Integrability breaking in the one-dimensional Bose gas: Atomic losses and energy loss

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The one-dimensional δ-function interacting Bose gas (the Lieb-Liniger model) is an integrable system, which can model experiments with ultra-cold atoms in one-dimensional traps. Even though the model is integrable, integrability breaking effects are always present in the real-world experiments. In this work we consider the integrability breaking due to atomic loss, which is the most relevant effect in the experiments. We set up a framework for the exact computation of the losses of the canonical charges of the model, and compute an exact result for the energy loss due to the local K-body processes, valid for arbitrary K. Our result takes the form of multiple integrals, which are explicitly factorized in the experimentally relevant cases of $K = 1, 2, 3$.

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I. INTRODUCTION

One-dimensional (1D) integrable models are special many-body systems, whose exact solution is possible with analytic methods. Their solvability depends on the existence of a large (typically infinite) number of conserved quantities, which constrain the dynamical processes in the system. As an effect, the scattering events in these models are purely elastic, and the many-body S-matrix factorizes into products of two-body S-matrices [1]. This property underlies the Bethe Ansatz solution of these models [2]. The exact solvability typically means that the eigenstates can be constructed analytically, nevertheless the computation of the physical observables is often quite challenging.

The experimental advances of the last 15 years made it possible to realize integrable systems in various cold atom experiments (see, for example, [3–5]), and this motivated the study of the nonequilibrium dynamics of integrable models [6]. A key result of the last 10 years was the understanding that the equilibration in isolated integrable models can be described by the generalized Gibbs ensemble (GGE) [7,8]. The GGE involves all the conserved charges of the model, possibly including the so-called quasilocal charges [9,10]. Regarding spatially inhomogeneous situations and quantum transport the theory of generalized hydrodynamics (GHD) was formulated in [11,12]. Within GHD it is possible to treat both the ballistic modes and also the diffusive corrections [13–16]. A series of recent works [17–22] also treated the phenomenon of superdiffusion. It is very important that GHD was successfully applied to describe real-world experiments [23,24].

One of the most interesting open problems within GHD is the treatment of the integrability breaking effects, which are necessarily present in the experiments. In the strict long time limit the integrability breaking effects completely spoil the applicability of the exact methods: the systems eventually thermalize, or in the presence of dissipation or driving they form nonequilibrium steady states. However, for small integrability breaking and/or for intermediate timescales it might be possible to handle these effects within GGE and GHD.

There are two main approaches to treat this problem. In the first approach the integrability breaking terms are added as a perturbation to the Hamiltonian [25]. Examples include situations with slowly varying potentials [26] or space-time inhomogeneities in the coupling constants of the model [27,28]. Weak integrability breaking and a special class of operators that do not lead to thermalization on the so-called Euler scale was considered recently in [29].

An other approach is to consider the interaction between the integrable model and its environment. If the environment is quickly thermalizing and its response is uncorrelated on the timescales of the integrable system, then the time evolution of its density matrix is well described by the Lindblad equation. It was first demonstrated in [30] that in certain Lindblad systems a time dependent GGE can give a very good approximation of the state of the system (see also [31,32]). Within this approach the effect of long wavelength noise was studied in [33]. In contrast, the effects of localized Lindbladian interactions were investigated in [34].

The model considered in [34] is the 1D δ-function interacting Bose gas (also known as the Lieb-Liniger model), which was already realized in a couple of experiments (for a relatively recent review see the corresponding section of [35]). In these experiments the most relevant integrability breaking effect is that of the particle losses, and in particular the local three-body loss [36,37] (see also [32]). The net particle loss is given by the local three-body correlation function (or more generally the K-body correlator for the K-body processes), for which a number of exact results were already computed in the literature [38–44]. However, as explained in [34] (and in the closely related work [45]) it is also important to know the changes in the rapidity distribution, and not only the net loss. At present there are no exact solutions available for this problem, and [34] developed a numerical summation method for the relevant quantities.

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In this work we set up a theoretical framework, which treats the effect of the K-body losses on the canonical conserved charges of the Lieb-Liniger model. These charges correspond to the moments of the rapidity distribution, and computing their time derivatives gives useful information about the root distribution itself. As a concrete example we consider the energy loss. The energy is the second moment of the rapidity distribution, and in parity symmetric cases it is the next simplest quantity after the total particle number. Up to now there have been no exact results for the energy loss in the literature.

The structure of the paper is as follows. In Sec. II we introduce the problem, and we explain our strategy for solving it. This section also includes some observations and ideas that can prove to be useful in other models as well. In Sec. III we introduce the q-boson model, which is used as a lattice regularization of the 1D Bose gas. Section IV includes the technical computations about the losses in the q-boson model. In Sec. V we perform the scaling limit towards the Lieb-Liniger model with a finite number of particles. The thermodynamic limit is taken afterwards in Sec. VI, where the factorized formulas for the final quantities are also presented. We discuss the results in Sec. VII. A number of technical computations are relegated to the Appendixes.

II. INTEGRABILITY BREAKING VIA THE LINDBLAD EQUATION

In this section we discuss integrability breaking effects in general, without specifying the concrete model. We discuss the key concepts such as the Lindblad approximation, the GGE and the so-called string charge relations in Bethe Ansatz solvable models. The aim of this section is to highlight the general ideas behind our computations. Sections II B and II C discuss a standard strategy which can be applied in a number of integrable models, such as the Heisenberg spin chains. However, our concrete model, the 1D Bose gas is rather special, and some of the standard methods do not work in this case. This is explained in Sec. II D, where we also highlight our strategy specially adapted to the Lieb-Liniger model.

A. The Lindblad equation

Let ϱ be the density matrix of our system. For simplicity let us assume here that our model has finite number of degrees of freedom, and let H be the Hamiltonian. Systems with continuous degrees of freedom (such as our main model, the interacting Bose gas) will be considered later.

We assume that our system is in contact with a Markovian environment: this means that the response of the environment is uncorrelated on the timescales of our model. In this case the evolution of the reduced density matrix of the system is well approximated by the Lindblad equation [46,47]:

$$\dot{\varrho} = -i[H,\varrho] + \sum_a \gamma_a \left(L_a \varrho L_a^\dagger - \frac{1}{2}\{L_a^\dagger L_a, \varrho\} \right). \tag{1}$$

Here the L_a are the so-called Lindblad or jump operators, which describe concrete processes of the system, mediated by the environment. The γ_a are nonnegative coupling constants. Within the Lindblad approach the time dependence of any quantity $\mathcal{O}$ is given by

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{O}\rangle &= \mathrm{Tr}(\mathcal{O}\dot{\varrho}) \\ &= -i\langle [\mathcal{O}, H]\rangle + \sum_a \gamma_a \left(\langle L_a^\dagger \mathcal{O} L_a\rangle - \frac{1}{2}\langle \{\mathcal{O}, L_a^\dagger L_a\}\rangle \right).\end{aligned} \tag{2}$$

It is our aim to compute the effect of the Lindblad terms on the equilibrated steady states of the integrable models. To this order first we discuss the nature of the equilibrium states.

B. GGE and the string-charge relations

Integrable models possess a set of conserved charges $\{Q_\alpha\}_{\alpha=1,2,\ldots}$, which commute with each other, and the Hamiltonian is a member of the series. The construction of these charges can depend on the concrete model, but for a large class of systems they are obtained from a commuting set of transfer matrices (TMs) [48].

The extra conservation laws restrict the possible dynamical processes in the system: they allow only elastic and completely factorized scattering events [1,49]. As an effect, the equilibrated steady states and the transport properties of these models are markedly different from a generic quantum mechanical model. Regarding equilibration it is now understood that in integrable systems which are sufficiently well isolated from the environment the steady states can be described by the GGE. The idea behind the GGE is to involve all conserved charges of the model, thus a GGE density matrix has the form

$$\varrho = \frac{e^{-\sum_j \beta_j Q_j}}{Z}, \quad Z = \mathrm{Tr}\, e^{-\sum_j \beta_j Q_j}. \tag{3}$$

Originally the GGE was devised for free models [7,50], where the charges can be chosen as the mode occupation numbers. In these cases the β_j parameters can be interpreted as mode-dependent inverse temperatures. In interacting models the situation is more complicated, and the GGE density matrix was an object of active interest for a couple of years. In the prototypical example of the Heisenberg spin chain it was first understood that it is not enough to include the canonical local charges [51,52] and a complete GGE requires also the recently discovered quasilocal charges [9,10]. Furthermore it was understood that the GGE is basically equivalent to specifying the so-called Bethe root densities [9,53,54]; this is known as the string-charge duality. Now we describe this connection, which is the basis of our computations.

In Bethe Ansatz solvable models the finite volume eigenstates are characterized by a set of Bethe roots, which describe the momenta of the particles within the interacting multiparticle state. In the thermodynamic limit (TDL) the states can be described by the root distribution functions $\rho_s(\lambda)$, where λ is the rapidity parameter and the discrete index s stands for the various particle types that exist in the model (for a precise definition see Sec. VI). It is generally understood that in the TDL the root densities completely determine the correlation functions, thus the root densities carry all relevant information about the equilibrated states. In accordance, a set of charges is complete for the construction of a GGE, if the

set of their eigenvalues completely specifies all Bethe root densities [9,51–55].

The charges in question are extensive and their eigenvalues are additive. In the TDL the eigenvalues are typically expressed as

$$\Lambda_\alpha = \int d\lambda\, \mathfrak{h}_\alpha(\lambda)\rho(\lambda), \tag{4}$$

where $\mathfrak{h}_\alpha(\lambda)$ is the one-particle eigenvalue function. For simplicity we assumed here that there is only one particle species in the spectrum. If the fundamental particles can form bound states (which is the case for the so-called string solutions in the Heisenberg chains or the attractive Lieb-Liniger model), then all the bound states have to be treated as different particles, and the above formula needs to be supplemented with a summation over the particle species.

It is convenient to introduce generating functions for the charges. For example, let us define

$$X(u) = \sum_{\alpha=2}^{\infty} \frac{u^{\alpha-2}}{(\alpha-2)!} Q_\alpha. \tag{5}$$

The eigenvalues of this operator are given by

$$\Lambda(u) = \int d\lambda\, \mathfrak{h}(u,\lambda)\rho(\lambda), \tag{6}$$

where

$$\mathfrak{h}(u,\lambda) = \sum_{\alpha=2}^{\infty} \frac{u^{\alpha-2}}{(\alpha-2)!} \mathfrak{h}_\alpha(\lambda). \tag{7}$$

Completeness of the charges means that the eigenvalue function $\Lambda(u)$ completely specifies the root density. In other words, a set of charges is complete, if the integral transform (6) can be inverted. The specific details of the formulas of the type (6) depend on the model and its particle content; in the Heisenberg chain the existence of the relations was called *string-charge duality* in [53].

There are a few concrete models where the relation (6) and its inversion is established. The most prominent example is the XXZ Heisenberg spin chain. That model allows a large variety of string solutions (bound states of spin waves) depending on the anisotropy parameter. In that model the relation (6) needs to be generalized to include all string types and all quasilocal charges. The upshot is that eventually the integral transforms are easily inverted (for details see [9,53,54]). A more simple situation is that of the so-called q-boson lattice model, where the relation (6) is basically just a Fourier transform [56] (see Sec. III).

The situation in the continuum Bose gas is more problematic, and it will be discussed in Sec. II D.

C. String-charge relations and the Lindblad equation

Our aim is to compute the changes in the Bethe root distributions, as an effect of the Lindblad jump operators. The most natural idea is to compute the time derivative of the charges via relation (2), and then to use the inversion of the relation (6) to find the time derivative of the root densities.

If the system is equilibrated at time t, then its density matrix commutes with the charges. In this case the time derivative of the charge generating function is given by

$$\frac{d}{dt}\langle X(u)\rangle = \sum_a \gamma_a \langle L_a^\dagger [X(u), L_a]\rangle. \tag{8}$$

The remaining step is the exact computation of the r.h.s. above, and the application of relation (6).

In integrable models the standard framework to treat objects like the r.h.s. above is the Algebraic Bethe Ansatz (ABA) [48]. This method naturally accommodates the conserved charges, and also the local operators of the models. Furthermore, commutation relations and mean values of operator products are relatively easily derived, thus ABA is a promising choice for the computation of (8).

In ABA the charges are derived from a commuting set of transfer matrices $t(u)$. These transfer matrices are constructed from local Lax operators; the specific construction will be given below. For the moment let us just note that the generating functions $X(u)$ are typically defined through relations like [9,10]

$$X(u) = \partial_\lambda \log[t(u)] = t^{-1}(u)t'(u). \tag{9}$$

Such a definition ensures that the $X(u)$ are extensive, and their mean values are additive.

In ABA it is relatively easy to compute the action of the transfer matrices or commutation relations with them. However, the definition (9) also involves the inverse of the transfer matrix, and generally it is not known how to handle that operator within the ABA.

One possibility is to insert a complete set of states; such a step was also used in the numerical summation scheme of [34]. An other possibility is to use an *asymptotic inverse* of the transfer matrix, which exists in a number of models including the Heisenberg chains.

Let us therefore assume that there exists an other transfer matrix $\bar{t}(u)$, which commutes with the original TM, and which satisfies the asymptotic inversion relation

$$\bar{t}(u)t(u) = 1 + O(e^{-\xi L}). \tag{10}$$

This relation should hold at least in some neighborhood of $u = 0$; the exponent ξ can depend on u, but it is required that $\xi(0) > 0$. If these conditions hold, then it is safe to substitute $\bar{t}(u)$ for $t^{-1}(u)$ in (9), and for the time derivatives in the TDL we obtain

$$\frac{d}{dt}\langle X(u)\rangle = \sum_a \gamma_a \langle L_a^\dagger [\bar{t}(u)t'(u), L_a]\rangle + \cdots, \tag{11}$$

where the dots denote exponentially small corrections that scale to zero in the TDL. Computation of the r.h.s. of (11) is a relatively standard task in ABA, and together with the inversion of (6) this can be considered the "canonical" way of approaching integrability breaking in ABA.

