Tilted elastic lines with columnar and point disorder, non-Hermitian quantum mechanics, and spiked random matrices: Pinning and localization

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We revisit the problem of an elastic line (such as a vortex line in a superconductor) subject to both columnar disorder and point disorder in dimension $d = 1 + 1$. Upon applying a transverse field, a delocalization transition is expected, beyond which the line is tilted macroscopically. We investigate this transition in the fixed tilt angle ensemble and within a "one-way" model where backward jumps are neglected. From recent results about directed polymers in the mathematics literature, and their connections to random matrix theory, we find that for a single line and a single strong defect this transition in the presence of point disorder coincides with the Baik–Ben Arous–Péché (BBP) transition for the appearance of outliers in the spectrum of a perturbed random matrix in the Gaussian unitary ensemble. This transition is conveniently described in the polymer picture by a variational calculation. In the delocalized phase, the ground state energy exhibits Tracy-Widom fluctuations. In the localized phase we show, using the variational calculation, that the fluctuations of the occupation length along the columnar defect are described by f_{KPZ}, a distribution which appears ubiquitously in the Kardar-Parisi-Zhang universality class. We then consider a smooth density of columnar defect energies. Depending on how this density vanishes at its lower edge we find either (i) a delocalized phase only or (ii) a localized phase with a delocalization transition. We analyze this transition which is an infinite-rank extension of the BBP transition. The fluctuations of the ground state energy of a single elastic line in the localized phase (for fixed columnar defect energies) are described by a Fredholm determinant based on a new kernel, closely related to the kernel describing the largest real eigenvalues of the real Ginibre ensemble. The case of many columns and many nonintersecting lines, relevant for the study of the Bose glass phase, is also analyzed. The ground state energy is obtained using free probability and the Burgers equation. Connections with recent results on the generalized Rosenzweig-Porter model suggest that the localization of many polymers occurs gradually upon increasing their lengths.

I. INTRODUCTION

A. General motivation and overview

Directed elastic lines have been used to model vortex lines in type II superconductors [1–3], aligned with an external magnetic field applied along the z axis. Point impurities, such as oxygen vacancies in high T_c superconductors, provide a short-range correlated random potential which tends to pin the vortex lines. Spatially correlated disorder may also arise, either planar, e.g., from twin boundaries, or columnar, e.g., from linear defects such as dislocation lines or damage tracks artificially created by heavy ion irradiation. In the presence of columnar disorder along z the vortex lines tend to localize along the columns leading to the so-called Bose glass phase (by analogy with the glass phase of interacting bosons [4–6]), with enhanced pinning and critical currents [7–10].

If the external field is weakly tilted away from the z direction, the response is zero; i.e., there is a threshold transverse field needed to tilt the lines (see Fig. 1). This effect is known as the transverse Meissner effect and has been observed in experiments in various geometries [11–13]. In the absence of point disorder, this transition has been described as a commensurate-incommensurate transition [7,14].

A continuum model for a single directed elastic line (also called directed polymer) in dimension $d = 1 + 1$, of coordinates $(u(z), z)$, is defined by the energy

$$\mathcal{E}[u] = \int_0^L dz \left[\frac{\gamma}{2} \left(\frac{du(z)}{dz} \right)^2 + U(u(z)) + V(u(z), z) - H \frac{du(z)}{dz} \right]. \tag{1}$$

The first term is the elastic energy cost of deforming the line away from the z axis, γ being the line tension, $U(u)$ is a columnar potential, $V(u, z)$ a random potential from point impurities. Written here in $d = 1 + 1$, the model extends to $d = 2 + 1$, with $u(z) \to \vec{u}(z)$. It is usually studied at temperature T, defining the canonical partition sum $Z = \int \mathcal{D}u(z) e^{-\frac{1}{T} \mathcal{E}[u]}$. Here H is the transverse part of the magnetic field, and the term $-H \int_0^L dz \frac{du(z)}{dz} = -H[u(L) - u(0)]$ in the energy (1) tends to tilt the elastic line away from the z axis. In the absence
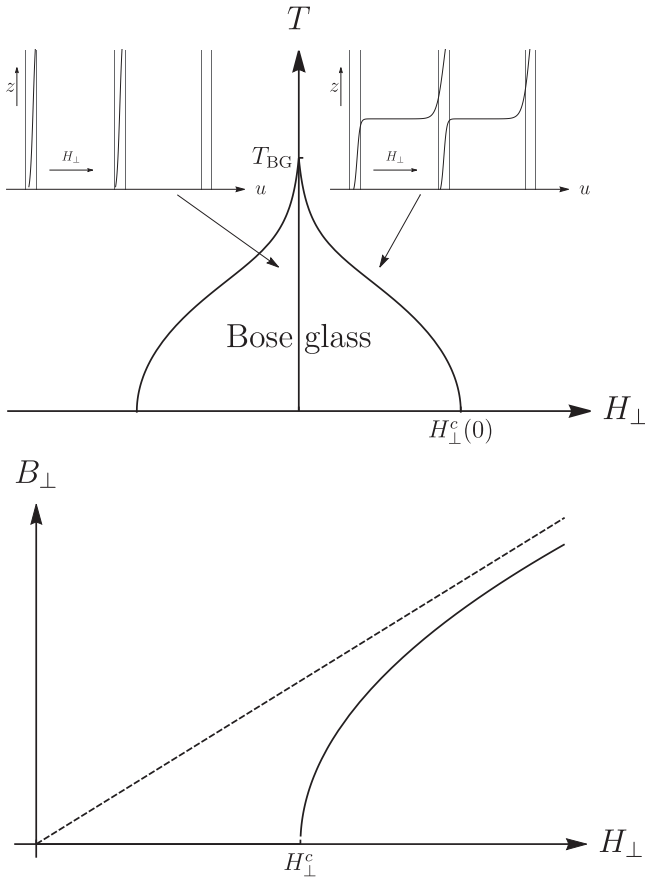
FIG. 1. Top: Typical schematic phase diagram for vortex lines in the presence of columnar defects, as a function of the temperature T and of the transverse external field $H_\perp$ (noted H in the text). Inset: Schematic picture of the line configurations (localized along the columns in the Bose glass phase, and delocalized in the tilted phase). Bottom: Behavior of the total transverse magnetic induction $B_\perp$ (proportional to the mean tilt angle of the vortex lines) as a function of $H_\perp$ at fixed temperature. $H_\perp^c$ is the critical field above which the vortex lines begin to tilt.

of external potentials, i.e., for $U = V = 0$, the preferred slope of the line is $\frac{du}{dz} = \tan\phi = H/\gamma$; see Fig. 1.

For $V = 0$ and $H = 0$, the model (1) at temperature T maps onto the quantum mechanics of a particle of position u in the potential $U(u)$, described by the Hamiltonian $\hat{H}_0 = -\frac{\gamma}{2}T^2\partial_u^2 + U(u)$ (Z being its imaginary time path integral version). When $U(u)$ is a random potential, the eigenstates of $\hat{H}_0$ are localized. The transverse field H acts as a non-Hermitian perturbation of $\hat{H}_0$ and leads to delocalized states above a certain threshold field, corresponding to tilted lines [15–18]. For the model (1) it is easily understood by a simple argument [19]. Consider a localized eigenfunction of $\hat{H}_0$, which decays typically as $\sim e^{-|u|/\xi}$, ξ being the localization length. Since the H term in (1) is a total derivative, for $H > 0$ this eigenfunction becomes $\sim e^{-|u|/\xi - \frac{H}{T}u}$, which is normalizable (no macroscopic tilt) for $H < H_c = T/\xi$. For $H > H_c$ this localized state (real eigenenergy) ceases to exist and is replaced by a delocalized state (with complex eigenenergy). Note that the higher energy, less localized states, i.e., with larger values of ξ, are the first ones to disappear upon

increasing H. This problem initiated a wave of interest for the so-called non-Hermitian quantum mechanics, in particular to study non-Hermitian localization to delocalization transitions, see, e.g., Refs. [20–24], and population dynamics [25].

The model (1) is extended to many interacting elastic lines to study the transverse Meissner effect in the Bose glass phase in $d = 2 + 1$ [7] or $d = 1 + 1$ dimension [6,26,27]. Schematically mimicking hard core interactions by Fermi exclusion, the threshold field for delocalization and macroscopic tilt is reached when the localized eigenstates at the "Fermi energy" start to disappear. Other situations have been studied, such as many interacting lines and a single columnar defect [10,28,29], as well as additional (non-Hermitian) Mott phases which arise upon commensuration of the number of lines and columns, or in the presence of an additional periodic potential [30–32].

The question of the additional effect of point disorder $V(u,z)$ is of great importance since point impurities are usually present in the experimental systems. The competition between extended and point defects was studied in the context of many interacting lines and many columnar defects in [14]. Weak point disorder was argued to weaken the pinning by the extended defects, with the possibility that the Bose glass phase is unstable to point disorder, but only beyond an astronomically large scale. Strong point disorder was shown to be stable to weak correlated disorder. The case of a single line and a single columnar defect (at $u = 0$) in dimension $d = 1 + 1$ is rather subtle. A one-sided version of the model (restricted to $u \geqslant 0$), natural in the context of wetting, leads to an unbinding transition [33], studied later in the context of the half-space Kardar-Parisi-Zhang (KPZ) equation [34–37] and of related models [38]. However in the full space model it was argued that the line is always pinned (i.e., localized on the column) at and below $d = 1 + 1$ [39–41]. The question is now settled in the mathematics literature; it is known as the slow bond problem [42], and it was numerically confirmed [43].

When the columnar defects are strong, the kink energy E_k, i.e., the energy cost from going from one column to its neighbor, is large. The polymer spends most of its length on the columns and the jumps are rare; see Appendix A. It it thus natural to study the discrete hopping model with N sites in one dimension,

$$\mathcal{H} = -\sum_j \frac{w_R}{2}|j+1\rangle\langle j| + \frac{w_L}{2}|j\rangle\langle j+1|$$
$$+ [\epsilon_j + \eta_j(t)]|j\rangle\langle j|, \qquad (2)$$

where $w_R = we^h$ and $w_L = we^{-h}$ are the hopping rates to the right and the left and the ϵ_j are the on-site attractive potentials of the columns, which we denote for convenience in the reminder of this paper

$$\epsilon_j = -a_j, \qquad (3)$$

where we often choose the column strength a_j positive. The point disorder is modeled by white noise $\eta_j(t)$ i.i.d. on each column. This model without the point disorder has been much studied [15–18,44–47] and the spectrum (in the complex plane) has been obtained exactly when the a_j are i.i.d. random variables from a Cauchy distribution [20–22]. In the absence of columnar disorder the spectrum is concentrated along an
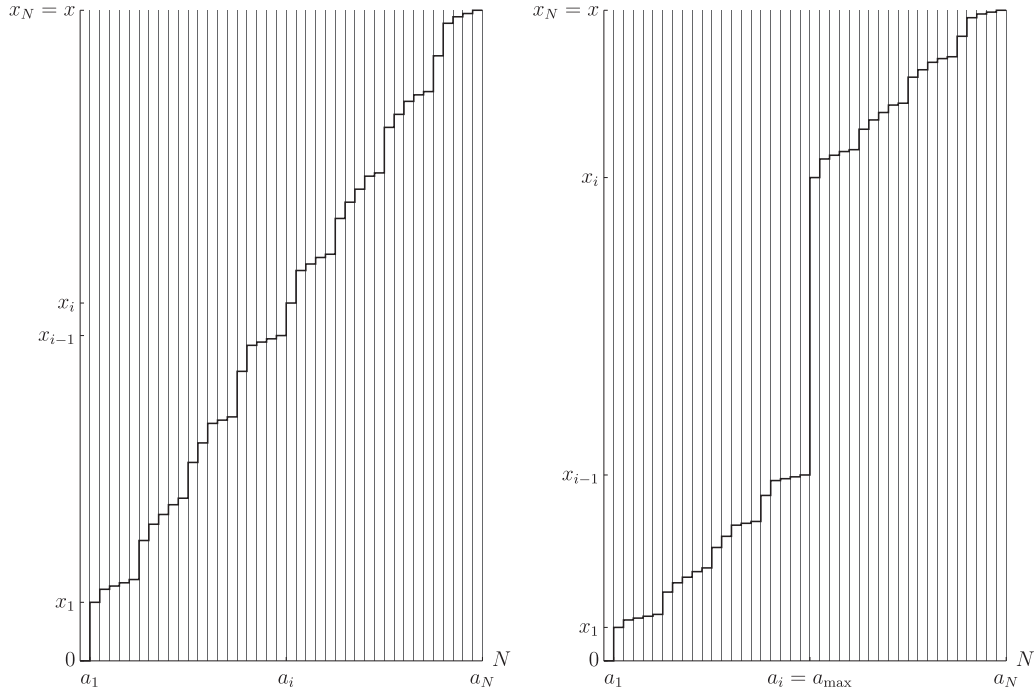
FIG. 2. Left: Single O'Connell-Yor polymer of total length x with N columnar defects, delocalized over the space. The x_i are the position of the jumps and the occupation length of column i is $\ell_i = x_i - x_{i-1}$. Right: Single O'Connell-Yor polymer localized over the column i, where the disorder is most favorable $a_i = a_{\max}$, and which has a macroscopic occupation length $\ell_i = O(x)$.

ellipse in the complex plane, corresponding to delocalized states, while in presence of columnar disorder it develops "wings" on the real axis corresponding to localized states [48]. In these works the boundary conditions at the ends of the chain of N sites are often chosen periodic.

Since the model (2) is quite difficult to analyze in the presence of point disorder we will consider the simpler limit $h \to +\infty$, $we^h \to 2$, where the lines can only jump to the right (i.e., $w_L \to 0$). In that case the operator $\mathcal{H}$ in (2) is the Markov generator of the so-called O'Connell-Yor polymer (at finite temperature), which we will study here with free boundary conditions. Note that this "one-way" limit model, also called maximally non-Hermitian, was also studied in Refs. [21] and [22] in the absence of point disorder, and retains some of the features of the full model (2). In particular, for Cauchy disorder these works found that there are also localized states.

Another motivation to study the "one-way" model is that one expects that near the transition at $H = H_c$, e.g., just above it, the lines start tilting and the backward jumps may have a subdominant effect; see Appendix A. Whether this model captures some of the universal features of the transition at $H = H_c$ remains to be understood. In this limit however we will present very detailed results.

B. Aim of the paper; model and observables

In this paper we study a model of lines (equivalently called polymers) in $d = 1 + 1$ in the presence of both columnar and point disorder, defined on a lattice with N sites and with jumps only to the right. It is called the O'Connell-Yor (OY) polymer and corresponds to the one-way limit of the model (2). The

OY model is related to random matrix theory (RMT) and many results are known in the mathematical context. A first aim of this paper is to review and translate these results in the language of localization to delocalisation transitions for the polymers, and to make them more widely known in the physics community. In addition we derive some new results, in particular in the case of continuous distributions of column strengths a_j, or concerning the macroscopic occupation length of the columns by the lines in the localized regime, little addressed in RMT; see, e.g., Fig. 2. Although we briefly address finite temperature, most of our study concerns the ground state energy, and its sample to sample fluctuations in the various phases due to point disorder.

The outline is as follows. In this subsection we first define the model and the observables for a single line. We then recall the connection to RMT in the simplest case and present a few immediate consequences for the physics of a single line. In Sec. II we study in more detail the case of a single line and a single "active" column (i.e., $a_1 > 0$ and $a_{j \neq 1} = 0$). For large N there is a localization to delocalisation transition related to the Baik–Ben Arous–Péché (BBP) transition in RMT for the appearance of outliers in the spectrum of a perturbed random matrix in the Gaussian unitary ensemble (GUE). In the delocalized phase, the fluctuations of the ground state energy due to point disorder are described by the Tracy-Widom distribution. We give a detailed description of the occupation length of the columnar defect by the line, see Fig. 2, and its fluctuations, in both phases and near the transition. In Sec. III we study one line and many columnar defects, in the case of a smooth density $\rho(a)$ of column strengths. It corresponds to a perturbation of *infinite rank* of a GUE random matrix. Only a few works have addressed infinite rank perturbations [49–51],

see also [[52], Remark 2] and [[53], Sec. 5.5.4], but not in the regime of interest here. We show that if $\rho(a)$ vanishes sufficiently fast near its (finite) upper edge there is a localization transition for the polymer. We obtain the fluctuations of the ground state energy (for fixed column strengths) in the localized phase, and around criticality, We show that it is described by a new one-parameter universal distribution, reminiscent of the one describing the largest real eigenvalue of the real Ginibre ensemble of random matrices. In Sec. IV we extend our study to many lines, first with a few active columnar defects, then with many columnar defects. The latter case can be studied using free probability and the Burgers equation, and has connections with the Rosenzweig-Porter model, a toy model for many-body localization much investigated recently. We find that many line localizations can occur for sufficiently long polymers, via an intermediate nonergodic delocalized phase.

The study in this paper is performed in the fixed tilt angle ensemble. In Appendix A we discuss the fixed transverse field H ensemble. We first recall the picture of the tilting transition for the continuum model (1) of an elastic line. We discuss a possible realization of the OY model by introducing a periodic array of columns of various strengths, and discuss the effect of point disorder.

Appendices B, D, and C recall useful results about the Dyson Brownian motion and the BBP kernel, and define the many-line model. Appendices F and E give more details about a variational calculation and the approach of the transition from the delocalized phase.

1. Definition of the model for a single line

The O'Connell-Yor polymer model [54], extended to arbitrary drifts, is defined as follows. The directed polymer path lives only on the columns $j = 1, \ldots, N$ and jumps from column j to $j + 1$ at a height x_j. There are no leftward jumps. The path is parametrized by the set $\mathbf{x} = \{x_j\}_{0 \leqslant j \leqslant N}$ with $x_0 = 0 < x_1 < \cdots < x_N = x$; see Fig. 2. One part of its energy is

$$E_N^p(\mathbf{x}) = B_1(x_1) + B_2(x_2, x_1) + \cdots + B_N(x, x_{N-1}), \quad (4)$$

where the $B_j(x)$ are independent unit Brownian motions with $B_j(0) = 0$. The $B_j(x, x') = B_j(x) - B_j(x')$ represent the total random energy from point impurity disorder collected along column j. Note that the end points are fixed at $(x, j) = (0, 1)$ and (x, N). In addition, there is a (negative) binding energy $\epsilon_j = -a_j \leqslant 0$ to the columnar defects

$$E_N^c(\mathbf{x}) = -\sum_{j=1}^N a_j \ell_j, \quad \ell_j = x_j - x_{j-1}, \quad (5)$$

where ℓ_j is the length along column j occupied by the directed polymer. The a_j are also called drifts since they can also be seen as (minus) the drifts for the Brownian B_j.

The model at temperature T is defined by its canonical partition sum

$$\mathcal{Z}_N(x, T) = \int_0^x dx_1 \int_{x_1}^x dx_2 \cdots \int_{x_{N-2}}^x dx_{N-1} e^{-E_N(\mathbf{x})/T}, \quad (6)$$

where $E_N(\mathbf{x}) = E_N^p(\mathbf{x}) + E_N^c(\mathbf{x})$ is the total disorder energy. Its free energy is $\mathcal{F}_N(x, T) = -T \ln \mathcal{Z}_N(x, T)$, and at $T = 0$ it

equals the ground state energy defined by the minimization problem

$$\mathcal{E}_N^0(x) = \mathcal{F}_N(x, T = 0) = \min_{\mathbf{x}} E_N(\mathbf{x}). \quad (7)$$

In general $\mathcal{F}_N(x, T)$ fluctuates from sample to sample with respect to the point disorder (the Brownian motions) as well as the columnar disorder (the $\epsilon_j = -a_j$). In this paper we will study the fluctuations with respect to the Brownian motions, for fixed values of the columnar strengths a_j. Hence we are interested in the mean value and probability distribution function (PDF), for a given set of a_j. Indeed these observables allow us to distinguish the various phases. Note that, remarkably, one can show that the PDF of $\mathcal{F}_N(x, T)$ is invariant by any permutation of the a_j [56].

To study a single line, we will be interested in the limit of both N and x taken large, with a fixed ratio $\theta = x/N$. Denoting ϕ the angle of the polymer with the columns (i.e., with the z axis) we consider a fixed ratio

$$\tan \phi \simeq \frac{Nr_0}{x} = \frac{r_0}{\theta}, \quad (8)$$

where we work in units where the lattice spacing $r_0 = 1$. The case of small ϕ corresponds to a field close to the z axis and to a vortex line almost localized by the columns. The case $\phi = \frac{\pi}{2} - \psi$ with ψ small corresponds to another situation (natural in layered superconductors) where the external field is almost perpendicular to the columns.

In the presence of columnar disorder $\epsilon_j = -a_j \leqslant 0$ one wants to study the possible localization of the directed polymer along the columns. This can be quantified by the occupation length ℓ_j (defined above) which the polymer spends on column j. The statistics of this observable is one of the focuses of this paper, and has not been addressed until very recently in the mathematics literature [57,58]. As we will see below the occupation fraction ℓ_j/x plays the role of an order parameter for the localization transition. Its expectation value can be obtained from the following important relation,

$$\partial_{\epsilon_j} \mathcal{F}_N(x, T) = -\partial_{a_j} \mathcal{F}_N(x, T) = \langle \ell_j \rangle_T, \quad (9)$$

in each disorder configuration, i.e., for each realization of the Brownian motions B_j, where $\langle \ldots \rangle_T$ denotes the thermal average.

