Effect of viscous friction on entropy, entropy production, and entropy extraction rates in underdamped and overdamped media

Mesfin Asfaw Taye*

West Los Angles College, Science Division 9000 Overland Ave, Culver City, California 90230, USA

(Received 21 October 2020; revised 9 March 2021; accepted 7 April 2021; published 23 April 2021)

Considering viscous friction that varies spatially and temporally, the general expressions for entropy production, free energy, and entropy extraction rates are derived to a Brownian particle that walks in overdamped and underdamped media. Via the well known stochastic approaches to underdamped and overdamped media, the thermodynamic expressions are first derived at a trajectory level then generalized to an ensemble level. To study the nonequilibrium thermodynamic features of a Brownian particle that hops in a medium where its viscosity varies on time, a Brownian particle that walks on a periodic isothermal medium (in the presence or absence of load) is considered. The exact analytical results depict that in the absence of load $f = 0$, the entropy production rate $\dot{e}_p$ approaches the entropy extraction rate $\dot{h}_d = 0$. This is reasonable since any system which is in contact with a uniform temperature should obey the detail balance condition in a long time limit. In the presence of load and when the viscous friction decreases either spatially or temporally, the entropy $S(t)$ monotonously increases with time and saturates to a constant value as t further steps up. The entropy production rate $\dot{e}_p$ decreases in time and at steady state (in the presence of load) $\dot{e}_p = \dot{h}_d > 0$. On the contrary, when the viscous friction increases either spatially or temporally, the rate of entropy production as well as the rate of entropy extraction monotonously steps up showing that such systems are inherently irreversible. Furthermore, considering a spatially varying viscosity, the nonequilibrium thermodynamic features of a Brownian particle that hops in a ratchet potential with load is explored. In this case, the direction of the particle velocity is dictated by the magnitude of the external load of f. Far from the stall load, $\dot{e}_p = \dot{h}_d > 0$ and at stall force $\dot{e}_p = \dot{h}_d = 0$ revealing the system is reversible at this particular choice of parameter. In the absence of load, $\dot{e}_p = \dot{h}_d > 0$ as long as a distinct temperature difference is retained between the hot and cold baths. Moreover, considering a multiplicative noise, we explore the thermodynamic features of the model system.

DOI: 10.1103/PhysRevE.103.042132

I. INTRODUCTION

Understanding the physics of systems out of equilibrium is challenging since unlike equilibrium thermodynamics, nonequilibrium thermodynamics deals with inhomogeneous systems where the systems thermodynamic relation complicatedly relies on the reaction rates. Recently, employing Boltzmann-Gibbs nonequilibrium entropy along with the entropy balance equation, the thermodynamic relations of systems which are far from equilibrium were explored [1–17]. The exactly solvable models presented in the works [18,19] not only exposed the factors that affect the entropy production and extraction rates for a Brownian particle that walks on a discrete lattice system but also uncovered how the free energy, entropy production, and entropy extraction rates behave in time. Furthermore, considering systems that operate in the quantum realm, the dependence of thermodynamic relations on the system parameters is explored in the works [20–22]. All of these studies are vital to comprehend the thermodynamic properties of systems such as intracellular transport of kinesin or dynein inside the cell, see for example the recent works by Bameta *et al.* [23], Oriola *et al.* [24], and Campas *et al.* [25].

More recently the general expressions for the free energy, entropy production, and entropy extraction rates to a Brownian particle that walks in an overdamped medium was derived [26]. Furthermore, considering a Brownian particle that walks in an underdamped medium, the dependence for entropy production, free energy, and entropy extraction rates on the system parameters was studied [27]. The results obtained by these two works indicate that as long as the system is driven out of equilibrium, it constantly produces entropy at the same time it extracts entropy out of the system. At steady state, the rate of entropy production $\dot{e}_p$ balances the rate of entropy extraction $\dot{h}_d$. At equilibrium, both entropy production and extraction rates become zero. Moreover, the entropy production and entropy extraction rates are also sensitive to time. As time progresses, both entropy production and extraction rates increase in time and saturate to constant values.

In the present study we consider a simple model where the single particle and its trajectory are considered to be the system as contrasted with the underdamped medium which provides friction and acts as a heat bath. Employing Boltzmann-Gibbs nonequilibrium entropy, the dependence for the free energy, entropy production $\dot{e}_p$, and entropy extraction rates $\dot{h}_d$ on the system parameter is explored to a Brownian particle moving in underdamped and overdamped media.

*tayem@wlac.edu

First, the role of viscous friction is explored by considering a viscous friction that varies spatially and temporally. Earlier, Seifert *et al* [6] introduced a way of calculating the entropy production and extraction rates at the ensemble level by first analyzing the thermodynamic relation at a trajectory level for a Brownian particle that operates in an overdamped medium. The alternative approach by Ge *et al.* [28] and Lee *et al.* [29] also indicate that under time reversal operation the total entropy production and extraction rates can be retrieved. In this work, extending these well known stochastic approaches to underdamped and overdamped media, the general expressions for different thermodynamics relations are derived. Unlike the previous studies, we show that the entropy production and extraction rates might not saturate to a constant value as the viscous friction dictates the dynamics of the system. Furthermore, to both underdamped and overdamped cases, the rate of entropy production as well as the rate of entropy extraction becomes zero in the long time limit when the detailed balance conditions are satisfied. On the other hand, our previous exact analytic work [27] as well as the result obtained in this work depicts that at steady state, the entropy production and extraction rates to the underdamped case quantitatively agree with the overdamped case. This is rather puzzling to the underdamped case since the heat exchange due to particle recrossing is unavoidable as long as a distinct temperature difference is retained between the hot and cold heat baths.

Some viscous fluids show a change in viscosity when time changes. This is because as the fluid shear stress changes in time, so does the viscosity. Often the dynamics of systems with self-organized criticality also can be explored by considering time dependent diffusion (viscous friction) and drift terms [30–32]. Some studies have also focused on calculating the mean first passage time by considering a time dependant diffusion term [33]. To explore the nonequilibrium thermodynamic features of a Brownian particle that hops in a medium where its viscosity depends on time, we consider a Brownian particle that walks on a periodic isothermal medium (in the presence or absence of load). The exact analytical results depicts that in the absence of load $f = 0$, $\dot{e}_p = \dot{h}_d = 0$. This is reasonable since any system which is in contact with a uniform temperature should obey the detail balance condition only in a long time limit. This can be intuitively comprehended on physical grounds. When the particle operates at a finite time, the system operates irreversibility, and in this regime the second law of thermodynamics states that the change in entropy $\Delta S(t) > 0$. As one can see that if the thermodynamic quantities are evaluated in the time interval between $t = 0$ and any time t, always the change in entropy, entropy production, and entropy extraction rates become greater than zero revealing such systems are inherently irreversible. Moreover, we show that when a distinct temperature difference is not retained between the hot and cold baths, in the absence of load, $\dot{e}_p = \dot{h}_d = 0$ showing that the system is reversible. In the presence of load and when the viscous friction decreases in time, we show that the entropy $S(t)$ monotonously increases with time and saturates to a constant value as t further steps up. The entropy production rate $\dot{e}_p$ decreases in time and at steady state (in the presence of load), $\dot{e}_p = \dot{h}_d > 0$ which agrees with the results shown in the works [27]. On the contrary, when the viscous friction increases in time, the rate of entropy production as well as the rate of entropy extraction monotonously steps up showing that such systems are inherently irreversible.

Most of the previous studies have also focused on exploring the thermodynamic feature of systems such as Brownian heat engines by assuming temperature invariance viscous friction. In reality, the viscous friction of a medium tends to decrease as the temperature of the medium increases. This is because as the intensity of the background temperature increases, the force of interaction between neighboring molecules decreases. In this paper, considering a spatially varying viscosity, the nonequilibrium thermodynamic features of a Brownian particle that hops in a ratchet potential with load is explored. The potential is also coupled with a spatially varying temperature. In this case, the direction of the particle velocity is dictated by the magnitude of the external load of f. As one can note the steady state velocity of the engine is positive when f is smaller and the engine acts as a heat engine. In this regime $\dot{e}_p = \dot{h}_d > 0$. When f steps up, the velocity of the particle steps down, and at stall force, we find that $\dot{e}_p = \dot{h}_d = 0$ revealing that the system is reversible at this particular choice of parameter. For large force, the current is negative and the engine acts as a refrigerator. In this region $\dot{e}_p = \dot{h}_d > 0$. In the absence of load, $\dot{e}_p = \dot{h}_d > 0$ as long as a distinct temperature difference is retained between the hot and cold baths.