An asymptotic inverse of the transfer matrix exist in many models, including the Heisenberg spin chains and their higher rank generalizations [57–59]; in the XXZ chain the asymptotic inverse is simply the original transfer matrix evaluated at a shifted rapidity [57,58]. However, such inversion relations seem to exist only in those models where the fundamental particles can form bound states [9,53]. As far as we know there is no asymptotic inverse for the TM of the repulsive Bose

gas. Thus the "canonical" strategy of ABA does not seem to be applicable in this model, and alternative ways are needed. This is discussed below.

D. The repulsive Lieb-Liniger model

The Lieb-Liniger model is a continuum theory which is given in second quantized formalism as

$$H = \int_0^L dx[\partial_x \Psi^\dagger(x)\partial_x \Psi(x) + c\Psi^{2\dagger}(x)\Psi^2(x)]. \tag{12}$$

Here $\Psi(x)$ is a canonical Bose field. The number c is the coupling constant of the model, and we treat here the repulsive case with $c > 0$.

The model is solved by the Bethe Ansatz [60]. The interacting multiparticle states can be characterized by a set of rapidities $\{p_1, \ldots, p_N\}$, and the (unnormalized) wave functions are given explicitly as

$$\chi_N(\{p\}|\{x\}) = \frac{1}{\sqrt{N!}} \sum_{\mathcal{P}\in S_N} e^{i\sum_j x_j(\mathcal{P}p)_j} \times \prod_{j>k} \frac{(\mathcal{P}p)_j - (\mathcal{P}p)_k - ic\epsilon(x_j - x_k)}{(\mathcal{P}p)_j - (\mathcal{P}p)_k}. \tag{13}$$

Here the summation runs over all permutations $\mathcal{P} \in S_n$ and $\epsilon(x)$ is the sign function.

Periodic boundary conditions imply that the rapidities satisfy the Bethe Ansatz equations

$$e^{ip_j L} \prod_{k\neq j} \frac{p_j - p_k - ic}{p_j - p_k + ic} = 1. \tag{14}$$

The energy and momentum of the multiparticle state is given by

$$E = \sum_j p_j^2, \quad P = \sum_j p_j.$$

Generally it is assumed that there are higher canonical charges Q_α with their finite volume eigenvalues being

$$\Lambda_\alpha = \sum_j p_j^\alpha. \tag{15}$$

In this notation we have $E = Q_2$ and $P = Q_1$.

These canonical charges can be found from a transfer matrix construction, appropriate for the continuous space [48]. However, the expansion of the transfer matrix into a discrete set of charges gives singular operators. This was discussed in detail in [61]. It was found there, that the operator expressions for the higher charges (starting from Q_4) contain a number of singular terms (for example, Dirac δ squared), which cannot be immediately canceled. However, the action of the charges remains finite on the eigenstates.

The reason for the apparent singularity of the higher operators lies in the special form of the Bethe wave function. It can be seen from (13) that the wave function is continuous in all of its variables, but due to the interaction there are jumps in its space derivatives whenever $x_j = x_k$ for some j, k. It follows that the higher space derivatives cannot be defined at all. The higher conserved charges naturally involve higher space derivatives, and this incompatibility with the wave function underlies the singularities discussed in [61]. On the other hand, a lattice regularization yields finite and well-defined action of the charges on the eigenstates.

It follows from (15) that in the TDL the densities of the charge eigenvalues are

$$\frac{\Lambda_\alpha}{L} = \int dp\, p^\alpha \rho(p), \tag{16}$$

where ρ is the density of the rapidities $\{p_k\}$ (for a precise definition see Sec. VI). In other words the charges measure the moments of the root distribution. If all the Λ_α are known, then in principle $\rho(p)$ can be reconstructed, although this might not be practical in concrete applications.

Let us now discuss the atomic losses within the Lindblad approach. In this model the experimentally relevant processes are the local K-body losses, the main contribution being the three-body loss [36,37]. The discrete Lindblad equation has to be replaced by a continuous version:

$$\dot{\varrho} = -i[H, \varrho] + G\int dx\left[L(x)\varrho L^\dagger(x) - \frac{1}{2}\left\{L^\dagger(x)L(x), \varrho\right\}\right], \tag{17}$$

where now G is an overall coupling constant. The Lindblad jump operators are

$$L(x) = \Psi^K(x), \quad L^\dagger(x) = \Psi^{\dagger K}(x). \tag{18}$$

We should compute the time derivative of the Bethe root densities under this Lindbladian time evolution. As explained above, we approach the problem by first looking at the canonical charges. Thus we intend to calculate

$$\frac{d}{dt}\langle Q_\alpha\rangle = G\int dx \langle L^\dagger(x)[Q_\alpha, L(x)]\rangle. \tag{19}$$

Assuming spatially homogeneous situations the time derivative of the density is found simply from a local action of the jump operators:

$$\frac{d}{dt}\frac{\langle Q_\alpha\rangle}{L} = G\langle L^\dagger(0)[Q_\alpha, L(0)]\rangle. \tag{20}$$

However, it is not immediately clear how to proceed from here. In principle the charges can be computed from a transfer matrix using the appropriate version of relations (5)–(9) [48,61], but there is no asymptotic inverse as far as we know. Thus it is not clear how to apply the ABA formalism to compute the r.h.s. above.

Instead of the ambitious goal of solving (19) for all α let us focus on the simplest charges. This can already give useful information, and it might be a starting point for the general case.

The simplest charge in the series is the net particle number

$$N = Q_0 = \int dx\, \Psi^\dagger(x)\Psi(x). \tag{21}$$

Its time derivative in equilibrium is given by

$$\frac{d}{dt}\frac{\langle N\rangle}{L} = G\langle L^\dagger(0)[N, L(0)]\rangle = -GKg_K, \tag{22}$$

where

$$g_K = \langle \Psi^{\dagger K}(0)\Psi^K(0)\rangle \tag{23}$$

is the local K-body correlation function. In deriving (22) we just used the commutation relations of the field operators.

The local correlation functions have been an object of active interest, and a number of exact results for g_K were computed in the literature [38–44]. Remarkably none of these works dealt directly with the Lieb-Liniger model; instead they considered various scaling limits of other models. The works [38,42] used the q-boson model and the XXZ Heisenberg chain as lattice regularizations, whereas [39–41,43,44] used a special nonrelativistic limit of the sinh-Gordon model.

Let us now turn to the next simplest charge. If the initial Bethe root distribution is parity symmetric, then the overall momentum is zero and the next relevant charge is the energy. We intend to compute

$$\frac{d}{dt}\frac{\langle H\rangle}{L} = G\langle \Psi^{\dagger K}(0)[H, \Psi^K(0)]\rangle. \tag{24}$$

To gain some insight to the problem, we can attempt a direct evaluation of the commutator above. However, this yields singular contributions. For example, a formal substitution of the interaction term of H into the r.h.s. gives

$$-GcK\int dx\,\delta(x)[\Psi^{\dagger K}(0)\Psi^{\dagger}(x)\Psi^{K-1}(0)\Psi^2(x) + \Psi^{\dagger K}(0)\Psi^{K-1}(0)\Psi^{\dagger}(x)\Psi^2(x)]. \tag{25}$$

This expression cannot be normal ordered, and we obtain a singular contribution. A different singular term is also obtained as we substitute the kinetic term in H. Altogether we observe a situation similar to that of the higher conserved charges discussed in [61]: we get singular contributions on the operator level, but we expect that their action on the Bethe states remains finite. In order to unambiguously obtain these finite terms a lattice regularization has to be applied. However, before turning to the lattice model let us compute the finite term in (25). Dropping the singular term and taking the mean value we get

$$-2GcK\langle \Psi^{\dagger(K+1)}(0)\Psi^{K+1}(0)\rangle = -2GcKg_{K+1}. \tag{26}$$

This means that in the final exact result there should be a term proportional to the local $K+1$-body correlation function. Below we show that this is indeed the case, and we obtain the proportionality factors as given above.

Regarding the lattice discretization we choose the so-called q-boson model, which is introduced in the next section. Two important advantages of the q-boson model are that the string-charge relations are simple (they consist of a mere Fourier transform, see below), and that the scaling limit towards the Lieb-Liniger model is rather straightforward.

To close this section we give more comments on the different regularization schemes. The singular terms obtained above do not seem to depend on the states, and it might be possible to subtract them directly in the continuum. However, the main advantage of the lattice regularization is that it yields all the higher charges and their action on the Bethe states. On the contrary, the regularization in the continuum would necessarily become more complicated, as already shown by the charges themselves [61].

III. THE q-BOSON MODEL AND ITS SCALING LIMIT

The q-boson model is a model of interacting bosons on the lattice, originally constructed in [62–64] and further analyzed in [65]. It can serve as a lattice regularization of the Lieb-Liniger model [38], and it has connections to combinatorics [66–69] and the theory of symmetric functions [70]. In this section we introduce the model and its Algebraic Bethe Ansatz solution, following the conventions of [65].

Consider a lattice consisting of L sites such that the configuration space of each site is a single bosonic space. We define the canonical Bose operators a_j, $a_j^\dagger$, and N_j acting on site j by the usual commutation relations

$$[a_j, a_k^\dagger] = \delta_{j,k},$$
$$[N_j, a_k] = -\delta_{j,k}a_k, \quad [N_j, a_k^\dagger] = \delta_{j,k}a_k^\dagger.$$

The action of these operators on the local states $|n\rangle_j$, $n = 0, \ldots, \infty$ is given by

$$a_j|n\rangle_j = \sqrt{n}|n-1\rangle_j, \quad a_j^\dagger|n\rangle_j = \sqrt{n+1}|n+1\rangle_j,$$
$$N_j|n\rangle_j = n|n-1\rangle_j.$$

Let us also define the local q-boson field operators $\psi_j^\dagger$, ψ_j through their action

$$\psi_j|n\rangle_j = \sqrt{[n]_q}|n-1\rangle_j \quad \psi_j^\dagger|n\rangle_j = \sqrt{[n+1]_q}|n+1\rangle_j,$$

where

$$[x]_q = \frac{1-q^{-2x}}{1-q^{-2}}.$$

The parameter q is a real number which is related to the interaction strength in the model. We will consider the regime $q \geqslant 1$, and we will use the parametrization $q = e^\eta$, $\eta > 0$.

These q-deformed operators satisfy the following commutation relations:

$$[N_k, \psi_k] = -\psi_k, \quad [N_k, \psi_k^\dagger] = \psi_k^\dagger, \quad [\psi_k, \psi_k^\dagger] = q^{-2N_k}.$$

These equations are the defining relations of the so-called q-boson algebra [71]. The canonical Bose operators are recovered in the $q \to 1$ limit:

$$\lim_{q\to 1}\psi_k = a_k, \quad \lim_{q\to 1}\psi_k^\dagger = a_k^\dagger.$$

The Hamiltonian of the q-boson model is defined as

$$H_{\mathrm{qb}} = \sum_{j=1}^{L}(2N_j - \psi_j^\dagger\psi_{j+1} - \psi_{j+1}^\dagger\psi_j). \tag{27}$$

We assume periodic boundary conditions.

The Hamiltonian (27) has the form of a free hopping model, but there are interactions between the particles due to the fact that the ψ and $\psi^\dagger$ are deformed annihilation and creation operators: their action depends on the local occupation numbers.

A. Algebraic Bethe Ansatz

The q-boson model can be solved by the Algebraic Bethe Ansatz (ABA) [72,73], which we now review. The main objects in ABA are the monodromy matrix and its matrix elements. They are constructed as follows.

Let us first consider an auxiliary space $V_a = \mathbb{C}^2$. The so-called Lax operator $\mathcal{L}(\lambda)$ is a linear operator acting on $V_a \otimes V_j$, where V_j is one of the local bosonic spaces. In this model the Lax operator is written in as

$$\mathcal{L}_j(\lambda) = \begin{pmatrix} e^{i\lambda} & \chi\psi_j^\dagger \\ \chi\psi_j & e^{-i\lambda} \end{pmatrix}. \tag{28}$$

Here the matrix structure corresponds to the auxiliary space, and the matrix elements are operators acting on V_j. The numerical parameter χ is given by $\chi^2 = 1 - q^{-2}$.

The Lax operator satisfies the RLL exchange relation

$$R(\lambda-\mu)\big[\mathcal{L}(\lambda)\otimes\mathcal{L}(\mu)\big] = \big[\mathcal{L}(\mu)\otimes\mathcal{L}(\lambda)\big]R(\lambda-\mu) \tag{29}$$

with the R-matrix

$$R(u-v) = \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & q & g(u,v) & 0 \\ 0 & g(u,v) & q^{-1} & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix}, \tag{30}$$

where

$$f(u,v) = \frac{\sin(u-v+i\eta)}{\sin(u-v)}, \quad g(u,v) = \frac{\sin(i\eta)}{\sin(u-v)}. \tag{31}$$

This representation of the R-matrix is slightly different from the conventional one, due to the factors of q and q^{-1}. Nevertheless it satisfies the usual Yang-Baxter relation

$$R_{1,2}(\lambda_{1,2})R_{1,3}(\lambda_{1,3})R_{2,3}(\lambda_{2,3}) = R_{2,3}(\lambda_{2,3})R_{1,3}(\lambda_{1,3})R_{1,2}(\lambda_{1,2}), \tag{32}$$

where it is understood that $\lambda_{j,k} = \lambda_j - \lambda_k$. The relation between this R-matrix and its conventional form is discussed in Appendix A.