As emphasized above the present study is performed in the fixed tilt angle ϕ ensemble. In order to connect to models such as (1) it is interesting to also consider the fixed external field H ensemble, where the exit position of the line, $u(x) = N$, can fluctuate. These two ensembles are related by a Legendre transform. Defining $f(\tan \phi) = \lim_{x \to +\infty} \mathcal{F}_{N=x \tan \phi}(x, T)/x$, the free energy per unit length at fixed H is $\min_{\phi \geqslant 0}[f(\tan \phi) - H \tan \phi]$. This ensemble, and the connection to elastic line models and to the transverse Meissner effect physics, is discussed in Appendix A. We argue that the localization to delocalisation transition which occurs at $\tan \phi_c = 1/\theta_c$ in the fixed angle ensemble may be associated with a first order jump in the tilt response at $H = H_c$, and that the localized phase discussed here for $\phi < \phi_c$, i.e., $\theta > \theta_c$, can be seen actually as a coexistence region.

2. Connection to random matrices: Zero temperature

It has been known for some time in mathematics that the PDF of the optimal energy of the O'Connell-Yor model (i.e., at zero temperature) is related to the one of the largest eigenvalues of a random matrix from the so-called deformed GUE [54,59–64]. Let V be an $N \times N$ random Hermitian matrix drawn from the GUE, i.e., with measure $Z^{-1} \exp(-\frac{1}{2}\text{Tr}V^2)dV$, where $dV = \prod_i dV_{ii} \prod_{i<j} d\text{Re}V_{ij}d\text{Im}V_{ij}$. With this normalization, the spectrum of V becomes, in the large-N limit, a semicircle with support $\simeq [-2\sqrt{N}, 2\sqrt{N}]$. Consider the matrix $\mathcal{M}(x)$ defined as

$$\mathcal{M}(x) = x \, \text{diag}(a_1, \ldots, a_N) + \sqrt{x}\,V, \qquad (10)$$

and denote $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_N(x)$ its eigenvalues in decreasing order. Then one has the equality in law for the ground state energy

$$\mathcal{E}_N^0(x) = \mathcal{F}_N(x, T=0) \overset{\text{in law}}{=} -\lambda_1(x). \qquad (11)$$

This characterizes the fluctuations over the point disorder, and it is valid for any fixed configuration of the column energies $\epsilon_j = -a_j$, and for any N.

As shown in Refs. [64–66], the equality in law at fixed x in (11) can be extended to an equality in law as a process in x (i.e., as random functions of x on both sides) if one replaces $\sqrt{x}\,V \to W(x)$ where $W(x)$ is the Dyson Brownian motion (DBM), i.e., a Brownian motion in the space of $N \times N$ Hermitian matrices; see [67–70] for the definition of the DBM, and Appendix B.

The formulas (10) and (11) allow us to make a bridge between the polymer representation and the random matrix representation. First note that the occupation lengths in the ground state, denoted ℓ_j^0, can be obtained as

$$\partial_{\epsilon_j} \mathcal{E}_N^0(x) = -\partial_{a_j} \mathcal{E}_N^0(x) = \ell_j^0 \qquad (12)$$

by taking the $T=0$ limit of (9). Now, denote ψ_i, $i = 1, \ldots, N$, the eigenvector associated with λ_i. A simple perturbation theory argument shows that

$$\partial_{a_j} \lambda_1(x) = x|\psi_1(j)|^2 \qquad (13)$$

for each realization of the matrix V. Consider now the average of (13) over V and compare it with the average of (12) over the Brownian motions B_j, using (11). One finds for any j and ϵ_j one has

$$\overline{\ell_j^0}^B = x \, \overline{|\psi_1(j)|^2}^V, \qquad (14)$$

where averages over the Brownian motions B and the matrix randomness V, respectively, are denoted by the corresponding overbars. Hence the average of the occupation length ℓ_j^0 of the polymer on the column j at $T=0$ can be related to the mean overlap with the column j of the eigenvector ψ_1 of $\mathcal{M}(x)$ with the largest eigenvalue. In this respect, note that recent studies address the distribution of the eigenvectors for such ensemble of random matrices either in the bulk [71], or near the edge but in the large deviation regime [72]. One would hope to determine the distribution of $|\psi_1(j)|^2$ in the typical fluctuation regime using (13), or to relate it to the distributions of ℓ_j^0 further using (11) and (12); however for this one would need

the knowledge of $\lambda_i(x)$ as a process with respect to the a_i's, which is not currently available. In fact the PDFs of ℓ_j^0 and of $|\psi_1(j)|^2$ differ in general, despite their means being related via (14), as seen in an explicit example in Appendix F 3.

There are many interesting consequences of the result (11); some will be explored in this paper, and others can be already stated here.

(i) In the absence of columnar disorder, $a_j = 0$, the matrix $\mathcal{M}(x) = \sqrt{x}\,V$ has the same one point distribution at fixed x as the Dyson Brownian motion $W(x)$. Hence the statistics of $\lambda_1(x)$ is known in the limit of large N, which implies that

$$\mathcal{F}_N(x, T=0) \overset{\text{in law}}{=} -2\sqrt{Nx} - \frac{\chi_2}{N^{1/6}}\sqrt{x}, \qquad (15)$$

where χ_2 is distributed according to the GUE Tracy-Widom distribution [73]. Furthermore, as discussed in the paragraph below (11), $\lambda_1(x)$ evolves as a function of x as the largest eigenvalue of the DBM. At large N, and in a window of values of x of width of order $\delta x \sim x/N^{1/3}$, it can be approximated as follows (see Appendix B 4),

$$\mathcal{F}_N(x + \delta x, T=0) \overset{\text{in law}}{=} -2\sqrt{N(x + \delta x)} - \frac{\mathcal{A}_2\left(\frac{N^{1/3}\delta x}{2x}\right)}{N^{1/6}}\sqrt{x}, \qquad (16)$$

where $z \mapsto \mathcal{A}_2(z)$ is the so-called Airy$_2$ process, a universal random function introduced in [74] in the context of a discrete growth model. It is a continuous stationary process (i.e., statistically invariant by translation) with a slow decay of correlations $1/z^2$. All its multipoint correlations are known and can be expressed as Fredholm determinants [75]. For $\delta x = 0$, (16) recovers (15) since $\mathcal{A}_2(0) = \chi_2$; i.e., the one point distribution of the Airy$_2$ process is the GUE Tracy-Widom distribution.

(ii) In the presence of columnar disorder, the problem maps onto determining the largest eigenvalue of the so-called *deformed or spiked* GUE. Study of that problem was pioneered in physics by Brezin and Hikami [76–80] and in mathematics by Johansson [81] and Tracy and Widom [82,83]. It can be reformulated in terms of the Dyson Brownian motion as follows. Upon redefining $xa_j = b_j$, the matrix $\mathcal{M}(x)$ in (10) can be interpreted, for fixed values of b_j, as performing a DBM in x, with initial condition $\lambda_j(x=0) = b_j$; see Appendix B.

Various initial conditions have been studied in the equivalent random matrix models with sources [84]. These also admit interesting representations as noncrossing random walks, also called watermelons [82,85,86], and DBM with wanderers [87].

The simplest case occurs for a single attractive columnar defect, $\epsilon_1 = -a_1 < 0$ and $a_{j\geqslant 2} = 0$. From (10) it corresponds to a rank-one perturbation of the GUE matrix $\sqrt{x}\,V$. This was studied in a celebrated work by Baik, Ben Arous, and Péché for spiked covariance matrices [88] and spiked GUE matrices [89]. In Ref. [89] it was shown that the largest eigenvalue λ_1' of the deformed GUE matrix $M' = \text{diag}(\pi_1, \{0\}) + \frac{V}{\sqrt{N}}$ exhibits two phases:

(i) If $\pi_1 < 1$, $\lambda_1' = 2 + N^{-2/3}\chi_2$.

(ii) If $\pi_1 > 1$, $\lambda_1' = \pi_1 + \frac{1}{\pi_1} + N^{-1/2}\mathcal{N}(0, \sigma^2)$ with $\sigma^2 = 1 - \frac{1}{\pi_1^2}$.

Here $\mathcal{N}(m, \sigma^2)$ denotes a Gaussian random variable with variance σ^2 and mean m. The correspondence with our notations leads to $\pi_1 = \sqrt{\frac{x}{N}} a_1$ and $\lambda_1 = \sqrt{Nx} \lambda_1'$. Thus, using (11), this predicts the following leading behavior for the free energy of the polymer at $T = 0$:

(a) If the column is weak, i.e., $a_1 < \sqrt{\frac{N}{x}}$, equivalently $\theta < \theta_c = \frac{1}{a_1^2}$ (or $\tan\phi > \tan\phi_c = a_1^2$, i.e., large angle from the z axis), then the rank-one perturbation (the columnar defect) has little effect; i.e., the largest eigenvalue of $\mathcal{M}(x)$ still behaves as $\lambda_1(x) \simeq 2\sqrt{Nx}$ at large N and the result (15) for the ground state free energy still holds.

(b) If the column is strong, i.e., $a_1 > \sqrt{\frac{N}{x}}$, equivalently $\theta > \theta_c = \frac{1}{a_1^2}$ (or $\tan\phi < \tan\phi_c = a_1^2$, i.e., small angle from the z axis), then the largest eigenvalue of $\mathcal{M}(x)$ detaches from the Wigner semicircle and becomes an outlier. This leads to

$$\mathcal{F}_N(x, T = 0) \simeq -\left(a_1 x + \frac{N}{a_1}\right) + \sqrt{N} \mathcal{N}\left(0, \theta - \frac{1}{a_1^2}\right). \quad (17)$$

As we discuss below, this BBP transition corresponds to a first order localization transition of the polymer on the columnar defect, which has some features of a freezing transition.

In the next section we will analyze in more detail this transition, and extract in particular information about the occupation length. This will prepare us to study the case of many columnar defects in the following section.

II. SINGLE LINE, SINGLE COLUMNAR DEFECT, ZERO TEMPERATURE

Let us consider now in more detail the case of a single line and a single column with an attractive potential $\epsilon_1 = -a_1 \leqslant 0$ (which with no loss of generality we can choose in position $j = 1$). All other potentials are set to $\epsilon_j = -a_j = 0$ for $j \neq 1$. We first present a variational calculation based on the polymer picture. In a second stage we recall the kernel which describes the largest eigenvalue of the matrix $\mathcal{M}(x)$ and study the phases using Fredholm determinants.

A. Approach by a variational calculation

We now obtain a physical derivation of the localization transition of the polymer, when a finite fraction of the length of the columnar defect becomes occupied. We first discuss the two phases and then the critical region.

1. Description of the two phases

For a single columnar defect, of energy $\epsilon_1 = -a_1 \leqslant 0$, the ground state energy $\mathcal{E}_N^0(x)$ is given by the following variational problem,

$$\mathcal{E}_N^0(x) = \min_{\ell_1 \in [0,x]} [B_1(\ell_1) - a_1 \ell_1 + G_{N-1}(\ell_1, x)], \quad (18)$$

where

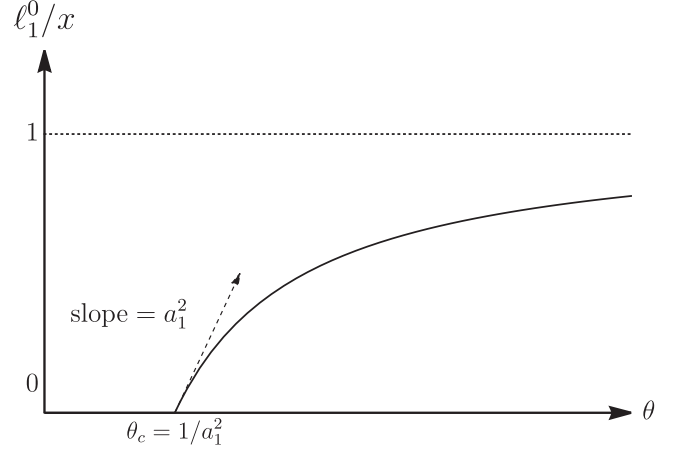$$G_{N-1}(\ell_1, x) = \min_{\ell_1 < x_2 < \cdots < x_{N-1} < x} \sum_{i=2}^{N} B_i(x_i, x_{i-1}), \quad (19)$$

FIG. 3. Occupation fraction ℓ_1^0/x at zero temperature for a single columnar defect as a function of $\theta = \frac{x}{N} = 1/\tan\phi$. It undergoes a first order localization transition at the critical value $\theta = \theta_c = \frac{1}{a_1^2}$.

where $x_1 = \ell_1$ is the occupation length of the first column, and $x_N = x$ the total length. As before, from (11), $G_{N-1}(\ell_1, x)$ for fixed ℓ_1 and x is distributed as the largest eigenvalue of $\sqrt{x - \ell_1}$ times an $(N - 1) \times (N - 1)$ GUE matrix. When varying ℓ_1 at fixed x it varies as the largest eigenvalue of a Dyson Brownian motion evolving during time $x - \ell_1$. Hence, for large N, the contribution of G_{N-1} is the sum of a deterministic part and a subdominant fluctuating part,

$$G_{N-1}(\ell_1, x) = -2\sqrt{N(x - \ell_1)} + O\left(\frac{(x - \ell_1)^{1/2}}{N^{1/6}}\right), \quad (20)$$

where for now, to determine the phases, we do not need to specify in more detail the fluctuating part in (20) (it will be important only near the transition). Similarly, to leading order, we can neglect the Brownian contribution $B_1(\ell_1)$ in (19), as well as the fluctuation term. This leads to the estimate

$$\mathcal{E}_N^0(x) \simeq \min_{\ell_1 \in [0,x]} [-a_1 \ell_1 - 2\sqrt{N(x - \ell_1)}]. \quad (21)$$

The optimal occupation length ℓ_1^0 of the first column is thus obtained to leading order as

$$\ell_1^0 \simeq \left(x - \frac{N}{a_1^2}\right)_+ = x\left(1 - \frac{\theta_c}{\theta}\right)_+, \quad (22)$$

where $(x)_+ = \max(0, x)$, $\theta = \frac{x}{N}$, and $\theta_c = 1/a_1^2$ is the critical angle; see Fig. 3. We see that a finite fraction of the column is occupied if and only if

$$a_1 = -\epsilon_1 > \sqrt{\frac{N}{x}}, \quad (23)$$

which corresponds to the localized phase $\theta > \theta_c$. Our result (22) coincides, using the occupation length–overlap connection given in (14), with the result for the overlap for the BBP transition obtained in [90]. In the localized phase the ground state energy is, to leading order,

$$\mathcal{E}_N^0(x) \simeq -\left(a_1 x + \frac{N}{a_1}\right) = -N a_1(\theta + \theta_c), \quad (24)$$

and, at the transition $\theta = \theta_c$, it reaches the value $\mathcal{E}_N^0(x) \simeq -2N\theta_c = -2\frac{N}{a_1}$.

It is easy to also obtain the leading fluctuations of the ground state energy in the localized phase. The leading fluctuating part of $\mathcal{E}_N^0(x)$ is clearly $B(\ell_1^0)$, which is a Gaussian random variable with variance equal to $\ell_1^0 = N(\theta - \frac{1}{a_1^2}) = N(\theta - \theta_c)$ in full agreement with (17). Note that from (20) we see that the fluctuations originating from the delocalized segment of the polymer of length $x - \ell_1$, i.e., the term $G_{N-1}(x, \ell_1)$ in (18), is only of order $O(N^{1/3})$, hence subleading as compared to the $O(N^{1/2})$ Gaussian fluctuations originating from the point disorder along the localized segment of length ℓ_1^0. The fluctuations of ℓ_1^0 are more subtle and we show, see (40) below, that in the localized phase there are KPZ-like fluctuations of order $N^{2/3} \ll \ell_1^0 = O(N)$ around the value given in (22).

If $a_1 < \sqrt{\frac{N}{x}}$, that is, $\theta < \theta_c$, the minimum in (21) is attained at $\ell_1^0 = 0$ and we then recover the results in the delocalized phase of Eqs. (15) and (16).

The above results are in agreement with the predictions from the BBP transition summarized in the previous subsection. In fact a related calculation was given in [[88], Sec. 6] for a fully discrete polymer model (last passage percolation on the square lattice) which in that case involves Wishart random matrices. These arguments provide an intuitive way to obtain the transition criterion for spiked random matrices for various ensembles. In the context of polymers this variational calculation, as we show below, allows us to also explore the fluctuations around the localization transition.

Note that the variational calculation can be extended to several columns located at different positions, and one can verify the property of invariance with respect to permutation of the columns; see Appendix F 1.

2. Fluctuations in the critical region

We can now refine the variational calculation to obtain the critical regime near the transition $\theta \approx \theta_c$. That region will be defined by

$$\theta = \frac{x}{N} = \theta_c\left(1 + \frac{\delta}{N^{1/3}}\right) \tag{25}$$

with $\delta = O(1)$. In that region, as we will see, the optimal occupation length ℓ_1^0 will fluctuate but with a typical magnitude $O(N^{2/3})$, corresponding to a vanishing occupied fraction $\ell_1^0/x \sim N^{-1/3} \ll 1$.

Let us go back to (18) and use the estimate (16) with $\delta x = -\ell_1$:

$$\mathcal{E}_N^0(x) \simeq \min_{\ell_1 \in [0, x]} \left[B_1(\ell_1) - a_1\ell_1 - 2\sqrt{N(x - \ell_1)} \right.$$
$$\left. - \frac{\sqrt{x - \ell_1}}{N^{1/6}} \mathcal{A}_2\left(-\frac{N^{1/3}\ell_1}{2x}\right) \right]. \tag{26}$$

We see that if we want the argument of the Airy$_2$ process $\mathcal{A}_2$ to be of order unity, we need indeed to choose $\ell_1 \sim N^{2/3}$; hence we will define the reduced length $\hat{\ell}_1$ by setting

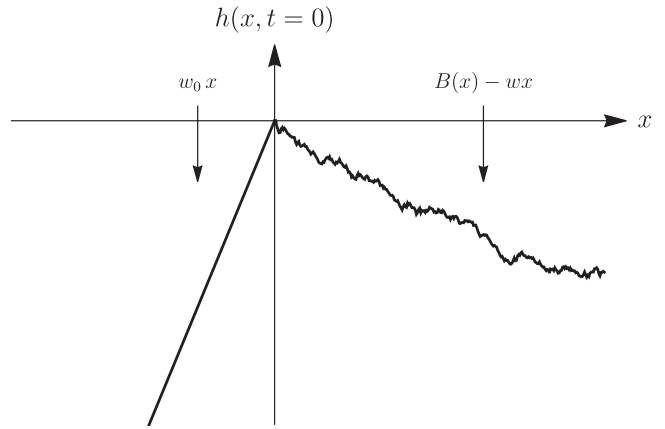$$\ell_1 = 2xN^{-1/3}\hat{\ell}_1 = 2N^{2/3}\theta\hat{\ell}_1. \tag{27}$$

FIG. 4. Half-Brownian motion with drift w as the initial condition $h(x, t = 0)$ of the Kardar-Parisi-Zhang equation (31). The fluctuations of the height at large time $h(0, t)$ relate to those of the ground state energy of the polymer (29) at the localization transition.

We can now insert (25) and (27) into (26) and expand at large N. We find

$$\mathcal{E}_N^0(x) = -2\sqrt{Nx} + \frac{N^{1/3}}{a_1} \min_{\hat{\ell}_1 \geqslant 0} [\sqrt{2}B(\hat{\ell}_1) - \delta\hat{\ell}_1 + \hat{\ell}_1^2$$
$$- \mathcal{A}_2(-\hat{\ell}_1)] + O(1), \tag{28}$$

where $B(s)$ is a Brownian motion obtained from $B_1(s)$ by the rescaling (27). Hence we find that the ground state energy in the critical region behaves as

$$\mathcal{E}_N^0(x) \simeq -2\sqrt{Nx} - \frac{N^{1/3}}{a_1}\upsilon_\delta, \tag{29}$$

where the random variable υ_δ is defined by the variational problem

$$\upsilon_\delta = \max_{z \geqslant 0}[\sqrt{2}B(z) + \delta z + \mathcal{A}_2(z) - z^2], \tag{30}$$

where B and $\mathcal{A}_2$ are statistically independent, and we have used that the process $\mathcal{A}_2(-z)$ is statistically identical to $\mathcal{A}_2(z)$. The PDF of this random variable appears in the problem of KPZ growth with a "half-Brownian" initial condition [91,92] which we now briefly recall. Consider for instance the KPZ equation [93] for the growth of the height field $h(x, t)$ as a function of time t,

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\xi(x, t), \tag{31}$$

where $\xi(x, t)$ is a space-time white noise. Denote $h_w(x, t)$ the solution with initial condition $h_w(x, t = 0) = [B(x) - wx]\Theta(x) + w_0x\Theta(-x)$ [where $\Theta(x)$ is the Heaviside function] which is represented in Fig. 4, in the limit where $w_0 \to +\infty$. The random variable υ_δ defined by (30) appears in the large time limit, $h_{w=-\delta/(2t^{1/3})}(0, t) = -\frac{t}{12} + \upsilon_\delta t^{1/3}$, where it is standard from the KPZ literature to scale the drift w with the observation time, as $w = -\delta/(2t^{1/3})$. The localized phase of the polymer problem, $\delta > 0$, corresponds to $w < 0$, leading to a Gaussian distribution in the limit $\delta \to +\infty$. The delocalized phase, $\delta < 0$, corresponds to $w > 0$, leading to the GUE Tracy-Widom distribution for $\delta \to -\infty$, i.e., $\upsilon_{-\infty} = \mathcal{A}_2(0) = \chi_2$. The PDF of υ_δ for any δ was obtained in [91,92]

and it was observed there that it coincides with the critical BBP distribution; see Sec. II B 3. This is expected from the relation of the polymer problem to the Baik–Ben Arous–Péché transition described in Sec. I B 2.