At this point we want also to stress that most of the previous works have focused on calculating the thermodynamic features of different model systems by considering additive noise. On the contrary, most realistic systems such as the neuron system can be also described by Langevin equations with multiplicative noise where in this case the noise amplitude varies spatially [34]. In this paper we study how thermodynamic features of such systems behave.

The rest of the paper is organized as follows: in Sec. II we derive the expression for various thermodynamic relations to a Brownian particle walking in overdamped and underdamped media. In Sec. III the role of viscous friction is studied by considering viscous friction that varies spatially and temporally. In Sec. IV we explore the model system in the presence of multiplicative noise. Section V deals with a summary and conclusion.

II. FREE ENERGY, ENTROPY PRODUCTION, AND ENTROPY EXTRACTION RATES

Recently the dependence for entropy production, free energy, and entropy extraction rates on the system parameters was explored [27] by considering a Brownian particle that walks in a medium where its viscous friction is insensitive to time or position. However, in most realistic systems, the viscous friction of the medium varies spatially or temporally. To address this issue, let us consider a Brownian particle that moves in an underdamped medium along the potential $U(x) = U_s(x) + fx$ where $U_s(x)$ and f are the periodic potential and the external force, respectively. Next, the relation for the entropy production and extraction rates will be derived considering a spatially varying viscous friction.

A. Underdamped case

Derivation for entropy production and entropy extraction rates. Let us consider a single Brownian particle that is arranged to undergo a random walk in an underdamped medium. Here the single particle and its trajectory are considered to be the system as contrasted with the underdamped medium which provides friction and acts as a heat bath. The dynamics of the system is governed by

$$m\frac{dv}{dt} = -\gamma(x,t)\frac{dx}{dt} + \frac{dU(x)}{dx} + \sqrt{2k_B\gamma(x,t)T(x)}\xi(t). \quad (1)$$

For simplicity, Boltzmann constant k_B is assumed to be unity. The random noise $\xi(t)$ is assumed to be Gaussian white noise satisfying the relations $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. The viscous friction $\gamma(x,t)$ and $T(x)$ are assumed to vary spatially along the medium.

For the underdamped case, the Fokker-Plank equation is given by

$$\frac{\partial P}{\partial t} = -\frac{\partial(vP)}{\partial x} - \frac{1}{m}\frac{\partial[U'(x)P]}{\partial v} + \frac{\gamma(x,t)}{m}\frac{\partial(vP)}{\partial v} + \frac{\gamma(x,t)T(x)}{m^2}\frac{\partial^2 P}{\partial v^2}, \quad (2)$$

where $P(x,v,t)$ is the probability of finding the particle at particular position x, velocity v, and time t.

For convenience, Eq. (2) can be rearranged as

$$\frac{\partial P}{\partial t} = -\left(k + \frac{\partial J'}{\partial v}\right), \quad (3)$$

where

$$k = v\frac{\partial P}{\partial x} = \frac{\partial J}{\partial x} \quad (4)$$

and

$$J' = -\frac{\gamma(x,t)}{m}vP + \frac{1}{m}(U'P) - \frac{\gamma(x,t)T(x)}{m^2}\frac{\partial P}{\partial v}. \quad (5)$$

From Eqs. (4) and (5) one gets

$$\frac{\partial P}{\partial v} = -\frac{m^2 J'}{\gamma(x,t)T(x)} + \frac{mU'P}{\gamma(x,t)T(x)} - \frac{mvP}{T(x)} \quad (6)$$

and

$$\frac{\partial P}{\partial x} = \frac{k}{v}. \quad (7)$$

Next we derive the expressions for the entropy production by considering two cases.

Case 1. Here we want to stress that the approach by Lee *et al.* [29] and Ge *et al.* [28] indicate that under time reversal operation, the total entropy production and extraction rate can be obtained. Particularly, the analysis by Ge *et al.* [28] shows that the entropy production rate is given by

$$\dot{e}_p = -\int \frac{1}{T(x)\gamma(x,t)}\left[F - \frac{T(x)\gamma(x,t)\nabla_v \ln(P)}{m}\right]^2 P\,dx\,dv$$
$$= -\int \frac{1}{T(x)\gamma(x,t)}\left[F - \frac{T(x)\gamma(x,t)\frac{\partial P}{\partial v}}{mP}\right]^2 P\,dx\,dv, \quad (8)$$

where $F = -T(x)\gamma(x,t) + U'$. Substituting Eq. (6) into Eq. (8), one gets

$$\dot{e}_p = \int \frac{m^2 J'^2}{PT(x)\gamma(x,t)}dx\,dv. \quad (9)$$

On the other hand, the entropy extraction rate can be found via the method developed by Ge *et al.* [28] as

$$\dot{h}_d = -\int \frac{1}{T(x)}\left[T(x)\gamma(x,t) + \frac{T(x)\gamma(x,t)\nabla_v \ln(P)}{m}\right]Pv\,dx\,dv$$
$$= -\int \frac{1}{T(x)}\left[T(x)\gamma(x,t) + \frac{T(x)\gamma(x,t)\frac{\partial P}{\partial v}}{mP}\right]Pv\,dx\,dv. \quad (10)$$

Substituting Eq. (6) into Eq. (10) leads to

$$\dot{h}_d = \int \frac{(U'J - vmJ')}{T(x)}dx\,dv. \quad (11)$$

Here $\dot{e}_p = \frac{de_p}{dt}$ and $\dot{h}_d = \frac{dh_d}{dt}$ denote the entropy production and extraction rates. The above expressions (for the entropy production and extraction rates) can be also derived at the ensemble level via the approach stated in the work [7].

Case 2. One can also re-derive the expressions for the entropy production and extraction rates at the ensemble level by first analyzing the entropy of the system at the trajectory level as

$$s(t) = -\ln P(x,v,t), \quad (12)$$

where $x(t)$ denotes the stochastic trajectory. The rate of entropy change at trajectory level is then given by

$$\dot{s}(t) = -\frac{\partial_t P(x,v,t)}{P(x,v,t)} - \frac{\partial_x P(x,v,t)}{P(x,v,t)}\dot{x} - \frac{\partial_v P(x,v,t)}{P(x,v,t)}\dot{v}. \quad (13)$$

Substituting Eqs. (6) and (7) into Eq. (13), the entropy production and dissipation rates at trajectory level are given as

$$\dot{e}_p^* = -\frac{\partial_t P(x,v,t)}{P(x,v,t)} + \frac{m^2 J'}{\gamma(x,t)T(x)P(x,v,t)}\dot{v} - \frac{k}{P(x,v,t)} \quad (14)$$

and

$$\dot{h}_d^* = \frac{mU'}{\gamma(x,t)T(x)}\dot{v} - \frac{m\gamma(x,t)v}{T(x)}\dot{v}. \quad (15)$$

Because averaging overall trajectories yields $\langle\dot{v}|x\rangle = \frac{J'}{P(x,t,v)}$ and $\int \partial_t P(x,v,t) = 0$, after some algebra one gets

$$\dot{e}_p = \int \frac{m^2 J'^2}{PT(x)\gamma(x,t)}dx\,dv - \int Pv\,dv \quad (16)$$

and

$$\dot{h}_d = \int \frac{m[U' - v\gamma(x,t)]J'}{T(x)\gamma(x,t)}dx\,dv, \quad (17)$$

respectively. When a periodic boundary condition is imposed, Eqs. (9) and (16) as well as Eqs. (11) and (17) converge to

$$\dot{e}_p = \int \frac{m^2 J'^2}{PT(x)\gamma(x,t)} dx\, dv \tag{18}$$

and

$$\dot{h}_d = -\int \frac{vmJ'}{T(x)} dx\, dv. \tag{19}$$

The heat dissipation rate $\dot{H}_d$ can be calculated [35,36] as

$$\begin{aligned} \dot{H}_d &= -\langle (-\gamma(x,t)\dot{x} + \sqrt{2k_B\gamma(x,t)T(x)})\dot{x}\rangle \\ &= -\left\langle m\frac{vdv}{dt} + vU'(x)\right\rangle. \end{aligned} \tag{20}$$

Our previous analysis also suggests [18,19,26] that the entropy extraction rate $\dot{h}_d$ can be expressed as

$$\dot{h}_d = -\int \left(\frac{m\frac{vdv}{dt} + vU'(x)}{T(x)}\right) P\, dx\, dv. \tag{21}$$

One should note that Eq. (21) is exact and does not depend on any boundary condition. Since $\frac{dS(t)}{dt}$ and $\dot{h}_d$ are computable, the entropy production rate can be readily obtained as

$$\dot{e}_p = \frac{dS(t)}{dt} + \dot{h}_d. \tag{22}$$

In the long time limit, $\frac{dS(t)}{dt} = 0$ which implies $\dot{e}_p = \dot{h}_d > 0$ at steady state and $\dot{e}_p = \dot{h}_d = 0$ at stationary state.