The monodromy matrix is constructed as

$$T(\lambda) = \mathcal{L}_L(\lambda)\mathcal{L}_{L-1}(\lambda)\cdots\mathcal{L}_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{33}$$

It follows from (29) that the monodromy matrix satisfies the RTT relation

$$R(\lambda-\mu)[T(\lambda)\otimes T(\mu)] = [T(\mu)\otimes T(\lambda)]R(\lambda-\mu). \tag{34}$$

A direct consequence of (34) is that the transfer matrices defined as

$$t(\lambda) = \operatorname{Tr} T(\lambda) = A(\lambda) + D(\lambda)$$

form a commuting family:

$$[t(\lambda), t(\mu)] = 0. \tag{35}$$

The asymptotic behavior at $\lambda = \pm i\infty$ is given by

$$\lim_{\lambda\to i\infty}[e^{iL\lambda}t(\lambda)] = \lim_{\lambda\to -i\infty}[e^{-iL\lambda}t(\lambda)] = 1. \tag{36}$$

In this model the commuting set of charges are obtained from the expansion of $t(\lambda)$ around the special points $\lambda = i\infty$; the natural expansion parameter is $e^{2i\lambda}$. We define a set of charges I_α with $m \geqslant 1$ as

$$I_\alpha = \frac{\alpha}{(2\alpha)!(1-q^{-2\alpha})}\left(\frac{\partial}{\partial x}\right)^{2\alpha}\log\{x^L t[(x)]\}|_{x=0}, \tag{37}$$

where $x = e^{i\lambda}$. The prefactors are chosen such that the eigenvalues of the charges will take a simple form.

It can be seen from the definition of the transfer matrix, that each operator I_α is extensive and it is a sum of local operators which span at most $\alpha + 1$ sites. In particular for the first charge we obtain

$$I_1 = \sum_j \psi_j^\dagger \psi_{j+1}. \tag{38}$$

Explicit formulas for I_2 and I_3 (in a slightly different multiplicative normalization) can be found in [38].

The operators I_α are not Hermitian. We can define the charges with negative indices as their adjoint:

$$I_{-\alpha} = (I_\alpha)^\dagger. \tag{39}$$

They can be obtained by expanding the transfer matrix around $\lambda = -i\infty$. Hermitian charges are then obtained as the combinations

$$\frac{I_\alpha + I_{-\alpha}}{2}, \quad \frac{I_\alpha - I_{-\alpha}}{2i}. \tag{40}$$

The particle number operator

$$N = \sum_j N_j$$

commutes with all of the charges, because the transfer matrix includes only terms with an equal number of $\psi^\dagger$ and ψ operators. We define $I_0 \equiv N$. The Hamiltonian can then be written as

$$H_{\mathrm{qb}} = -I_1 - I_{-1} + 2I_0. \tag{41}$$

We build the following set of vectors (called *Bethe vectors*) via the B-operators of the monodromy matrix:

$$|\{\lambda\}\rangle \equiv \prod_{j=1}^N B(\lambda_j)|0\rangle. \tag{42}$$

Here $|0\rangle = \otimes_{j=1}^L |0\rangle_j$ is the vacuum state. The parameters λ_j are the rapidities of the interacting bosons.

The dual vectors can be defined as

$$\langle\{\lambda\}| \equiv \langle 0|\prod_{j=1}^N C(\lambda_j), \quad \langle 0| = |0\rangle^\dagger. \tag{43}$$

A state of the form (42) is an eigenstate of the transfer matrix if the rapidities satisfy the Bethe equations:

$$e^{2i\lambda_j L}\prod_{k\neq j}\frac{\sin(\lambda_j-\lambda_k-i\eta)}{\sin(\lambda_j-\lambda_k+i\eta)} = 1. \tag{44}$$

All solutions to this equation consist of purely real rapidities.

If the Bethe equations are satisfied, then we call the states (42) and (43) on-shell; in other cases we refer to them as off-shell Bethe states.

The eigenvalues $\Lambda(u|\{\lambda\})$ of the transfer matrix $t(u)$ on the Bethe states are

$$\Lambda(u|\{\lambda\}) = \frac{1}{q^N}\left[e^{iLu}\prod_{j=1}^N f(u,\lambda_j) + e^{-iLu}\prod_{j=1}^N f(\lambda_j,u)\right]. \tag{45}$$

Eigenvalues of the local charges are easily obtained using the definition (37). It is easy to see that they can be expressed as sums of single particle eigenfunctions:

$$I_\alpha|\{\lambda\}\rangle = \sum_{j=1}^{N} i_\alpha(\lambda_j)|\{\lambda\}\rangle, \tag{46}$$

where

$$i_\alpha(\lambda) = \frac{\alpha}{(2\alpha)!(1-q^{-2\alpha})}\left(\frac{\partial}{\partial\xi}\right)^{2\alpha} \log\left[f(\lambda, \log(\xi))\right]|_{\xi=0}.$$

In (46) we used that the charge I_α exists only in lattices with $L > m$, therefore it is enough to keep the second term from (45). Using the substitution $e^{i\lambda} = a$ the derivatives are calculated easily, and we find

$$i_\alpha(\lambda) = e^{-2i\alpha\lambda}.$$

It follows from (41) that the one-particle energy is

$$e(\lambda) = 2[1-\cos(2\lambda)], \tag{47}$$

which is always nonnegative for the physical rapidities when λ is purely real.

B. Integrability breaking

We wish to compute the lattice regularization of relation (19), which describes the atomic losses of the Lieb-Liniger model. We choose the Lindblad jump operators

$$L_j = \psi_j^K, \quad L_j^\dagger = \psi_j^{\dagger K}, \tag{48}$$

where now j is a site index.

The time derivative of the charge densities is then given by

$$\frac{d}{dt}\frac{\langle I_\alpha\rangle}{L} = G\langle\psi_1^{\dagger K}\left[I_\alpha, \psi_1^K\right]\rangle. \tag{49}$$

Here we assumed a homogeneous situation and chose $j = 1$ for the site index.

The expectation value above concerns any equilibrium state. We approach it from a finite volume situation, and we intend to compute the normalized amplitude

$$\mathfrak{O}_\alpha = \frac{\langle\{\lambda\}|\psi_1^{\dagger K}\left[I_\alpha, \psi_1^K\right]|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle}, \tag{50}$$

where the states are given by (42) and (43).

The simplest case is that of $I_0 = N$, for which we find

$$\mathfrak{O}_0 = -Kg_{K,\mathrm{qb}}, \tag{51}$$

where we introduced the local correlation function of the q-boson model as

$$g_{K,\mathrm{qb}} = \frac{\langle\{\lambda\}|\psi_1^{\dagger K}\psi_1^K|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle}. \tag{52}$$

In deriving (51) we used

$$[N, \psi_1^K] = -K\psi_1^K. \tag{53}$$

The next simplest Hermitian charges are the lattice momentum and the energy. We focus on space reflection symmetric configurations, thus we are interested only in the energy. It follows from (41) that

$$\frac{\langle\{\lambda\}|\psi_1^{\dagger K}\left[H_{\mathrm{qb}}, \psi_1^K\right]|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle} = -2Kg_K - \mathfrak{O}_1 - \mathfrak{O}_1^*. \tag{54}$$

The star denotes complex conjugation.

In the next section we will show that g_K can be computed within the ABA; the concrete steps of the present derivation are different from those of [42]. Afterwards, the next step is the computation of $\mathfrak{O}_1$, which requires the treatment of I_1 within ABA. As explained in the previous section, the quantities $\mathfrak{O}_\alpha$ would be relatively easy to compute for arbitrary α, if there were an asymptotic inverse for the transfer matrix. However, to our best knowledge there is no asymptotic inverse in the q-boson model, and we need an alternative approach.

Our solution is to expand the transfer matrix itself to get the charges. Using the asymptotic behavior (36) we find the first nontrivial term as

$$I_1 = \chi^{-2}\lim_{u\to i\infty} e^{-2iu}[e^{iLu}t(u) - 1]. \tag{55}$$

Thus the charge I_1 and therefore also the Hamiltonian can be treated with a single insertion of a transfer matrix into the commutator (50).

Higher charges could be obtained by further expanding $t(u)$ into powers of e^{2iu}, and subtracting the contributions from lower order terms. For example, the expansion of $t(u)$ at order e^{4iu} includes terms proportional to I_2 and $(I_1)^2$. The latter term can be subtracted using (55). This way the resulting expression for I_2 would involve only a product of at most two transfer matrices.

In this work we content ourselves with the treatment I_1, and we give some further comments about the higher charges in the Discussion.

C. Scaling to the Lieb-Liniger model

The q-boson model can serve as a lattice regularization of the Lieb-Liniger model. Its scaling limit was already studied in a number of works, for example, [38,74]. Here we summarize the scaling procedure, including the scaling of the integrability breaking amplitudes.

The idea of the scaling is to perform the continuum limit and the $q \to 1$ limit simultaneously (a simple $q \to 1$ limit would result in free bosons on the lattice). To this order let us choose a small parameter ε. The q-boson model with a coupling constant $q = e^\eta$ and a finite volume L is scaled to the Lieb-Liger model with coupling c in a finite volume l, if the parameters are connected via

$$L = l/\varepsilon, \quad \eta = \frac{c}{2}\varepsilon, \tag{56}$$

and then the $\varepsilon \to 0$ limit is taken. In this procedure the parameters l and c of the Lieb-Liniger model are kept finite, the length of the lattice is increased (continuum limit) and the coupling is scaled to zero. During this process the rapidities λ of the q-boson model will be scaled as

$$\lambda = \frac{p\varepsilon}{2}, \tag{57}$$

where p is the rapidity parameter of the Lieb-Liniger model. Furthermore, the Lieb-Liniger space coordinate x is obtained from the site index j as

$$x = \varepsilon j. \tag{58}$$

The scaling of the constant χ is given by

$$\chi^2 = 1 - e^{-2\eta} \to c\varepsilon. \tag{59}$$

It can be seen that this scaling works on the level of wave functions and even in ABA. For example, the scaling limit of the Bethe equations (44) becomes

$$e^{ip_j l} \prod_{k\neq j} \frac{p_j - p_k - ic}{p_j - p_k + ic} = 1. \tag{60}$$

The one-particle energy (47) is scaled as

$$e(\lambda) \longrightarrow \varepsilon^2 e(p), \tag{61}$$

where $e(p)$ is the single particle energy of the Lieb-Liniger model given by

$$e(p) = p^2. \tag{62}$$

Regarding the K-body annihilation and creation processes we have

$$\psi_1^{\dagger K} \psi_1^K \longrightarrow \varepsilon^K \Psi^{\dagger K}(0)\Psi^K(0). \tag{63}$$

This follows simply from the normalization of the field operators. As an effect, the scaling of the local correlator becomes

$$g_{K,\mathrm{qb}} \longrightarrow \varepsilon^K g_K. \tag{64}$$

Finally the scaling of the Lindblad amplitude for the energy becomes

$$\frac{\langle\{\lambda\}|\psi_1^{\dagger K}\big[H_{\mathrm{qb}}, \psi_1^K\big]|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle} \longrightarrow \varepsilon^{K+2} \frac{\langle\{p\}|\Psi^{\dagger K}(0)[H_{\mathrm{LL}}, \Psi^K(0)]|\{p\}\rangle}{\langle\{p\}|\{p\}\rangle}. \tag{65}$$

Our goal is to compute the l.h.s. above exactly, in a finite volume and with a finite number of particles. Afterwards we perform the scaling limit, and extract the leading terms, which are of the order ε^{K+2}. We take the thermodynamic limit only at the end of the computation, when the finite volume result in the Lieb-Liniger model is already obtained.

IV. PARTICLE LOSSES IN THE q-BOSON MODEL

In this section we compute the loss amplitudes for the first two charges I_0 and I_1 of the q-boson model.

A. Notations

The computations to be presented are rather technical, with the formulas becoming quite lengthy. In order to shorten the formulas we introduce the following special notations.

Sets and ordered sets are to be denoted as $\bar{x} = \{x_1, \ldots, x_N\}$. Omission of elements is denoted as $\bar{x}_k = \bar{x} \setminus x_k$, $\bar{x}_{k,\ell} = \bar{x} \setminus \{x_k, x_\ell\}$.

For arbitrary functions $G(x)$, $F(x, y)$ and arbitrary sets $\bar{x}, \bar{y}$ we define

$$G(\bar{x}) = \prod_{i=1}^N G(x_i), \quad F(\bar{x}, y) = \prod_{i=1}^N F(x_i, y), \quad \text{etc.} \tag{66}$$

Similarly, for products with omissions we define

$$G(\bar{x}_k) = \prod_{\substack{i=1\\ i\neq k}}^N G(x_i), \quad F(\bar{x}_k, y) = \prod_{\substack{i=1\\ i\neq k}}^N F(x_i, y), \quad \text{etc.} \tag{67}$$

Furthermore, for any function $g(x, y)$ of two variables we define

$$\Delta_g(\bar{x}) = \prod_{k>j} g(x_k, x_j), \quad \Delta'_g(\bar{x}) = \prod_{k<j} g(x_k, x_j). \tag{68}$$

Now we give a few simple examples for the use of these notations. For example, in these notations the Bethe vectors (42) are defined as

$$|\{\lambda\}\rangle \equiv B(\bar{\lambda})|0\rangle. \tag{69}$$

The Bethe equations (44) can be written as

$$e^{-2i\lambda_j L} \frac{f(\lambda_j, \bar{\lambda})}{f(\bar{\lambda}, \lambda_j)} = -1. \tag{70}$$

Furthermore, the transfer matrix eigenvalues (45) are expressed as

$$\Lambda(u|\{\lambda\}) = \frac{1}{q^N}[e^{iLu} f(u, \bar{\lambda}) + e^{-iLu} f(\bar{\lambda}, u)]. \tag{71}$$

The strength of the bar notation becomes evident as the formulas become more and more complicated.

