Comment on "Non-Markovian harmonic oscillator across a magnetic field and time-dependent force fields"

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In a recent paper Das *et al.* [J. Chem. Phys. **147**, 164102 (2017)] proposed the Fokker-Planck equation (FPE) for the Brownian harmonic oscillator in the presence of a magnetic field and the non-Markovian thermal bath, respectively. This system has been studied very recently by Hidalgo-Gonzalez and Jiménez-Aquino [Phys. Rev. E **100**, 062102 (2019)] and the Fokker-Planck equation was derived using the characteristic function. It includes a few extra terms in the FPE and the authors conclude that their method is accurate compared to the calculation by Das *et al.* Then we reexamine our calculation and which is present in this comment. The revised calculation shows that both methods give the same result.

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In a recent paper [1], we derived the Fokker-Planck equations using an alternative method for the non-Markovian dynamics for a free particle and the harmonic oscillator, respectively. Then we extend this method for the non-Markovian dynamics in the presence of a magnetic field. Very recently, the Fokker-Planck equation (FPE) equation for a non-Markovian harmonic oscillator across a magnetic field has been derived by the characteristic function in Ref. [2]. Here it has been shown that a few extra terms appear in the FPE compared to Ref. [1]. Then to reexamine our calculation we started with the relevant Langevin equations of motion [1,2],

$$\dot{u}_x = -\omega^2 x - \int_0^t \gamma(t-\tau) u_x(\tau) d\tau + \Omega u_y + f_x(t), \quad (1)$$

and

$$\dot{u}_y = -\omega^2 y - \int_0^t \gamma(t-\tau) u_y(\tau) d\tau - \Omega u_x + f_y(t), \quad (2)$$

where ω is the frequency of the harmonic oscillator and Ω corresponds to the cyclotron frequency. The random forces, f_x and f_y are independent Gaussian noises, and they are related with the frictional memory kernel $\gamma(t-t')$ by the standard fluctuation-dissipation relation $\langle f_i(t) f_j(t')\rangle = k_B T \gamma(t-t')\delta_{ij}$ where $i = x, y$ and $j = x, y$.

Since the equations of motion correspond to the Gaussian noise driven linear system then the phase space distribution function is a Gaussian one [3]. It can be written as

$$P[x, x(0); y, y(0); u_x, u_x(0); u_y, u_y(0); t] = (2\pi)^{-2}[\sigma(t)]^{-(1/2)} \exp\left[-\tfrac{1}{2}\mathbf{g}\dagger(t)\boldsymbol{A}'^{-1}(t)\mathbf{g}(t)\right], \quad (3)$$

where $\mathbf{g}(t)$ is a column matrix with the elements $g_1(t) = x - \langle x\rangle(t)$, $g_2(t) = y - \langle y\rangle(t)$, $g_3(t) = u_x - \langle u_x\rangle(t)$, and $g_4(t) = u_y - \langle u_y\rangle(t)$, respectively. $\langle x\rangle(t) = A(t)x(0) - B(t)y(0) + C(t)u_x(0) + D(t)u_y(0)$, $\langle y\rangle(t) = A(t)y(0) + B(t)x(0) + C(t)u_y(0) - D(t)u_x(0)$, $\langle u_x\rangle(t) = \dot{A}(t)x(0) - \dot{B}(t)y(0) + \dot{C}(t)u_x(0) + \dot{D}(t)u_y(0)$, $\langle u_y\rangle(t) = \dot{A}(t)y(0) + \dot{B}(t)x(0) + \dot{C}(t)u_y(0) - \dot{D}(t)u_x(0)$ with $A(t) = \chi_0(t) + \Omega^2\omega^2\chi(t)$, $B(t) = \Omega\omega^2 H'(t)$, $C(t) = H_0(t) - \Omega^2 H_0'(t)$, $D(t) = \Omega H(t)$, $\chi_0(t) = 1 - \omega^2\int_0^t H_0(\tau)d\tau$, and $\chi(t) = \int_0^t H_0'(\tau)d\tau$. $H_0(t)$, $H_0'(t)$, $H(t)$, and $H'(t)$ which appear in these relations are the inverse Laplace transformation of $\tilde{H}_0(s) = \frac{1}{s^2+s\tilde{\gamma}(s)+\omega^2}$, $\tilde{H}_0'(s) = s^2[\frac{1}{[s^2+s\tilde{\gamma}(s)+\omega^2]\{[s^2+s\tilde{\gamma}(s)+\omega^2]^2+(\Omega s)^2\}}]$, $\tilde{H}(s) = s[\frac{1}{[s^2+s\tilde{\gamma}(s)+\omega^2]^2+(\Omega s)^2}]$, and $\tilde{H}'(s) = \frac{1}{[s^2+s\tilde{\gamma}(s)+\omega^2]2+(\Omega s)^2}$, respectively. Here $\tilde{\gamma}(s)$ is the Laplace transform of $\gamma(t)$. For further details one may go through Ref. [4]. $\boldsymbol{A}'^{-1}$ in the above equation is the inverse of matrix $\boldsymbol{A}'$. The element of this matrix is defined by $A'_{ij} = \langle g_i(t) g_j(t)\rangle$. Finally, $\sigma(t) = A_1 A_2 - A_3^2 - A_4^2$ with $A'_{11} = A'_{22} = A_1$, $A'_{33} = A'_{44} = A_2$, $A'_{13} = A'_{31} = A'_{24} = A'_{42} = A_3$, $A'_{14} = A'_{41} = A_4$, and $A'_{23} = A'_{32} = -A_4$. It is to be noted here that the rest of the off diagonal elements of matrix $\boldsymbol{A}'(t)$ are zero.