Once the expressions for $\dot{S}(t)$, $\dot{e}_p(t)$, and $\dot{h}_d(t)$ are computed as a function of time t, the analytic expressions for the change in entropy production, heat dissipation, and total entropy can be found analytically via $\Delta h_d(t) = \int_0^t \dot{h}_d(t)dt$, $\Delta e_p(t) = \int_0^t \dot{e}_p(t)dt$, and $\Delta S(t) = \int_0^t \dot{S}(t)dt$ where $\Delta S(t) = \Delta e_p(t) - \Delta h_d(t)$.

Derivation for free energy. Our next objective is to write the expression for the free energy in terms of $\dot{E}_p(t)$ and $\dot{H}_d(t)$ where $\dot{E}_p(t)$ and $\dot{H}_d(t)$ are the terms that are associated with $\dot{e}_p(t)$ and $\dot{h}_d(t)$. As discussed before, the heat dissipation rate is either given by Eq. (20) (for any cases) or if a periodic boundary condition is imposed, $\dot{H}_d(t)$ is given by

$$\dot{H}_d = -\int m(v)J'\, dx\, dv, \tag{23}$$

which is notably different from Eq. (19), due to the term $T(x)$. The term associated with $\dot{e}_p$ is given by

$$\dot{E}_p = -\int \frac{m^2 J'^2}{P\gamma(x,t)} dx\, dv. \tag{24}$$

The entropy balance equation

$$\frac{dS^T(t)}{dt} = \dot{E}_p - \dot{H}_d \tag{25}$$

is associated with Eq. (11) or (17) except the term $T(x)$. Once again, employing the expressions for $\dot{S}^T(t)$, $\dot{E}_p(t)$, and $\dot{H}_d(t)$, one can get $\Delta H_d(t) = \int_0^t \dot{H}_d(t)dt$, $\Delta E_p(t) = \int_0^t \dot{E}_p(t)dt$, and $\Delta S(t)^T = \int_0^t \dot{S}(t)^T dt$ where $\Delta S(t)^T = \Delta E_p(t) - \Delta H_d(t)$.

On the other hand, the expression for the internal energy has a form

$$\dot{E}_{\rm in} = \int [\dot{K} + vU_s'(x)]P(x,v,t)dv\, dx, \tag{26}$$

where $\dot{K} = m\frac{vdv}{dt}$ and U_s' denote the rate of kinetic and potential energy, respectively. The network work done by the system

$$\dot{W} = \int vfP(x,v,t)dv\, dx \tag{27}$$

explicitly depends on the velocity V and the load f. In terms of $\dot{H}$ and $\dot{W}$, the rate of the internal energy is given by

$$\dot{E}_{\rm in} = -\dot{H}_d(t) - \dot{W}, \tag{28}$$

and after some algebra, the first law of thermodynamics can be written as

$$\Delta E_{\rm in} = -\int_0^t [\dot{H}_d(t) + \dot{W}]dt. \tag{29}$$

Rearranging some terms, one gets the rate of free energy as $\dot{F} = \dot{E} - T\dot{S}$ for the isothermal case and $\dot{F} = \dot{E} - \dot{S}^T$ for the nonisothermal case where $\dot{S}^T = \dot{E}_p - \dot{H}_d$. The rate of free energy dissipation

$$\begin{aligned} \dot{F} &= \dot{E}_{\rm in} - \dot{S}^T \\ &= \dot{E}_{\rm in} - \dot{E}_p + \dot{H}_d \end{aligned} \tag{30}$$

can be expressed as a definite integral as

$$\Delta F(t) = -\int_0^t (\dot{W} + \dot{E}_p(t))dt. \tag{31}$$

For the isothermal case, at quasistatic limit where the velocity approaches zero $v = 0$, $\dot{E}_p(t) = 0$ and $\dot{H}_d(t) = 0$ and far from quasistatic limit $E_p = \dot{H}_d > 0$ which is expected as the particle operates irreversibly.

B. Overdamped case

Derivation for free energy, entropy production, and entropy extraction rates. For the overdamped case, as discussed by Sancho *et al* [37] and Jayannavar *et al* [38], Eq. (1) converges to

$$\begin{aligned} \gamma(x,t)\frac{dx}{dt} &= \frac{-\partial U(x)}{\partial x} - \frac{[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)} \\ &\quad + \sqrt{2k_B\gamma(x,t)T(x)}\xi(t), \end{aligned} \tag{32}$$

which corresponds to the Stratonovich interpretation [39,40]. The corresponding Fokker Planck equation is given by

$$\begin{aligned} &\frac{\partial P(x,t)}{\partial t} \\ &= \frac{\partial}{\partial x}\left(\frac{U'(x)}{\gamma(x,t)} + \frac{[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)^2}\right)P(x,t) \\ &\quad + \frac{\partial}{\partial x}\left(\frac{T(x)}{\gamma(x,t)}\frac{\partial P(x,t)}{\partial x}\right), \end{aligned} \tag{33}$$

which can be rewritten as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J}{\partial x}, \tag{34}$$

where

$$J = -\left(\frac{U'(x)P(x,t)}{\gamma(x,t)} + \frac{P(x,t)[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)^2}\right) - \left(\frac{T(x)}{\gamma(x,t)}\frac{\partial P(x,t)}{\partial x}\right). \tag{35}$$

The rate of entropy change at trajectory level is given by

$$\dot{s}(t) = -\frac{\partial_t P(x,t)}{P(x,t)} - \frac{\partial_x P(x,t)}{P(x,t)}\dot{x}. \tag{36}$$

On the other hand, from Eq. (35) one gets

$$\frac{\partial P}{\partial x} = -\frac{\gamma(x,t)J}{T(x)} - \frac{U'(x)P(x,t)}{T(x)} - \frac{P(x,t)[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)T(x)}. \tag{37}$$

Substituting Eq. (37) into Eq. (36), the entropy production and dissipation rates at trajectory level are given as

$$\dot{e}_p^* = -\frac{\partial_t P(x,t)}{P(x,t)} + \frac{\gamma(x,t)J}{T(x)P(x,t)}\dot{v} \tag{38}$$

and

$$\dot{h}_d^* = \frac{U'(x)}{T(x)} + \frac{[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)T(x)}. \tag{39}$$

