mathbf{p}}-\mathcal{Q}_3(t)\mathbf{p}]\cdot\nabla_{\mathbf{u}}P+\mathcal{Q}_4(t)\,[\mathbf{p}\times\nabla_{\mathbf{u}}P]_z\\
&\quad=-\mathbf{u}\cdot\nabla_{\mathbf{x}}P-\mathcal{Q}_1(t)\,\mathbf{x}\cdot\nabla_{\mathbf{u}}P-\mathcal{R}_1(t)\,[\mathbf{x}\times\nabla_{\mathbf{u}}P]_z-\mathcal{R}_3(t)\,[\mathbf{u}\times\nabla_{\mathbf{u}}P]_z\\
&\quad\quad-\mathcal{Q}_3(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P-\mathcal{S}_2(t)\,\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P+\mathcal{S}_3(t)\,[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z-\mathcal{S}_1(t)\,\nabla_{\mathbf{u}}^2P, \qquad (7)
\end{aligned}$$

where the vectors $\mathbf{x}=(x,y)$ and $\mathbf{u}=(v_x,v_y)$.

III. COMPARISON WITH ANOTHER METHOD

In Sec. V of the submitted paper by Das *et al.* [6], and cited in the second submitted comment by Das and Bag [7], the authors construct the GFPE associated with the above GLE for a Brownian harmonic oscillator across a magnetic field and time-dependent force fields. According to the proposed method, the GFPE is established on the xy plane and given in Eq. (100) of the paper. It reads as

$$\begin{aligned}
\frac{\partial P}{\partial t}&=-\frac{\partial u_xP}{\partial x}-\frac{\partial u_yP}{\partial y}+H_1(t)\left[x\frac{\partial P}{\partial u_x}+y\frac{\partial P}{\partial u_y}\right]-G_1(t)\frac{\partial P}{\partial u_x}-G_2(t)\frac{\partial P}{\partial u_y}+K_1(t)\frac{\partial P}{\partial x}-K_2(t)\frac{\partial P}{\partial y}+H_2(t)\left[\frac{\partial u_xP}{\partial u_x}+\frac{\partial u_yP}{\partial u_y}\right]\\
&\quad-H_3(t)\left[\frac{\partial u_yP}{\partial u_x}-\frac{\partial u_yP}{\partial u_y}\right]+H_4(t)\left[x\frac{\partial P}{\partial y}-y\frac{\partial P}{\partial x}\right]+H_5(t)\left[u_x\frac{\partial P}{\partial y}-u_y\frac{\partial P}{\partial x}\right]-H_6(t)\left[x\frac{\partial P}{\partial v_y}-y\frac{\partial P}{\partial v_x}\right]\\
&\quad+H_7(t)\left[\frac{\partial}{\partial x}\frac{\partial P}{\partial u_y}-\frac{\partial}{\partial y}\frac{\partial P}{\partial u_x}\right]+H_8(t)\left[\frac{\partial}{\partial x}\frac{\partial P}{\partial u_x}+\frac{\partial}{\partial y}\frac{\partial P}{\partial u_y}\right]+H_9(t)\left[\frac{\partial^2P}{\partial x^2}+\frac{\partial^2P}{\partial y^2}\right]+H_{10}(t)\left[\frac{\partial^2P}{\partial u_x^2}+\frac{\partial^2P}{\partial u_y^2}\right]. \qquad (8)
\end{aligned}$$

After a long and tedious algebra but not "shortcut," the authors conclude that

$$\frac{\partial P}{\partial t} = -\dot{\mathbf{p}} \cdot \nabla_{\mathbf{u}} P - H_1(t)\mathbf{q} \cdot \nabla_{\mathbf{u}} P + H_6(t)[\mathbf{q} \times \nabla_{\mathbf{u}} P]_z - H_2(t)\nabla_{\mathbf{p}} \cdot \mathbf{u}P - H_3(t)[\mathbf{p} \times \nabla_{\mathbf{x}} P]_z$$
$$- \mathbf{u} \cdot \nabla_{\mathbf{x}} P + H_1(t)\mathbf{x} \cdot \nabla_{\mathbf{u}} P + H_2(t)\nabla_{\mathbf{u}} \cdot \mathbf{u}P - H_3(t)[\mathbf{u} \times \nabla_{\mathbf{x}} P]_z$$
$$- H_6(t)[\mathbf{x} \times \nabla_{\mathbf{u}} P]_z + H_7(t)[\nabla_{\mathbf{x}} \times \nabla_{\mathbf{u}} P]_z + H_9(t)\nabla_{\mathbf{u}} \cdot \nabla_{\mathbf{x}} P + H_{10}(t)\nabla^2_{\mathbf{u}} P. \quad (9)$$

However, there are again some inaccuracies in this equation, and they are the following: (i) the terms $[\frac{\partial u_y P}{\partial u_x} - \frac{\partial u_y P}{\partial u_y}]$ and $[\mathbf{u} \times \nabla_{\mathbf{x}} P]_z$ which multiply $H_3(t)$ in Eqs. (8) and (9), respectively, are not consistent. (ii) The same occurs with the terms $[x\frac{\partial P}{\partial v_y} - y\frac{\partial P}{\partial v_x}]$ and $[\mathbf{x} \times \nabla_{\mathbf{u}} P]_z$ which multiply $H_6(t)$. (iii) The coefficient $H_9(t)$ must be $H_8(t)$. (iv) The term $(p_x\frac{\partial P}{\partial v_x} + p_y\frac{\partial P}{\partial v_y})$ which multiplies the coefficient $\mathcal{Q}_3(t)$ in Eq. (6) is the same as $\mathbf{p} \cdot \nabla_{\mathbf{u}} P$, does not appear in Eqs. (9).

In conclusion, when the time-dependent force field is considered in the method proposed by Das *et al.*, one cannot know *a priori* the exact number of terms which have to be taken into account in the GFPE. For instance, in Eq. (8) the sum $-G_1(t)\frac{\partial P}{\partial u_x} - G_2(t)\frac{\partial P}{\partial u_y}$ gives the impression to be a divergence, and the difference $K_1(t)\frac{\partial P}{\partial x} - K_2(t)\frac{\partial P}{\partial y}$ to be a rotational. It seems that these terms have been included because the authors now know the form of the expressions arising in a natural way in the GFPE (6), see the second–fifth terms on the left-hand side of Eq. (6). In general, the proposed method by Das *et al.* seems to be established without solid foundations.

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