Now following the procedure as reported in the recent paper [1] for several linear systems, one may read the Fokker-Planck equation with the solution (3) as

$$\begin{aligned}\frac{\partial P}{\partial t} = &-\frac{\partial u_x P}{\partial x} - \frac{\partial u_y P}{\partial y} + H_1(t)\left[x\frac{\partial P}{\partial u_x} + y\frac{\partial P}{\partial u_y}\right] \\ &+H_2(t)\left[\frac{\partial u_x P}{\partial u_x} + \frac{\partial u_y P}{\partial u_y}\right] - H_3(t)\left[\frac{\partial u_y P}{\partial u_x} - \frac{\partial u_x P}{\partial u_y}\right] \\ &+H_4(t)\left[x\frac{\partial P}{\partial y} - y\frac{\partial P}{\partial x}\right] + H_5(t)\left[u_x\frac{\partial P}{\partial y} - u_y\frac{\partial P}{\partial x}\right] \\ &-H_6(t)\left[x\frac{\partial P}{\partial u_y} - y\frac{\partial P}{\partial u_x}\right] + H_7(t)\left[\frac{\partial}{\partial x}\frac{\partial P}{\partial u_y} - \frac{\partial}{\partial y}\frac{\partial P}{\partial u_x}\right] \\ &+H_8(t)\left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right] + H_9(t)\left[\frac{\partial}{\partial x}\frac{\partial P}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial P}{\partial u_y}\right] \\ &+H_{10}(t)\left[\frac{\partial^2 P}{\partial u_x^2} + \frac{\partial^2 P}{\partial u_y^2}\right], \end{aligned} \quad (4)$$

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where $H_1(t)$, $H_2(t)$, $H_3(t)$, $H_4(t)$, $H_5(t)$, $H_6(t)$, $H_7(t)$, $H_8(t)$, $H_9(t)$, and $H_{10}(t)$ are relevant time dependent quantities to account for the non-Markovian dynamics properly. The first two terms on the right hand side of the above equation are usual drift terms in the phase space description for both Markvian [3] and non-Markovian dynamics [1,5,6], respectively. The next term is corresponding to the harmonic force field [1,5,6]. Then contribution from the dissipative force is considered by the fourth term [1,5,6]. The next drift term may be identified as due to the magnetic force [1,7]. Although additional drift terms in the presence of a magnetic field do not appear for the Markovian dynamics [3,8], but the non-Markovian dynamics may modify the probability flux. Keeping it in mind and the cross effect of the magnetic force, one may include additional all possible drift and diffusion terms. Thus, sixth–eighth and ninth to tenth are the additional drift and diffusion terms, respectively. It is to be noted here that the calculation of the second moment also implies to include the ninth and tenth terms. If the proposed Fokker-Planck equation is a correct one then fifth–tenth terms should disappear in the absence of the magnetic field. We will check it after the determination of all coefficients. Finally, the 11th and 12th terms are the usual diffusion terms in the phase space description [1,5,6]. To avoid any confusion we would mention here that the diffusion terms with other possible cross derivatives are not considered since the cross correlation of the fluctuations is zero for the respective case.