Because averaging overall trajectories yields $\langle \dot{x}|x\rangle = \frac{J}{P(x,t)}$ and $\int \partial_t P(x,t) = 0$, after some algebra one gets

$$\dot{e}_p = \int \frac{\gamma(x,t)J^2}{PT(x)}dx \tag{40}$$

and

$$\dot{h}_d = \int \left(\frac{JU'(x)}{T(x)} + \frac{J[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)T(x)}\right)dx, \tag{41}$$

respectively. One should note that Eqs. (18) and (19) (underdamped case) as well as Eqs. (40) and (41) (overdamped case) approach

$$\dot{h}_d = \dot{e}_p = \int \left(\frac{JU'(x)}{T(x)}\right)dx \tag{42}$$

at steady state ($v\frac{dv}{dt} = 0$), as long as a periodic boundary condition is imposed. From Eqs. (40) and (41) it is evident that when detailed balance conditions are satisfied the velocity of equivalently the current $J = 0$ and as a result $\dot{e}_p = 0$ and $\dot{h}_d = 0$. Far from equilibrium, $J > 0$, and in this case when the viscous friction decreases either spatially or temporally, $\dot{e}_p$ and $\dot{h}_d$ approach to a constant value. When the system is driven out of equilibrium and when viscous friction increases spatially and temporally, $\dot{e}_p$ and $\dot{h}_d$ monotonously step up. Moreover, from Eq. (41), the heat dissipation rate is derived as

$$\dot{H}_d = \int \left(JU'(x) + \frac{J[\gamma'(x,t)T(x) + \gamma(x,t)T'(x)]}{2\gamma(x,t)}\right)dx. \tag{43}$$

On the other hand, the term $\dot{E}_p$ is related to $\dot{e}_p$ and it is given by

$$\dot{E}_p = \int \frac{\gamma(x,t)J^2}{P}dx. \tag{44}$$

The new entropy balance equation has a simple form

$$\frac{dS^T(t)}{dt} = \dot{E}_p - \dot{H}_d. \tag{45}$$

Furthermore, the internal energy

$$\dot{E}_{\rm in} = \int JU_s'(x)dx \tag{46}$$

has functional dependence on the current J and the potential profile U_s. The total work done is then given by

$$\dot{W} = \int \left(Jf + \frac{J[\gamma'(x,t)T(x)]}{2\gamma(x,t)} + \frac{JT'(x)}{2}\right)dx. \tag{47}$$

The first law of thermodynamics can be written as

$$\dot{E}_{\rm in} = -\dot{H}_d(t) - \dot{W}. \tag{48}$$

The change in the internal energy reduces to $\Delta E_{\rm in} = -\int_0^t [\dot{H}_d(t) + \dot{W}]dt$. Once again the rate of free energy dissipation can be written as $\dot{F} = \dot{E}_{\rm in} - \dot{S}^T = \dot{E}_{\rm in} - \dot{E}_p + \dot{H}_d$. The change in the free energy is then given by $\Delta F(t) = -\int_0^t (\dot{W} + \dot{E}_p(t))dt$.

III. TIME DEPENDENT VISCOUS FRICTION

Some viscous fluids show a change in viscosity when time changes. This is because as the fluid shear stress changes in time, so does the viscosity. Often the dynamics of systems with self-organized criticality also can be explored by considering time dependent diffusion (viscous friction) and drift terms [30–32]. Some studies have also focused on calculating the mean first passage time by considering a time dependant diffusion term [33]. To explore the nonequilibrium thermodynamic features of a Brownian particle that hops in a medium where its viscosity depends on time, we consider a Brownian particle that walks on a periodic isothermal medium (in the presence or absence of load) where its viscosity is given by

$$\gamma(t) = \frac{1}{g(1+t^z)}. \tag{49}$$

In this case, the corresponding Fokker Planck equation in overdamped medium is given as

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{f}{\gamma(t)}\right)P(x,t) + \frac{\partial}{\partial x}\left(\frac{T}{\gamma(t)}\frac{\partial P(x,t)}{\partial x}\right). \tag{50}$$

Imposing a periodic boundary condition $P(0,t) = P(L,t)$ and let us choose a Fourier cosine series

$$P(x,t) = \sum_{n=0}^{\infty} b_n(t)\cos\left[\frac{n\pi}{L_0}(x + \frac{f}{\gamma(t)})\right] \tag{51}$$

as a possible solution. After some algebra we get the probability distribution as

$$P(x,t)=\sum_{n=0}^{\infty}\cos\left\{\frac{n\pi}{L_0}\left[x+f\left(gt+\frac{gt^{z+1}}{(z+1)}\right)\right]\right\}\zeta, \quad (52)$$

where

$$\zeta=e^{-\frac{(n\pi)^2T\left(gt+\frac{gt^{z+1}}{(z+1)}\right)}{L^2}}. \quad (53)$$

Here f is the external load and T is the temperature of the medium. The current is then given by

$$J(x,t)=-\left[\frac{fP(x,t)}{\gamma(t)}+\frac{T}{\gamma(t)}\frac{\partial P(x,t)}{\partial x}\right]. \quad (54)$$

The current $J(x,t)>0$, only when $f\neq 0$ since $\gamma(x)$ is not the necessary parameter to keep the system out of equilibrium. As stated before, $\dot{e}_p=\dot{h}_d+\frac{dS(t)}{dt}$ where $\frac{dS(t)}{dt}=-\int\frac{J}{P(x,t)}\frac{\partial}{\partial x}P(x,t)dx$. After some algebra we write

$$\frac{dS(t)}{dt}=-\int J\sum_{n=0}^{\infty}\frac{\frac{n\pi}{2}\cos\left\{\frac{n\pi}{L_0}\left[x+f\left(gt+\frac{gt^{z+1}}{(z+1)}\right)\right]\right\}\zeta}{\sum_{n=0}^{\infty}\cos\left\{\frac{n\pi}{L_0}\left[x+f\left(gt+\frac{gt^{z+1}}{(z+1)}\right)\right]\right\}\zeta}dx. \quad (55)$$

The entropy production and entropy extraction rates are given by the relations

$$\dot{e}_p=\int\frac{J^2}{P(x,t)Tg(1+t^z)}dx \quad (56)$$

and

$$\dot{h}_d=\int\left(\frac{Jf}{T}\right)dx. \quad (57)$$

Substituting Eqs. (52) and (54) into Eqs. (56) and (57), one can explore how $\dot{e}_p$ and $\dot{h}_d$ depend on time. For $z\geqslant 0$, $\dot{e}_p$ and $\dot{h}_d$ decrease and approach a constant value at steady state. When $z<0$, $\dot{e}_p$ and $\dot{h}_d$ step up continuously. Hereafter, for simplicity, the parameter g is considered to be a constant. Furthermore, the heat dissipation rate is given by

$$\dot{H}_d=\int(Jf)dx, \quad (58)$$

while the term $\dot{E}_p$ is given by

$$\dot{E}_p=\int\frac{J^2}{P(x,t)g(1+t^z)}dx. \quad (59)$$

On the other hand, the internal energy has a form

$$\dot{E}_{\rm in}=\int JU_s'(x)dx. \quad (60)$$

The total work done is then given by

$$\dot{W}=\int(Jf)dx. \quad (61)$$

The first law of thermodynamics can be written as

$$\dot{E}_{\rm in}=-\dot{H}_d(t)-\dot{W}. \quad (62)$$

Hereafter, whenever we plot any figures, we use the following dimensionless load $\bar{f}=fL_0/T_c$, $\bar{U}=U/T_c$, temperature

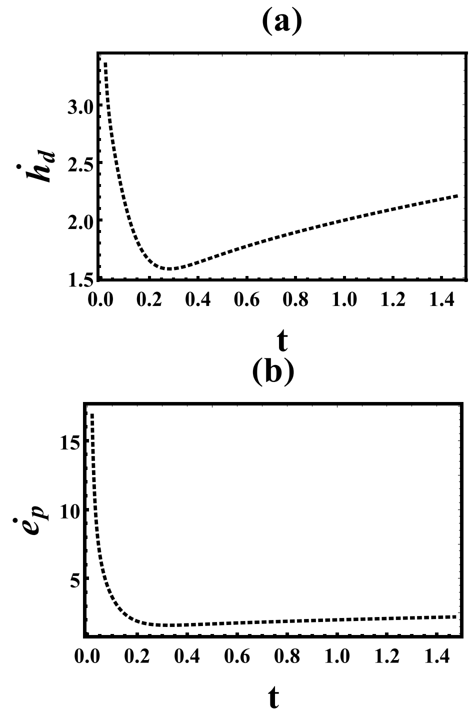

FIG. 1. (a) The entropy extraction rate $\dot{h}_d(t)$ as a function of t evaluated analytically by substituting Eqs. (52) and (54) into Eq. (57). (b) The plot of entropy production rate $\dot{e}_p(t)$ as a function of t. $\dot{e}_p(t)$ is analyzed analytically by substituting Eqs. (52) and (54) into Eq. (56). The two figures exhibit that $\dot{e}_p(t)$ and $\dot{h}_d(t)$ decrease in time and as time further steps up, $\dot{e}_p(t)$ and $\dot{h}_d(t)$ increase. In both figures, the parameters are fixed as $f=1.0$, $\tau=1.0$, and $z=-0.5$.