Interestingly, we now obtain a new distribution if we study the occupation length ℓ_1^0 in the ground state. Indeed the PDF of ℓ_1^0 is given by

$$\frac{\ell_1^0}{2N^{2/3}/a_1^2} \overset{\text{in law}}{=} \arg\max_{z \geqslant 0}[\sqrt{2}B(z) + \delta z + \mathcal{A}_2(z) - z^2]. \quad (32)$$

To our knowledge this distribution is not known exactly. We can study two limits:

(i) *Limit $\delta \to -\infty$: The side of the delocalized phase.* The argmax is obtained for small values of z in (32), so we set $z = y/\delta^2$. One can then approximate the Airy process by a Brownian motion since [94] $\mathcal{A}_2(\frac{y}{\delta^2}) - \mathcal{A}_2(0) \simeq \frac{\sqrt{2}}{|\delta|}\tilde{B}(y)$ as $\delta \to -\infty$, where $\tilde{B}$ is a Brownian motion independent of B. Hence we find

$$\frac{\ell_1^0}{2N^{2/3}/a_1^2} \overset{\text{in law}}{=} \frac{1}{\delta}\arg\max_{y \geqslant 0}[2B(y) - y]. \quad (33)$$

We can now use the known result that the PDF of the time of the maximum ω of a Brownian motion of variance σ^2 and negative drift $-\mu$, with $\mu > 0$, defined as

$$\omega = \arg\max_{y \geqslant 0}[\sigma B(y) - \mu y], \quad (34)$$

is given by (e.g., [[95], Chap. IV, item 32], or taking the limit $T \to +\infty$ in [96] or [[97], Eq. (30)]; see also Appendix F 3)

$$P_{\mu,\sigma}(\omega) = \frac{\mu}{\sigma}\sqrt{\frac{2}{\pi\omega}}e^{-\frac{\mu^2\omega}{2\sigma^2}} - \frac{\mu^2}{\sigma^2}\text{erfc}\left(\frac{\mu\sqrt{\omega}}{\sqrt{2}\sigma}\right). \quad (35)$$

Hence we find, setting $\sigma = 2$ and $\mu = 1$, that for $\delta \to -\infty$,

$$\ell_1^0 \simeq \frac{2N^{2/3}}{a_1^2}\frac{\omega}{\delta^2}, \quad (36)$$

where ω is a positive random variable distributed with $p(\omega)d\omega = P_{1,2}(\omega)d\omega$.

It is shown in Appendix F 3 that inside the delocalized phase far from the critical region, i.e., for $\theta < \theta_c$, the occupation length fluctuates as $\ell_1^0 = O(1)$ with the same distribution characterized by ω, scaled by an amplitude which diverges as $\sim 1/(\theta_c - \theta)^2$ at the transition, and which matches smoothly with the result (36) in the critical regime, i.e., for $\delta = O(1)$.

(ii) *Limit $\delta \to +\infty$: The side of the localized phase.* To study that limit let us rewrite in an equivalent way the argmax in (32) by defining

$$z = \frac{\delta}{2} + \tilde{z}. \quad (37)$$

We can rewrite

$$\frac{\ell_1^0}{2N^{2/3}/a_1^2} \overset{\text{in law}}{=} \arg\max_{\tilde{z} \geqslant -\delta/2}[\sqrt{2}B(\tilde{z}) + \mathcal{A}_2(\tilde{z}) - \tilde{z}^2],$$

$$(38)$$

where we have used (i) that the $\mathcal{A}_2$ process is statistically invariant by translation (ii) that $B(\frac{\delta}{2} + \tilde{z}) - B(\frac{\delta}{2})$ is equivalent in process to a two-sided Brownian $B(\tilde{z})$ for $\tilde{z} \geqslant -\delta/2$.

In the limit $\delta \to +\infty$ we see that it becomes a (two-sided) optimization over the real axis,

$$\frac{\ell_1^0}{2N^{2/3}/a_1^2} \overset{\text{in law}}{=} \arg\max_{\tilde{z} \in \mathbb{R}}[\sqrt{2}B(\tilde{z}) + \mathcal{A}_2(\tilde{z}) - \tilde{z}^2]. \quad (39)$$

The PDF of the right-hand side was obtained in [98], where it was shown that it equals f_{KPZ}, a function introduced in [99,100] to describe the two space-time point stationary correlations of the Burgers velocity field, associated with the KPZ height field. It also describes the midpoint distribution of a directed polymer in a stationary regime as obtained in [101]. Hence we obtain that for $\delta \to +\infty$

$$\ell_1^0 \simeq \frac{2N^{2/3}}{a_1^2}\left(\frac{\delta}{2} + \omega\right) = x\left(1 - \frac{\theta_c}{\theta}\right) + \frac{2N^{2/3}}{a_1^2}\omega, \quad (40)$$

where ω is a real random number distributed with $p(\omega)d\omega = f_{\text{KPZ}}(\omega)d\omega$. We recall that $f_{\text{KPZ}}(\omega)$ is an even function of ω with cubic exponential decay at large value of the argument $|\omega| \to +\infty$,

$$f_{\text{KPZ}}(\omega) \simeq e^{-0.295|\omega|^3}, \quad (41)$$

and standard deviation 0.714 and fourth moment 0.733 [99].

Using the definition (25), we notice that the first term in (40) is precisely the leading estimate of ℓ_1^0 in the localized phase obtained in (22). Hence our result (40) matches smoothly the critical region with the localized phase. We thus discover that inside the localized phase there are nontrivial KPZ-like fluctuations of the occupation length around its typical value $x(1 - \frac{\theta_c}{\theta})$.

Using (18) and the estimate (16) (with $x \to x - \ell_1$), one can indeed check that these fluctuations are described by f_{KPZ} in the whole localized phase; i.e., the result (40) in the form $\ell_1^0 \simeq x(1 - \frac{\theta_c}{\theta}) + \frac{2N^{2/3}}{a_1^2}\omega$, where ω is distributed with f_{KPZ}, holds for any $\theta > \theta_c$.

In conclusion we have found that the fluctuations of the occupation length around its typical value are of the order $N^{2/3}$ inside the localized phase, and up to and including the transition. In the delocalized phase these fluctuations are of order $O(1)$.

B. Approach using a Fredholm determinant

We now perform the calculation of the ground state energy and its fluctuations using the method of Fredholm determinants. The manipulations follow the analysis of the BBP transition in RMT in [89] and [88], expressed here in the language of the polymer. Although they are classical they allow us to prepare the ground for the generalization to an infinite rank perturbation in the next section.

1. Kernel at any N

A classical calculation [81,88,89] obtains the cumulative distribution function (CDF) of the largest eigenvalue $\lambda_1(x)$ of the matrix $\mathcal{M}(x)$ in (10) for an arbitrary set of a_j, $j = 1, \ldots, N$, as a Fredholm determinant

$$\mathbb{P}(\lambda_1(x) \leqslant \Lambda) = \text{Det}(I - K_N)_{\mathbb{L}^2(\Lambda,+\infty)}, \quad (42)$$
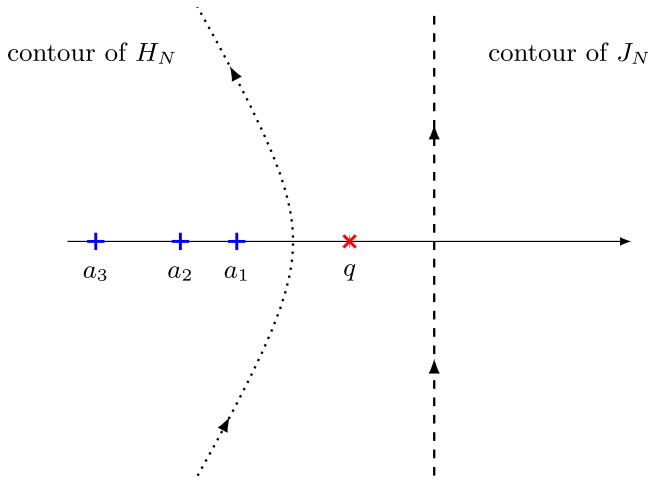
FIG. 5. Integration contours in the integrals H_N and J_N in (44).

where the kernel $K_N(v, v')$ is given for $v, v' \in [\Lambda, +\infty[$ by

$$K_N(v, v') = \int_0^{+\infty} dr \, H_N(v + r) J_N(v' + r) \qquad (43)$$

along with

$$H_N(v) = \oint \frac{dz}{2\mathbf{i}\pi} e^{v(z-q)-xz^2/2} \prod_{j=1}^{N} \frac{1}{z - a_j},$$

$$J_N(v) = \int_{\mathbf{i}\mathbb{R}+\varepsilon} \frac{dz}{2\mathbf{i}\pi} e^{-v(z-q)+xz^2/2} \prod_{j=1}^{N} (z - a_j), \qquad (44)$$

where q is introduced for convergence purpose (it does not change the value of the Fredholm determinant). The contour in $H_N(v)$ is taken to enclose all $\{a_j\}$'s counterclockwise. The contour in $J_N(v)$ is chosen such that $\varepsilon > \max_j a_j$, and passes to the right of the contour for $H_N(v)$; see Fig. 5. For more details on the kernel (43) and its derivation see [[81], Proposition 2.3 and Eq. (2.18)], [[78], Eq. (3.19)], and [[89], Proposition 1.3].

Note that the functions H_N, J_N, and the kernel K_N are invariant under permutations of the a_j, which shows the property mentioned in the introduction.

We first use this result to study the ground state energy in the case of a single active column $a_1 > 0$ and $a_{j>1} = 0$. The general case will be studied in Sec. III. We recall that we focus on the limit of large x, N at fixed $\theta = x/N$ and determine the asymptotic form of the kernel K_N in that limit. We start with the delocalized phase.

2. Delocalized phase

In the delocalized phase one anticipates that at large N, $\lambda_1 - \mu N = O(N^{1/3})$, where μ is to be determined later. Hence we rewrite the function H_N and J_N in (44) as

$$H_N(\mu N + \sigma N^{1/3}) = \oint \frac{dz}{2\mathbf{i}\pi} e^{-N\varphi(z)+z\sigma N^{1/3}},$$

$$J_N(\mu N + \sigma N^{1/3}) = \int_{\mathbf{i}\mathbb{R}+\varepsilon} \frac{dz}{2\mathbf{i}\pi} e^{N\varphi(z)-z\sigma N^{1/3}}, \qquad (45)$$

where, using $a_j = a_1 \delta_{j1}$,

$$\varphi(z) = -\mu z + \theta \frac{z^2}{2} + \ln z + \frac{1}{N} \ln \left(1 - \frac{a_1}{z}\right). \qquad (46)$$

Here and below we omit the factor q from (44) since it plays no role in the results. Let us first examine the function $J_N(z)$, for which the integration contour obeys $\text{Re}(z) > a_1$. The first two derivatives read

$$\varphi'(z) = -\mu + \theta z + \frac{1}{z} + \frac{1}{N} \left(\frac{1}{z - a_1} - \frac{1}{z} \right), \qquad (47)$$

$$\varphi''(z) = \theta - \frac{1}{z^2} + \frac{1}{N} \left(\frac{1}{z^2} - \frac{1}{(z - a_1)^2} \right). \qquad (48)$$

We now look for a degenerate saddle point $z = z^*$ such that $\varphi'(z^*) = 0$ and $\varphi''(z^*) = 0$. Inside the delocalized phase one can neglects the $1/N$ terms in (47) and one finds that

$$z^* = \frac{1}{\sqrt{\theta}}, \quad \mu^* = 2\sqrt{\theta}. \qquad (49)$$

Hence we choose $\mu = \mu^*$. Since one must have $z^* > a_1$, this is possible only until the transition point

$$\theta < \theta_c = \frac{1}{a_1^2}. \qquad (50)$$

One can now expand around the saddle point up to the third order and write

$$\varphi(z) = \varphi(z^*) + \frac{1}{3} \left(\frac{z}{z^*} - 1 \right)^3 + O\left((z - z^*)^4, \frac{1}{N} \right). \qquad (51)$$

Performing the change of variable

$$z = z^*(1 + yN^{-1/3}) \qquad (52)$$

and inserting into (45) we obtain

$$J_N(\mu N + \sigma N^{1/3}) \simeq z^* N^{-1/3} e^{N\varphi(z^*)-z^*\sigma N^{1/3}} \text{Ai}(z^*\sigma), \qquad (53)$$

where

$$\text{Ai}(w) = \int_{\mathbf{i}\mathbb{R}+\varepsilon} \frac{dy}{2\mathbf{i}\pi} e^{\frac{y^3}{3}-yw} \qquad (54)$$

and the neglected terms are subdominant at large N. One can show [88,89] that the counterclockwise contour integral for the function $H_N(\mu N + \sigma N^{1/3})$ is dominated by the same saddle point at $z = z^*$. This leads to an expression for $H_N(\mu N + \sigma N^{1/3})$ which is identical to (53) (taking into account the reversal of the contour) but with a prefactor $e^{-N\varphi(z^*)+z^*\sigma N^{1/3}}$; see Fig. 6. Putting these expressions together in (43) and rescaling $r \to rN^{1/3}/z^*$ one obtains

$$K_N(\mu N + \sigma N^{1/3}, \mu N + \sigma' N^{1/3}) \simeq \frac{z^*}{N^{1/3}} K_{\text{Ai}}(z^*\sigma, z^*\sigma') \qquad (55)$$

in terms of the Airy kernel

$$K_{\text{Ai}}(v, v') = \int_0^{+\infty} dr \, \text{Ai}(v + r) \text{Ai}(v' + r). \qquad (56)$$
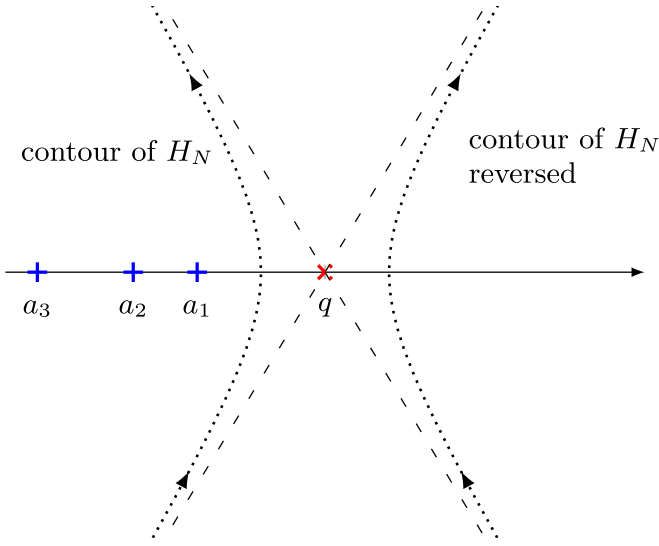
FIG. 6. Contour reversal: The original contour for H_N is on the left of q, as in Fig. 5. The reversal is performed by a rotation around q of angle π followed by a reversal of orientation of the contour. This leads to the contour on the right. This allows to have the same contours for H_N and J_N asymptotically (near the saddle point) leading to the formula in the text.

We have discarded a factor $e^{z^* N^{1/3}(\sigma-\sigma')}$ which is immaterial in the Fredholm determinant. From (42) we now obtain that

$$-\mathcal{E}_N^0(x) = \lambda_1(x) = \mu^* N + \frac{N^{1/3}}{z^*}\chi_2 = 2\sqrt{xN} + \sqrt{x}N^{-1/6}\chi_2, \tag{57}$$

where $\mathbb{P}(\chi_2 \leqslant s) = F_2(s) = \mathrm{Det}(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(s,+\infty)}$. This is the standard result in the delocalized phase.

3. Critical region

As $\theta \to \theta_c = \frac{1}{a_1^2}$ the saddle point z^* of the previous calculation moves to a_1^+. Hence one cannot neglect the term $\frac{1}{N}\ln(1 - \frac{a_1}{z})$. It is now more convenient to leave it outside the exponential

$$J_N(\mu N + \nu N^{1/3}) = \int_{i\mathbb{R}+\varepsilon} \frac{dz}{2i\pi}\left(1 - \frac{a_1}{z}\right)e^{N\varphi_0(z) - z\nu N^{1/3}} \tag{58}$$

with

$$\varphi_0(z) = -\mu z + \theta\frac{z^2}{2} + \ln z \tag{59}$$

and $\varepsilon > a_1$. There is a similar expression for H_N with a contour which encloses a_1 counterclockwise. The critical regime is defined by

$$\theta = \frac{1}{a_1^2} + \tau N^{-1/3} \tag{60}$$

which is consistent with (25) obtained via the variational calculation, with the correspondence in notations $\tau = \theta_c\delta$.

Let us again denote $z^* = \frac{1}{\sqrt{\theta}}$ the degenerate point where $\varphi_0'(z^*) = 0$ and $\varphi_0''(z^*) = 0$. Its position is now

$$z^* = \frac{1}{\sqrt{\theta}} = a_1 - \frac{\tau a_1^3}{2}N^{-1/3}. \tag{61}$$

One performs again the change of variable $z = z^*(1 + yN^{-1/3})$, and one obtains

$$J_N(\mu N + \nu N^{1/3}) = \frac{z^*}{a_1}N^{-1/3}e^{N\varphi_0(z^*) - z^*\sigma N^{1/3}}\mathrm{Ai}^{b_1}(z^*\nu) \tag{62}$$

with

$$\mathrm{Ai}^{b_1}(w) = \int_{i\mathbb{R}+\epsilon'} \frac{dy}{2i\pi}(y - b_1)e^{\frac{1}{3}y^3 - wy},$$

where $\epsilon' > b_1$ and $b_1 = \frac{\tau a_1^2}{2}$, which can be checked to also equal $b_1 = \lim_{N\to+\infty} N^{1/3}(a_1 - \frac{1}{\sqrt{\theta}})/a_1$. A very similar expression holds for H_N,

$$H_N(\mu N + \nu N^{1/3}) = a_1 z^* N^{-1/3}e^{-N\varphi_0(z^*) + z^*\sigma N^{1/3}}\mathrm{Ai}_{b_1}(z^*\nu), \tag{63}$$

taking into account the reversal of the contour, see Fig. 6,

$$\mathrm{Ai}_{b_1}(w) = \int_{i\mathbb{R}+\epsilon'} \frac{dy}{2i\pi}\frac{1}{-y - b_1}e^{\frac{1}{3}y^3 - wy},$$

where $\epsilon' < -b_1$. Putting these expressions together in (43) and rescaling $r \to rN^{1/3}/z^*$ one obtains from (43)

$$K_N(\mu N + \sigma N^{1/3}, \mu N + \sigma'N^{1/3}) \simeq \frac{z^*}{N^{1/3}}K_{\mathrm{BBP},b_1}(z^*\sigma, z^*\sigma') \tag{64}$$

in terms of the rank-1 BBP kernel

$$K_{\mathrm{BBP},b_1}(v, v') = \int_0^{+\infty} dr\,\mathrm{Ai}_{b_1}(v + r)\mathrm{Ai}^{b_1}(v' + r). \tag{65}$$

This is a particular case of the general rank-m BBP kernel recalled in the Appendix D. In particular (65) is identical to (D2), upon simple integrations. Hence one obtains, using (42), the ground state energy and its fluctuations in the critical region as

$$-\mathcal{E}_N^0(x) = \lambda_1(x) = \mu^* N + \frac{N^{1/3}}{z^*}\chi_{\mathrm{BBP},b_1} \tag{66}$$

$$= 2\sqrt{xN} + \sqrt{x}N^{-1/6}\chi_{\mathrm{BBP},b_1}, \tag{67}$$

where [102]

$$\mathbb{P}(\chi_{\mathrm{BBP},b_1} \leqslant \Lambda) = \mathrm{Det}(I - K_{\mathrm{BBP},b_1})_{\mathbb{L}^2(\Lambda,+\infty)} \tag{68}$$

is the CDF of the BBP random variable. Comparing with Sec. II A 2 we see that $\chi_{\mathrm{BBP},b_1} = \upsilon_\delta$ with $b_1 = \delta/2$. As discussed there, this random variable, and the kernel (65), also describes the half-Brownian initial condition for the KPZ class [[91], Formula (6.23)].

4. Localized phase

In the localized phase $\theta > \theta_c = 1/a_1^2$, there is no condition to determine z^*; hence we will choose $z^* = a_1$. We start with (58) and we choose $\mu = \mu^*$ so that $\varphi_0'(z^*) = 0$. Now the second derivative does not vanish and the rescaling involves now $N^{1/2}$. This gives values for μ^* and z^* different from those in the delocalized phase,

$$\mu^* = (\theta + \theta_c)a_1, \quad \varphi_0''(z^*) = \theta - \theta_c. \tag{69}$$

Note that the interpretation is that the occupation of the column a_1 is $\theta - \theta_c$. We insert the Taylor expansion of $\varphi_0(z)$

around $z = z^* = a_1$ inside formula (58), and change integration variable as $z = a_1 + \frac{y}{\sqrt{N(\theta-\theta_c)}}$, and obtain

$$
\begin{aligned}
&J_N(\mu N + v[(\theta - \theta_c)N]^{1/2}) \\
&\simeq \frac{e^{N\varphi_0(z^*) - z^*[(\theta-\theta_c)N]^{1/2}}}{N(\theta-\theta_c)a_1} \int_{i\mathbb{R}+\varepsilon} \frac{dy}{2i\pi} y \, e^{\frac{y^2}{2} - yv},
\end{aligned} \tag{70}
$$

where the higher orders in the Taylor expansion are subdominant at large N. Similarly,

$$
\begin{aligned}
&H_N(\mu N + v[(\theta - \theta_c)N]^{1/2}) \\
&\simeq a_1 e^{-N\varphi_0(z^*) + z^*[(\theta-\theta_c)N]^{1/2}} \int_\Gamma \frac{dy}{2i\pi} \frac{1}{y} e^{-\frac{y^2}{2} + yv},
\end{aligned} \tag{71}
$$

where Γ passes to the right of zero and goes upward (it encircles zero). Putting these expressions together in (43) and rescaling $r \to r[(\theta - \theta_c)N]^{1/2}$ one obtains

$$
\begin{aligned}
&K_N(\mu N + \sigma[(\theta-\theta_c)N]^{1/2}, \mu N + \sigma'[(\theta-\theta_c)N]^{1/2}) \\
&\simeq [(\theta-\theta_c)N]^{-1/2} \int_0^{+\infty} dr \\
&\times \int_\Gamma \frac{dw}{2i\pi} \frac{1}{w} e^{-\frac{w^2}{2} + w(\sigma+r)} \int_{i\mathbb{R}+\varepsilon} \frac{dy}{2i\pi} y e^{\frac{y^2}{2} - y(\sigma'+r)}.
\end{aligned} \tag{72}
$$

Explicit calculation, using the residue at $w = 0$ from the first integral and Gaussian integration from the second, gives

$$
\begin{aligned}
&K_N(\mu N + \sigma[(\theta-\theta_c)N]^{1/2}, \mu N + \sigma'[(\theta-\theta_c)N]^{1/2}) \\
&\simeq [2\pi(\theta-\theta_c)N]^{-1/2} e^{-\frac{(\sigma')^2}{2}}.
\end{aligned} \tag{73}
$$

This leads to

$$
\lambda_1(x) = (\theta + \theta_c)a_1 N + N^{1/2} \mathcal{N}(0, \theta - \theta_c) \tag{74}
$$

$$
= xa_1 + \frac{N}{a_1} + N^{1/2} \mathcal{N}\left(0, \theta - \frac{1}{a_1^2}\right), \tag{75}
$$

in agreement with (17).