Then we have determined the coefficients and presented them in detail in Ref. [4]. The determination of the coefficients automatically implies that the distribution function (3) is a solution of the Fokker-Planck equation,

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\mathbf{u}\cdot\nabla_{\mathbf{x}}P + H_1(t)\mathbf{x}\cdot\nabla_{\mathbf{u}}P + H_2(t)\nabla_{\mathbf{u}}\cdot\mathbf{u}P\\ &\quad + H_3(t)[\mathbf{u}\times\nabla_{\mathbf{u}}P]_z - H_6(t)[\mathbf{x}\times\nabla_{\mathbf{u}}P]_z\\ &\quad + H_7(t)[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z + H_9(t)\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P + H_{10}(t)\nabla_{\mathbf{u}}^2 P,\end{aligned} \tag{5}$$

with $H_1 = \frac{\{-\ddot{A}a_x(t)-\ddot{B}a_y(t)-\ddot{C}a_{v_x}(t)-\ddot{D}a_{v_y}[(t)]\}}{\Delta_m}$, $H_2 = \frac{\{\ddot{A}b_x(t)-\ddot{B}b_y(t)-\ddot{C}b_{v_x}(t)-\ddot{D}b_{v_y}[(t)]\}}{\Delta_m}$, $H_3 = \frac{\{-\ddot{A}d_x(t)+\ddot{B}d_y(t)-\ddot{C}d_{v_x}(t)+\ddot{D}d_{v_y}[(t)]\}}{\Delta_m}$, $H_6 = \frac{\{-\ddot{A}c_x(t)+\ddot{B}c_y(t)+\ddot{C}c_{v_x}(t)-\ddot{D}c_{v_y}[(t)]\}}{\Delta_m}$, $H_7 = [\dot{A}_4 + H_2A_4 + H_3A_3 - H_6A_1]$, $H_9 = [\dot{A}_3 - A_2 - H_3A_4 + H_1A_1 + H_2A_3]$, and $H_{10} = \frac{1}{2}[\dot{A}_2 + 2H_1A_3 + 2H_2A_2 - 2H_6A_4]$. Here we have used $\Delta_m = (A^2+B^2)(\dot{C}^2+\dot{D}^2) + (C^2+D^2)(\dot{A}^2+\dot{B}^2) - 2(AC-BD)(\dot{A}\dot{C}-\dot{B}\dot{D}) - 2(AD+BC)(\dot{A}\dot{D}+\dot{B}\dot{C})$, $a_x(t) = A(\dot{C}^2+\dot{D}^2) - C(\dot{A}\dot{C}-\dot{B}\dot{D}) - D(\dot{A}\dot{D}+\dot{B}\dot{C})$, $b_x(t) = B(C\dot{D}-\dot{C}D) + C(A\dot{C}-\dot{A}C) + D(A\dot{D}-D\dot{A})$, $c_x(t) = B(\dot{C}^2+\dot{D}^2) + D(\dot{A}\dot{C}-\dot{B}\dot{D}) - C(\dot{A}\dot{D}+\dot{B}\dot{C})$, $d_x(t) = B(C\dot{C}+D\dot{D}) - C(A\dot{D}+C\dot{B}) + D(A\dot{C}-D\dot{B})$, $a_y(t) = B(\dot{C}^2+\dot{D}^2) + D(\dot{A}\dot{C}-\dot{B}\dot{D}) - C(\dot{A}\dot{D}+\dot{B}\dot{C})$, $b_y(t) = A(C\dot{D}-\dot{C}D) - C(B\dot{C}-\dot{C}B) - D(B\dot{D}-\dot{B}D)$, $c_y(t) = A(\dot{C}^2+\dot{D}^2) - C(\dot{A}\dot{C}-\dot{B}\dot{D}) - D(\dot{A}\dot{D}+\dot{B}\dot{C})$, $d_y(t) = A(C\dot{C}+\dot{D}D) + C(B\dot{D}-\dot{A}C) - D(B\dot{C}+\dot{A}D)$, $a_{v_x}(t) = C(\dot{A}^2+\dot{B}^2) - A(\dot{A}\dot{C}-\dot{B}\dot{D}) - B(\dot{A}\dot{D}+\dot{B}\dot{C})$, $b_{v_x}(t) = A(A\dot{C}-\dot{A}C) + B(B\dot{C}-\dot{B}C) - D(A\dot{B}-\dot{A}B)$, $c_{v_x}(t) = D(\dot{A}^2+\dot{B}^2) + B(\dot{A}\dot{C}-\dot{B}\dot{D}) - A(\dot{A}\dot{D}+\dot{B}\dot{C})$, $d_{v_x}(t) = A(A\dot{D}+\dot{B}C) + B(B\dot{D}-\dot{A}C) - D(A\dot{A}+\dot{B}B)$, $a_{v_y}(t) = D(\dot{A}^2+\dot{B}^2) + B(\dot{A}\dot{C}-\dot{B}\dot{D}) - A(\dot{A}\dot{D}+\dot{B}\dot{C})$, $b_{v_y}(t) = A(A\dot{D}-\dot{A}D) + B(B\dot{D}-\dot{B}D) - C(A\dot{B}+\dot{A}B)$, $c_{v_y}(t) = C(\dot{A}^2+\dot{B}^2) - A(\dot{A}\dot{C}-\dot{B}\dot{D}) - B(\dot{A}\dot{D}+\dot{B}\dot{C})$, and $d_{v_y}(t) = A(A\dot{C}-\dot{B}D) + B(B\dot{C}+\dot{A}D) - C(A\dot{A}+\dot{B}B)$. Then one may check easily that the distribution function (3) is a solution of the above Fokker-Planck equation. It constitutes the necessary and sufficient check of the present calculation. Now we have to compare the above equation with the Fokker-Planck equation which is derived recently in Ref. [2] for the same equations of motion and the associated distribution function. Then one can easily find out that the FPE in Ref. [2] contains additional three terms with the coefficients, H_4, H_5, and H_8, respectively. The remaining terms exactly correspond with each other. At this circumstance our check of the coefficients, H_4, H_5, and H_8 in the respective Fokker-Planck equation in Ref. [2] suggests that $H_4 = H_5 = H_8 = 0$. Thus, taking care of all the comments (including the typographical error and the rearrangement of the Fokker-Planck equation) in Sec. V (in Ref. [2]) which is devoted to Ref. [1] we conclude that both methods give the same result. Another point is to be noted here. From the independent relations among the time dependent coefficients the present method automatically requires that one of the coefficients in Eq. (5) must be zero. Then we have chosen that the coefficient in the diffusion term (which appears in the Fokker-Planck equation in the configuration space) may be zero. Because it is well known that this term does not appear usually [1,5–8] in the probabilistic description in velocity space or phase space for the Gaussian noise driven dynamical systems. With this choice the present method predicts automatically other coefficients exactly as the distribution function satisfies the above equation. To derive the same equation, the method [2] with the characteristic function does not need such a kind of any choice which may offer a shortcut way for the same destination (as shown in the present case). In other words, all the terms in Eq. (5) and the other case appear automatically in Ref. [2]. But the above discussion does not mean that the present method always needs to include a choice, such as the present case. For examples one may go through Ref. [1]. Finally, to avoid any confusion we would mention here that if any choice appears in the method, such as in the present case, that may not be an arbitrary one as mentioned above.