$\bar{\tau}=T(x)/T_c$ where T_c is the reference temperature. We also introduced dimensionless parameter $\bar{x}=x/L_0$, $\bar{v}=vm/\gamma L_0$, and $\bar{t}=t\gamma/m$. Hereafter the bar will be dropped. From now on all the figures will be plotted in terms of the dimensionless parameters.

The expression for the rate of entropy production as well as entropy extraction rate can be readily calculated via Eqs. (56) and (57). In the absence of load $f=0$, $\dot{e}_p=\dot{h}_d=0$. This is reasonable since any system which is in contact with a uniform temperature should obey the detail balance condition in a long time limit. When a distinct temperature difference is retained between the hot and cold baths, in the absence of load, $\dot{e}_p=\dot{h}_d=0$ showing that such a system is inherently reversible. In the presence of load and when the viscous friction increases in time (see Fig. 1), $\dot{e}_p$ and $\dot{h}_d$ decrease in time and as time further steps up $\dot{e}_p$ and $\dot{h}_d$ monotonously increase in time as shown in Fig. 1. Figure 1 is plotted by fixing $\tau=1.0$, $f=1.0$, and $z=-0.5$. On the contrary, in the presence of load and when the viscous friction decreases in time (see Fig. 2), $\dot{e}_p$ and $\dot{h}_d$ monotonously decrease with time and saturate to a constant value as t further steps up. Figure 2 is plotted by fixing $\tau=1.0$, $f=1.0$, and $z=1.0$.

IV. SPATIALLY VARYING VISCOUS FRICTION

Most of the previous studies have focused on exploring the thermodynamic feature of systems such as Brownian heat engines by assuming temperature invariance viscous friction.

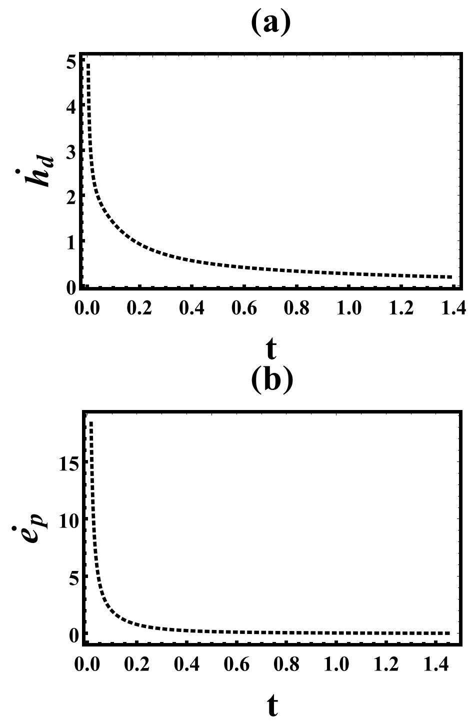

FIG. 2. (a) $\dot{h}_d(t)$ as a function of t evaluated analytically by substituting Eqs. (52) and (54) into Eq. (57). (b) $\dot{e}_p(t)$ as a function of t. $\dot{e}_p(t)$ is analyzed analytically by substituting Eqs. (52) and (54) into Eq. (56). The figure exhibits that $\dot{e}_p(t)$ and $\dot{h}_d(t)$ decrease in time and as time further steps up, $\dot{e}_p(t)$ and $\dot{h}_d(t)$ approach a constant value. In both figures, the parameters are fixed as $f = 1.0$, $\tau = 1.0$, and $z = 1.0$.

However, various studies have indicated that the viscous friction of a medium tends to decrease as the temperature of the medium increases [41–43]. Particularly in the liquid medium, the viscosity decreases as the intensity of the background temperature steps up. This is because when the temperature of the medium increases, more molecules start vibrating, and as a result their speed increases. This speedy motion of the molecules creates a reduction in interaction time between neighboring molecules. At the macroscopic level, there will be a reduction in the intermolecular force, and hence reduced viscosity of the fluid. Consequently, when the temperature of the viscous medium decreases, the viscous friction in the medium decreases.

In this paper, considering a spatially varying viscosity, the nonequilibrium thermodynamic features of a Brownian particle that hops in a ratchet potential with load is explored. The potential is also coupled with a spatially varying temperature.

The model. Let us consider a Brownian particle that walks in a piecewise linear potential with an external load $U(x) = U_s(x) + fx$ (as shown in Fig. 3), where the ratchet potential $U_s(x)$ is given by

$$U_s(x) = \begin{cases} 2U_0\left(\frac{x}{L_0}\right), & \text{if } \ 0 \leqslant x \leqslant \frac{L_0}{2}; \\ 2U_0\left(1 - \frac{x}{L_0}\right), & \text{if } \ \frac{L_0}{2} \leqslant x \leqslant L_0. \end{cases} \tag{63}$$

Here U_0 and L_0 denote the barrier height and the width of the ratchet potential, respectively. f designates the external force. The potential exhibits its maximum value U_0 at $x = \frac{L_0}{2}$ and its minima at $x = 0$ and $x = L_0$. The spatially varying temperature is arranged as

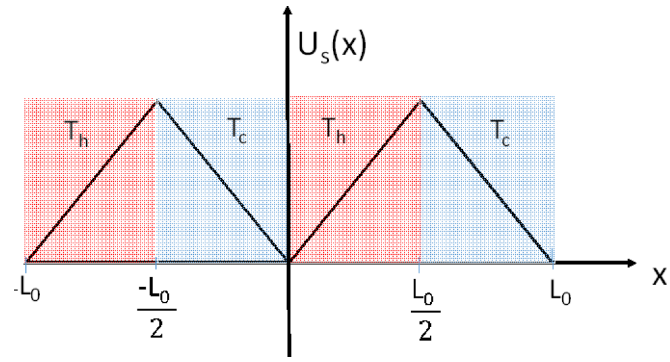

FIG. 3. Schematic diagram for a particle that walks in a piecewise linear potential in the absence of an external load.

$$T(x) = \begin{cases} T_h, & \text{if } \ 0 \leqslant x \leqslant \frac{L_0}{2}; \\ T_c, & \text{if } \ \frac{L_0}{2} \leqslant x \leqslant L_0 \end{cases} \tag{64}$$

as shown in Fig. 1. The potential $U_s(x)$ and $T(x)$ are assumed to be periodic with a period L_0, $U_s(x + L_0) = U_s(x)$, and $T(x + L_0) = T(x)$.

For a fluid such as blood, it is reasonable to assume that when the temperature of the blood sample increases by 1.0 degree Celsius, its viscosity steps down by 2.0 degree Celsius [43]. Thus let us consider viscous friction that varies as

$$\gamma(x) = \gamma' - C[T(x) - T_c], \tag{65}$$

where C is a constant which is less than one. In the next two sections we will explore the model system in the overdamped and underdamped limits.

A. Underdamped case in the absence of ratchet potential

In this section we consider an important model system where a colloidal particle that undergoes a biased random walk in a spatially varying thermal arrangement in the presence of external load f with no potential. Solving Eq. (3) at steady state, the general expression for the probability distribution is obtained as

$$P(x, v) = \frac{e^{-\frac{m[f-(\gamma+cT_c)v+cvT[x]]^2}{2T[x](\gamma+cT_c-cT[x])^2}}\sqrt{\frac{m}{T[x]}}}{\sqrt{2\pi}}. \tag{66}$$

The average velocity is found to be

$$v = \frac{f}{\gamma(x)}. \tag{67}$$

In the absence of force, the velocity of the particle approaches zero.