B. Action of ψ and $\psi^\dagger$

In order to compute the loss amplitudes the first step is to derive of the action of multiple ψ operators on the Bethe states. This is a straightforward procedure, but for this particular model it has not yet been done in the literature. Therefore we detail some of the steps.

From the explicit form of Lax operator (28) it is easy to establish

$$\chi\psi_1 = \lim_{v\to -i\infty} e^{-iv(L-1)} C(v),$$
$$\chi\psi_1^\dagger = \lim_{v\to i\infty} e^{iv(L-1)} B(v). \tag{72}$$

Thus we first need the action of C operators on the Bethe states.

From the RTT relation (34) the following commutation relation can be found:

$$C(\lambda)B(\mu) - e^{-2\eta} B(\mu)C(\lambda) = e^{-\eta} g(\lambda, \mu)[A(\lambda)D(\mu) - A(\mu)D(\lambda)]. \tag{73}$$

Furthermore the exchange of the A and D operators with the B operators is given by

$$e^\eta A(\lambda)B(\mu) = f(\lambda, \mu)B(\mu)A(\lambda) + g(\mu, \lambda)B(\lambda)A(\mu),$$
$$e^\eta D(\lambda)B(\mu) = f(\mu, \lambda)B(\mu)D(\lambda) + g(\lambda, \mu)B(\lambda)D(\mu). \tag{74}$$

Using (73) the formula of action of ψ operators on the Bethe vectors can be found explicitly. The computation is presented in Appendix B 1. The result is

$$\psi_1^K|\{\mu\}\rangle = \frac{\chi^K e^{\eta[K(K+1)/2-NK]}}{[2\sinh(\eta)]^K}\prod_{\ell=1}^{K}(1-e^{-2\ell\eta})\sum a(\bar\mu_{\rm A})e^{i\bar\mu_{\rm A}} \times f(\bar\mu_{\rm A},\bar\mu_{\rm C})|\{\mu_{\rm C}\}\rangle. \tag{75}$$

The summation here is taken over all possible partitions $\bar\mu \to \{\bar\mu_{\rm A},\bar\mu_{\rm C}\}$ such that $|\bar\mu_{\rm A}| = K$. We expanded our agreement for the shorthand notation as

$$e^{\bar\mu} = \prod_{\mu_j\in\bar\mu} e^{\mu_j}. \tag{76}$$

Furthermore we will use the shorthand notation for the prefactor

$$C_K = \frac{1}{[2\sinh(\eta)]^K}\prod_{\ell=1}^{K}(1-e^{-2\ell\eta}). \tag{77}$$

C. Norm

The norm of an eigenvector $|\{\mu\}\rangle$ of the q-boson model is given by [48]

$$\langle\{\mu\}|\{\mu\}\rangle = \Delta_g(\bar\mu)\Delta_g'(\bar\mu)h(\bar\mu,\bar\mu)\sinh^N(\eta)\det\mathcal{G}(\{\mu\}), \tag{78}$$

with

$$h(\mu,\lambda) = f(\mu,\lambda)g^{-1}(\mu,\lambda). \tag{79}$$

The so-called Gaudin matrix is defined as

$$\mathcal{G}_{jk}(\{\mu\}) = \delta_{jk}\left[L+\sum_{\ell=1}^{a}\varphi_q(\mu_k-\mu_\ell)\right] - \varphi_q(\mu_k-\mu_j),\ j,k=1,\dots,N, \tag{80}$$

with

$$\varphi_q(\mu) = \frac{\sinh(2\eta)}{\sin(\mu+i\eta)\sin(\mu-i\eta)}. \tag{81}$$

D. Scalar products

As a next step we compute normalized scalar products of the type

$$\frac{\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_{\rm C}\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}, \tag{82}$$

where $\bar\mu_{\rm C}$ is a subset of $\bar\mu$ with $|\bar\mu_{\rm C}| = N-K$. In the actual computation of the loss amplitudes we will need such scalar products with the number of field operators given by K and $K+1$.

The normalized scalar products above can be found from the definition (72) by using the famous Slavnov formula for the overlaps between on-shell and off-shell Bethe states. The detailed computation is given in Appendix B. The result is

$$\frac{\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_{\rm C}\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle} = \prod_{j=1}^{K}(1-e^{-2j\eta})\frac{\chi^{-K}e^{KN\eta}}{[2\sinh(\eta)]^{K(K-1)/2}} \times \frac{e^{iK\bar\mu_{\rm A}}}{d(\bar\mu_{\rm A})\Delta_g(\bar\mu_{\rm A})f(\bar\mu_{\rm C},\bar\mu_{\rm A})h(\bar\mu_{\rm A},\bar\mu_{\rm A})} \times \frac{\det\tilde{\mathcal{M}}(\{\mu_{\rm C}\}|\{\mu\})}{\det\mathcal{G}(\{\mu\})}. \tag{83}$$

Here $\tilde{\mathcal{M}}$ is a matrix obtained after a slight modification of $\mathcal{G}$ [see (B23)].

E. Local correlation function

Let us first compute the quantity $g_{K,\rm qb}$ defined in (52). To do this we take the scalar product between $\langle\{\mu\}|\psi_1^{\dagger K}$ and the r.h.s. of (75). This gives

$$g_{K,\rm qb} = C_K\chi^K e^{\eta[K(K+1)/2-NK]} \times e^{-\eta K}\sum a(\bar\mu_{\rm A})e^{i\bar\mu_{\rm A}}f(\bar\mu_{\rm A},\bar\mu_{\rm C})\frac{\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_{\rm C}\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}. \tag{84}$$

Summation is taken over partitions $\bar\mu \to \{\bar\mu_{\rm A},\bar\mu_{\rm C}\}$, such that $|\bar\mu_{\rm A}| = K$.

Substituting (83) into (84) we obtain after elementary simplifications

$$g_{K,\rm qb} = \prod_{j=1}^{K}(1-e^{-2j\eta})\frac{C_K 2^K}{\chi^{K(K+1)}} \times \sum s(\{\mu_{\rm A}\})\frac{\det\mathcal{G}^{(f)}}{\det\mathcal{G}}, \tag{85}$$

where

$$\mathcal{G}^{(f)}_{jk} = \sinh(\eta)e^{-i(2k-K-1)\mu_j}, \quad k=1,\dots,K, \qquad \mathcal{G}^{(f)}_{jk} = \mathcal{G}_{jk}, \quad k=K+1,\dots,N, \tag{86}$$

and $s(\{\mu\})$ is a function defined as

$$s(\{\mu\}) = \frac{1}{\Delta_g(\bar\mu)\Delta_h(\bar\mu)\Delta_h'(\bar\mu)} = \prod_{j<k}\frac{\sin(\mu_{jk})\sin(i\eta)}{\sin(\mu_{jk}+i\eta)\sin(\mu_{jk}-i\eta)}. \tag{87}$$

The elements of the matrix $\mathcal{G}^{(f)}$ are complex, but the phases are arranged symmetrically, and it can be seen that the answer is manifestly real.

The result (85) is analogous to similar formulas found in [42]. Regarding the q-boson model it is new.

F. Action of transfer matrix

In order to compute the loss amplitudes for the conserved charges we also need the action of TM's on off-shell Bethe states. For the charge I_1 (and thus the energy) it is enough to take a single TM, which acts as

$$t(u)|\{\mu\}\rangle = \Lambda(u|\{\mu\})|\{\mu\}\rangle + \sum_j \tilde\Lambda_j(u|\{\mu\})|\{\mu_j,u\}\rangle. \tag{88}$$

Here the first term on the r.h.s. is the so-called "wanted term," which gives the TM eigenvalue $\Lambda(u|\bar\mu)$ if the Bethe vector is on-shell. The remaining terms are the "unwanted terms" for which the prefactors are

$$\tilde\Lambda_j(u|\{\mu\}) = e^{-\eta N}[a(\mu_j)g(\mu_j,u)f(\mu_j,\bar\mu_j) + d(\mu_j)g(u,\mu_j)f(\bar\mu_j,\mu_j)]. \tag{89}$$

Now plugging (75), (83), and (88) into (50) and (55) we can write

$$\begin{aligned}\mathfrak{O}_1 = &\lim_{\omega\to i\infty} e^{i(L-2)\omega}\Bigg\{\frac{\langle\{\mu\}|\psi_1^{\dagger K}t(\omega)\psi_1^K|\{\mu\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}\\ &-\frac{\langle\{\mu\}|\psi_1^{\dagger K}\psi_1^K|\{\mu\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}\Lambda(\omega|\{\mu\})\Bigg\}\\ = C_K\chi^K e^{-[N-(K+1)/2]K\eta}&\sum a(\bar\mu_{\rm A})e^{i\bar\mu_{\rm A}}f(\bar\mu_{\rm A},\bar\mu_{\rm C})\\ \times\lim_{\omega\to i\infty} e^{i(L-2)\omega}&\Bigg\{\frac{\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_{\rm C}\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}[\Lambda(\omega|\{\mu_{\rm C}\})\\ -\Lambda(\omega|\{\mu\})]+&\sum_j\tilde\Lambda_j(\omega|\{\mu_{\rm C}\})\frac{\langle\{\mu\}|\psi_1^{\dagger K}|\{\bar\mu_{\rm C,j},\omega\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}\Bigg\}.\end{aligned} \tag{90}$$

Here the outer summation is taken over partitions $\bar\mu\to\{\bar\mu_{\rm A},\bar\mu_{\rm C}\}$ with $|\bar\mu_{\rm A}|=K$, and the inner summation is over the partitions $\bar\mu_{\rm C}\to\{\bar\mu_{\rm C,j},\mu_j\}$ where $|\mu_j|=1$ and $\bar\mu_{\rm C,j}=\bar\mu_{\rm C}\setminus\mu_j$.

The limit $w\to i\infty$ in the direct amplitude follows from (55). The limit $w\to i\infty$ of the indirect amplitude can be computed using

$$\begin{aligned}\lim_{u\to i\infty} e^{iu}\tilde\Lambda_j(u|\bar\mu_{\rm C}) = 2e^{-(N-K)\eta}\sinh(\eta)e^{i\mu_j}\\ \times\left[a(\mu_j)\frac{f(\mu_j,\bar\mu_j)}{f(\mu_j,\bar\mu_{\rm A})}-d(\mu_j)\frac{f(\bar\mu_j,\mu_j)}{f(\bar\mu_{\rm A},\mu_j)}\right].\end{aligned} \tag{91}$$

G. Loss amplitude of I_1

We can now rewrite (90) as the sum of two parts

$$\mathfrak{O}_1=\mathcal{Y}+\mathcal{Z}, \tag{92}$$

where we call $\mathcal{Y}$ and $\mathcal{Z}$ the *direct term* and *indirect term*, respectively. The direct term stems from the direct action of the transfer matrix on the eigenstate, and it is given by

$$\begin{aligned}\mathcal{Y} = -C_K e^{-[N-(K+1)/2]K\eta}&\sum a(\bar\mu_{\rm A})e^{i\bar\mu_{\rm A}}\\ \times f(\bar\mu_{\rm A},\bar\mu_{\rm C})&\frac{\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_{\rm C}\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}I_1(\{\mu_{\rm A}\}).\end{aligned} \tag{93}$$

Here the summation is taken over partitions $\bar\mu\to\{\bar\mu_{\rm A},\bar\mu_{\rm C}\}$, such that $|\bar\mu_{\rm A}|=K$, and for the eigenvalues of I_1 we use the following shorthand notation

$$I_1(\{\mu_{\rm A}\})=\sum_{\mu_j\in\bar\mu_{\rm A}}e^{2i\mu_j}. \tag{94}$$

The indirect term results from the "unwanted terms" of the action of the transfer matrix, and it reads

$$\begin{aligned}\mathcal{Z} = C_K e^{-[N-(K+1)/2]K\eta}&\chi e^{-(N-K-1)\eta}\\ \times\sum a(\bar\mu_{\rm A})a(\mu_b)&e^{i\bar\mu_{\rm A}}e^{i\mu_b}f(\bar\mu_{\rm A},\bar\mu_{\rm C})f(\mu_b,\bar\mu_{\rm C})\\ \times[f(\bar\mu_{\rm A},\mu_b)-f(\mu_b,\bar\mu_{\rm A})]&\frac{\langle\{\mu\}|\psi_1^{\dagger(K+1)}|\{\mu_{\rm C}\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}.\end{aligned} \tag{95}$$

The summation is taken over all partitions $\bar\mu\to\{\bar\mu_{\rm A},\bar\mu_{\rm C},\bar\mu_b\}$, such that $|\bar\mu_{\rm A}|=K$, μ_b is a single rapidity, and finally $|\bar\mu_{\rm C}|=N-K-1$. Note that the set $\bar\mu_{\rm C}$ here is different than in the previous formula: now it has one less element.

To proceed we substitute the scalar products (83) with K and $K+1$ into the formulas above.

For the direct term (93) we obtain after elementary simplifications

$$\begin{aligned}\mathcal{Y} = -C_K\prod_{j=1}^K(1-e^{-2j\eta})&\frac{2^K}{[2\sinh(\eta)]^K\chi^{K(K+1)}}\\ \times\sum s(\{\mu_{\rm A}\})&\frac{\det\mathcal{G}^{(f)}}{\det\mathcal{G}}I_1(\{\mu_{\rm A}\}).\end{aligned} \tag{96}$$

Here summation is taken over the same partitions as in (93).