Before leaving this issue we would mention that the above equation reduces to the standard results at the appropriate limits, such as at $\Omega = 0$ [4]. For a further check, one may consider the condition $\omega = 0$. For this condition we have shown in Ref. [4] that in the absence of the harmonic force field the Fokker-Planck Eq. (5) reduces to the Fokker-Planck equation which was derived in Ref. [7] using the characteristic function. Thus, the accuracy of the present method is well justified with the check of the calculation for appropriate limiting conditions. Very recently, using the Fokker-Planck equation has been derived in Ref. [9] for the non-Markovian dynamics in the presence of the magnetic field and time dependent conservative force. This equation reduces to all the standard results at appropriate limits. Thus, the present method may be applicable for any kind of linear Langevin equation of motion which describes additive colored noise driven non-Markovian dynamics with or without the frictional memory kernel.

In conclusion, the present calculation suggests that the drift terms, $H_4(t)[x\frac{\partial P}{\partial y} - y\frac{\partial P}{\partial x}]$, $H_5(t)[u_x\frac{\partial P}{\partial y} - u_y\frac{\partial P}{\partial x}]$, and the diffusion term, $H_8(t)[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}]$ are not relevant quantities in the Fokker-Planck description of the Brownian motion of a harmonic oscillator in the presence of a magnetic field and the non-Markovian thermal bath. At the same, it contradicts the claim made in Ref. [2] in the context of the comment on Ref. [1]. The authors in Ref. [2] claimed that their method is accurate compared to the calculation by Das *et al.* [1]. In other words, the present calculation justifies that both methods give the same result.

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