Note that this is generalized in [89] to the case of m columns equal to a_1. The power factors in the integrals in (72) generalize to $y \to y^m$ and $w^{-1} \to w^{-m}$. One then obtains the kernel associated with an $m \times m$ GUE random matrix, i.e., Eqs. (43) and (44) with $N = m$ and $a_j = 0$.

III. SINGLE LINE, MANY COLUMNAR DEFECTS, ZERO TEMPERATURE

In this section we study the OY model with a single line or polymer and many columnar defects at $T = 0$, specifically with a continuum distribution of energies $\epsilon_j = -a_j$ described by a density $\rho(a)$. We assume that in the limit $N \to \infty$ the number of columns with energies $a_j \in [a, a + da]$ is $\rho(a)da$, with $\int da \rho(a) = 1$. We will assume that this density has an upper (right) edge a_e, where it vanishes for $a \to a_e^-$ as

$$
\rho(a) \sim A(a_e - a)^{2k+\frac{1}{2}}, \tag{76}
$$

with $\rho(a) = 0$ for $a > a_e$. We also assume that there are no columns with $a_j > a_e$. We recall that we are interested in the limit $x, N \to +\infty$ with $\theta = x/N$ fixed and for a given configuration of the a_i.

It is convenient for the analysis below to assume the following convergence at large N: for $\mathrm{Re}(z) > \max_i a_i$,

$$
\sum_{j=1}^N \ln(z - a_j) \simeq N \int_\Omega da \, \rho(a) \ln(z - a) + o(N^{1/3}), \tag{77}
$$

where Ω is the support of ρ. Although this condition appears a bit restrictive (it excludes the case of i.i.d. random variables a_i), we believe it is simply a technical restriction and does not impact our main results [103].

We can now define the rate function for the many columns case as

$$
\varphi(z) = -\mu z + \theta \frac{z^2}{2} + \int_\Omega da \, \rho(a) \ln(z - a), \tag{78}
$$

so that the functions H_N and J_N from (44) become, at large N (setting for simplicity the convergence factor $q = 0$)

$$
\begin{aligned}
H_N(v + \mu N) &= \oint \frac{dz}{2i\pi} e^{vz - N\varphi(z)}, \\
J_N(v + \mu N) &= \int_{i\mathbb{R}+\varepsilon} \frac{dz}{2i\pi} e^{N\varphi(z) - vz}.
\end{aligned} \tag{79}
$$

We start by studying the extensive part of the ground state energy, and then we proceed to obtain its fluctuations.

A. Ground state energy

In this section we determine the ground state energy, $\mathcal{F}_N(x, T = 0) = \mathcal{E}_N^0(x)$, to leading order in N as a function of the angle θ [equivalently the position of the edge of the spectrum of $\mathcal{M}(x)$ in (10)]. It reads

$$
\mathcal{E}_N^0(x) \simeq -\mu N, \quad \mu = \mu(\theta). \tag{80}
$$

To determine μ we study the integrals in (79) setting $v = 0$, and look for a saddle point at large N.

The critical points of the rate function (78) are defined as the solutions of $\varphi'(z^*) = 0$, i.e., with $\mathrm{Re}(z^*) > a_i$,

$$
-\mu + \theta z^* + \int_\Omega da \frac{\rho(a)}{z^* - a} = 0. \tag{81}
$$

For finite N this is a polynomial equation of degree $N + 1$ and studying the graph of the function one finds that there is a unique real solution z^* in the interval $]a_e, +\infty[$. Since we have two unknowns, z^* and μ, we need an extra condition. This extra condition is the degeneracy condition given by the condition that $\varphi''(z^*) = 0$, i.e.,

$$
\theta - \int_\Omega da \frac{\rho(a)}{(z^* - a)^2} = 0. \tag{82}
$$

There are thus two cases; see Table I. Either
(i) the following integral diverges,

$$
\int_\Omega da \frac{\rho(a)}{(a_e - a)^2} = +\infty, \tag{83}
$$

which happens if $k \leqslant 1/4$, and there is always a solution to (82), in which case there is only a delocalized phase.

(ii) Or, the integral in (83) is finite, which happens for $k > 1/4$; then there exists a critical angle $\theta_c < +\infty$ such that

$$
\forall \theta < \tilde\theta_c, \exists(z^*, \mu) \text{ such that } \varphi'(z^*) = \varphi''(z^*) = 0 \tag{84}
$$

TABLE I. Existence and nature of the phases for a single line and a continuous column density which vanishes near its upper edge as $\rho(a) \sim (a_e - a)^{2k+\frac{1}{2}}$. The delocalized phase with Tracy-Widom fluctuations of the ground state energy always exists. "Airy" means that critical fluctuations are described by the new one-parameter distribution described in (123)–(126).

	Nature of the transition	Existence of localized phase	Exponent of $\mu - \mu_e$ vs $\theta_c - \theta$
$k \leqslant \frac{1}{4}$	No transition	No	$\emptyset$
$k \in]\frac{1}{4}, \frac{3}{4}[$	Anomalous	Yes	$\frac{4k+1}{4k-1} \in]2, +\infty[$
$k > \frac{3}{4}$	Airy	Yes	2

and θ_c is determined by

$$\theta_c = \int_\Omega da \, \frac{\rho(a)}{(a_e - a)^2}, \quad k > 1/4. \quad (85)$$

For $\theta < \theta_c$ this is a delocalized phase, while for $\theta > \theta_c$ it is a localized phase, as we will discuss below.

In the delocalized phase, the ground state energy is given by $\mathcal{E}_N^0(x) \simeq -\mu N$, with μ the solution of (81) where z^* is the unique root of (82) in the interval $]a_e, +\infty[$.

In the localized phase, $\theta > \theta_c$, the saddle point freezes at $z^* = a_e$, with now $\varphi''(z^*) = \theta - \theta_c$, which plays the role of an order parameter for this freezing transition. The ground state energy is now given by $\mathcal{F}_N(T=0) \simeq -\mu N$ with $\mu = \mu(\theta)$ determined from (81) as

$$\mu(\theta) = a_e \theta + \int_\Omega da \, \frac{\rho(a)}{a_e - a} = (\theta - \theta_c)a_e + \mu(\theta_c) \quad (86)$$

and grows ballistically as a function of $\theta = x/N$ in the localized phase, consistent with the fact that a long enough polymer, i.e., with $x > x_c = N\theta_c$, is localized on columns of energy $-a_e$ (their localized length is $x - x_c$). In fact in both phases one has

$$\frac{d\mu(\theta)}{d\theta} = z^*. \quad (87)$$

It is important to note that upon approaching the transition from inside the delocalized phase there are two cases, leading to different critical behaviors; see Table I.

(a) If the following integral is finite,

$$\int_\Omega da \, \frac{\rho(a)}{(a_e - a)^3} < +\infty, \quad (88)$$

i.e., $k > 3/4$, then $z^* - a_e$ vanishes linearly at the transition

$$z^* - a_e \simeq \frac{\theta_c - \theta}{2 \int_\Omega da \, \frac{\rho(a)}{(a_e-a)^3}}. \quad (89)$$

From (87) one also has $\mu - \mu(\theta_c) \simeq -(\theta_c - \theta)a_e$ near the transition. In addition, defining $\mu_e(\theta) = (\theta - \theta_c)a_e + \mu(\theta_c)$ the continuation to the delocalized phase of the formula valid in the localized phase one finds for $\theta < \theta_c$

$$\mu(\theta) - \mu_e(\theta) \simeq -\frac{(\theta_c - \theta)^2}{4 \int_\Omega da \, \frac{\rho(a)}{(a_e-a)^3}}; \quad (90)$$

i.e., $z^* - a_e$ vanishes linearly iff $\int_\Omega da \, \frac{\rho(a)}{(a_e-a)^3} < +\infty$ which is the case if $k > 3/4$.

(b) If the integral (88) diverges, i.e., for $1/4 < k < 3/4$, the transition is in a different, anomalous, universality class.

Indeed one finds in that case (see Appendix E for details)

$$z^* - a_e \sim (\theta_c - \theta)^{\frac{2}{4k-1}}, \mu(\theta) - \mu_e(\theta) \simeq -(\theta_c - \theta)^{\frac{4k+1}{4k-1}}. \quad (91)$$

The behavior of the fluctuations of the ground state energy in this new universality class is at present open. Below, we will only study the critical region in the case $k > 3/4$.

These results are summarized in Table I, and for some choice of $\rho(a)$, plotted in Fig. 7.

B. Occupation length

Let us now discuss the occupation length of the columnar defects in the ground state in both phases. In the case of a continuum distribution of column strength the occupation length becomes a measure. Let us define $\ell(a)$ such that the total occupation length of columns in any interval $[a_-, a_+]$ is given by

$$\int_{a_-}^{a_+} \ell(a)da = \sum_j \ell_j^0 \, \Theta(a_- < a_j < a_+). \quad (92)$$

To obtain $\ell(a)$ we can use Eq. (12) in the ground state, which gives that $\ell_j^0 \simeq N\partial_{a_j}\mu$, together with the expression of μ given above from (81) and (82). This leads to $N\partial_{a_j}\mu = \frac{1}{(z^*-a_j)^2}$. We thus obtain the following:

(i) In the delocalized phase $\ell(a)da$ is a continuous measure given by

$$\ell(a)da = N\frac{\rho(a)da}{(z^* - a)^2}, \quad \int_\Omega \ell(a)da = N\theta = x, \quad (93)$$

where z^* is the solution of Eq. (82). Since $z^* > a_e$, this measure is bounded everywhere.

(ii) In the localized phase, the occupation length measure, in the large-N limit, exhibits a smooth part for $a < a_e$, and an atomic part at $a = a_e$,

$$\ell(a)da = N\frac{\rho(a)da}{(a_e - a)^2} + (x - x_c)\delta(a - a_e)da. \quad (94)$$

Using (85) we see that

$$\int^{a_e^-} \ell(a)da = N\theta_c = x_c < x. \quad (95)$$

The occupation length measure is plotted for illustration in Fig. 7 for some choice of $\rho(a)$.

In summary, the localization transition here is a "condensation" on the (few) columns of lowest energy very close to the edge $a = a_e$. The existence of this transition requires that the density $\rho(a)$ vanishes sufficiently fast near the edge, $k > 1/4$ (see Table I); in other words it requires an "elitist"
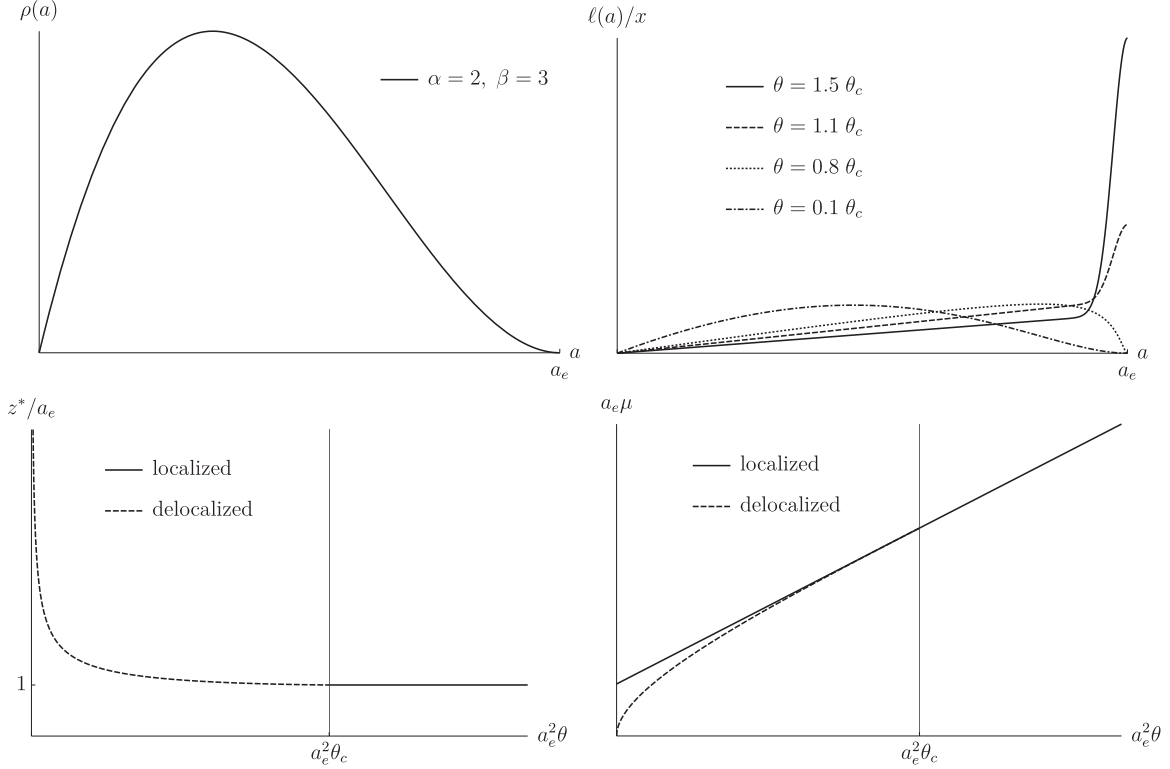
FIG. 7. Results for the model (96) for $\alpha = 2$ and $\beta = 3$, corresponding to $k = 3/4$. Top left: The density $\rho(a)$ of column strengths a. Top right: The occupation length measure $\ell(a)da/x$ normalized to unity (defined in the text) plotted as a function of a for various angles θ. The localized phase corresponds to $\theta > \theta_c$ and the expected delta function peak in Eq. (94) is represented for clarity as a narrow half-Gaussian with the same weight. Bottom left: Plot of the saddle point $z^* = d\mu(\theta)/d\theta$ as a function of θ (in the two phases). Its value freezes at $z^* = a_e$ at the localization transition. Bottom right: (Minus) the ground state energy per column $\mu(\theta)$ as a function of θ. The dashed line is the result in the delocalized phase $\theta < \theta_c$. The solid line is the result in the localized phase for $\theta > \theta_c$ and its continuation for $\theta < \theta_c$ denoted $\mu_e(\theta)$ in the text.

population of columns. If there are too many columns near the edge, the overwhelming competition between them results in a delocalized phase only. A workable example of this situation is given in Appendix F 2.

Application. A nice tractable example is provided by a beta distribution of column strengths with parameters (α, β):

$$\rho(a) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{a^{\alpha-1}(a_e - a)^{\beta-1}}{a_e^{\alpha+\beta-1}} \mathbb{1}_{[0,a_e]}. \quad (96)$$

The correspondence with the decay exponent (76) of the density at the upper edge is $\beta = 2k + \frac{3}{2}$. We consider the case $\beta > 2$, which corresponds to $k > 1/4$ such that there is a localization transition. Using the above formula one finds that all integrals can be evaluated in terms of hypergeometric functions. One finds the critical point and the ground state energy per column

$$\theta_c = \frac{1}{a_e^2}\left(1 + \frac{\alpha}{\beta - 1}\right)\left(1 + \frac{\alpha}{\beta - 2}\right), \quad (97)$$

$$\mu(\theta_c) = \frac{1}{a_e}\left(1 + \frac{\alpha}{\beta - 1}\right)\left(2 + \frac{\alpha}{\beta - 2}\right). \quad (98)$$

The results are plotted in Fig. 7 for $\alpha = 2$ and $\beta = 3$, which corresponds to $k = 3/4$, the limiting case between the Tracy-Widom and anomalous localized phase. It is interesting to note that for $\beta \to +\infty$ one recovers from (97) the results of a single column for θ_c and $\mu(\theta_c)$.

C. Fluctuations of the ground state energy

Let us go back to Eqs. (79). To study the fluctuations of the ground state energy we set $v = \nu N^\beta$, with as yet unknown β, and study $J_N(\mu N + \nu N^\beta)$ and $H_N(\mu N + \nu' N^\beta)$. As in the previous section we consider the saddle point at $z = z^*$, such that $\varphi'(z^*) = 0$, where $\varphi(z)$ is given in (78). We can generally perform an expansion of the rate function $\varphi(z)$ around z^* as

$$\varphi(z^* + w) = \varphi(z^*) + \frac{1}{2}\varphi''(z^*)w^2 + \frac{1}{6}\varphi^{(3)}(z^*)w^3 + \cdots, \quad (99)$$

where the successive derivatives, obtained from (78), read

$$\varphi''(z^*) = \theta - \int_\Omega da \frac{\rho(a)}{(z^* - a)^2},$$

$$\varphi^{(3)}(z^*) = 2\int_\Omega da \frac{\rho(a)}{(z^* - a)^3}. \quad (100)$$

The same value of z^* is chosen in both J_N and H_N; hence the first term in (99) cancels out in the product and we can ignore it and write

$$H_N(\mu N + \nu N^\beta) = \oint \frac{dw}{2\mathbf{i}\pi} e^{\nu N^\beta w - \frac{N}{2}\varphi''(z^*)w^2 - \frac{N}{6}\varphi^{(3)}(z^*)w^3 + \cdots},$$

$$J_N(\mu N + \nu N^\beta) = \int_{\mathbf{i}\mathbb{R}+\varepsilon} \frac{dw}{2\mathbf{i}\pi} e^{\frac{N}{2}\varphi''(z^*)w^2 + \frac{N}{6}\varphi^{(3)}(z^*)w^3 - \nu N^\beta w + \cdots}. \quad (101)$$

1. Localized phase

Consider now the localized phase. From the previous section we choose $z^* = a_e$. Hence the second derivative in the expansion in (101) is $\varphi''(z^*) = \theta - \theta_c > 0$. We can thus rescale the integration variable w by $N^{-1/2}$ and choose $\beta = \frac{1}{2}$. In the large-N limit the higher order terms in the expansion can be neglected. The saddle point integral reads

$$J_N(\mu N + \nu N^{1/2}) \simeq \frac{1}{N^{1/2}} \int_{i\mathbb{R}} \frac{dw}{2i\pi} e^{\frac{\theta - \theta_c}{2} w^2 - \nu w} \quad (102)$$

$$= \frac{1}{\sqrt{2\pi(\theta - \theta_c)N}} e^{-\frac{\nu^2}{2(\theta - \theta_c)}}. \quad (103)$$

The same result (103) holds for H_N since, under rescaling, the contour opens to be parallel to the imaginary axis.

The kernel (43) then becomes, after rescaling $r \to N^{1/2}r$,

$$K_N(\mu N + \nu' N^{1/2}, \mu N + \nu N^{1/2})$$

$$= N^{1/2} \int_0^{+\infty} dr$$

$$\times H_N(\mu N + (\nu' + r)N^{1/2}) J_N(\mu N + (\nu + r)N^{1/2})$$

$$= \frac{1}{2\pi(\theta - \theta_c)\sqrt{N}} \int_0^{+\infty} dr \, e^{-\frac{(\nu'+r)^2}{2(\theta-\theta_c)} - \frac{(\nu+r)^2}{2(\theta-\theta_c)}} \quad (104)$$

$$= \frac{1}{4\pi^2 \sqrt{N(\theta - \theta_c)}} e^{-\frac{(\nu-\nu')^2}{4(\theta-\theta_c)}} \mathrm{Erfc}\left(\frac{\nu' + \nu}{2\sqrt{\theta - \theta_c}}\right). \quad (105)$$

Hence, defining the kernel

$$\mathcal{T}(x, y) = \frac{1}{\pi} \int_0^\infty du \, e^{-(x+u)^2} e^{-(y+u)^2}, \quad (106)$$

upon rescaling, we find that the ground state energy in the localized phase fluctuates as

$$-\mathcal{E}_N^0(x) = \lambda_1(x) = \mu N + N^{1/2}\sqrt{2(\theta - \theta_c)} \, \omega, \quad (107)$$

where $\mu = \mu(\theta)$ was obtained in (86) and ω is a random variable whose CDF is given by the Fredholm determinant,

$$\mathbb{P}(\omega \leqslant s) = \mathrm{Det}(I - \mathcal{T})_{\mathbb{L}^2(s, +\infty)}. \quad (108)$$

Although this distribution is, to our knowledge, novel, the kernel $\mathcal{T}(x, y)$ already appeared in the study of the largest real eigenvalue of the real Ginibre ensemble [104–107]. More precisely the kernel which appears there is $\mathcal{T}(x, y) + g(x)G(y)$, where $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $G(y) = \int_{-\infty}^y g(x)dx$. The asymptotics are obtained as

$$\mathbb{P}(\omega \leqslant s) \simeq_{s \to +\infty} 1 - \frac{e^{-2s^2}}{16\pi s^2} \simeq_{s \to -\infty} e^{-\frac{\zeta(3/2)}{\sqrt{2\pi}}|s|}, \quad (109)$$

combining in the second case [[105], Eqs. (1.9) and (1.11)].