Substituting (B25) into (95) we get

$$\begin{aligned}\mathcal{Z} =C_K\prod_{j=1}^{K+1}(1-e^{-2j\eta})&\frac{2^{K+1}}{\chi^{(K+1)(K+2)}}\\ \times\sum&\frac{e^{2i\mu_b}}{\Delta_g(\bar\mu_{\rm A})h(\bar\mu_{\rm A},\bar\mu_{\rm A})h(\mu_b,\bar\mu_{\rm A})}\\ \times\left[1-\frac{f(\mu_b,\bar\mu_{\rm A})}{f(\bar\mu_{\rm A},\mu_b)}\right]&\frac{\det\mathcal{G}^{(f)}}{\det\mathcal{G}},\end{aligned} \tag{97}$$

where

$$\begin{aligned}\mathcal{G}^{(f)}_{jk}&=\sinh(\eta)e^{-i(2k-K-2)\mu_j},\quad k=1,\dots,K,\\ \mathcal{G}^{(f)}_{jk}&=\sinh(\eta)e^{-i(2k-K-2)\mu_b},\quad k=K+1,\\ \mathcal{G}^{(f)}_{jk}&=\mathcal{G}_{jk},\quad k=K+2,\dots,N.\end{aligned} \tag{98}$$

Here rapidities $\{\mu_k\}$ in the first K columns belong to the set $\bar\mu_{\rm A}$. Summation is taken over the same partitions as in (95).

H. Energy loss

Using (41) and (39) we can present the energy loss amplitude as a sum of two terms:

$$\frac{\langle\{\mu\}|\psi^{\dagger K}\big[H_{\rm qb},\psi_1^K\big]|\{\mu\}\rangle}{\langle\{\mu\}|\{\mu\}\rangle}=-(\mathcal{Y}_{E,K}+\mathcal{Z}_{E,K}). \tag{99}$$

Here the subscript E denotes that the quantities describes the energy loss, and K stands for the K-body processes.

The new direct term is

$$\begin{aligned}\mathcal{Y}_{E,K} =C_K\prod_{j=1}^K(1-e^{-2j\eta})&\frac{2^K}{\chi^{K(K+1)}}\\ \times\sum E(\{\mu_{\rm A}\})s(\{\mu_{\rm A}\})&\frac{\det\mathcal{G}^{(f)}}{\det\mathcal{G}},\end{aligned} \tag{100}$$

where $E(\{\mu_{\rm A}\})$ is given by the sum of the one-particle eigenvalues (47)

$$E(\{\mu_{\rm A}\})=\sum_{\mu_j\in\bar\mu_{\rm A}}^N e(\mu_j). \tag{101}$$

The new indirect term is given by

$$\mathcal{Z}_{E,K} = C_K \prod_{j=1}^{K+1}(1-e^{-2j\eta})\frac{2^{K+1}}{\chi^{(K+1)(K+2)}}\sum s(\{\mu_{\rm D}\})\frac{\det \mathcal{G}^{(f)}}{\det \mathcal{G}} \times \left\{\sum [f(\bar{\mu}_{\rm A},\mu_b)-f(\mu_b,\bar{\mu}_{\rm A})](e^{2i\mu_b}-e^{-2i\mu_b})\right\}, \tag{102}$$

where we denote set $\bar{\mu}_{\rm D}=\{\bar{\mu}_{\rm A},\mu_b\}$, $|\bar{p}_{\rm D}|=K+1$ and use the rewriting $h(\bar{\mu}_{\rm A},\mu_b)h(\mu_b,\bar{\mu}_{\rm A})h(\bar{\mu}_{\rm A},\bar{\mu}_{\rm A})=h(\bar{\mu}_{\rm D},\bar{\mu}_{\rm D})$. The first sum is taken over partitions $\bar{\mu}\to\{\bar{\mu}_{\rm C},\bar{\mu}_{\rm D}\}$, $|\bar{\mu}_{\rm D}|=K+1$ and the second over partitions $\bar{\mu}_{\rm D}\to\{\bar{\mu}_{\rm A},\mu_b\}$.

This way the indirect term becomes

$$\mathcal{Z}_{E,K} = C_K \prod_{j=1}^{K+1}(1-e^{-2j\eta})\frac{2^{K+1}}{\chi^{(K+1)(K+2)}} \times \sum_{\bar{\mu}\to\{\bar{\mu}_{\rm C},\bar{\mu}_{\rm D}\}} s(\{\mu_{\rm D}\})F(\{\mu_{\rm D}\})\frac{\det \mathcal{G}^{(f)}}{\det \mathcal{G}}, \tag{103}$$

where the sum is taken over partitions $\bar{\mu}\to\{\bar{\mu}_{\rm C},\bar{\mu}_{\rm D}\}$, $|\bar{\mu}_{\rm D}|=K+1$. The function F is given by

$$F(\{\mu_{\rm D}\})=\sum (e^{2i\mu_b}-e^{-2i\mu_b})[h(\bar{\mu}_{\rm A},\mu_b)-h(\mu_b,\bar{\mu}_{\rm A})]. \tag{104}$$

The sum in (104) is taken over partitions $\bar{\mu}_{\rm D}\to\{\bar{\mu}_{\rm A},\mu_b\}$, where μ_b is a single rapidity.

V. SCALING LIMIT TOWARDS THE LIEB-LINIGER MODEL

Now we derive the scaling limit of the formulas of the previous section, using the scaling rules given in Sec. III C.

A. Local correlation function

As a first step we derive the value of the local correlation function.

The scaling of the Gaudin matrix is given by

$$\frac{\det \mathcal{G}^{(f)}}{\det \mathcal{G}^{LL}} \to 2^{-K}(i\varepsilon)^{K(K-1)/2}\varepsilon^K\frac{\det \mathcal{G}^{(e)}}{\det \mathcal{G}^{LL}}, \tag{105}$$

where $\mathcal{G}^{LL}$ now is the Gaudin matrix of the Lieb-Liniger model

$$\mathcal{G}^{LL}_{jk}=\delta_{jk}\left[l+\sum_m \varphi(p_j-p_m)\right]-\varphi(p_j-p_k) \tag{106}$$

with

$$\varphi(p)=\frac{2c}{p^2+c^2}, \tag{107}$$

and $\mathcal{G}^{(e)}$ coincides with $\mathcal{G}^{LL}$ except the first K columns that are given by

$$\mathcal{G}^{(e)}_{jk}=(p_j)^{k-1}, \quad k\leqslant K, \quad j=1,\dots,N. \tag{108}$$

Collecting all factors we obtain

$$g_K=(K!)^2\sum s(\{p_{\rm A}\})\frac{\det \mathcal{G}^{(e)}}{\det \mathcal{G}^{LL}}. \tag{109}$$

Here the function $s(\{p\})$ is given by

$$s(\{p\})=\prod_{j>k}\frac{p_j-p_k}{(p_j-p_k)^2+c^2}, \tag{110}$$

and the summation runs over the partitions $\bar{p}\to\{\bar{p}_{\rm A},\bar{p}_{\rm C}\}$ with $|\bar{p}_{\rm A}|=K$. During the derivation we also used $C_K\to K!$.

The formula above is completely identical to the earlier results of [42]; in particular it agrees with Eq. (5.6) of that work.

B. Energy loss rate

Let us now focus on the scaling limit of the energy loss amplitude. In this case we expect an additional factor of ε^2 according to (65).

We compute the amplitude as

$$\frac{\langle\{p\}|\Psi^{\dagger K}(0)[H_{\rm LL},\Psi^K(0)]|\{p\}\rangle}{\langle\{p\}|\{p\}\rangle}=-(\mathcal{Y}_{E,K}+\mathcal{Z}_{E,K}), \tag{111}$$

where now $\mathcal{Y}_{E,K}$ and $\mathcal{Z}_{E,K}$ are the direct and indirect terms in the Lieb-Liniger model.

Collecting all factors we find for the direct term

$$\mathcal{Y}_{E,K}=(K!)^2\sum E(\{p_{\rm A}\})s(\{p_{\rm A}\})\frac{\det \mathcal{G}^{(e)}}{\det \mathcal{G}^{LL}}, \tag{112}$$

where summation is taken over partitions $\bar{p}\to\{\bar{p}_{\rm A},\bar{p}_{\rm C}\}$, $|\bar{p}_{\rm A}|=K$ and

$$E(\{p_{\rm A}\})=\sum_{p_j\in\bar{p}_{\rm A}}(p_j)^2. \tag{113}$$

Regarding the indirect term we get

$$\mathcal{Z}_{E,K}=K!(K+1)!\sum F(\{p_{\rm D}\})s(\{p_{\rm D}\})\frac{\det \mathcal{G}^{(e)}}{\det \mathcal{G}^{LL}}, \tag{114}$$

where summation in (114) is taken over partitions $\bar{p}\to\{\bar{p}_{\rm C},\bar{p}_{\rm D}\}$, $|\bar{p}_{\rm D}|=K+1$. The matrix $\mathcal{G}^{(e)}$ coincides with $\mathcal{G}^{LL}$ except the first $K+1$ columns where

$$\mathcal{G}^{(e)}_{jl}=(p_j)^{l-1}, \quad l=1,\dots,K+1, \quad j=1\dots N, \tag{115}$$

and $p_j\in\bar{p}_{\rm D}$. The function $F(\bar{p}_{\rm D})$ after scaling is given by

$$F(\{p_{\rm D}\})=\sum_{\bar{p}_{\rm D}\to\{\bar{p}_{\rm A},p_b\}}2ip_b[f(\bar{p}_{\rm A},p_b)-f(p_b,\bar{p}_{\rm A})]. \tag{116}$$

It is shown in Appendix C that this function is in fact equal to the constant $2K(K+1)c$. Then we arrive at the simplified formula

$$\mathcal{Z}_{E,K}=2cK[(K+1)!]^2\sum s(\{p_{\rm D}\})\frac{\det \mathcal{G}^{(e)}}{\det \mathcal{G}^{LL}}. \tag{117}$$

The summation here is taken as in (114). We can see that this expression is proportional to the $K+1$-body correlator. In fact, comparing to (109) we get

$$\mathcal{Z}_{E,K}=2cKg_{K+1}. \tag{118}$$

This result has to be compared to equation (26), which was obtained from the commutation relation of the field operators. Indeed, we see that the expected higher local correlation function emerges at the end of the computation, with the same prefactor as obtained by (26).

VI. THERMODYNAMIC LIMIT AND FACTORIZATION

Here we take the thermodynamic limit of the loss amplitudes. We take the infinite volume limit such that the Bethe rapidities become dense in rapidity space. We introduce the root density $\rho(p)$, such that the number of rapidities between p and $p+dp$ becomes $L\rho(p)dp$ in a large volume L.

We do not specify the nature of the equilibrium state; it can be the ground state with a given particle density, a finite temperature state, or any other excited state which is relevant for experiments. Therefore we also introduce the hole density $\rho_h(p)$, the total density $\rho_t(p)=\rho(p)+\rho_h(p)$ and the filling fraction

$$f(p)=\frac{\rho(p)}{\rho_t(p)}. \tag{119}$$

It follows from the Bethe equations that the root and hole densities satisfy the linear integral equation

$$\rho_t(p)=\frac{1}{2\pi}+\int\frac{dp'}{2\pi}\varphi(p-p')\rho(p'). \tag{120}$$

The filling fraction can be use to characterize the equilibrium states. In the case of the ground state $f(p)$ is such that it is 1 within the Fermi zone $|p|\leqslant p_F$ and zero otherwise. In the finite temperature case we have $f(p)=(1+e^{\varepsilon(p)})^{-1}$, where $\varepsilon(p)$ is the solution of the nonlinear integral equation [75]

$$\varepsilon(p)=\frac{p^2-\nu}{T}-\int dp'\ \varphi(p-p')\log(1+e^{-\varepsilon(p')}), \tag{121}$$

where T is the temperature and ν is the chemical potential. In quantum quench problems the filling fraction can be sometimes be found using the Quench-Action method; see, for example, [76–78]. In this work we do not specify $f(p)$; we leave it as an arbitrary function that enters the final formulas.

In order to take the thermodynamic limit of the formulas of the previous section we apply the methods of [42], which were based on earlier results in the literature. There are two key steps in this procedure: the summation over partitions is turned into a multiple integral, and the ratios of determinants are expressed using certain auxiliary functions.

Let us consider a summation over partitions $\bar p=\{\bar p_{\rm A},\bar p_{\rm C}\}$, where $|\bar p_{\rm C}|=K$ is fixed, and $|\bar p_{\rm A}|=N-K$ is taken to infinity. We can write simply

$$\sum_{\bar p=\{\bar p_{\rm A},\bar p_{\rm C}\}}\cdots\longrightarrow\frac{L^K}{K!}\int d\bar p_{\rm C}\ \rho(\bar p_{\rm C})\cdots. \tag{122}$$

The factor of $K!$ in the denominator cancels the overcounting of the different permutations of a given $\bar p_{\rm C}$ in the multiple integral.

Let us now investigate the ratios of the determinants. We apply the standard trick

$$\frac{\det\mathcal{G}^{(e)}}{\det\mathcal{G}^{LL}}=\det[(\mathcal{G}^{LL})^{-1}\mathcal{G}^{(e)}]. \tag{123}$$

The matrices in the numerator are such that they are modified only in a limited number of elements; thus after multiplication with $\mathcal{G}^{-1}$ we obtain a matrix which is equal to the identity except for the rows affected.

In the thermodynamic limit the action of the Gaudin matrix can be transformed into an integral equation, and we obtain the result

$$\det[(\mathcal{G}^{LL})^{-1}\mathcal{G}^{(e)}]=\frac{1}{(2\pi L)^K\rho_t(\bar p_{\rm C})}\times\det I, \tag{124}$$

where I is a matrix of size $K\times K$ with elements given by

$$I_{jl}=h^{(l-1)}(p_{{\rm C},j}). \tag{125}$$

Here $h^{(l)}(p)$ are functions that are defined as

$$h^{(l)}(p)=p^l+\int\frac{dp'}{2\pi}\varphi(p-p')f(p')h^{(l)}(p'). \tag{126}$$

Notice that for $l=0$ we have $h^{(0)}(p)=2\pi\rho_t(p)$.