Via Eqs. (9) and (11), the entropy production and extraction rates are calculated as

$$\begin{aligned} \dot{h}_d(t) &= \dot{e}_p(t) \\ &= \frac{1}{2} f^2 L_0 \left(\frac{1}{\gamma T_c} + \frac{1}{[\gamma + c(T_c - T_h)]T_h} \right). \end{aligned} \tag{68}$$

One can see that in the limit where the load approaches the stall force $\dot{h}_d(t) = \dot{e}_p(t) = 0$. Exploiting Eq. (62), it is evident that $\dot{e}_p(t)$ and $\dot{h}_d(t)$ increase when the temperature difference between the hot and cold baths decreases. This is feasible since when the temperature difference between the heat baths steps up, the magnitude of the viscose friction decreases.

The rate of heat dissipation is calculated employing Eq. (20) and it converges to

$$\dot{H}_d(t) = \dot{E}_p(t)\frac{1}{2}f^2L_0\left(\frac{1}{\gamma} + \frac{1}{\gamma + c(T_c - T_h)}\right). \tag{69}$$

In the limit where the load approach zero, $\dot{H}_d(t) = \dot{E}_p(t) = 0$ showing that at quasistatic limit the system is reversible. On the other hand, the rate of work done is given by

$$\dot{W}(t) = \dot{E}_p(t) = \frac{1}{2}f^2L_0\left(\frac{1}{\gamma} + \frac{1}{\gamma + c(T_c - T_h)}\right). \tag{70}$$

For isothermal case $T_h = T_c$, one gets $v = f/\gamma$, $\dot{h}_d(t) = \dot{e}_p(t) = f^2L_0/\gamma T_c$, and $\dot{H}_d(t) = \dot{E}_p(t) = f^2L_0/\gamma$.

B. Overdamped case

In the presence of ratchet potential, in the overdamped limit, the closed-form expression for the steady-state current can be given as

$$J = -\frac{\varsigma_1}{\varsigma_2\varsigma_3 + (\varsigma_4 + \varsigma_5)\varsigma_1}, \tag{71}$$

where the expressions for ς_1, ς_2, ς_3, and ς_4 are given by

$$\begin{aligned}
\varsigma_1 &= -1 + e^{\frac{L_0(f - \frac{2U_0}{L_0})}{2T_c} + \frac{L_0(f + \frac{2U_0}{L_0})}{2T_h}},\\
\varsigma_2 &= \frac{e^{-\frac{\frac{fL_0(T_c+T_h)}{T_c} + 2U_0}{2T_h}}\left(e^{\frac{fL_0}{2T_c}} - e^{\frac{U_0}{T_c}}\right)L_0}{fL_0 - 2U_0}\\
&\quad - \frac{\left(e^{-\frac{fL_0+2U_0}{2T_h}} - 1\right)L_0}{fL_0 + 2U_0},\\
\varsigma_3 &= \frac{e^{-\frac{U_0}{T_c} + \frac{fL_0+2U_0}{2T_h}}\left(e^{\frac{fL_0}{2T_c}} - e^{\frac{U_0}{T_c}}\right)L_0T_c\gamma}{fL_0 - 2U_0}\\
&\quad + \frac{\left(e^{\frac{fL_0+2U_0}{2T_h}} - 1\right)L_0T_h[\gamma + c(T_c - T_h)]}{fL_0 + 2U_0},\\
\varsigma_4 &= [\gamma + c(T_c - T_h)]L_0^2\left(\frac{fL_0 + 2[(-1 + e^{-\frac{fL_0+2U_0}{2T_h}})T_h + U_0]}{2(fL_0 + 2U_0)^2}\right),
\end{aligned} \tag{72}$$

and $\varsigma_5 = L_0^2(t_1 + t_2 + t_3t_4)$. Here t_1, t_2, t_3, and t_4 are given by

$$\begin{aligned}
t_1 &= \frac{\gamma}{2fL_0 - 4U_0},\\
t_2 &= \frac{\gamma T_c\left(-1 + e^{-\frac{fL_0-2U_0}{2T_c}}\right)T_c}{(fL_0 - 2U_0)^2},\\
t_3 &= \frac{\left(-1 + e^{-\frac{fL_0-2U_0}{2T_h}}\right)T_h}{\left(f^2L_0^2 - 4U_0^2\right)},\\
t_4 &= \frac{e^{-\frac{fL_0(T_c+T_h)}{2T_cT_h} - \frac{U_0}{T_h}}\left(e^{\frac{fL_0}{2T_c}} - e^{\frac{U_0}{T_c}}\right)[\gamma + c(T_c - T_h)]}{\left(f^2L_0^2 - 4U_0^2\right)}.
\end{aligned} \tag{73}$$

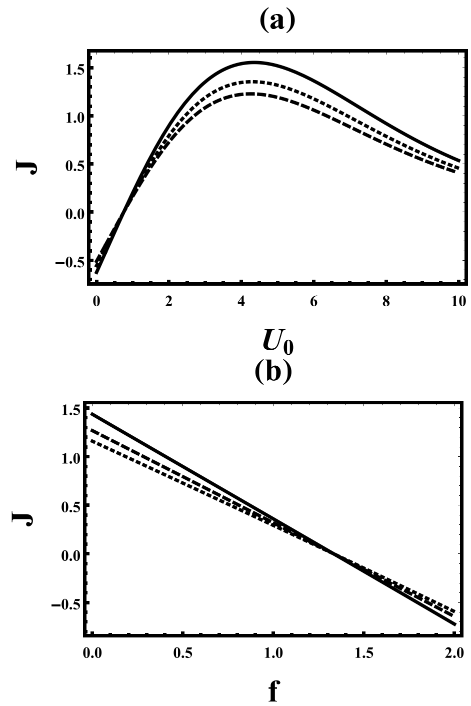

FIG. 4. (a) The dependence of J on U_0 for fixed $\tau = 2.0$, $\gamma' = 1$, and $f = 0.5$. The parameter C is also fixed as 0.4 (solid line), 0.2 (dashed line), and 0.04 (dotted line). (b) The plot J as a function of f for parameter choice $U_0 = 2.0$ and $\tau = 2.0$. The parameter C is fixed as 0.4 (solid line), 0.2 (dashed line), and 0.04 (dotted line).

The steady state current converges to zero (at quasistatic limit) when

$$f' = \frac{2U_0(T_h - T_c)}{[L_0(T_h + T_c)]}. \tag{74}$$

Next, let us explore the dependence for the thermodynamic quantities on the model parameters. In Fig. 4(a) the current as a function of potential height is plotted. The current exhibits a maximum value at a particular barrier height. As shown in Fig. 4(b), the current monotonously decreases with the load. When $f < f'$, $J > 0$ while when $f > f'$, $J < 0$.

Once the expression for steady-state current is obtained, the values for $\dot{h}_d$ and $\dot{e}_p$ can be readily evaluated via Eq. (42). At steady state ($v\frac{dv}{dt} = 0$), to both underdamped and overdamped cases, one finds

$$\dot{h}_d = \dot{e}_p = \int\left(\frac{JU'(x)}{T(x)}\right)dx. \tag{75}$$

The rate of heat extraction is given by

$$\dot{H}_d = = \int[JU'(x)]dx. \tag{76}$$

At quasistatic limit($f \to f'$), $\dot{h}_d = \dot{e}_p = 0$ as well as $\dot{H}_d = 0$ since at this limit $J = 0$. Let us explore how the rate of entropy production $\dot{e}_p(t)$ and the rate of entropy extraction $\dot{h}_d(t)$ behave. The plot of $\dot{e}_p(t)$ and $\dot{h}_d(t)$ as a function of U_0 is depicted in Fig. 5(a). The entropy production and extraction rates take a zero value at the stall force (zero velocity), $\dot{e}_p(t) = \dot{h}_d(t) = 0$ which implies that at the stall force the system is reversible. The plot $\dot{e}_p(t)$ and $\dot{h}_d(t)$ as a function of f