We note that to obtain this result we have (i) assumed that θ_c is finite, i.e., that $\int_\Omega da \frac{\rho(a)}{(a_e-a)^2} < +\infty$ (equivalent to the existence of a localized phase) which holds for $k > 1/4$, and (ii) neglected, after rescaling, the higher derivatives in the saddle point. To be more precise one can check that the

condition for the above analysis to work is that for fixed w

$$\lim_{N \to +\infty} N \int_\Omega da \rho(a) \left[\ln\left(1 + \frac{wN^{-1/2}}{a_e - a}\right) \right.$$

$$\left. - \frac{wN^{-1/2}}{a_e - a} + \frac{1}{2}\left(\frac{wN^{-1/2}}{a_e - a}\right)^2 \right] = 0, \quad (110)$$

which is weaker than the condition of the existence of the third derivative $\varphi'''(z^*)$.

2. Delocalized phase

Consider now the delocalized phase, $\theta < \theta_c$. Then one can choose $z^* > a_e$ so that $\varphi''(z^*) = 0$. The saddle point will then be of cubic type. One now rescales $w \to wN^{-1/3}$ and chooses $\beta = \frac{1}{3}$ so that the function J_N in (101) takes the form

$$J_N(\mu N + \nu N^{1/3}) = \frac{1}{N^{1/3}} \int_{i\mathbb{R}+\epsilon} \frac{dz}{2i\pi} e^{\frac{\varphi^{(3)}(z^*)}{6} z^3 - \nu z}$$

$$= \left(\frac{2}{N\varphi^{(3)}(z^*)}\right)^{1/3} \mathrm{Ai}\left(\frac{2^{1/3}}{\varphi^{(3)}(z^*)^{1/3}} \nu\right). \quad (111)$$

Similarly one finds that $H_N(\mu N + \nu' N^{1/3})$ is also given by (111).

The kernel (43) then becomes, after rescaling $r \to N^{1/3}r$,

$$K_N(\mu N + \nu' N^{1/3}, \mu N + \nu N^{1/3})$$

$$= \left(\frac{2}{N\varphi^{(3)}(z^*)}\right)^{1/3} K_{\mathrm{Ai}}\left(\frac{2^{1/3}}{\varphi^{(3)}(z^*)^{1/3}} \nu, \frac{2^{1/3}}{\varphi^{(3)}(z^*)^{1/3}} \nu'\right). \quad (112)$$

Hence we find Tracy-Widom fluctuations for the ground state energy at leading order,

$$-\mathcal{E}_N^0(x) = \lambda_1(x) = \mu N + \left(N \frac{\varphi^{(3)}(z^*)}{2}\right)^{1/3} \chi_2. \quad (113)$$

We recall that to obtain the coefficients μ and $\varphi^{(3)}(z^*)$ we must first find z^* as a function of θ from the second equation of the parametric system and then insert its value in the first and third equations of the parametric system,

$$\mu = \theta z^* + \int_\Omega da \frac{\rho(a)}{z^* - a},$$

$$\theta = \int_\Omega da \frac{\rho(a)}{(z^* - a)^2},$$

$$\varphi^{(3)}(z^*) = 2 \int_\Omega da \frac{\rho(a)}{(z^* - a)^3}. \quad (114)$$

3. Critical region

We now study the critical region between the localized and delocalized phases, near $\theta \approx \theta_c$. We assume that $k > 3/4$ so that $\theta_c < +\infty$ and

$$\varphi^{(3)}(a_e) = 2 \int_\Omega da \frac{\rho(a)}{(a_e - a)^3} < +\infty. \quad (115)$$

To be able to describe the critical region starting from the localized phase, we choose $z^* = a_e$. We also impose $\varphi'(a_e) = 0$

by choosing

$$\mu = \mu_e(\theta) = \theta a_e + \int_\Omega da \, \frac{\rho(a)}{(a_e - a)}, \qquad (116)$$

so that the first derivative is zero. In (101) we choose $\beta = 1/3$ and obtain

$$J(\mu N + \nu N^{1/3})$$
$$= \int_{i\mathbb{R}+\epsilon} \frac{dz}{2i\pi} e^{\frac{N}{2}(\theta - \theta_c)(z-a_e)^2 + \frac{N}{6}\varphi^{(3)}(a_e)(z-a_e)^3 - \nu N^{1/3}z + \cdots}, \qquad (117)$$

where we used that $\varphi''(a_e) = \theta - \theta_c$. Here Γ is an appropriate contour parallel to the imaginary axis where the integral converges (see below). It is clear from (45) that in order to balance the quadratic and cubic terms, the critical region is defined at large N as

$$\theta - \theta_c = \tau N^{-1/3}, \qquad (118)$$

where τ is fixed. We write $z = a_e + w N^{-1/3}$ and obtain

$$J_N(\mu N + \nu N^{1/3}) \simeq N^{-1/3} \int_{i\mathbb{R}+\epsilon} \frac{dw}{2i\pi} e^{\frac{1}{2}\tau w^2 + \frac{1}{6}\varphi^{(3)}(a_e)w^3 - \nu w}, \qquad (119)$$

where, after the rescaling, the higher order derivatives can be (naively) neglected. Note that this integral is convergent only for $\tau \geqslant 0$, which we assume for now.

Let us recall the useful formula, for any $a \in \mathbb{R}^*$, which we use here for $b \geqslant 0$,

$$\int_{i\mathbb{R}+\epsilon} \frac{dz}{2i\pi} e^{a\frac{z^3}{3} + bz^2 + cz} = \frac{e^{\frac{2b^3}{3a^2} - \frac{bc}{a}}}{|a|^{1/3}} \mathrm{Ai}\left(\frac{b^2}{|a|^{4/3}} - \frac{c\,\mathrm{sgn}(a)}{|a|^{1/3}}\right), \qquad (120)$$

using the definition of the Airy function (54) and shifting the integration contour.

Hence we find for $\tau \geqslant 0$

$$J_N(\mu N + \nu N^{1/3}) \simeq \left(\frac{2}{N\varphi^{(3)}(a_e)}\right)^{1/3} e^{\frac{\tau^3}{3\varphi^{(3)}(a_e)^2} + \frac{\tau\nu}{\varphi^{(3)}(a_e)}}$$
$$\times \mathrm{Ai}\left(\frac{\tau^2}{2^{2/3}\varphi^{(3)}(a_e)^{4/3}} + \frac{2^{1/3}\nu}{\varphi^{(3)}(a_e)^{1/3}}\right). \qquad (121)$$

One can show that H_N is given by the same formula, using contour reversal and orientation reversal [see discussion around (54)]. It amounts to using the same formula (120) with $(a, b, c) \to (-a, b, -c)$, which leaves it invariant.

This leads to the kernel in the form

$$K(\mu N + \nu' N^{1/3}, \mu N + \nu N^{1/3})$$
$$\simeq \frac{1}{N^{1/3}}\left(\frac{2}{\varphi^{(3)}(a_e)}\right)^{2/3} e^{2\frac{\tau^3}{3\varphi^{(3)}(a_e)^2}} \int_0^{+\infty} dr \, e^{\frac{\tau(\nu+\nu'+2r)}{\varphi^{(3)}(a_e)}}$$
$$\times \mathrm{Ai}\left(\frac{\tau^2}{2^{2/3}\varphi^{(3)}(a_e)^{4/3}} + \frac{2^{1/3}}{\varphi^{(3)}(a_e)^{1/3}}[\nu' + r]\right)$$
$$\times \mathrm{Ai}\left(\frac{\tau^2}{2^{2/3}\varphi^{(3)}(a_e)^{4/3}} + \frac{2^{1/3}}{\varphi^{(3)}(a_e)^{1/3}}[\nu + r]\right). \qquad (122)$$

Hence the final result is as follows. For $\tau \geqslant 0$, i.e., on the localized side of the critical region, the ground state energy fluctuates to leading order as

$$-\mathcal{E}_N^0(x) = \lambda_1(x) = \mu N + \left(\frac{\varphi^{(3)}(a_e)}{2}N\right)^{1/3}\chi, \qquad (123)$$

where the CDF of the random variable χ is given by the following Fredholm determinant,

$$\mathbb{P}(\chi \leqslant s) = \mathrm{Det}(I - K_{\tilde{\tau}})_{\mathbb{L}^2(s,+\infty)}, \qquad (124)$$

where the dimensionless parameter $\tilde{\tau}$ measures the distance to criticality,

$$\tilde{\tau} = \frac{N^{1/3}(\theta - \theta_c)}{2^{1/3}\varphi^{(3)}(a_e)^{2/3}}, \qquad (125)$$

and the kernel $K_{\tilde{\tau}}$ is given for $\tilde{\tau} \geqslant 0$ as

$$K_{\tilde{\tau}}(v, v')$$
$$= e^{\frac{4}{3}\tilde{\tau}^3} \int_0^{+\infty} dr \, e^{\tilde{\tau}(v+v'+2r)} \mathrm{Ai}(v + r + \tilde{\tau}^2)\mathrm{Ai}(v' + r + \tilde{\tau}^2). \qquad (126)$$

One now notes that for $\tilde{\tau} = 0$, this kernel recovers simply the Airy kernel $K_{\tilde{\tau}=0} = K_{\mathrm{Ai}}$. This indicates that for $\tilde{\tau} < 0$, on the delocalized side, one should instead choose, as usual to describe the Tracy Widom phase, z^* such that $\varphi''(z^*) = 0$. Hence for $\tilde{\tau} < 0$, the result (113) holds; i.e., the fluctuations are GUE Tracy-Widom, $\chi = \chi_2$. It is quite remarkable that there is no precursor of the transition in the leading fluctuations of the ground state energy on the delocalized side (fluctuations remain Tracy-Widom all the way to $\tilde{\tau} = 0$), while their CDF varies continuously on the localized side.

The above kernel $K_{\tilde{\tau}}(v, v')$ interpolates between the Airy kernel for $\tilde{\tau} = 0$ and the kernel $\mathcal{T}$ in (106), which describes the fluctuations in the localized phase. It happens as follows. In the limit $\tilde{\tau} \to +\infty$ one can use in (126) the asymptotics of the Airy function for large positive argument

$$\mathrm{Ai}(z) \sim \frac{1}{\sqrt{4\pi}z^{1/4}} e^{-\frac{2}{3}z^{3/2}} \qquad (127)$$

and obtain, expanding up to quadratic order in the exponential,

$$K_{\tilde{\tau}}(v, v') \underset{\tilde{\tau}\to+\infty}{\to} \frac{1}{\sqrt{4\tilde{\tau}}}\mathcal{T}\left(\frac{v}{\sqrt{4\tilde{\tau}}}, \frac{v'}{\sqrt{4\tilde{\tau}}}\right). \qquad (128)$$

We thus find that in this limit the random variable χ in (123)

$$\chi \to \sqrt{4\tilde{\tau}}\,\omega, \qquad (129)$$

where ω is the random variable in (107); hence given the definition of $\tilde{\tau}$ in (125) we find that (123) and (107) match deep in the localized side of the transition.

D. Additional relations to RMT: Ground state energy as a function of the polymer end point position

One can ask how the ground state energy $\mathcal{E}_N^0(x)$, for a polymer of length x and fixed entry point position on the first column $j = 1$, depends on the choice of the exit point position $j = N$. This is asking about $\mathcal{E}_N^0(x)$ at fixed x, as a process in N (in the same disorder environment). It is indeed important to study how the ground state responds to a small
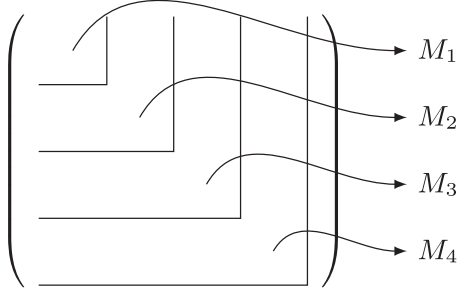
FIG. 8. Representation of the principal minors M_n of the matrix $\mathcal{M}$.

perturbation (here moving the end point by one unit). Indeed, glassy systems often have broadly distributed, intermittent response, i.e., rare but large response called avalanches.

It turns out that this question is also related to random matrices. More precisely, for a given configuration of the point disorder, i.e., the Brownian motions $B_j(x)$, and the column strengths, the a_j, the joint PDF (JPDF) of $-\mathcal{E}_{N-1}^0(x)$ and $-\mathcal{E}_N^0(x)$ is the same as the JPDF of the largest eigenvalue $\lambda_1^{(N)}$ of an $N \times N$ GUE random matrix and of $\lambda_1^{(N-1)}$, the largest eigenvalue of its $N \times N$ minor matrix (obtained by erasing one line and one column).

It is known that $\mathcal{E}_N^0(x)$, as a process in N, is identical to the so-called minor GUE process [59], which is determinantal (i.e., all its correlations are given by determinants involving a kernel) and that these properties extend in the presence of drifts, in relation to the deformed GUE minor process [108].

Let us recall the properties of the deformed GUE minor process described in Ref. [108]. Define the nth principal minor (top left) as shown in Fig. 8,

$$M_n = [\mathcal{M}(x=1)]_n = [\text{diag}(a_1, \ldots, a_N) + V]_n, \quad (130)$$

where V is the same GUE matrix as in (10).

For simplicity we consider here a polymer of fixed length $x = 1$ but arbitrary x is easily obtained by rescaling. We denote the eigenvalues of M_n as $\lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$. Note that here we ordered them in increasing order.

A first result is the transition probability. Suppose the eigenvalues with $N - 1$ columns are known, called $\mu = (\mu_1, \ldots, \mu_{N-1})$. Then the JPDF of the $(\lambda_1, \ldots, \lambda_N)$ satisfy [[108], Theorem 1]

$$P_\mu(\lambda) = C_\mu \Delta_N(\lambda) e^{\sum_{i=1}^N (a_N \lambda_i - \frac{\lambda_i^2}{2})} \mathbf{1}_{\mu \preceq \lambda}, \quad (131)$$

where the normalization constant $C_\mu = \frac{e^{-a_N^2/2}}{\sqrt{2\pi}\Delta_{N-1}(\mu)} e^{\sum_{i=1}^{N-1} \frac{1}{2}\mu_i^2 - a_N \mu_i}$. Here $\mu \preceq \lambda$ stands for the interlacing property $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, since an important property of the eigenvalues of a matrix and of its largest minor is that they are interlaced. Another property is that the JPDF of the complete interlacing set of eigenvalues is given by [[108], Corollary 1]

$$P(\lambda^{(1)}, \ldots, \lambda^{(n)}, \ldots, \lambda^{(N)}) \quad (132)$$

$$= \Delta_N(\lambda^{(N)})(2\pi)^{-N/2} e^{-\frac{1}{2}\sum_{n=1}^N a_n^2} \prod_{n=1}^N e^{-\frac{1}{2}(\lambda_n^{(N)})^2 + a_N \lambda_n^{(N)}}$$

$$\times e^{\sum_{n=1}^N (a_n - a_{n+1}) \sum_{i=1}^n \lambda_i^{(n)}} \mathbf{1}_{\lambda^{(n)} \preceq \lambda^{(n+1)}}. \quad (133)$$

In order to perform actual calculations it is useful to note that the deformed GUE minor process is a so-called extended determinantal process. The definitions and the explicit expression for its extended kernel $K(n_1, x; n_2, y) \equiv K_{n_1,n_2}(x, y)$ are given in [[108], Theorem 3]. It is an extended version of the kernel (43), for different values of N.

The standard formula for a determinantal process then allows us to write the joint CDF of minus the ground state energies for two polymer end point positions at N_1 and N_2, i.e., of $\lambda_n^{(N_1)} = -\mathcal{E}_{N_1}^0$ and $\lambda_n^{(N_2)} = -\mathcal{E}_{N_2}^0$ with $N_2 < N_1$, as a matrix Fredholm determinant,

$$\mathbb{P}\left(-\mathcal{E}_{N_1}^0 < s_1, -\mathcal{E}_{N_2}^0 < s_2\right)$$

$$= \text{Det}\begin{pmatrix} I - P_{s_1} K_{N_1,N_1} P_{s_1} & -P_{s_1} K_{N_1,N_2} P_{s_2} \\ P_{s_2} K_{N_2,N_1} P_{s_1} & I - P_{s_2} K_{N_2,N_2} P_{s_2} \end{pmatrix}, \quad (134)$$

where $P_s(x) = \Theta(x - s)$ is the projector of $[s, +\infty[$. We will not attempt here to analyze this formula, but in principle it can be done along similar lines to those in this paper.

IV. MANY LINES, ZERO TEMPERATURE

In this section we explore the problem of several interacting elastic lines in the same columnar and point impurity disorder. The "solvable" case studied here corresponds to imposing an infinite hard core repulsion between the lines, enforced by a noncrossing condition. In Ref. [7] a hard core repulsion between the lines was also considered, leading to the prediction of a "Bose glass" phase. In the absence of point disorder, it was implemented in a heuristic way by filling the lowest energy columns (also called the best localized states) one by one, very much like noninteracting fermions, until reaching a "Fermi energy." In the case of a commensuration between the number of lines and of (active) columns it was predicted that one can reach a "Mott insulator" phase.

Here we describe exact results in the presence of point disorder, for the "one-way" model with many lines. Since it neglects all jumps backward this model is *a priori* more relevant to describe the delocalization transition away from the Bose glass which occurs upon tilting the lines (see Fig. 5 in [14] and Fig. 1). The results described below also assume that the end points of the lines are closely packed on neighboring columns. In practice, it means that the entry and exit positions of the flux lines are constrained within a narrow region. That could be enforced in experiments by inducing channels where the superconductivity, i.e., the critical field H_{c1} and/or the vortex core energy, is locally reduced. Note that inside the sample the lines will expand and form a limit shape. This is similar to the limit shape of a set of nonintersecting random walks, also called watermelons, but here in the presence of a columnar and point disorder. For usual watermelons the shape can be inferred from the connection with the DBM and the GUE, as a semicircle with a time-varying width, while in presence of disorder much less is known (see discussion of related problems and continuum limits in [109–111]). Note that some properties for more general end point configurations have also been obtained [112].
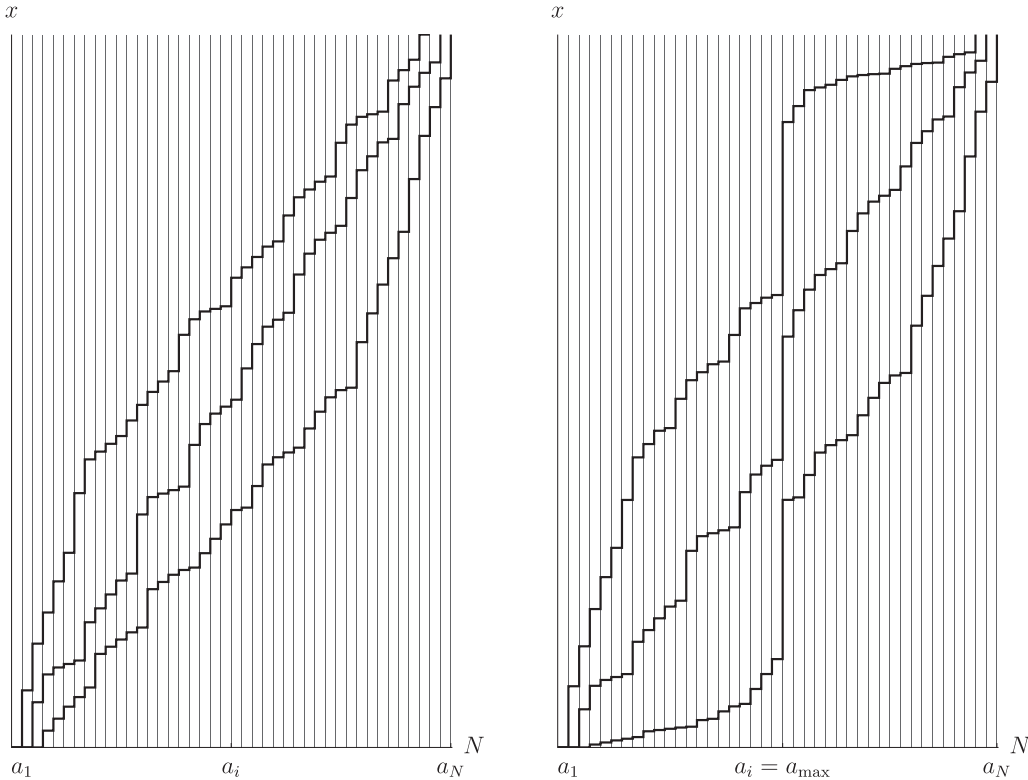
FIG. 9. Left: Three lines ($M = 3$) in the O'Connell-Yor polymer model (i.e., constrained not to cross) of length x with N columnar defects, in the phase where they are delocalized over the space. Right: Three lines localized over the column i where the disorder is most favorable. The total occupation length of column i, L_i, is the sum of the occupation lengths of each line.