Let us now write down the multiple integrals that arise from the computation. In order to further simplify the formulas we introduce one more notation:

$$\int(d\bar p)\equiv\int\frac{d\bar p\ f(\bar p)}{(2\pi)^K}=\int\prod_{j=1}^K\left[\frac{dp_j}{2\pi}f(p_j)\right]. \tag{127}$$

In this notation the cardinality of the set $\bar p$ is suppressed, but it is always given in the text.

In the case of the local correlator we get

$$g_K=(K!)\int(d\bar p)\ s(\{p\})\det I \tag{128}$$

with $|\bar p|=K$. The prefactor $s(\{p\})$ is completely antisymmetric in its variables. After an expansion of the determinant we can use this property to write

$$g_K=(K!)^2\int(d\bar p)\ s(\{p\})\prod_{j=1}^K h^{(j-1)}(p_j). \tag{129}$$

This agrees with the final result of [42].

For the indirect term the relation (118) was already established in a finite volume; the relation clearly holds also in the TDL. Thus the only remaining task is to take the limit of the direct term. We get

$$\mathcal{Y}_{E,K}=(K!)^2\times\int(d\bar p)\ s(\{p\})E(\{p\})\prod_{j=1}^K h^{(j-1)}(p_j). \tag{130}$$

Using (24) the energy loss rate is eventually given by

$$\frac{d}{dt}\frac{H}{L}=-G(\mathcal{Y}_{E,K}+2cKg_{K+1}). \tag{131}$$

We have thus obtained the energy loss rate as a sum of a K-fold and a $K+1$-fold integral.

A. Factorization

The multiple integrals can be factorized. Factorization of the local correlator (129) was already performed in [42]. The factorization of the direct term (130) is made in Appendix D. Here we present just the final results, and for the sake of completeness we also repeat here the results of [42].

The idea of the factorization is to express the multiple integral as a combination of single integrals. It turns out that the building blocks for the final formulas are the following simple integrals:

$$\{n,m\}=\frac{1}{c^{n+m+1}}\int\frac{dp}{(2\pi)}f(p)p^nh^{(m)}(p). \tag{132}$$

The coupling constant c has the same dimension as the rapidity parameter, thus the quantities $\{n,m\}$ are dimensionless.

The addition of the prefactor before the integral is new here; it was not included in the corresponding formula of [42]. It can be proven that the following symmetry relation holds [42]:

$$\{n, m\} = \{m, n\}. \tag{133}$$

Let us now first present the factorized formulas for the local correlators. In the simplest case we have

$$g_1 = c\{0, 0\} = \frac{N}{L}. \tag{134}$$

For $K = 2$ we have

$$g_2 = 2c^2(\{0, 2\} - \{1, 1\}). \tag{135}$$

This result can be obtained simply using the Hellman-Feynman theorem [39–41].

For $K = 3$ the result was first found in [41], and it reads

$$g_3 = c^3[-4\{1, 3\} + 3\{2, 2\} + \{0, 4\} + \{0, 2\} - \{1, 1\} + 2(\{0, 1\}^2 - \{0, 0\}\{1, 1\})]. \tag{136}$$

For $K = 4$ the result was obtained in [42]:

$$g_4 = \tfrac{2}{5}c^4[8(\{0, 1\}^2 - \{0, 0\}\{1, 1\}) + 32(\{0, 1\}\{0, 3\} - \{0, 0\}\{1, 3\}) + 24(\{0, 2\}\{1, 1\} - \{0, 1\}\{1, 2\}) + 30(\{0, 0\}\{2, 2\} - \{0, 2\}^2) + 4(\{0, 2\} - \{1, 1\}) + 5(\{0, 4\} - 4\{1, 3\} + 3\{2, 2\}) + \{0, 6\} - 6\{1, 5\} + 15\{2, 4\} - 10\{3, 3\}]. \tag{137}$$

Let us now turn to the factorization of the direct terms. The computation is presented in Appendix D, and the results are as follows.

For $K = 1$ we have a simple integral, which gives

$$\mathcal{Y}_{E,1} = c^3\{0, 2\}. \tag{138}$$

In case $K = 2$ the answer is given by

$$\mathcal{Y}_{E,2} = \frac{2c^4}{3}[2(\{0, 4\} - \{1, 3\} + \{0, 0\}\{1, 1\} - \{0, 1\}^2) + \{1, 1\} - \{0, 2\}]. \tag{139}$$

For $K = 3$ we find

$$\mathcal{Y}_{E,3} = \frac{c^5}{10}[28(\{0, 0\}\{1, 1\} - \{0, 1\}^2) + 84(\{0, 1\}\{1, 2\} - \{0, 2\}\{1, 1\}) + 14(\{1, 1\} - \{0, 2\}) - 5\{0, 4\} + 9\{0, 6\} + 90(\{0, 2\}^2 - \{0, 0\}\{2, 2\}) + 72(\{0, 0\}\{1, 3\} - \{0, 1\}\{0, 3\}) + 50\{1, 3\} - 24\{1, 5\} - 45\{2, 2\} + 15\{2, 4\}]. \tag{140}$$

The total rate of the energy loss is given by formula (131), where the above results for g_K and $\mathcal{Y}_{E,K}$ have to be substituted.

It is important that in deriving the factorized formulas we assumed that the integrals in the definition (132) are well defined. This is certainly true for the ground states and the finite temperature states. However, in certain nonequilibrium situations the root distribution function can acquire algebraic tails, making the single integrals ill defined. In these cases an appropriate cut-off procedure is needed; we leave this question to future work.

B. Numerical evaluation

In this paper we contain ourselves with the derivation of the exact results, and we leave the numerical analysis of concrete cases to a future work. Nevertheless let us make some comments about the numerical methods.

Our factorized results are expressed using the building blocks (132) which use the auxiliary functions (126). These are the same objects that appeared in the previous work [42], where it was already demonstrated that they can be computed very efficiently. The auxiliary functions are found by solving a simple linear equation, and the objects (132) are simple integrals over them. Therefore the quantities $\{n, m\}$ can be computed with arbitrary numerical precision. The input to this numerical procedure is the filling fraction $f(p)$, which enters the linear equations (126). The steps of this numerical procedure are standard by now, and publicly available codes can also be used as an aid [79].

VII. DISCUSSION

In this work we treated the atomic losses in the repulsive Lieb-Liniger model. We explained that the losses of the canonical charges can be computed via the q-boson model, which serves as a lattice regularization of the problem. In the concrete computation we considered the energy loss, which we expressed as a sum of two multiple integrals. The final formula is (131), which uses (129) and (130). The multiple integrals are factorized explicitly for $K = 1, 2, 3$, and the formulas are presented in Sec. VI A.

Our work leaves a number of open questions: Is it possible to compute explicit formulas for the losses of the higher moments? What types of integral formulas can we expect? Is it always possible to factorize the multiple integrals? Is there a deeper algebraic structure behind the factorization? Can we expect to find the time derivative of the full root density, perhaps in some approximative scheme such as a large coupling expansion? At present we do not know the answer to these questions, but we wish to give some comments about them.

It is clear from the structure of our computation, that in principle all conserved charges can be treated in the q-boson

model. Even though we do not have an asymptotic inverse in this model, the higher charges can be expressed from products of transfer matrices, thus they can be treated within ABA. We expect multiple integral formulas with an increasing number of integrals as we move to higher and higher charges. It is not clear whether there should be any pattern in the resulting formulas: it is possible that the higher charges need to be treated on a case by case basis, making the approach impractical. Nevertheless we believe that the next charge I_2 could be treated with reasonable effort: this charge can be obtained with the insertion of two transfer matrices, leading to more complicated but still manageable formulas. It is very natural to expect that from I_2 we would the fourth moment of the root distribution in the Lieb-Liniger model; this can be seen easily on the level of one-particle eigenvalues, after performing a careful scaling procedure, and subtracting some lower order terms.

Regarding factorization it is quite likely that all expected multiple integrals can eventually be factorized. However, at present there is no algebraic theory for this procedure. Based on the results of this work and [42], and also on experience with the Heisenberg chain, it is very likely that factorization needs to be performed on a case-by-case basis, and there will not be any closed formulas applicable to all the higher moments.

Summarizing these expectations, the search for the time derivative of the full root density is indeed quite challenging. An alternative approach would be to perform a systematic large coupling expansion of the quantities treated in this work, which could perhaps lead to the time derivatives of all the moments. We note that in the case of $K = 1$ the leading term of such an expansion was already given in [34]. Our results could be used as a benchmark for such a large coupling expansion: It is relatively easy to expand the multiple integrals into powers of $1/c$, in order to facilitate future comparisons.

It would be interesting to consider the attractive Lieb-Liniger model as well. In that model the fundamental particles can form bound states of arbitrary size. This might seem like a serious complication as opposed to the repulsive case, however, certain properties of that model are actually simpler. We expect to find relatively simple string-charge relations and also asymptotic inverse operators for the transfer matrices. We note that already for quantum quenches it was found that the attractive case displays simpler properties: for example, in the quench from the Bose-Einstein condensate state the root densities were found to be polynomials [77,78] as opposed to the special functions that appear in the repulsive case [76].

It is desirable to compare our present results to those of [43,44], which treated the local correlations of the 1D Bose gas using a nonrelativistic limit of the sinh-Gordon model. There the K-body correlator is expressed using a different family of auxiliary functions, such that a practical and closed form result is found for all K. Thus no factorization was needed in [43,44], and this is a considerable advantage over the earlier approaches. If the methods of [43,44] could be extended to treat the particle losses, then this could perhaps lead to more compact exact formulas.

Finally, it would be interesting to compare our formulas to the numerical results of [34].

We believe that our methods could be generalized to other models, for example, to the integrable two-component Bose gas. A direct application of the present ideas will be possible, if a two-component version of the q-boson model is formulated. As far as we know such a lattice model has not yet been established, but we believe it is within reach of the standard methods.

We hope to return to these open questions in a future work.

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APPENDIX A: SIMILARITY TRANSFORMATION

The existing literature dealing with scalar products and correlation functions, and in particular Slavnov's seminal work [80] uses an R-matrix, which is not identical to (30), and which is given by

$$\tilde{R}(u-v) = \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & 1 & g(u,v) & 0 \\ 0 & g(u,v) & 1 & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix}, \quad \text{(A1)}$$

up to multiplicative normalization. It can be seen that the difference between (30) and (A1) is *not* a similarity (or gauge) transformation. Instead it is given by a linear transformation

$$\tilde{R}_{1,2}(u) = G_1 G_2^{-1} R_{1,2}(u) G_1 G_2^{-1}, \quad \text{(A2)}$$

where G_j with $j = 1, 2$ is a linear operator given for both spaces by

$$G = q^{S_z/2} = \begin{pmatrix} q^{1/4} & \\ & q^{-1/4} \end{pmatrix}. \quad \text{(A3)}$$

Note that both $R(u)$ and $\tilde{R}(u)$ preserve the $U(1)$ charge, thus

$$R_{12}(u) G_1 G_2 = G_1 G_2 R_{12}(u), \quad \text{(A4)}$$

and similarly for $\tilde{R}(u)$. It can be shown using this identity that the Yang-Baxter relation (32) holds for $R(u)$ if and only if it holds for $\tilde{R}(u)$. The computation consists of mere substitution and commuting through the appropriate factors.

In the main text it is stated that the Lax operator (28) satisfies the RLL relation (29) with the R-matrix (30). This exchange relation can be checked by direct computation.

Similar to the relation (A2) let us also construct a new Lax operator. Let

$$\tilde{L}_j(u) = G_a q^{N_j/2} L_j(u) G_a q^{N_j/2}. \quad \text{(A5)}$$

The Lax operator also conserves the $U(1)$-charge:

$$L_j G_a q^{-N_j/2} = G_a q^{-N_j/2} L_j. \quad \text{(A6)}$$

Using this property it can be shown that the $\tilde{L}_j(u)$ operators satisfy the RLL relation (29) with the R-matrix (A1).

An important observation is that in (A5) the effect of the G_a factors is easily canceled with a shift in the rapidity parameter:

$$\tilde{L}_j(u - \eta/2) = q^{N_j/2} L_j(u) q^{N_j/2}. \quad \text{(A7)}$$

The R-matrix depends only on the rapidity differences; thus the operators $\tilde{L}_j(u-\eta/2)$ also satisfy the RLL relation with the R-matrix (A1).

Let us now construct a monodromy matrix using the deformed and shifted Lax operators:

$$\tilde{T}(\lambda) = \tilde{L}_L(\lambda-\eta/2)\cdots\tilde{L}_1(\lambda-\eta/2). \tag{A8}$$

Making use of (A7) it is seen that

$$\tilde{T}(\lambda) = q^{N/2}T(\lambda)q^{N/2}, \tag{A9}$$

where $T(\lambda)$ is the monodromy matrix defined in (33) and

$$N = \sum_{j=1}^{L} N_j, \tag{A10}$$

and we used that the N_j commute with the q-boson operators at site k for every $k \neq j$.

The relation (A9) holds for every matrix element of the monodromy matrix separately. These relations can be used to relate the scalar products computed with the two different conventions. It is easy to see that the proportionality factors between Bethe states depend only on the particle number and not the rapidities; thus for any two sets $\bar{\lambda}$ and $\bar{\mu}$ we have

$$\frac{\langle 0|C(\bar{\lambda})B(\bar{\mu})|0\rangle}{\langle 0|C(\bar{\mu})B(\bar{\mu})|0\rangle} = \frac{\langle 0|\tilde{C}(\bar{\lambda})\tilde{B}(\bar{\mu})|0\rangle}{\langle 0|\tilde{C}(\bar{\mu})\tilde{B}(\bar{\mu})|0\rangle}. \tag{A11}$$

This is guaranteed by the relations (A9).