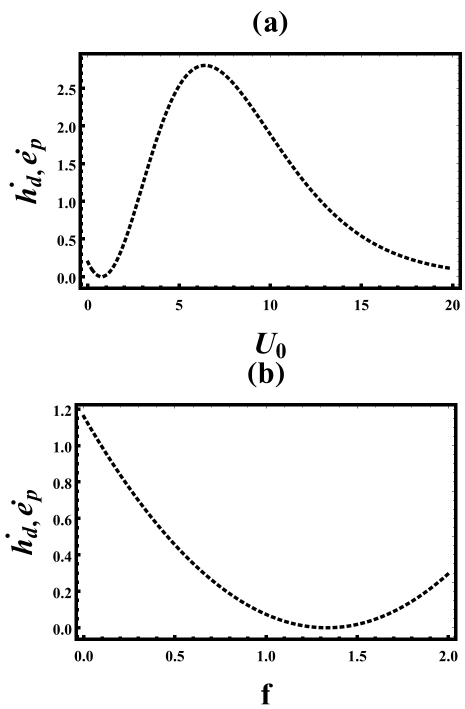

FIG. 5. (a) The plot for $\dot{e}_p(t)$ and $\dot{h}_d(t)$ as a function of U_0 for parameter choice of $\tau = 2.0$, $C = 0.04$, and $f = 0.5$. (b) The plot $\dot{e}_p(t)$ and $\dot{h}_d(t)$ as a function of f for parameter choice of $U_0 = 2.0$, $C = 0.04$, and $\tau = 2.0$.

is depicted in Fig. 5(b). As depicted in the figure, the entropy production and extraction rates decrease as the load increases and attains a zero value at the stall force. As the load further increases, $\dot{e}_p(t)$ and $\dot{h}_d(t)$ step up. The entropy production and extraction rates increase as C and the temperature difference between the two baths decreases. On the other hand, entropy production and extraction rates decrease as τ increases and attain a zero value at a particular τ. As the temperature further increases, $\dot{e}_p(t)$ and $\dot{h}_d(t)$ increase

If one considers a periodic boundary condition at steady state in the absence of ratchet potential $U_0 = 0$, the results obtained quantitatively agree with the underdamped case (Sec. IV) and one gets

$$\begin{aligned}\dot{h}_d(t) &= \dot{e}_p(t)\\ &= \frac{1}{2} f^2 L_0 \left(\frac{1}{\gamma T_c} + \frac{1}{[\gamma + c(T_c - T_h)]T_h} \right)\end{aligned} \tag{77}$$

and

$$\begin{aligned}\dot{H}_d(t) &= \dot{E}_p(t)\\ &= \frac{1}{2} f^2 L_0 \left(\frac{1}{\gamma} + \frac{1}{\gamma + c(T_c - T_h)} \right).\end{aligned} \tag{78}$$

Let now explore the energetics of the model system. When the Brownian particle along the reaction coordinate, the heat that is taken from the hot heat bath Q_h is given as

$$Q_h = U_0 + \frac{f L_0}{2} \tag{79}$$

while the rate of heat flow into the cold heat bath Q_h can be found as

$$Q_c = U_0 - \frac{f L_0}{2}, \tag{80}$$

which implies the work done is given by

$$W = Q_h - Q_c = f L_0. \tag{81}$$

Let us now explore how the efficiency η and the coefficient of performance of the refrigerator P_{ref} behave. When the engine acts as a heat engine, the efficiency is given by

$$\eta = \frac{W}{Q_h} = \frac{f L_0}{U_0 + f L_0/2}. \tag{82}$$

At quasistatic limit, plugging Eq. (76) into Eq. (84), one gets

$$\eta = 1 - \frac{T_c}{T_h}, \tag{83}$$

which is the efficiency of the Carnot heat engine. When the engine performs as a refrigerator, the coefficient of performance of the refrigerator P_{ref} is given by

$$P_{\text{ref}} = \frac{Q_c}{W} = \frac{U_0 - f L_0/2}{f L_0} \tag{84}$$

and at quasistatic limit, plugging Eq. (76) into Eq. (86), P_{ref} approaches Carnot refrigerator

$$P_{\text{ref}} = \frac{T_c}{T_h - T_c}. \tag{85}$$

V. MULTIPLICATIVE NOISE

Most of the previous works have focused on calculating the thermodynamic features of different model systems by considering additive noise. Most realistic systems such as a neuron system can be also described by Langevin equations with multiplicative noise where in this case the noise amplitude varies spatially [34]. Considering multiplicative noise, the intrinsic noise-induced ordering phase transition has been also studied in the work [44]. In this work we study how entropy, entropy production, and extraction rate depend on the strength of the background noise by solving the model exactly.

For the case where the temperature is position-dependent $T(x) = \sqrt{D}|x|^{\frac{-z}{2}}$, in the absence any external potential, the corresponding Fokker Planck equation is given as

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{T'(x)}{2\gamma}\right) P(x,t)\\ &\quad + \frac{\partial}{\partial x}\left(\frac{T(x)}{\gamma}\frac{\partial P(x,t)}{\partial x}\right).\end{aligned} \tag{86}$$

The probability current is given as

$$\begin{aligned}J = &-\left(\frac{P(x,t)T'(x)}{2\gamma}\right)\\ &-\left(\frac{T(x)}{\gamma}\frac{\partial P(x,t)}{\partial x}\right).\end{aligned} \tag{87}$$

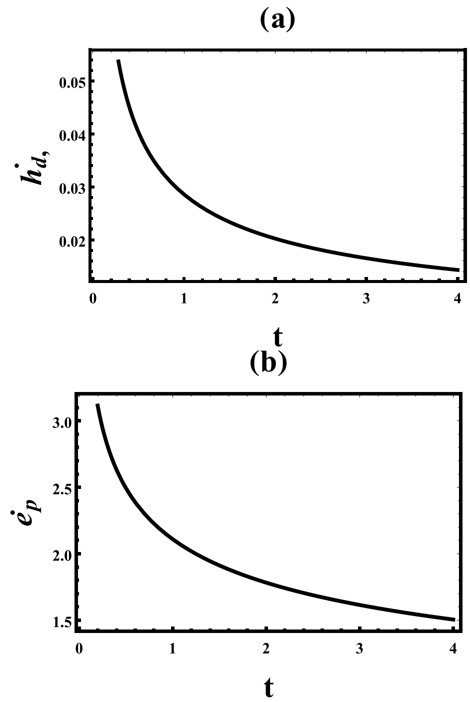

FIG. 6. The plot $\dot{h}_d(t)$ and $\dot{e}_p(t)$ as a function of t for parameter choice $\tau = 1$, $D = 1.0$, and $z = -4.0$ is depicted in (a) and (b), respectively. The figures depict that $\dot{h}_d(t)$ and $\dot{e}_p(t)$ decrease as time increases and in long time limit it approaches its stationary value $\dot{h}_d(t) = \dot{e}_p(t) = 0$.

The solution for the probability distribution is well known [45] and it is given by

$$P(x,t) = \frac{|x|^{\frac{z}{2}} e^{-\frac{|x|^{z+2}}{D(z+2)^2 t}}}{\sqrt{4\pi D t}}. \tag{88}$$

From Eqs. (42) and (43) one gets

$$\dot{e}_p = \int \frac{\gamma J^2}{PT(x)} dx \tag{89}$$

and

$$\dot{h}_d = \int \left(\frac{J[T'(x)]}{2T(x)} \right) dx. \tag{90}$$

The expression for the entropy production and extraction rates can be found by substituting Eqs. (87) and (88) into Eqs. (89) and (90). The plot $\dot{h}_d(t)$ and $\dot{e}_p(t)$ as a function of t for parameter choice $\tau = 1$, $D = 1.0$, and $z = -4.0$ is depicted in Figs. 6(a) and 6(b). The figures depict that $\dot{h}_d(t)$ and $\dot{e}_p(t)$ decrease as time increases and in long time limit it approaches its stationary value $\dot{h}_d(t) = \dot{e}_p(t) = 0$. Only in the long time limit $t \to \infty$, $\frac{dS(t)}{dt} = 0$ since $\dot{e}_p(t) = \dot{h}_d(t) = 0$. This can be intuitively comprehended on physical grounds. For the isothermal case, in the long time limit, the system approaches stationary state and only at this particular state, $\Delta h_d = 0$, $\Delta S = 0$, or $\Delta e_p = 0$ (at stationary state). However when the particle operates at finite time, the system operates irreversibility and in this regime, the second law of thermodynamics states that $\Delta S(t) > 0$. As it can be seen from Fig. 6 that if the thermodynamic quantities are evaluated in the time interval between $t = 0$ and any time t, always the inequality $\Delta h_d(t) = h_d(t) - h_d(0) > 0$, $\Delta S(t) = S(t) - S(0) > 0$, or $\Delta e_p(t) = e_p(t) - e_p(0) > 0$ holds true and as time progresses the change in this parameters increases. In fact, in small t regimes, $\dot{e}_p(t)$ becomes much larger than $\dot{h}_d(t)$ [see Figs. 6(a) and 6(b)] showing that the entropy production is higher (than entropy extraction) in the first few periods of time. When time increases, more entropy will be extracted $\dot{h}_d(t) > \dot{e}_p(t)$. Overall, since the system produces enormous amount of entropy at initial time, in latter time or any time t, $\Delta e_p(t) > \Delta h_d(t)$ and hence $\Delta S(t) > 0$.