A. Relation to RMT

The relation between the O'Connell-Yor polymer at $T = 0$ and the deformed GUE extends to several noncrossing polymers. Consider now $M \geqslant 1$ O'Connell-Yor polymers which see the same $N \geqslant M$ columns, with energies $\epsilon_j = -a_j$, and the same Brownian impurity disorder defined on each column, $B_j(x)$, $j = 1, \ldots, N$. The polymers are furthermore constrained not to intersect. If one denotes $0 < x_j^{(k)} < x$ the successive jump positions of polymer $k = 1, \ldots, M$, they are thus constrained by an interlacing condition (see the definition of the model in Appendix C 2).

The boundary conditions are the following. The polymer end points at $x = 0$ are in column positions $j = 1, \ldots, M$. The end points at x are in column positions $j = N - M + 1, \ldots, N$. This is illustrated in Fig. 9. Note that as a consequence there is a global "tilt" of the lines by an angle ϕ_M such that

$$\tan \phi_M = \frac{N - M}{x} = \frac{1}{\theta_M}. \tag{135}$$

The total energy to be minimized is the sum of the energies $\sum_{k=1}^{M} E_N^{(k)}$ of each polymer. Let us call $\mathcal{E}_{N,M}^0(x)$ the minimum energy (ground state energy) under the above constraints (nonintersection and boundary conditions). The theorem states that [64,108]

$$\mathcal{E}_{N,M}^0(x) \overset{\text{in law}}{=} -\sum_{\alpha=1}^{M} \lambda_\alpha(x), \tag{136}$$

where $\lambda_1(x) > \lambda_2(x) > \ldots > \lambda_M(x)$ are the M largest eigenvalues of the deformed $N \times N$ GUE matrix $\mathcal{M}(x)$ defined in (10).

It turns out that the joint PDF of these eigenvalues $\lambda_j(x)$ for the matrix $\mathcal{M}(x)$ at fixed value of x is known explicitly, so we indicate it here. As shown in [79,81], the symmetrized joint PDF of the $\lambda_j(x)$ (i.e., here with no ordering), $j = 1, \ldots, N$, is given by

$$P_x(\{\lambda\}) = \frac{1}{(2\pi x^N)^{N/2}} \frac{\Delta_N(\lambda)}{\Delta_N(a)} \det(e^{-(\lambda_i - xa_j)^2/2x})_{1 \leqslant i, j \leqslant N}, \tag{137}$$

where the Vandermonde determinant is defined with the following convention $\Delta_N(\lambda) = \prod_{1 \leqslant i < j \leqslant N} (\lambda_i - \lambda_j)$. This JPDF has a determinantal form, with a kernel given in [[81], Proposition (2.3) and formula (2.18)]. This result is the starting point for obtaining the formula (42) and (43) for the PDF of the largest eigenvalue $\max_j \lambda_j(x)$ [which above and below is denoted $\lambda_1(x)$].

Again, as for the case $M = 1$, one can show that (136) holds as a process in x, replacing $\sqrt{x}V \rightarrow W(x)$ the Hermitian Brownian motion [64]. Again the $\{\lambda_i(x)\}_{i=1,\ldots,N}$ form a determinantal point process and its extended kernel is known [116] following earlier works [81,83], and can be found in explicit form in, e.g., [[122], Proposition 2.1] and in [[129], Eq. (1.13)] (see also [117]).

From the above property (136) we see that if M is fixed and N becomes large (i.e., few lines, many columns) one is

probing the edge of the spectrum of the matrix $\mathcal{M}(x)$. By contrast if M and N grow with a fixed ratio, one is probing the bulk of the spectrum. We will study both cases below.

One can define the total occupation length of a given column j by all lines as $L_j = \sum_{k=1}^{M} \ell_j^k$ where $\ell_j^k = x_j^k - x_{j-1}^k$ is the occupation length of column j by the kth line. These occupation lengths satisfy the sum rule $\sum_{j=1}^{N} L_j = Mx$. The optimal total occupation length, denoted L_j^0, can again be obtained as a derivative of the ground state energy

$$-\partial_{a_j} \mathcal{E}_{N,M}^0(x) = L_j^0 = \sum_{k=1}^{M} \ell_j^{k,0}, \qquad (138)$$

where $\ell_j^{k,0}$ denote the corresponding values in the ground state.

Note that there is again a nontrivial property of permutation invariance of the column strengths a_j (see remark after Theorem 8.3 of [64]).

Finally, note that the model can also be solved at finite temperature [119,120] for M noncrossing O'Connell-Yor polymers, which define a hierarchy of partition sums.

B. Few lines and few columnar defects: Independent BBP transitions

Suppose first that there are only a fixed and finite number n of active columns, with $a_1 > a_2 > \ldots > a_n$, and all other $a_{j>n} = 0$. One can ask how the system of M lines will localize on these n columns.

Using the above relation (136) we see that this problem corresponds to a rank-n perturbation of an $N \times N$ GUE matrix. It is known that as $N \to +\infty$, there are in that case n successive and distinct BBP transitions as $\theta = x/N$ is increased from 0 (see [[90], Theorem 2.1 and Sec. 3.1] with the correspondence $\sigma \to \sqrt{N}x$ and $\theta_i \to xa_i$). They occur successively at $\theta = \theta_c^{(i)} = 1/a_i^2$, $i = 1, \ldots, n$. For $\theta < \theta_c^{(1)}$ the density of eigenvalues $\lambda_i(x)$ is given by the semicircle law with an upper edge at $2\sqrt{xN}$. At the first transition the largest eigenvalue detaches from the semicircle, at the second transition the second largest eigenvalue detaches, and so on. Thus one has, to leading order in N (up to subleading fluctuations),

$$\lambda_i(x) \simeq \begin{cases} 2\sqrt{xN}, & \theta < 1/a_i^2, \\ a_i x + \dfrac{N}{a_i}, & \theta > 1/a_i^2. \end{cases} \qquad (139)$$

These transitions are "decoupled" from each other as long as the column strengths verify $|a_i - a_{i+1}| \gg N^{-1/3}$.

Let us consider now a fixed number of lines M, which remains finite as $N \to +\infty$. From (136) and (139) the ground state energy (to leading order in N) reads

$$-\mathcal{E}_{N,M}^0(x) \simeq N\left[2M\sqrt{\theta} + \sum_{i=1}^{\min(M,n)} a_i\left(\sqrt{\theta} - \frac{1}{a_i}\right)_+^2 \right], \quad (140)$$

where we recall that $(x)_+ = \max(x, 0)$. The second derivative of the free energy as a function of θ will thus have the form of a staircase, with a jump at each transition point $\theta = \theta_c(a_i)$. The total occupation length of each active column $i = 1, \ldots, n$ in the ground state is obtained from differentiation

according to (138) to leading order in N as, for $i \leqslant \min(M, n)$,

$$L_i^0 = \sum_{k=1}^{M} \ell_i^{k,0} \simeq \left(x - \frac{N}{a_i^2}\right)_+ = N\left(\theta - \frac{1}{a_i^2}\right)_+, \quad (141)$$

and $L_i^0 = 0$ for $i > \min(M, n)$. This coincides with the result for the single line, single column problem (22). Hence the total occupation length of a given column is completely independent of the values of the other column strength a_i's (as long as all strengths are distinct). One can also define the effective number of active columns for a given θ, i.e., the number of columns where localization occurs (that is, those which have a macroscopic occupation, $L_i^0/N > 0$ at large N), as $n(\theta) = \sum_{i=1}^{\min(n,M)} \Theta(\theta - \frac{1}{a_i^2})$ which is always smaller or equal to M. Note that the sum of the occupation lengths of the macroscopically occupied columns is smaller than the total length,

$$\sum_{i=1}^{\min(n,M)} L_i^0 \simeq \sum_{i=1}^{\min(n,M)} N\left[\theta - \frac{1}{a_i^2}\right]_+ < Mx = \sum_{i=1}^{N} L_i^0. \quad (142)$$

This is because each of the other columns has occupation length $L_j^0 = O(1)$; these sum up to the remainder. In the case of a single line $M = 1$, the PDF of the $O(1)$ occupation length was obtained in Appendix F 3 in the simpler case where all $a_j = 0$.

If the positions of the n active columns are permuted among N, the ground state energy is unchanged but the actual optimal configuration of the polymers, i.e., the set of $\ell_i^{k,0}$, is different. The constraint however is the total occupation lengths for each column if given by the above formula (141) (for a_i any of the active columns).

Finally, if instead the active columns are all close in energies, within $a_1 - a_n \sim N^{-1/3}$, there is a single localization transition, described by the M largest eigenvalues of the rank n deformed GUE, that is, upon rescaling, by the M largest points of the determinantal point process described by the kernel BBP$_n$ recalled in Appendix D.

C. The limit $M, N \to \infty$ with a fixed "density" $r = \frac{M}{N}$

In the limit $M, N \to \infty$ with a fixed "line density" $r = \frac{M}{N}$ one can calculate the leading term in the ground state energy. We consider the case of a continuous density of column strengths $\rho(a)$. First one obtains the density of eigenvalues of $\mathcal{M}(x)$ to leading order, from free probability. Let us define the scaled matrix, from (10),

$$\tilde{M}(x) = \frac{\mathcal{M}(x)}{x} = \frac{\mathcal{M}(x)}{\theta N} = \mathrm{diag}(a_1, \ldots, a_N) + \sqrt{\tau}\,\tilde{V},$$

$$\tau = \frac{1}{\theta}, \quad \theta = \frac{x}{N}, \qquad (143)$$

where $\tilde{V} = V/\sqrt{N}$ is a GUE matrix with the semicircle density $v_{sc}(y) = \frac{1}{2\pi}\sqrt{4 - y^2}$, and we recall that we study $N, x \to +\infty$ with θ fixed. We introduce the variable $\tau = 1/\theta$ which is more convenient. The eigenvalues of $\mathcal{M}(x)$ are thus $\lambda_i(x) = N\theta\tilde{\lambda}_i(x)$, where $\tilde{\lambda}_i(x)$ are the eigenvalues of $\tilde{M}(x)$. Their density in the large-N limit, $v_\tau(\tilde{\lambda}) = \lim_{N \to +\infty} \frac{1}{N} \sum_{i=1}^{N} \delta(\tilde{\lambda} - \tilde{\lambda}_i(x))$, is determined by the free

additive convolution (see [121,123] and references therein),

$$\nu_\tau = \rho \boxplus \nu_{sc,\tau}, \qquad (144)$$

where $\nu_{sc,\tau}(y) = \frac{1}{2\pi\tau}\sqrt{4\tau - y^2}$. Let us define $G_\tau(z)$ as the Stieljes transform of the density $\nu_\tau(\tilde{\lambda})$,

$$G_\tau(z) = \lim_{N\to+\infty} \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \tilde{\lambda}_i(x)} = \int_{-\infty}^{+\infty} d\tilde{\lambda}\frac{\nu_\tau(\tilde{\lambda})}{z - \tilde{\lambda}}, \quad (145)$$

from which the density can be extracted as

$$\nu_\tau(\tilde{\lambda}) = -\frac{1}{\pi}\mathrm{Im}\, G(\tilde{\lambda} + \mathbf{i}0^+) = \frac{1}{\pi}\mathrm{Im}\, G(\tilde{\lambda} - \mathbf{i}0^+). \quad (146)$$

Then, (143) and (144) imply that $G_\tau(z)$ satisfies the Burgers equation,

$$\partial_\tau G_\tau(z) = -G_\tau(z)\partial_z G_\tau(z), \qquad (147)$$

see Appendix B, of solution [70,121,123–126], [[127], Theorem 5], which obeys for $z \in \mathbb{C}^+$

$$G_\tau(z) = \int_\Omega \frac{da\,\rho(a)}{z - \tau G_\tau(z) - a}. \qquad (148)$$

For $\tau \to 0$, equivalently for $\theta \to +\infty$, one can neglect the term $\tau G_\tau(z)$ inside the denominator of (148) and the corresponding initial condition of the Burgers equation is $G_{\tau=0}(z) = \int_\Omega da\frac{\rho(a)}{z-a}$, i.e., $\nu_{\tau=0} = \rho$, and the spectrum of $\tilde{M}(x)$ is given by the $\{a_i\}$, the strengths of the columnar defects, i.e., as clear from (143). In the opposite limit $\tau \to +\infty$, equivalently $\theta \to 0$ [a limit equivalent to choosing $\rho(a) \to \delta(a)$], Eq. (148) becomes $G_\tau(z)[z - \tau G_\tau(z)] = 1$ and the solution takes the form

$$G_\tau(z) \simeq \frac{1}{\sqrt{\tau}}G_{sc}\left(\frac{z}{\sqrt{\tau}}\right) \qquad (149)$$

in terms of the resolvent associated to the unit semicircle density [128]

$$G_{sc}(z) = \frac{1}{2}\left(z - z\sqrt{1 - \frac{4}{z^2}}\right). \qquad (150)$$

Note that (148) can be written as $G_\tau(z) = G_0(w)$ with, equivalently, $z = w + \tau G_0(w)$ or, $w = z - \tau G_\tau(z)$.

As discussed in Appendix B, the scaled eigenvalues $\tilde{\lambda}_i$ perform a Dyson Brownian motion as a function of the parameter $\tau = 1/\theta$. This is valid for any N, and leads to the above equations at large N. The initial condition is $\tilde{\lambda}_i = a_i$ at $\tau = 0$, and at large τ the density converges to the semicircle shape.

One can ask how the density ν_τ evolves between these two limits. We will assume again that the columnar defect energies have a density $\rho(a)$ with a soft right edge a_e where it vanishes as in (76), i.e., $\rho(a) = A(a_e - a)^{2k+\frac{1}{2}}$, with $\rho(a) = 0$ for $a > a_e$. In [123] it is proved that $\nu_\tau(\tilde{\lambda})$ vanishes at its edge with the same exponent as long as

$$\tau < \tau_c, \quad \frac{1}{\tau_c} = \theta_c = \int_\Omega da\frac{\rho(a)}{(a_e - a)^2}. \qquad (151)$$

This corresponds to the localized phase for $\theta > \theta_c$, that we have studied by other methods in Sec. III A with exactly the same value for θ_c in (85) (there we focused on the *largest*

eigenvalue $\tilde{\lambda}_1$). More precisely in [[123], Theorem 1.3] it is shown that, denoting $\tilde{\lambda}_e = \tilde{\lambda}_e(\tau)$ the upper edge of ν_τ, one has

$$\nu_\tau(\tilde{\lambda}) \simeq A\left(\frac{\tau_c}{\tau_c - \tau}\right)^{2k+3/2}[\tilde{\lambda}_e(\tau) - \tilde{\lambda}]^{2k+1/2}, \quad (152)$$

$$\tilde{\lambda}_e(\tau) = a_e + \tau\int_\Omega da\frac{\rho(a)}{a_e - a}, \quad \tau = \frac{1}{\theta}. \quad (153)$$

The position of the edge is exactly what was found in Sec. III A from the study of the largest eigenvalue λ_1, i.e., for the single line problem $M = 1$, with the correspondence $\tilde{\lambda}_e(\tau) = \mu(\theta)/\theta \simeq \lambda_1/(N\theta)$, where $\mu(\theta)$ is given in (86) in the localized phase and recalling that $\tau = 1/\theta$. The results (152) are established in [123] for integer $k \geqslant 1$, but in view of our results in Sec. III A it is reasonable to conjecture them to be valid for any real $k > 1/4$, so that the integral in (151) converges and $\tau_c > 0$ (i.e., so that there is a localized phase for a single line).

At criticality $\tau = \tau_c$, Ref. [[123], Theorem 1.4. (d)] states that the semicircle shape holds near the edge,

$$\nu_\tau(\tilde{\lambda}) \simeq \frac{1}{\pi\tau_c^{3/2}\sqrt{g}}|\tilde{\lambda}_e(\tau_c) - \tilde{\lambda}|^{1/2}, \qquad (154)$$

where

$$g = \int_\Omega da\frac{\rho(a)}{(a_e - a)^3}. \qquad (155)$$

Note that this parameter g is identical to the one defined in (115), i.e., $g = \frac{1}{2}\varphi^{(3)}(a_e)$ in the study of the localization transition for a single line $M = 1$. There it was assumed to be finite, equivalent to the convergence of the integral which holds for $k > 3/4$, and implied a critical regime for the fluctuations of λ_1 (equal to minus the ground state energy) interpolating between the Tracy-Widom distribution and a new distribution. Again we can conjecture that the above results at criticality, shown in [123] for integer $k \geqslant 1$, extend to any real $k > 3/4$.

In conclusion, we see that the localization transition obtained for a single line $M = 1$ shows up in the many line problem $M > 1$ as a transition in the behavior of the density of eigenvalues $\tilde{\lambda}_i$ near its upper edge: it vanishes with the same exponent $2k + 1/2$ as the column strengths in the (single-line localized) phase $\theta > \theta_c$, and with the semicircle exponent $1/2$ in the (single-line delocalized) phase $\theta < \theta_c$.

We can now apply these results to make predictions about the ground state energy for a fixed density $r = M/N$ of lines versus columns. One introduces the "Fermi" level $\tilde{\lambda}_f$ as the solution of the equation

$$\int_{\tilde{\lambda}_f}^{\tilde{\lambda}_e} d\tilde{\lambda}\,\nu_\tau(\tilde{\lambda}) = r = \frac{M}{N}, \qquad (156)$$

where we recall that $\tilde{\lambda}_e$ is the upper edge of the eigenvalue density $\nu_\tau(\tilde{\lambda})$ of the matrix $\tilde{M}(x) = \mathcal{M}(x)/x$. The ground state energy of the system of M lines is then given for large N, with $r = M/N$ fixed, as

$$-\mathcal{E}_{N,M}^0(x) \simeq \theta N^2\int_{\tilde{\lambda}_f}^{\tilde{\lambda}_e} d\tilde{\lambda}\,\tilde{\lambda}\,\nu_\tau(\tilde{\lambda}), \qquad (157)$$

where we recall that $\nu_\tau(\tilde{\lambda})$ is determined from solving (148) and (146) and $\tilde{\lambda}_f$ from (156). Using the above results one finds the following:

(i) *In the (single-line localized) phase $\tau < \tau_c$, i.e., $\theta > \theta_c$,* one has, to leading order for small $r = O(1)$, from (152),

$$\tilde{\lambda}_e - \tilde{\lambda}_f \simeq a_k \left(1 - \frac{\theta_c}{\theta} \right) \times r^{\frac{1}{2k+\frac{3}{2}}}, \qquad (158)$$

with $a_k = \left(\frac{2k+\frac{3}{2}}{A} \right)^{\frac{1}{2k+\frac{3}{2}}}$. This leads to the small $r = O(1)$ expansion of the ground state energy

$$-\mathcal{E}^0_{N,M}(x) \simeq \theta N^2 \left[\lambda_e r - b_k \left(1 - \frac{\theta_c}{\theta} \right) \times r^{\frac{2k+\frac{5}{2}}{2k+\frac{3}{2}}} \right], \quad (159)$$

with $b_k = \frac{1}{2k+\frac{5}{2}} (2k+\frac{3}{2})^{\frac{2k+\frac{5}{2}}{2k+\frac{3}{2}}} A^{-\frac{1}{2k+\frac{3}{2}}}$. The first term linear in r corresponds to independent lines, and the second, singular term arises from the nonintersection constraint (i.e., the interactions between the lines).

(ii) *At criticality $\tau = \tau_c = \frac{1}{\theta_c}$,* one finds, from (154),

$$\tilde{\lambda}_e - \tilde{\lambda}_f \simeq \left(\frac{3}{2} \pi \sqrt{g} \right)^{\frac{2}{3}} \frac{1}{\theta_c} \times r^{2/3}, \qquad (160)$$

leading to the small $r = O(1)$ expansion of the ground state energy at the critical point

$$-\mathcal{E}^0_{N,M}(x) \simeq \theta N^2 \left[\lambda_e r - \frac{2}{5} \left(\frac{3}{2} \right)^{\frac{5}{3}} (\pi \sqrt{g})^{\frac{2}{3}} \frac{1}{\theta_c} \times r^{5/3} \right]. \quad (161)$$

We have not attempted to study the crossover near criticality between (159) and (161).

(iii) *Inside the (single-line delocalized) phase $\tau > \tau_c$, i.e., $\theta < \theta_c$,* the density of states vanishes as a semicircle and again one has $\lambda_e - \lambda_f \sim r^{2/3}$ and $\mathcal{E}^0_{N,M} + \theta N^2 \lambda_e r \propto N^2 r^{5/3}$. In the limit $\tau \gg 1$, i.e., $\theta \ll 1$, the density converges to the semicircle, $\nu_\tau(y) \simeq \nu_{sc,\tau}(y) = \frac{1}{2\pi\tau} \sqrt{4\tau - y^2}$, which leads to $\tilde{\lambda}_e \simeq 2\sqrt{\tau}$ and

$$\tilde{\lambda}_e - \tilde{\lambda}_f \simeq \left(\frac{3\pi}{2} \right)^{\frac{2}{3}} \frac{1}{\sqrt{\theta}} r^{2/3},$$

$$-\mathcal{E}^0_{N,M}(x) \simeq N^2 \left[2\sqrt{\theta} r - \frac{2}{5\pi} \left(\frac{3\pi}{2} \right)^{\frac{5}{3}} \theta^{3/2} \times r^{5/3} \right]. \quad (162)$$

Note that from Eqs. (148), (146), (156), and (157) one could in principle access, using (12), the total occupation lengths L_i in the ground state. We have not attempted that calculation.