The ratio of scalar products on the r.h.s. above can be computed with the Slavnov determinant, because the operators involved satisfy the exchange relations given by the R-matrix (A1). This is the R matrix which was used in [80]. On the other hand, the equality tells us that we can use the same formulas also in our construction, given that we consider such normalized scalar products.

APPENDIX B: ACTION OF ψ AND $\psi_1^\dagger$

1. Homogeneous limit for the action of ψ_1

Using the standard steps from [48] we find that the action of a single C operator is given by

$$C(\xi)B(\bar{\mu})|0\rangle = \sum_{k=1}^{N}\left[d(\xi)a(\mu_k)X_k + a(\xi)d(\mu_k)\tilde{X}_k\right]B(\bar{\mu}_k)|0\rangle + \sum_{j<k}\left[a(\mu_j)d(\mu_k)Y_{jk} + d(\mu_j)a(\mu_k)Y_{kj}\right]\times B(\{\bar{\mu}_{jk},\xi\})|0\rangle, \tag{B1}$$

where

$$X_k = g(\xi,\mu_k)f(\bar{\mu}_k,\mu_k)f(\xi,\bar{\mu}_k)e^{-2\eta N+\eta}, \quad \tilde{X}_k = g(\mu_k,\xi)f(\mu_k,\bar{\mu}_k)f(\bar{\mu}_k,\xi)e^{-2\eta N+\eta}, \tag{B2}$$

and

$$Y_{jk} = g(\xi_j,\mu_k)g(\mu_j,\xi)\times f(\mu_k,\mu_j)f(\mu_k,\bar{\mu}_{jk})f(\bar{\mu}_{jk},\mu_j)e^{-2\eta N+\eta}. \tag{B3}$$

Using now

$$\lim_{\xi\to-i\infty} g(\mu,\xi) \sim -2e^{-|\xi|}e^{i\mu}i\sin(i\eta), \quad \lim_{\xi\to-i\infty} f(\mu,\xi) \sim e^{-\eta}, \tag{B4}$$

it is easy to obtain from (B1) and (B2)–(B3) the following action of a single operator ψ_1:

$$\psi_1|\{\mu\}\rangle = \chi e^{-(N-1)\eta}\sum_j a(\mu_j)e^{i\mu_j}f(\mu_j,\bar{\mu}_j)|\{\mu_j\}\rangle. \tag{B5}$$

The multiple action is found by iterative procedure. Let us consider the double action of ψ, then

$$\psi_1^2|\{\mu\}\rangle = \chi^2 e^{-(2N-3)\eta}\sum_j a(\mu_j)e^{i\mu_j}f(\mu_j,\bar{\mu}_j)\times\sum_{j\neq k} a(\mu_k)e^{i\mu_k}f(\mu_k,\bar{\mu}_{jk})|\{\mu_{jk}\}\rangle. \tag{B6}$$

Let us rewrite the double sum in (B6) as a sum over partition of $\bar{u}\to\{\bar{u}_{\rm A},\bar{u}_{\rm C}\}$ with $|\bar{u}_{\rm A}|=2$, then

$$\psi_1^2|\{\mu\}\rangle = \chi^2 e^{-(2N-3)\eta}\sum_{\bar{u}\to\{\bar{u}_{\rm A},\bar{u}_{\rm C}\}} a(\bar{\mu}_{\rm A})e^{i\bar{\mu}_{\rm A}}f(\bar{\mu}_{\rm A},\bar{\mu}_{\rm C})\times\sum_{\bar{u}_{\rm A}\to\{u_j,u_k\}} f(u_j,u_k)|\{\mu_{\rm C}\}\rangle. \tag{B7}$$

We can compute the last sum over partition explicitly. Indeed,

$$\sum_{\bar{x}_{\rm A}\to\{x_1,x_2\}} f(x_1,x_2) = e^{\eta}+e^{-\eta}. \tag{B8}$$

Similarly, in a general case

$$\psi_1^K|\bar{\mu}\rangle = \chi^2 e^{-(2N-3)\eta}\sum_{\bar{\mu}\to\{\bar{\mu}_{\rm A},\bar{\mu}_{\rm C}\}} a(\bar{\mu}_{\rm A})e^{i\bar{\mu}_{\rm A}}f(\bar{\mu}_{\rm A},\bar{\mu}_{\rm C})\times\sum_{\bar{\mu}_{\rm A}\to\{\bar{\mu}_i,\bar{\mu}_{ii}\}} f(\bar{\mu}_i,\bar{\mu}_{ii})|\{\mu_{\rm C}\}\rangle, \tag{B9}$$

with $|\bar{\mu}_i| = 1$, $|\bar{\mu}_{ii}| = K - 1$ it can be checked that

$$\sum_{\bar{\mu}_A \to \{\mu_i, \bar{\mu}_{ii}\}} f(\bar{\mu}_i, \bar{\mu}_{ii}) = (e^{K\eta} + \cdots + e^{-K\eta}). \tag{B10}$$

Thus in the case of $\psi_1^K |\{\bar{\mu}\}\rangle$ we have a common factor

$$\left(\chi^K \prod_{s=1}^{K-1} e^{\eta s}\right) \frac{e^{-\eta NK}}{[2\sinh(\eta)]^K} \prod_{\ell=1}^{K} (1 - e^{-2\eta\ell}) = \frac{\chi^K}{[2\sinh(\eta)]^K} e^{\eta[K(K+1)/2 - NK]} \prod_{\ell=1}^{K} (1 - e^{-2\eta\ell}). \tag{B11}$$

Here we used $2\sinh(\eta) \sum_{\ell=1}^{k} e^{-2\ell\eta} = e^{\eta}(1 - e^{-2k\eta})$. Thus we arrive at (75).

2. Scalar product

Here we compute the scalar products relevant to the main computation. As explained in Appendix A, the normalized scalar products can be obtained using the known formulas in the literature that use the R-matrix as given by (A1). Additional factors of q which arise in the unnormalized scalar products eventually cancel.

The scalar product of an eigenvector $\langle\{\mu^C\}|$ and an arbitrary vector $|\{\mu^B\}\rangle$ of form (42) is given by [80]

$$\langle\{\mu^C\}|\{\mu^B\}\rangle = d(\bar{\mu}^B) d(\bar{\mu}^C) \Delta_g'(\bar{\mu}^C) \Delta_g(\bar{\mu}^B) \times h(\bar{\mu}^B, \bar{\mu}^C) \det \mathcal{M}(\{\mu^B\}|\{\mu^C\}), \tag{B12}$$

where $d(v) = e^{-iLv}$, $a(v) = e^{iLv}$, $r(v) = a(v)/d(v)$, and the corresponding matrix is given by

$$\mathcal{M}(\{\mu^B\}|\{\mu^C\}) = t\left(\mu_k^B, \mu_j^C\right) + t\left(\mu_j^C, \mu_k^B\right) r\left(\mu_k^B\right) \frac{h\left(\bar{\mu}^C, \mu_k^B\right)}{h\left(\mu_k^B, \bar{\mu}^C\right)}, \quad j = 1, \ldots, N, \quad k = 1, \ldots, N. \tag{B13}$$

Here we also used the function

$$t(\mu, \lambda) = g(\mu, \lambda)/h(\mu, \lambda). \tag{B14}$$

Note that in the limit $\bar{\mu}^C \to \bar{\mu}^B$ the matrix $\mathcal{M}$ becomes identical to the Gaudin matrix (80).

Using the definition (72) the action of $\psi_1^{\dagger K}$ can be presented as

$$\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_C\}\rangle = \chi^{-K} \lim_{\bar{\xi} \to i\infty} [e^{i\bar{\xi}(L-1)} \langle\{\mu\}|\{\xi, \mu_C\}\rangle]. \tag{B15}$$

Here $|\bar{\mu}_C| + K = |\bar{\mu}|$. We take the limits such that we first send $\xi_1 \to \cdots \xi_K \to \xi$ and afterwards we take $\xi \to i\infty$. In the homogeneous limit $\xi_i \to \xi_j$ the determinant of $\mathcal{M}$ becomes zero, while the prefactor becomes singular. A finite limit is reached, whose value needs separate treatment.

It is clear that only the first term in each of the first K columns of (B13) will survive, since the second term contains the factor $r(\xi) = e^{2iL\xi} \to 0$ at $\xi \to i\infty$. Rewrite the first K columns of $\mathcal{M}$ as the series w.r.t. $z = e^{|\xi|}$:

$$t(\xi_p, \mu_j) = \frac{4i^2 \sin^2(i\eta)}{z_p^2 a_j b_j [1 - (z_p a_j)^{-2}][1 - (z_p b_j)^{-2}]} = i^2 \sin^2(i\eta) \frac{4}{z_p^2 a_j b_j} \sum_{\ell,n} (a_j z_p)^{-2\ell} (b_j z_p)^{-2n}. \tag{B16}$$

Here we denote $z_p = e^{-i\xi_p}$, $a_j = e^{i\mu_j}$ and $b_j = e^{i\mu_j + \eta}$. We expand (B16) in the Taylor series w.r.t. z_p^2 around some point z:

$$4i^2 \sin^2(i\eta) \sum_k \frac{1}{k!} \left(z_p^2 - z^2\right)^k \times \frac{\partial^k}{\partial \left(z_p^2\right)^k} \left[\sum_{\ell,n} a_j^{-2\ell-1} b_j^{-2n-1} z_p^{-2(n+l+1)}\right]. \tag{B17}$$

In order to keep the columns of determinant linearly independent we take in the first K columns and extract factor $(z_p^2 - z^2)^k z_p^{-2k}$ $(k = 1, \ldots, K)$ from the each column. This factor in the limit $z_1 \to z_2 \to \cdots \to z_K \to \infty$ cancels with the factor $\Delta_g(\bar{z})$ such that

$$\prod_{k=1}^{K} \left(z^2 - z_p^2\right)^k / z_p^{2k} \, \Delta_g(\bar{z}) = [2i \sin(i\eta)]^{K(K-1)/2}. \tag{B18}$$

The limit in the determinant can be also taken. Extracting form each column $z_p^{2(\ell+n+1)}$ and taking the homogeneous limit $z_1 \to z_2 \to \cdots \to z_K \to z$ (and further $z = e^{\xi} \to \infty$) we get that the total order is $\xi^{-K(K+1)}$. We have in each column $a_j^{-2k-2} e^{-(2n+1)\eta}$, $k = \ell + n$, and a different power of k should be taken in each column otherwise the columns will be dependent again (in the kth column we keep only the kth power of e^{μ_j} since all lower powers can be canceled by subtraction of linear combination of different columns), thus we need to take in the first column $\ell = n = 0$, in the second $\ell + n = 1$, in the third $n + \ell = 2$, etc. Then the determinant can be written as

$$\det \mathcal{M}(\{\mu_C, \xi\}|\{\mu\}) = \det \tilde{M}(\{\mu_C\}|\{\mu\}) e^{-\xi K(K+1)}, \quad k = 1, \ldots, K, \tag{B19}$$

with matrix elements of $\tilde{M}(\{\mu_{\rm A}\}|\{\mu\})$ are given by

$$\tilde{M}_{jk} = C_k e^{-2ik\mu_j} \sum_{\ell=1}^{k} e^{-2\ell\eta}, \quad \tilde{M}_{jk} = t(\mu_p, \mu_j) + t(\mu_j, \mu_p) r(\mu_p) \frac{h(\bar{\mu}, \mu_p)}{h(\mu_p, \bar{\mu})}, \tag{B20}$$

where $C_k = 4e^{-\eta}(-1)^{k-1} i^2 \sin^2(i\eta)$.

On the other hand, the prefactor of the determinant gives at the corresponding order z^{K^2}:

$$h(\bar{\mu}_{\rm A}, \bar{\xi}) h(\bar{\mu}_{\rm C}, \bar{\xi}) g(\bar{\mu}_{\rm C}, \bar{\xi}) \to h(\bar{\mu}_{\rm A}, \bar{\xi}) e^{\eta NK} \to (-z)^{K^2} [2i \sin(i\eta)]^{-K^2} e^{\eta KN} e^{iK\bar{\mu}_{\rm A}}. \tag{B21}$$

Thus collecting all powers of $z = e^{\xi}$ from (B19) and (B21), extracting factors C_k from the first K columns of the determinant and taking into account the additional power of $\sin(i\eta)$ from (B18) we obtain the following behavior of the scalar product:

$$\lim_{\bar{\xi}\to i\infty} \langle\{\mu\}|\{\xi, \mu_{\rm C}\}\rangle = e^{iK\bar{\mu}_{\rm A}} \prod_{k=1}^{K} \left(1 - e^{-2k\eta}\right) \frac{e^{\eta KN}}{[2\sinh(\eta)]^{K(K-1)/2}} d(\bar{\mu}) d(\bar{\mu}_{\rm C}) \Delta_g(\bar{\mu}) \Delta_g(\bar{\mu}_{\rm C})\, h(\bar{\mu}, \bar{\mu}_{\rm C}) \det \tilde{\mathcal{M}}(\{\mu_{\rm C}\}|\{\mu\}) d(\bar{\xi}) e^{-\xi K}, \tag{B22}$$

where

$$\tilde{\mathcal{M}}_{jk} = e^{-2ik\mu_j}, \quad k = 1, \ldots, K,$$
$$\tilde{\mathcal{M}}_{jk} = \mathcal{G}_{jk}, \quad k = K+1, \ldots, N. \tag{B23}$$

Above we also used the equality $2\sinh(\eta) \sum_{\ell=1}^{k} e^{-2\ell\eta} = e^{\eta}(1 - e^{-2k\eta})$.