VI. SUMMARY AND CONCLUSION

The influence of viscous friction on the thermodynamic properties of a Brownian particle that walks in overdamped and underdamped media is studied. The viscous friction is considered to vary either spatially or temporally. By extending Seifert stochastic approach to underdamped and overdamped media, the general expressions for entropy production, free energy, and entropy extraction rates are derived. To explore the nonequilibrium thermodynamic features of a Brownian particle that hops in medium where its viscosity depends on time, a Brownian particle that walks on a periodic isothermal medium (in the presence or absence of load) is considered. The analytical results depict that in the absence of load, the entropy production rate balances the entropy extraction rate which is reasonable since any system which is in contact with a uniform temperature should obey the detail balance condition in a long time limit. It is shown that when a distinct temperature difference is not retained between the hot and cold baths, in the absence of load, the entropy production still balances the entropy extraction rate revealing the system is reversible. When the external load is zero and when the viscous friction decreases in time, the entropy monotonously increases with time and saturates to a constant value as t further steps up. The entropy production rate decreases in time and at steady state (in the presence of load), $\dot{e}_p = \dot{h}_d > 0$ which agrees with the results shown in the works [27]. On the contrary, when the viscous friction increases in time, the rate of entropy production as well as the rate of entropy extraction monotonously steps up showing that such systems are inherently irreversible.

For a system where the viscous friction of a medium tends to decrease as the temperature of the medium increases, the nonequilibrium thermodynamic features of the model system are explored. In this case, the load f dictates the direction of the particle velocity. The steady-state velocity of the engine is positive when f is smaller and the engine acts as a heat engine. In this regime the entropy production and extraction rates become nonzero. When f steps up, the velocity of the particle steps down and at stall force, the entropy production rate balances the entropy extraction rate revealing the system is reversible at this particular choice of parameter. For large loads, the current is negative and the engine acts as a refrigerator. In this region the entropy production and extraction rates become nonzero. In the absence of load, the entropy production and extraction rates become larger than zero as long as a distinct temperature difference is retained between the hot and cold baths. We further explore the thermodynamic features of such systems by considering a multiplicative noise wherein case the noise amplitude varies spatially.

In conclusion, in this work we derive several thermodynamic relations to a Brownian particle moving in underdamped and overdamped media by considering viscous friction that varies temporally and spatially. We believe that the present theoretical work serves as a basic tool to understand the nonequilibrium thermodynamics.

ACKNOWLEDGMENTS

I would like to thank B. Bezabih and M. Zebene for their constant encouragement.

[1] H. Ge and H. Qian, Phys. Rev. E **81**, 051133 (2010).
[2] T. Tome and M. J. de Oliveira, Phys. Rev. Lett. **108**, 020601 (2012).
[3] J. Schnakenberg, Rev. Mod. Phys. **48**, 571 (1976).
[4] T. Tome and M. J. de Oliveira, Phys. Rev. E **82**, 021120 (2010).
[5] R. K. P. Zia and B. Schmittmann, J. Stat. Mech. (2007) P07012.
[6] U. Seifert, Phys. Rev. Lett. **95**, 040602 (2005).
[7] T. Tome, Braz. J. Phys. **36**, 1285 (2006).
[8] G. Szabo, T. Tome, and I. Borsos, Phys. Rev. E **82**, 011105 (2010).
[9] B. Gaveau, M. Moreau, and L. S. Schulman, Phys. Rev. E **79**, 010102(R) (2009).
[10] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).
[11] D. Andrieux and P. Gaspar, J. Stat. Phys. **127**, 107 (2007).
[12] R. J. Harris and G. M. Schutz, J. Stat. Mech. (2007) P07020.
[13] T. Tomé and M. J. de Oliveira, Phys. Rev. E **91**, 042140 (2015).
[14] J.-L. Luo, C. Van den Broeck, and G. Nicolis, Z. Phys. B **56**, 165 (1984).
[15] C. Y. Mou, J.-L. Luo, and G. Nicolis, J. Chem. Phys. **84**, 7011 (1986).
[16] C. Maes and K. Netocny, J. Stat. Phys. **110**, 269 (2003).
[17] L. Crochik and T. Tome, Phys. Rev. E **72**, 057103 (2005).
[18] M. Asfaw, Phys. Rev. E **89**, 012143 (2014).
[19] M. Asfaw Taye, Phys. Rev. E **92**, 032126 (2015).
[20] K. Brandner, M. Bauer, M. Schmid, and U. Seifert, New J. Phys. **17**, 065006 (2015).
[21] B. Gaveau, M. Moreau, and L. S. Schulman, Phys. Rev. E **82**, 051109 (2010).
[22] E. Boukobza and D. J. Tannor, Phys. Rev. Lett. **98**, 240601 (2007).
[23] T. Bameta, D. Das, D. Das, R. Padinhateeri, and M. M. Inamdar, Phys. Rev. E **95**, 022406 (2017).
[24] D. Oriola and J. Casademunt, Phys. Rev. Lett. **111**, 048103 (2013).
[25] O. Campàs, Y. Kafri, K. B. Zeldovich, J. Casademunt, and J.-F. Joanny, Phys. Rev. Lett. **97**, 038101 (2006).
[26] M. A. Taye, Phys. Rev. E **94**, 032111 (2016).
[27] M. A. Taye, Phys. Rev. E **101**, 012131 (2020).
[28] H. Ge, Phys. Rev. E **89**, 022127 (2014).
[29] H. K. Lee, C. Kwon, and H. Park, Phys. Rev. Lett. **110**, 050602 (2013).
[30] C. Adami, Phys. Lett. A **203**, 29 (1995).
[31] P. Bak and M. Paczuski, Proc. Natl. Acad. Sci. **92**, 6689 (1995).
[32] H. J. Jensen, *Cambridge Lecture Notes in Physics* (Cambridge University Press, Cambridge, 1998).
[33] A. Molini, P. Talkner, G. G. Katul, and A. Porporato, Physica A **390**, 1841 (2011).
[34] J. Bauermann and B. Lindner, Biosystems **178**, 25 (2019).
[35] K. Sekimoto, J. Phys. Soc. Jpn. **66**, 1234 (1997).
[36] K. Sekimoto, Prog. Theor. Phys. Suppl. **130**, 17 (1998).
[37] J. M. Sancho, M. S. Miguel, and D. Duerr, J. Stat. Phys. **28**, 291 (1982).
[38] A. M. Jayannavar and M. C. Mahato, Pramana J. Phys. **45**, 369 (1995).
[39] P. Hänggi, Helv. Phys. Acta **51**, 183 (1978).
[40] P. Hänggi, Helv. Phys. Acta **53**, 491 (1980).
[41] O. Reynolds, Philos. Trans. R. Soc. London **177**, 157 (1886).
[42] M. A. Taye and S. Duki, Eur. Phys. J. B **88**, 322 (2015).
[43] R. E. Klabunde, *Cardiovascular Physiology Concepts* (Lippincott Williams and Wilkins, Hagerstown, MD, 2005).
[44] O. Carrillo, M. Ibañes, J. García-Ojalvo, J. Casademunt and J. M. Sancho, Phys. Rev. E **67**, 046110 (2003).
[45] K. S. Fa, Phys. Rev. E **84**, 012102 (2011).