Another case of great interest is when there are two families of columnar defects such that the support of $\rho(a)$ consists of two intervals, separated by a gap. It is known in the context of the deformed GUE random matrix $\mathcal{M}(x)$ that there is usually a critical value $\tau_c = 1/\theta_c$ at which the gap of $\nu_\tau(\tilde{\lambda})$ closes, and such that the two half supports merge for $\tau > \tau_c$, i.e., $\theta < \theta_c$. The behavior around that point is quite nontrivial; for a recent review see [129]. In Refs. [123] and [51] there are results concerning the case where the support of $\rho(a)$ has an interior singular point a^* where the density vanishes. If $\rho(a)$

vanishes as $|a - a^*|^\kappa$ with $\kappa > 1$, this singular point survives for $\tau < \tau_c$, while if $\kappa < 1$ it immediately disappears. These critical phenomena can be explored in the present problem, by varying the filling $r = M/N$ and the tilt angle of the lines [we recall that at large N, $\theta \simeq x/N$, and $\theta_M = x/(N-M) \simeq \theta/(1-r)$]. One needs to vary r near the critical filling where one of the half supports is fully occupied and the other empty. It would be of great interest to see whether it gives some description of the Mott insulator phase predicted in [7], in the presence of point disorder and upon tilting the lines near the critical transverse field. Note that the stability of a band insulator for the two-way model in the presence of columnar disorder was studied in [32].

D. The Rosenzweig-Porter model and fractal delocalization of interacting lines

1. The generalized Rosenzweig-Porter model

The problem studied here is closely related to the so-called generalized Rosenzweig-Porter (GRP) model recently studied in physics [130–133] and mathematics [71,134–136]. The GRP model is a cousin of the Anderson models on the Bethe lattice and on the random regular graph, themselves studied [137] as simpler settings for investigating the many-body localization transition [138]. In particular the GRP allows us to investigate the existence of a nonergodic delocalized phase, or bad metal, predicted in this context [137,138]. These phases are also of great interest for glassy quantum dynamics in models such as the quantum random energy model [139], with applications to quantum computing [140,141].

The GRP model studied in [71,130,131,134–136] is defined by the deformed GUE matrix

$$H = \text{diag}(a_1, \ldots, a_N) + \frac{\sqrt{t}}{N^{\gamma/2}} V, \qquad (163)$$

where the GUE matrix V has the same distribution as in (10). The a_i are i.i.d. random variables drawn from a distribution $\rho(a)$ with a compact support. The connection with (10) and (143) is thus

$$H = \tilde{M}(x) = \frac{\mathcal{M}(x)}{x}, \quad x = \frac{N^\gamma}{t}, \quad \tau = t N^{1-\gamma}. \quad (164)$$

The GRP model is studied for $N \to +\infty$ at fixed t and $\rho(a)$, equivalently for $\tau = O(N^{1-\gamma})$. Note that some works consider instead V as drawn from the Gaussian orthogonal ensemble (GOE), but there are no important differences in the main features discussed below.

The case $\gamma = 1$ is thus the same as studied until now in this paper, with the correspondence $t = \tau = 1/\theta$; see also [76,142–144]. As we discussed above, at large N the mean eigenvalue density of H, ν_τ, interpolates from $\rho(a)$ at small τ to a semicircle at large τ, as described by (146) and the self-consistent equation (148) or the Burgers equation (147). If $\rho(a)$ vanishes fast enough near its upper edge, it retains its shape for $\tau < \tau_c$ and exhibits a transition to a semicircle shape at $\tau = \tau_c$, i.e., $\theta = \theta_c$.

It is thus not surprising that for $\gamma < 1$ at large N, which corresponds to $\tau \gg 1$, the spectrum of H is a semicircle, while for $\gamma > 1$, i.e., $\tau \ll 1$, is it exactly $\rho(a)$ [131,134–136]. However the transition in the local spectral correlations of H

between the Wigner-Dyson statistics and the Poisson statistics takes place at a different value, $\gamma = 2$. If $\gamma > 2$, i.e. $\tau \ll 1/N$, the local level statistics falls into the Poisson class. On the other hand, if $\gamma < 2$, i.e., $\tau \gg 1/N$, the local level statistics falls into the Wigner-Dyson class [136,145,146].

The most interesting case is $1 < \gamma < 2$, i.e., $1/N \ll \tau \ll 1$ (see Fig. 10). Although the mean density of H is $\rho(a)$, the local level statistics is Wigner-Dyson. It was conjectured in [130] that the eigenvectors are delocalized, but only in $N^{2-\gamma}$ sites close in energy, leading to a "fractal dimension" for the eigenstates. The mass of each eigenfunction was predicted to spread to a large number of sites, which nevertheless form a vanishing fraction of the entire volume, the "sites" $\{1, \ldots, N\}$. This phase was called a nonergodic delocalized phase. It was then proved in [134] (for GOE matrices, T there being τ here) that each normalized eigenfunction ψ_α delocalizes across a set of approximately $N\tau \sim N^{2-\gamma} \gg 1$ sites for which a_i is closest to λ. More precisely these sets are such that $|\tilde{\lambda}_\alpha - a_i|$ is of order $N^{1-\gamma}$; hence they contain $\sim \tau N \sim N^{2-\gamma}$ sites, on which $|\psi_\alpha(x)|^2 \leqslant N^{\gamma-2} = (N\tau)^{-1}$. Hence they are maximally delocalized on these sites. In [131], DBM and perturbative arguments were used to explain why the abrupt transition in the local statistics does not contradict the gradual transition in the degree of eigenfunction localization, by arguing that the statistics retain a Poissonian character on mesoscopic scales greater than τ. Other results concerning the eigenvalue statistics can be found in [147,148] and near the edge in [129,149]. Note that as $\gamma \to 2$, and for $\gamma > 2$, the eigenstates become localized on one site, while as $\gamma \to 1$ they become fully delocalized over the N sites.

Some of these properties can be understood using the DBM [131,134–136]. As recalled in Appendix B, from (B5) the eigenvalues $\tilde{\lambda}_i$ of H expressed as functions of τ satisfy the standard $\beta = 2$ DBM,

$$d\tilde{\lambda}_i = \frac{1}{N} \sum_{j \neq i} \frac{d\tau}{\tilde{\lambda}_i - \tilde{\lambda}_j} + \frac{1}{\sqrt{N}} d\tilde{b}_i(\tau) \qquad (165)$$

$$= \frac{1}{N^\gamma} \sum_{j \neq i} \frac{dt}{\tilde{\lambda}_i - \tilde{\lambda}_j} + \frac{1}{N^{\gamma/2}} db_i(t), \qquad (166)$$

with initial condition $\tilde{\lambda}_i(0) = a_i$, and b_i and $\tilde{b}_i$ are i.i.d. Brownian motions. The DBM expressed in the variable t thus has parameters $a = N^{-\gamma}$ and $b = N^{-\gamma/2}$.

The resolvant expressed as a function of t, $\hat{G}_t(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \tilde{\lambda}_i}$ obeys

$$\partial_t \hat{G}_t(z) = -\frac{1}{N^{\gamma-1}} \hat{G}_t(z) \partial_z \hat{G}_t(z) - \frac{1}{N^{\frac{1+\gamma}{2}}} \partial_z \eta_{z,t} \qquad (167)$$

with $\mathrm{Cov}(\eta_{z,t}\eta_{z,t}) = -\delta(t-t') \frac{\hat{G}_t(z) - \hat{G}_t(z')}{\hat{z} - \hat{z}'}$. Note the misprint in the amplitude of the noise in Eq. (18) in [131]. The resolvant $G_\tau(z) = \hat{G}_t(z)$ as a function of τ also obeys (167), but with $\gamma = 1$.

From the DBM equation (165) it is clear that at very short times τ the eigenvalues $\tilde{\lambda}_i$ remain close to their starting points a_i. As long as they have not moved by more than the typical interparticle distance $\sim 1/N$ [since $\rho(a)$ has a compact support of order unity] they simply perform independent diffusion $\tilde{\lambda}_i - a_i \simeq (\tau/N)^{1/2}$. Equating both scales one sees
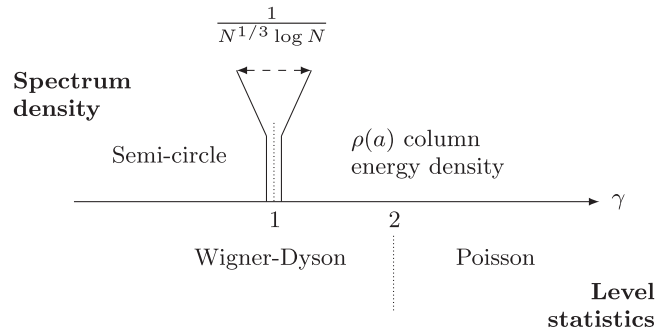

FIG. 10. Sketch of the different phases of the generalized Rosensweig-Porter (GRP) model in terms of the spectral density and the level statistics in the bulk. There are two crossovers: (i) at $\gamma = 1$ from a delocalized phase to a delocalized nonergodic phase where the order parameter is the spectrum density, and (ii) at $\gamma = 2$ from the delocalized nonergodic phase to a standard localized phase where the order parameter is the local level statistics. The critical region associated with the phase transition studied in Sec. III C 3, which occurs at the edge of the spectrum near $\gamma = 1 + O(1/N^{1/3} \ln N)$, is also indicated schematically.

that it corresponds to $\tau = O(1/N)$, i.e., $\gamma = 2$, and it is thus natural to expect local Poisson statistics below that time and Wigner Dyson above, when the neighboring particles start interacting to avoid collisions. Then it takes a much longer time $\tau = O(1)$ to reach a steady state at the global scale, and for the density to change from $\rho(a)$ to the semicircle, which corresponds to the transition $\gamma = 1$. More detailed arguments using eigenvectors are required to understand the nature of these regimes [71,131,134–136].

Note that the above analysis deals with eigenstates in the bulk. Very near the upper edge of the spectrum of H it may be a bit different. Indeed, one cannot find $N^{2-\gamma}$ columns too close to the edge, more precisely when $a > a_e - \delta a$ where $N^{2-\gamma} \simeq N \int_{a_e-\delta a}^{a_e} \rho(a) da \simeq N(\delta a)^{2k+3/2}$. Hence the eigenstates near the edge should localize on fewer columns.

Remark. For $\gamma = 1$ we studied in Sec. III C 3 the transition near the edge (for $M = 1$, i.e., for the largest eigenvalue), at $\theta = \theta_c$. From (118) the critical region is defined by the scaling variable $u = N^{1/3}(\theta - \theta_c) = O(1)$ (denoted by the letter u to avoid confusion). We note that this critical region can also be explored in the GRP model, at fixed $t = 1/\theta_c$, if one chooses, as N becomes large,

$$\gamma = 1 + \frac{u}{\theta_c} \frac{1}{N^{1/3} \ln N}, \qquad (168)$$

which is equivalent to $\theta = \frac{N^{\gamma-1}}{t_c} \simeq \theta_c + u N^{-1/3}$.

2. From the eigenvectors to the polymers

Let us go back to the polymer picture, using the relation (136) between the ground state energies of M polymers or lines and the M largest eigenvalues of H. The GRP model corresponds to polymers with $x = N^\gamma/t = O(N^\gamma)$, equivalently an "angle" variable $\tan \phi = \frac{1}{\theta} = \frac{N}{x} = \tau = tN^{1-\gamma}$ at fixed t and N large. Thus $\gamma > 1$ corresponds to very long polymers, i.e., very small angle ϕ, which tend to be more localized along the columns, while $\gamma < 1$ corresponds to large angles which lead to delocalization.

One should distinguish between the localization of the eigenstates ψ_α, $\alpha = 1, \ldots, N$, of H, which states how the normalized measure $|\psi_\alpha(j)|^2$ tends to concentrate on a few sites (i.e., columns) j, and the localization of the M polymers on the columns, measured by the total occupation length $L_j^0 = \sum_{k=1}^M \ell_j^{k,0}$ of each column j in the ground state. There are however connections between the two. Indeed a perturbation theory argument as in (13) shows that each eigenvalue $\tilde{\lambda}_\alpha$ of H (here ordered as a decreasing sequence) obeys

$$\partial_{a_j} \tilde{\lambda}_\alpha = |\psi_\alpha(j)|^2 \tag{169}$$

for each realization of the matrix V (and any given set of $\{a_i\}$). We can now combine this with (138) and (136) and obtain the relation between the averages over respectively the Brownian point disorder (indicated by B) and the GUE matrix V

$$\overline{L_j^0}^B = \sum_{k=1}^M \overline{\ell_j^{k,0}}^B = x \sum_{\alpha=1}^M \overline{|\psi_\alpha(j)|^2}^V, \tag{170}$$

which generalizes (14). It is valid for any j and for any given set of $\{a_i\}$. We recall that ℓ_j^k is the occupation length along column j of the polymer starting at column $i = k$ and ending at column $i' = N - M + k$. Equation (170) implies that the change in the mean occupation length in the ground state, when adding one polymer, is given by $\overline{L_j^0}^B|_M - \overline{L_j^0}^B|_{M-1} = x\overline{|\psi_M(j)|^2}^V$. It does not imply however a relation between $\overline{\ell_j^{k,0}}^B$ and $\overline{|\psi_k(j)|^2}^V$ because the joint distributions of the $\overline{\ell_j^{k,0}}^B$ also depend on the number of polymers M.

For a single polymer $M = 1$ we already know from Sec. III A that for $\gamma = 1$ there is a localization transition of the polymer at $\theta = \theta_c$ [if $\rho(a)$ vanishes fast enough at a_e]. The polymer visits all columns with the occupation length measure $\ell(a)da$ given in (94), which for $\theta > \theta_c$ develops a delta peak on the few best columns at the edge $a = a_e$. From (170), i.e., (14), it indicates that ψ_1, the eigenstate of H with the largest eigenvalue, is localized on one or a few $O(N^0)$ sites or columns for $\theta > \theta_c$, and delocalized for $\theta < \theta_c$. As one increases M the eigenstates of highest eigenvalues are successively "filled." Presumably for finite $M = O(1)$ the picture is similar to $M = 1$ with localized eigenstates near the edge.

When M increases such that $M/N = r = O(1)$, the above results about the eigenvectors in the bulk of the GRP model at large N apply. Let us give some qualitative arguments. For $\gamma < 1$, i.e., $\theta \ll 1$, the eigenvectors are fully delocalized so that typically $|\psi_\alpha(j)|^2 = O(1/N)$. From (170) one thus surmises that the L_j are all of order xM/N and the set of M polymers are delocalized. For $\gamma > 2$, i.e., $\theta \sim N^{\gamma-1} \gg 1$, the eigenvectors ψ_α are typically localized on a single site or column, say j_α. From (170) one expects that the $L_{j_\alpha} = O(x)$ for $\alpha = 1, \ldots, M$, and that the set of M polymers are well localized on the best columns. For $1 < \gamma < 2$, i.e., the nonergodic phase, $1 \ll \theta \ll N$, adding an extra polymer leads to a reorganization of the total occupation lengths L_j given by $\overline{|\psi_M(j)|^2}^V$; i.e., this excitation is localized on $N^{2-\gamma}$ sites around $|\lambda_M - a_j| \sim N^{1-\gamma}$. Hence it is reasonable to expect that the set of lines are again localized, but on a macroscopic set of columns. The above heuristic arguments assume that

the typical and average $|\psi_\alpha(j)|^2$ behave in a similar way, and it would be interesting to make them more rigorous.

In conclusion, when the number of polymers is $M = O(N)$, we can learn from the GRP model in the bulk. We see that because of the (noncrossing) interactions the set of polymers tends to be more easily delocalized. It seems that the polymer localization transition which, as we found, occurs for a single (or a few) polymers for $x \sim N$, i.e., $\theta = \frac{x}{N} = \theta_c = O(1)$, now requires much longer polymers. The nature of the transition changes significantly. For $\tan\phi = O(1)$ the set of $M = O(N)$ polymers are delocalized. For $\tan\phi \ll 1$ they start to localize, but the polymer localization occurs gradually. For $N \ll x \sim N^\gamma \ll N^2$, one can say that the set of $M = O(N)$ polymers are localized but in the weaker sense, related to the nonergodic phase of the GRP, and excitations are delocalized on subsets of $N^{1-\gamma}$ sites. One then expects to recover the linear response when varying H or $\tan\phi$ on scales $N^{1-\gamma}$. For $\gamma \geqslant 2$, i.e., $\tan\phi \lesssim 1/N$, they are fully localized.

V. SOME KNOWN RESULTS AT FINITE TEMPERATURE

We finish this paper by mentioning a few available results at finite temperature for the single polymer OY model $M = 1$. The free energy is $\mathcal{F}_N(x, T) = -T \ln \mathcal{Z}_N(x, T)$ where $\mathcal{Z}_N(x, T)$ is defined in (6). First let us note that the rescaling $x \to xT^2$ and $a_j \to a_j T$ in the partition function (6) leads to the relation

$$\mathcal{F}_N(x, T, \{a\}) = T\mathcal{F}_N(x/T^2, 1, \{aT\}) - 2(N-1)T \ln T. \tag{171}$$

Exact results for the free energy at $T > 0$ and its fluctuations were obtained in [119,150,151]. Let us discuss now the results of [120,152] in the presence of drifts a_j which correspond to columnar defects. They reproduce the ones discussed in Secs. I B 2 and II in the $T \to 0$ limit.

Consider a single active columnar defect $a_1 > 0$ and $a_{j>1} = 0$ and ask about the localization transition. We again consider the limit $x, N \to +\infty$ with $\theta = x/N$ fixed. Let us denote t^* the unique positive solution of

$$\psi'(t^*) = \frac{x}{T^2 N} = \frac{\theta}{T^2}, \tag{172}$$

where $\psi(t) = \frac{d}{dt} \ln \Gamma(t)$ is the digamma function, and denote $c = [-\frac{1}{2}\psi''(t^*)]^{1/3}$ a temperature dependent constant. Then, it was shown in [152] that in the region where

$$a_1 T < t^*, \tag{173}$$

the following asymptotics holds at large N for the free energy:

$$\mathcal{F}_N(x, T, a_1) \simeq -TN\left[\frac{\theta}{T^2}t^* - \psi(t^*) + 2\ln T\right] - cTN^{1/3}\chi_2. \tag{174}$$

The region delimited by (173) thus corresponds to the delocalized phase, where the free energy exhibits Tracy-Widom fluctuations. When one approaches the boundary of this phase, more precisely in the critical region near the transition defined by [153]

$$a_1 T = t^* + \frac{b_1}{c}N^{-1/3}, \tag{175}$$

with a fixed value of b_1, the following asymptotics for the free energy holds [152]:

$$\mathcal{F}_N(x, T, a_1) \simeq -TN\left[\frac{\theta}{T^2}t^* - \psi(t^*) + 2\ln T\right]$$
$$- cTN^{1/3}\chi_{\text{BBP},b_1}, \qquad (176)$$

where χ_{BBP,b_1} is the BBP random variable, distributed according to the BBP(b_1) distribution (of CDF $F_1(x; b_1)$ in [[88], Definition 1.3]) which we recalled in (68) when we studied $T = 0$. Hence we see that the critical behavior of the delocalization or localization transition at finite temperature is the same as for $T = 0$.

One recovers the results for $T \to 0$ as follows. Using the small-t behavior $\psi(t) \simeq -\frac{1}{t}$, we obtain that as $T \to 0$ for a fixed θ one has

$$t^* \simeq T/\sqrt{\theta}. \qquad (177)$$

In the delocalized phase $\theta < \theta_c = 1/a_1^2$, this leads to

$$\mathcal{F}_N(x, T, a_1) \simeq -2N\sqrt{\theta} - N^{1/3}\sqrt{\theta}\chi_2 \qquad (178)$$

recovering (57), equivalently (15). The critical region is defined by taking the limit $T \to 0$ of (175), using $c \simeq 1/t^* \simeq 1/(a_1T)$ which leads to $a_1 = \frac{1}{\sqrt{\theta}} + a_1b_1N^{-1/3}$, equivalent to the $T = 0$ definitions in (25) and (60) (with $\delta = 2b_1 = \tau a_1^2$). In that region one obtains

$$\mathcal{F}_N(x, T, a_1) \simeq -2N\sqrt{\theta} - N^{1/3}\sqrt{\theta}\chi_{\text{BBP},b_1}, \qquad (179)$$

which recovers the $T = 0$ result (67).

Although there are no results for the localized phase, the methods in [152] can be used to investigate this phase as well. In the case where there is a finite number k of columns with strictly negative energies $\epsilon_j = -a_j$ within the critical region, the above result generalizes with the replacement [152] with the rank-k BBP distribution, $\chi_{\text{BBP},\mathbf{b}}$; see Appendix D.

At finite temperature for a single line there is another interesting regime: in a different scaling limit, $x \sim \sqrt{N}$, one obtains the continuum directed polymer, i.e., the model in (1), equivalent to the KPZ equation, as (in the absence of drifts)

$$\mathcal{Z}_N(\sqrt{\tau N} + \tilde{x}, T = 1) \sim Z^{\text{KPZ}}(\tau, \tilde{x}). \qquad (180)$$

The case of many noncrossing lines at finite temperature can also be investigated. One interesting question is whether there is phase transition as a function of temperature.

VI. CONCLUSION

In this paper we have analyzed results originating from random matrix theory in the context of the localization to delocalisation of polymers and lines in $d = 1 + 1$ dimension in the presence of both columnar and point disorder. The central solvable model that we revisited is the O'Connell-Yor (OY) polymer for either one line or $M > 1$ lines, in the presence of N columns of arbitrary strengths a_j. At zero temperature this model has a direct connection to the deformed GUE random matrices and to the Dyson Brownian motion. An immediate consequence, that we have explored in detail, is that for one line and one column there is a localization to delocalisation phase transition in the universality class of the BBP spiked

random matrix transition. In the localized phase the occupation length of the column by the line is macroscopic. We have developed a variational calculation to calculate the occupation length in the localized phase and across the transition, which led to new results for its fluctuations. We found that these fluctuations are of order $N^{2/3}$ and described in the localized phase by the distribution f_{KPZ}, which appears ubiquitously in the KPZ class.