Note now, that the definition (72) can be written as

$$\psi_1^{\dagger} = \chi^{-1} \lim_{\xi\to i\infty} \{[d^{-1}(\xi) B(\xi)] e^{\xi}\}, \tag{B24}$$

thus factor $d(\bar{\xi}) e^{-\xi K}$ from (B22) cancels with the ξ-dependent factors in (B24). Hereby we observe the cancellation of all singular factors in the formula for the action of the field $\psi_1^{\dagger}$.

Thus, finally we arrive at

$$\langle\{\mu\}|\psi_1^{\dagger K}|\{\mu_{\rm C}\}\rangle = \chi^{-K} e^{iK\bar{\mu}_{\rm A}} \prod_{k=1}^{K} (1 - e^{-2k\eta}) e^{\eta KN} \times \frac{\Delta_g(\bar{\mu}) \Delta_g(\bar{\mu}_{\rm C})}{[2\sinh(\eta)]^{K(K-1)/2}} h(\bar{\mu}, \bar{\mu}_{\rm C}) \det \tilde{\mathcal{M}}(\{\mu_{\rm C}\}|\{\mu\}). \tag{B25}$$

APPENDIX C: COMPUTATION OF THE FUNCTION *F*

Here we consider the function $F(\{p\})$ defined in (116). It is easy to see that this function is in fact a constant. First note that $F(\{p\})$ is a symmetric function of the rapidities in $\bar{p}$. Thus in order to prove that $F(\{p\})$ is a constant it is enough to prove that it does not depend on one of the variables. Let us consider the poles of (116). The function has simple poles w.r.t. some fixed variable p_k at points $p_1, p_2, \ldots$. Let us calculate residues at these poles. We rewrite $F(\{p\})$ in the following form:

$$F(\{p\}) = \sum [f(\bar{p}_k, p_b) f(p_k, p_b) - f(p_b, p_k) f(p_b, \bar{p}_k)] 2ip_b + 2i[f(\bar{p}, p_k) - f(p_k, \bar{p})] p_k, \tag{C1}$$

where summation in the first term is taken over partitions $\bar{p} \to \{\bar{p}_{\rm A}, p_b\}$, $p_b \neq p_k$ and we write separately the term with $p_b = p_k$. We explicitly write in the first term $f(\bar{p}_{\rm A}, x) = f(\bar{p}_k, x) f(p_k, x)$ where the notation $\bar{p}_k = \bar{p}_{\rm A} \setminus p_k$ is used. Taking the residue of $F(\{p\})$ at $p_k = p_s$ for arbitrary $s \neq k$ (note that in the sum in $F(\{p\})$ only one term with $p_b = p_s$ survives) we arrive at

$$\operatorname*{Res}_{p_k = p_s} F(\{p\}) = 2ic[f(\bar{p}_k, p_s) + f(p_s, \bar{p}_b)] p_s - 2ic[f(\bar{p}_k, p_s) + (p_s, \bar{p}_k)] p_s = 0. \tag{C2}$$

Thus $F(\{p\})$ does not have a pole w.r.t. p_k at p_s for arbitrary $k \neq s$. Also it is easy to see that residue of $F(\{p\})$ at infinity is zero. Then we conclude that $F(\{p\})$ is a constant.

This constant can be found by recursion. Let us denote by F_K the value of this constant, which refers to the evaluation on the set with $|\bar{p}| = K + 1$. It can be checked that the initial condition is $F_1 = 4c$. For the value of F_K let us take the $p_1 \to \infty$ limit of the formula. Let us separate the terms and write

$$F_K = \lim_{p_1\to\infty} \sum_{j=2}^{K+1} 2ip_j [f(\bar{p}_{j,1}, p_j) f(p_1, p_j) - f(p_j, \bar{p}_{j,1}) f(p_j, p_1)] + \lim_{p_1\to\infty} 2ip_1 [f(\bar{p}_1, p_1) - f(p_1, \bar{p}_1)]. \tag{C3}$$

The first terms can be found using the limit

$$\lim_{u\to\pm\infty} f(u, x) = 1. \tag{C4}$$

The second term can be found by using the subleading contributions in the f functions. Finally we obtain the recursion

$$F_K = F_{K-1} + 4cK. \tag{C5}$$

The solution of this equation with the given initial condition is $F_K = 2K(K+1)c$.

APPENDIX D: FACTORIZATION

Here we consider the factorization of multiple integrals given by Eq. (130). For convenience we omit here some of the prefactors and factorize the multiple integrals of the form

$$J_K = \int (d\bar{p})\, s(\{p\})E(\{p\}) \prod_{j=1}^{K} h^{(j-1)}(p_j), \tag{D1}$$

where the short notation for the integrals was defined in (127). We will see that the factorized expressions become combinations of the quantities $\{n, m\}$ defined in (132).

Before turning to the details, let us explain the key idea of the factorization, as it was first laid out in [42]. The goal is to manipulate the rational functions in the integrand such that a certain degree of separation is achieved. We intend to write the integrand as sums of terms of the type

$$\cdots \frac{1}{(p_j - p_k)^2 + c^2} h^{(l)}(p_k), \tag{D2}$$

where j, k and l are arbitrary indices, and the dots stand for a product of further $h^{(m)}$ functions and an arbitrary rational function of the rapidities, such that this product does not depend on p_k. In such a case the integral over p_k can be performed using the definition (126), by noting that

$$\frac{1}{(p_j - p_k)^2 + c^2} = \frac{1}{2c}\varphi(p_j - p_k). \tag{D3}$$

After such a step the number of integrals (and thus the number of variables) is reduced by one. In the next step further algebraic manipulations might be required, but the process can be repeated with the same strategy. Eventually one ends up with the simple integrals given by (132).

At present there is no algebraic theory behind this procedure, and the computation need to be performed on a case-by-case basis.

1. $K = 1$

For $K = 1$ we have a simple integral given by

$$J_1 = \int dp\, f(p) p^2 h^{(0)}(p). \tag{D4}$$

Using the definition (132) this is equal to $c^3\{0, 2\}$.

2. $K = 2$

Consider now the case $K = 2$:

$$J_2 = \int (d\bar{p}) \frac{(p_2 - p_1)\left(p_1^2 + p_2^2\right)}{(p_2 - p_1)^2 + c^2} h^{(0)}(p_1) h^{(1)}(p_2). \tag{D5}$$

We write the rational factors of the integrand as

$$\frac{(p_2 - p_1)\left(p_1^2 + p_2^2\right)}{(p_2 - p_1)^2 + c^2} = \frac{1}{3}\left\{ \frac{2\left(p_2^3 - p_1^3\right) + c^2(p_1 - p_2)}{(p_1 - p_2)^2 + c^2} - (p_1 - p_2) \right\}. \tag{D6}$$

Now we can perform the integrals separately, using the definitions of the auxiliary functions. After the manipulations described above we obtain

$$J_2 = \frac{c^4}{6}[2(\{0, 4\} - \{1, 3\}) + \{1, 1\} - \{0, 2\} + 2(\{0, 0\}\{1, 1\} - \{0, 1\}^2)]. \tag{D7}$$

3. $K = 3$

Let us consider now the case $K = 3$. We use the following expansion:

$$\prod_{i<j}^{3} \frac{p_j - p_i}{(p_j - p_i)^2 + c^2} \sum_{k=1}^{3} p_k^2 = \sum_{\sigma \in \mathbb{P}} (-1)^{[\mathbb{P}]} D(p_{\sigma_1}, p_{\sigma_2}, p_{\sigma_3}), \tag{D8}$$

where the sum is taken over the permutations of spectral parameters, $[\mathbb{P}]$ denotes the parity of permutations and $D(x, y, z)$ is defined as

$$D(x, y, z) = \frac{z^3 - z^2 y + z c^2/3}{[(z - y)^2 + c^2][(y - x)^2 + c^2]}. \tag{D9}$$

The equality (D8) can be checked by direct computation.

Applying this expansion we perform the integration over the first variable in each term:

$$\int (d\bar{p}) \frac{p_3^3 - p_3^2 p_2 + c^2 p_3/3}{[(p_3 - p_2)^2 + c^2][(p_2 - p_1)^2 + c^2]} h^{(0)}(p_1) h^{(1)}(p_2) h^{(2)}(p_3) + \text{perm}\,. = \frac{1}{2c} \int (d\bar{p}) \frac{p_3^3 - p_3^2 p_2 + c^2 p_3/3}{(p_3 - p_2)^2 + c^2} [h^{(0)}(p_2) - 1] h^{(1)}(p_2) h^{(2)}(p_3) + \text{perm}\,. \tag{D10}$$

In the second line perm stand for analogous terms, with the appropriate replacements.

It can be seen that the terms proportional to $h^{(0)}(p_i)h^{(1)}(p_j)h^{(2)}(p_k)$ for $i, j, k = 1, \ldots, 3$ disappear, and the remaining terms are

$$J_3 = J_3^{(a)} + J_3^{(b)} + J_3^{(c)} \tag{D11}$$

with

$$J_3^{(a)} = -\frac{1}{2c}\int (d\bar{p})\, h^{(1)}(p_2)h^{(2)}(p_3)\frac{\left(p_3^3 - p_2^3\right) + \left(p_2^2 p_3 - p_3^2 p_2\right) + c^2(p_3 - p_2)/3}{(p_3 - p_2)^2 + c^2}, \tag{D12}$$

$$J_3^{(b)} = \frac{1}{2c}\int (d\bar{p})\, h^{(0)}(p_1)h^{(1)}(p_2)\frac{2\left(p_2^2 p_1^3 - p_1^2 p_2^3\right) + c^2\left(p_1 p_2^2 - p_1^2 p_2\right)/3}{(p_1 - p_2)^2 + c^2}, \tag{D13}$$

and

$$J_3^{(c)} = \frac{1}{2c}\int (d\bar{p})\frac{p_1 p_3^3 - p_1^3 p_3}{(p_1 - p_3)^2 + c^2}h^{(0)}(p_1)h^{(2)}(p_3). \tag{D14}$$

Now the problem of the factorization of (D12)–(D14) is reduced to the factorization of integrals

$$K_{\alpha\beta}^{(sr)} = \int dx\, dy \frac{x^\alpha y^\beta - x^\beta y^\alpha}{(x-y)^2 + c^2}\frac{h^{(s)}(x)h^{(r)}(y)}{c^{\alpha+\beta+s+r}}, \tag{D15}$$

with $\alpha, \beta = 1, \ldots, 4$, $r, s = 0, \ldots, 2$. Note that we introduced factors of c in the denominator so that each $K_{\alpha\beta}^{(sr)}$ is dimensionless.

Regarding $J^{(a,b,c)}$ we have

$$J_3^{(a)} = \frac{c^5}{4}\left[2K_{12}^{(23)} + \{2,4\} - \{1,5\} + \frac{1}{3}(\{2,2\} - \{1,3\})\right], \quad J_3^{(b)} = \frac{c^5}{2}\left(\frac{1}{3}K_{12}^{(12)} - 2K_{23}^{(12)}\right), \quad J_3^{(c)} = \frac{c^5}{2}K_{13}^{(13)}. \tag{D16}$$

For the rational functions appearing in the double integrals above we perform the following manipulations:

$$\frac{xy^2 - x^2 y}{(x-y)^2 + 1} = \frac{1}{3}(x-y) + \frac{1}{3}\frac{(y-x) + (y^3 - x^3)}{(x-y)^2 + 1}, \tag{D17}$$

$$\frac{xy^3 - x^3 y}{(x-y)^2 + 1} = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}\frac{(y^2 - x^2) + (y^4 - x^4)}{(x-y)^2 + 1}, \tag{D18}$$

$$\frac{x^2y^3 - x^3y^2}{(x-y)^2 + 1} = 2/15(x-y) + 2/5(x^2y - y^2x) + 1/5(x^3 - y^3) + \frac{2/15(y-x) + 1/3(y^3 - x^3) + 1/5(y^5 - x^5)}{(x-y)^2 + 1}. \tag{D19}$$

Here the variable x is be understood as the dimensionless combination $x = p/c$, with some p, and similarly for y.

This separation of terms leads to the results

$$K_{12}^{(12)} = \tfrac{1}{6}(\{0,4\} - \{1,3\} + \{0,2\} - \{1,1\} + 2\{0,1\}^2 - \{0,0\}\{1,1\}), \tag{D20}$$

$$K_{12}^{(23)} = \tfrac{1}{6}[\{2,4\} - \{1,5\} + \{2,2\} - \{1,3\} + 2(\{2,1\}\{1,0\} - \{1,1\}\{2,0\})], \tag{D21}$$

$$K_{13}^{(13)} = \tfrac{1}{4}[2(\{0,2\}^2 - \{2,2\}\{0,0\}) + \{0,4\} - \{2,2\} + \{0,6\} - \{2,4\}], \tag{D22}$$

and

$$\begin{aligned} K_{23}^{(12)} = \tfrac{1}{30}[&4(\{0,1\}^2 - \{0,0\}\{1,1\}) + 3\{0,6\} - 3\{1,5\} + 12(\{0,2\}\{1,1\} - \{0,1\}\{1,2\}) \\ &+ 2(\{0,2\} - \{1,1\}) + 5(\{0,4\} - \{1,3\}) + 6(\{0,3\}\{0,1\} - \{1,3\}\{0,0\})]. \end{aligned} \tag{D23}$$

Substituting the equations (D20)–(D23) to (D16) and finally into (D11) gives the final answer for J_3. The direct term for the energy loss is given by $\mathcal{Y}_{E,3} = (3!)^2 J_3$, and this leads to formula (140).

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