We have recalled and then extended classical methods using Fredholm determinants, which allowed us to solve the case of one line and many columns with a continuous distribution of column strengths $\rho(a)$. We have found that if $\rho(a)$ vanishes sufficiently fast near its upper edge, there is a localization to delocalisation transition. We have shown that this transition belongs to a new universality class (hence the same applies for spiked random matrices with a full rank perturbation). We have shown that the fluctuations of the ground state energy in the localized phase are nontrivial, at variance with the usual BBP case where they are Gaussian. We have obtained the universal distribution for these fluctuations and those in the critical region. It is expressed in terms of a Fredholm determinant involving a new one-parameter kernel. This kernel is reminiscent of the one appearing in the elliptic Ginibre ensemble.

In the case of many nonintersecting lines with specific (packed) boundary conditions, we could use some known results about the OY model. The case of a few active columns and a few lines was discussed and leads to superpositions of BBP type transitions. In the case of a thermodynamic number of lines $r = M/N > 0$, we calculated the ground state energy. It exhibits a change of behavior in the very dilute limit $r \to 0$, but no true phase transition at fixed angle $\theta = x/N$ and $r > 0$. As we argued, based on known results on the generalized Rosenzweig-Porter (GRP) model, full polymer localization on the columns occurs for long polymers, $x \sim N^\gamma$ with $\gamma \geqslant 2$, and only partial localization occurs for $1 \leqslant \gamma < 2$. The latter case is related to the celebrated "nonergodic" delocalized phase of the GRP model. Translated to the polymer side of the model, it suggests that the polymers delocalize over subsets of $\sim N^{2-\gamma}$ columns, retaining some glassy features.

Many other interesting questions remain such as the crossover from finite-rank to infinite-rank perturbation (in the RMT context), the fluctuations of the ground state energy as a function of the column strengths a_j (we have worked here for a fixed set of a_j). We have also unveiled transitions with an anomalous critical behavior quite different from the Tracy-Widom or Airy family, which we were not able to analyze. Although our discussion focused on zero temperature we analyzed the existing results at $T > 0$ in the case of one line and one (or a few) columns. The conclusion there was that temperature does not change the universality class of the $T = 0$ localization to delocalisation transition. It would be interesting to generalize the present study with many columns, and possibly many lines, in the case of finite temperature, where the thermal effects are expected to be more subtle.

In a broader context, the results in this paper concern a one-way model of a polymer or line (i.e., which can jump only to the right) in $d = 1 + 1$. One expects on heuristic grounds that this model could serve as some kind of approximation of the more general two-way model [such as (2) in the

introduction, or the continuum model (1)] near its delocalization (i.e., tilting) transition (e.g., in the presence of an applied field) since left jumps may be subdominant there. Such one-way models have been investigated in the context of non-Hermitian localization but not in the presence of point disorder. In Appendix A we have presented a first step in that direction, by studying the OY model in the fixed transverse field ensemble H (rather than the fixed angle ensemble) where the tilt angle can fluctuate. We have presented some connections to the models of elastic lines discussed in the introduction which exhibit a tilt angle transition at $H = H_c$ and the transverse Meissner effect. Although more details remain to be understood, it appears, roughly, that for a single line the localization to delocalisation transition which occurs at $\tan \phi_c = 1/\theta_c$ in the fixed angle ensemble is naturally associated with a first order jump in the tilt response at $H = H_c$, and that the localized phase discussed here for $\phi < \phi_c$, i.e., $\theta > \theta_c$, can be seen as a coexistence region. In the many line case, the fact that the polymer localization occurs gradually on scales $x \sim N^\gamma$ as γ varies within $1 \leqslant \gamma \leqslant 2$ suggests that the Bose-glass transition at H_c is (i) continuous since for any fixed $\tan \phi = N/x$ the system is delocalized, and has (ii) nontrivial features for vanishingly small angle scales $\phi \sim N^{1-\gamma}$. It remains an outstanding question whether some universal property of this transition in the presence of point disorder (for one or many lines) may be captured by the present model, for which many analytical results have been obtained.

of Bristol and University College Dublin mathematics departments for hospitality, where part of this work was completed.

APPENDIX A: FIXED H ENSEMBLE AND MODEL OF VORTEX LINES

In this Appendix we discuss a possible realization of the polymer model of the text in the context of vortex lines. We focus on a single line. We first recall in an elementary way the tilting transition (transverse Meissner effect) for an elastic line, upon an applied field H. Then we apply the results obtained for the one-way model to investigate the tilting transition and the localization to delocalisation transition in the presence of columnar and point disorder.

1. Single columnar defect

Let us consider the model (1) for a single continuum vortex line (without point disorder $V = 0$) and a single columnar defect modeled by a local potential well $U(u)$ centered at $u = 0$ of depth $-U_0$ as in Fig. 11 (left). The line enters at $u(0) = 0$ (which is fixed) and exits at $u(x) = u_f > 0$. The tilt, i.e., the mean angle with the z axis, is ϕ with $t = \tan \phi = u_f/x$. Since the line consists of a pinned segment of length ℓ and a depinned segment of length $x - \ell$ the ground state energy is approximately

$$\mathcal{E}^0 \simeq -U_0 x + \min_{0 \leqslant \ell \leqslant x} \left[U_0(x - \ell) + \frac{\gamma u_f^2}{2(x - \ell)} - H u_f \right].$$

$$(A1)$$

Consider first the fixed angle ϕ ensemble discussed in this paper (setting $H = 0$). Minimizing (A1) over ℓ, one finds that the depinned fraction of the line is $1 - \frac{\ell^0}{x} = \sqrt{\gamma/(2U_0)} \tan \phi$. This solution is valid for $\phi < \phi_c$ with $\tan \phi_c = \sqrt{2U_0/\gamma}$. For $\phi \geqslant \phi_c$ the line is fully depinned with $\ell^0 = 0$, and $\mathcal{E}^0 = \gamma u_f^2/(2x)$. The first solution is an analog of the localized phase
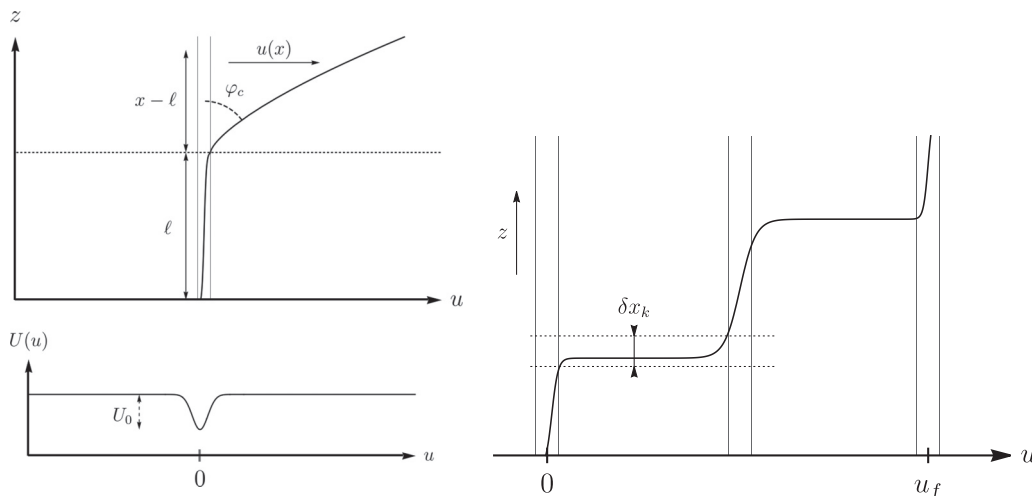

FIG. 11. Left: Elastic line of length x, ending in $u(x) = u_f$, in the presence of a columnar defect studied in (A1). At $T = 0$, for a fixed tilt angle ϕ with the z axis, the line is partially depinned when $\tan \phi = u_f/x < \tan \phi_c$. This is the localized phase (analog to the one discussed in the text for the OY model) where the pinned and depinned segments coexist. The tilt angle of the depinned segment is also ϕ_c. Right: Cartoon of a series of kinks of width δx_k and energy E_k, for an elastic line with a positive tilt angle in the periodic or disordered columnar model which mimics the OY model, as studied in (A5).
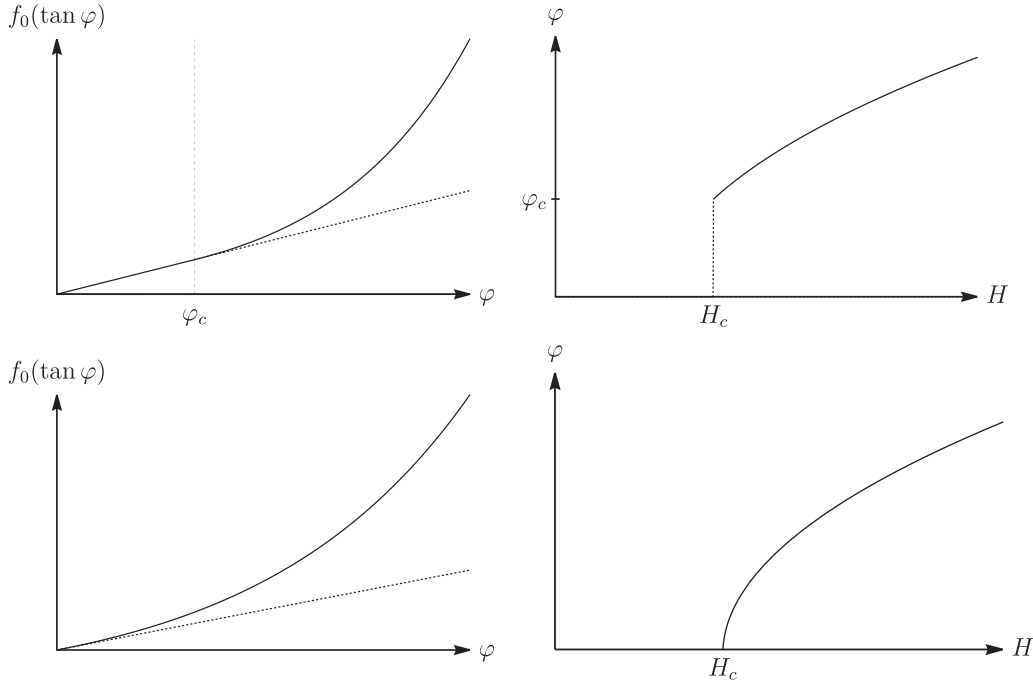
FIG. 12. Left: Free energy $f_0(\tan \phi)$ as a function of the tilt angle ϕ with the z axis, in two cases: (i) Top: It is linear along a segment $[0, \phi_c]$, as in (A2). (ii) Bottom: It is slightly curved. The slope at the origin is H_c indicated in dashed. Right: Corresponding tilt angle versus H curve in the fixed H ensemble. (i) Top: The tilt exhibits a first order jump at H_c, from $\phi = 0$ to $\phi = \phi_c$. (ii) Bottom: The tilt raises sharply at H_c but continuously, and $\phi_c = 0$; see Eq. (A14).

as defined in the text, and the second of the delocalized phase (although for quite different models, and there is no point disorder here). The energy at fixed angle is

$$
\frac{\mathcal{E}^0}{x} = f_0(\tan \phi)
$$
$$
\simeq \begin{cases} -U_0 + \sqrt{2\gamma U_0} \tan \phi, & \tan \phi < \sqrt{2U_0/\gamma}, \\ \frac{\gamma}{2}(\tan \phi)^2, & \tan \phi > \sqrt{2U_0/\gamma}. \end{cases} \quad \text{(A2)}
$$

In the fixed H ensemble the energy is given by minimizing $\min_{\phi \geqslant 0}[f_0(\tan \phi) - H \tan \phi]$ over $\tan \phi$, i.e., by the Legendre transform of f_0. It gives here

$$
\tan \phi = \begin{cases} 0, & H < H_c = \sqrt{2U_0/\gamma}, \\ \tan \phi_c + \frac{H - H_c}{\gamma} = \frac{H}{\gamma}, & H > H_c = \sqrt{2U_0/\gamma}. \end{cases} \quad \text{(A3)}
$$

Hence there is a tilting transition at H_c (and transverse Meissner effect for $H < H_c$), a generic feature in presence of columnar disorder. Since $f_0'(t)$ is generically increasing, it happens when $f_0'(0) > 0$, so that there is no solution to the minimization equation $H = f_0'(t)$ for $H < H_c = f_0'(0)$. However there is a second feature in the model (A1); i.e., the tilt angle ϕ jumps from 0 to a finite value ϕ_c at H_c. This first order jump of the tilt angle was also obtained in the finite temperature $T > 0$ version of the model (1), using the mapping to non-Hermitian quantum mechanics ($\hbar$ playing the role of T) [16,154]. In that language the localized state (around the column) disappears at $H \geqslant H_c$ and is replaced by a pair of eigenstates (with complex conjugated energies) with finite imaginary current (analog to the tilt). This was shown

for some solvable cases like the delta potential. In the above $T = 0$ estimate (A2), $f_0(t)$ is linear on an interval, leading to a jump in the Legendre transform. This calculation however can be made in a more exact manner (see Appendix A 4) and one finds that if $U(u)$ is smooth near its minimum $f_0(t)$ is not exactly linear and the jump is rounded into a very steep smooth curve with singularity [see, e.g., (A14) below]

$$
\tan \phi \sim \frac{\Theta(H - H_c)}{|\ln(H - H_c)|}; \quad \text{(A4)}
$$

i.e., ϕ_c vanishes. The (partially localized) optimal configuration at small fixed angle will still look like in Fig. 11 (left), the line being simply slightly shifted with respect to $u = 0$. These two cases (jump or no jump) are illustrated in Fig. 12.

Note that in the OY model studied in the text for a single active column, there is also a localized phase for $\theta > \theta_c$, i.e., $\phi < \phi_c$ (with some different, unrelated value for ϕ_c) where a finite fraction of the column is occupied. While in (A1) it is simply the (trivial) elastic energy of a free depinned segment which is in competition with the columnar energy, in the OY model the competition is with the point disorder energy. Although the competition has a different origin we note that in both cases the tilt angle of the segment which is depinned from the column is precisely ϕ_c.

(i) For (A1) it follows since at the optimum: $\frac{u_f}{x - \ell^0} = \sqrt{2U_0/\gamma} = \tan \phi_c$.

(ii) For the OY model it can be seen in (22): $\frac{N}{x - \ell^0} = 1/\theta_c = \tan \phi_c$.

Some of the above considerations extend to the many column cases. Jumps in the tilt angle were also observed for model (2) [with $\eta_j(t) = 0$]. As discussed in Sec. III B, in the

fixed tilt angle ensemble the signature of the localized phase in this case is that the occupation length density $\ell(a)$ acquires a delta function peak on the best columns. Below we will see an analog phenomenon for the model (1) at $T = 0$ in the absence of point disorder.

The OY model does not contain "free space" elastic energy. To connect to the elastic line model (1) one must introduce there a columnar potential $U(u)$ with a lattice structure. Elastic deformations occur as kinks, and one can then define an extension of the OY model in the fixed H ensemble (with fluctuating N) by adding the additional energy term $(E_k - H)r_0 N$. Let us describe it in more detail.

2. Periodic columnar potential

Consider now the model (1) in the presence of a periodic potential $U(u)$ of period r_0 which is nearly zero in between the columns and has a depth of order $U_{\min} = -U_0$ at each column location. We first assume the absence of point disorder $V(u, z) = 0$. Let us recall that ϕ such that $\tan\phi = \frac{(N-1)r_0}{x} \simeq \frac{Nr_0}{x} = 1/\theta$ is the angle of the line with the z axis. In the presence of a macroscopic tilt, i.e., $\phi, \theta > 0$, we want to approximate this model by the lattice polymer model of the text with all column energies identical $\epsilon_i = -U_0$, $L = x$, $u(0) = 0$, and $u(x) = u_f = (N-1)r_0$.

In the continuum model the jumps from column to column occur as kinks, whose shape is obtained by the minimization of the energy function (1) for a single jump, and depend on $U(u)$ and γ; see Fig. 11 (right). Its precise shape can be calculated (see Appendix A 4 below) but is not important here: if $U(u)$ has a single scale r_0, dimensional estimates show that the "kink energy" cost is $E_k \sim r_0\sqrt{\gamma U_0}$ and the "kink width" (i.e., its length along x) is $\delta x_k \sim r_0\sqrt{\gamma/U_0}$. The analogy with the OY lattice model is closest when the kink width is "small" so that the jump can be considered as almost instantaneous: it is much smaller than the mean interkink distance, whenever $\delta x_k \ll x/N$, i.e., $1/\theta \simeq \tan\phi \ll \sqrt{U_0/\gamma}$. The ground state energy per unit length for a fixed tilt angle is

$$\frac{\mathcal{E}^0}{x} \simeq f_0(\tan\phi) - H\tan\phi, \quad f_0(\tan\phi)$$

$$\simeq \begin{cases} -U_0 + \frac{E_k}{r_0}\tan\phi, & \tan\phi \ll \sqrt{\frac{U_0}{\gamma}}, \\ \frac{\gamma}{2}(\tan\phi)^2, & \tan\phi \gg \sqrt{\frac{U_0}{\gamma}}, \end{cases} \quad (A5)$$

where the convex function $f_0(t)$ can be calculated from Appendix A 4 below, but its asymptotics are easy to obtain, as indicated. In the first regime the minimum energy configuration is made of N well separated kinks, i.e., one has $N\delta x_k \ll x$, while in the second regime the line spends little time on the columns and one obtains the usual elastic energy [as in the absence of $U(u)$]. The general form of (A5) is quite similar to (A2), leading to a similar behavior in the fixed H ensemble [when $u(L)$ is free to move]. The minimization of (A5) over the angle leads to $\phi = 0$ for $H < H_c^0 = \frac{E_k}{r_0}$ and a tilting transition at H_c^0 (with transverse Meissner effect for $H < H_c^0$). For large $H - H_c^0$ one can use the asymptotics of f_0 and $\tan\phi \simeq \frac{H - H_c^0}{\gamma}$. The precise form of the curve of the optimal $\tan\phi$ versus $H - H_c^0$ is obtained by studying in more

detail the minimization equation $f_0'(\tan\phi) = H$. Depending on $U(u)$ it leads to either a jump in ϕ, or, for a smooth potential, a singularity of the form (A4) [see (A14) below].

Note that this "delocalization transition" for a periodic potential $U(u)$ in $d = 1$ exists only at $T = 0$. For any finite $T > 0$ the eigenstates in the associated quantum mechanics for $H = 0$ are Bloch waves; hence they are delocalized and there is no localized phase. For a localized phase to exist one needs quenched disorder in the column strength, i.e., a random potential $U(u)$; see below.

We now use the results of the present paper on the one-way OY model. Let us first continue with the above model of a periodic $U(u)$ and add point disorder. It is difficult to estimate when the effect of backward jumps can be neglected. As an attempt, consider the insertion of a kink-antikink pair (which costs an energy $2E_k$) of size (i.e., the separation) x_p between two forward kinks. The typical energy gain from the point disorder by forming this pair is $2\sigma\sqrt{x_p}$ [here σ measures the strength of the point disorder; i.e., the Brownian motions in the effective OY are now taken as $B_j(x) \to \sigma B_j(x)$]. Comparing the gain and the cost one sees that it becomes favorable to create such an excitation in the ground state if $x_p > x_p^* = E_k^2/\sigma^2$. If x_p^* is itself larger than the mean interkink distance x/N, then such defects may not be favored, which leads to the condition (i) $\tan\phi \gg \tan\phi_0 = r_0\sigma^2/E_k^2$. For $\phi \lesssim \phi_0$ the ground state is unstable to backward jumps [155].

Let us assume that the kink energy E_k is large, equivalently the point disorder is weak, so that $\phi_0 \ll 1$. In the presence of point disorder we must now minimize, approximately,

$$\frac{\mathcal{E}^0}{x} \simeq f_0(\tan\phi) - 2\frac{\sigma}{\sqrt{r_0}}\sqrt{\tan\phi} - H\tan\phi, \quad (A6)$$

where we used the result (15). Examining the minimization equation, $H = f_0'(t) - \frac{\sigma}{\sqrt{r_0 t}}$ with $t = \tan\phi$, we see that now there is no transition. In fact there is a solution, $\phi = \phi_0 > 0$ even at $H = 0$, where $\phi_0 \ll 1$ was defined above. This is not too surprising: the point disorder favors the wandering of the line and in $d = 1 + 1$ we expect that it overcomes at large scale the effect of the periodic potential (a bit as $T > 0$ does). In the OY model the only way to wander that is by tilting. The spontaneous tilt is an artifact of the one-way model, and $\phi \sim \phi_0 \ll 1$ is precisely the region where we cannot use it because in the two-way model kink-antikink pairs will proliferate [from condition (i) above]. However, we can use it for $\phi = O(1)$. From the minimization equation, we find that, in the whole region $H \lesssim H_c^0 = E_k/r_0$, $\tan\phi \approx \frac{\sigma^2}{r_0(H_c^0 - H)^2}$, and crossover for $H \gtrsim H_c^0$ to a tilt response nearly identical to the one in the absence of point disorder.

3. Columnar disorder on top of periodic

Let us now smoothly deform the above periodic potential so that the local minima remain at positions $u = jr_0$ but with a random value $U(jr_0) = U_0 - a_j$, introducing columnar disorder, so as to mimic the OY model with drifts. Let us consider the case of a single active column $a_1 > 0$, $a_{j\neq1} = 0$. From the results in Secs. I B 2 and II and restoring units, the energy to minimize is either given by (A6) if $\tan\phi > a_1^2 r_0/\sigma^2$ or, if

$\tan \phi < a_1^2 r_0/\sigma^2$, it is given by [from (17)]

$$\frac{\mathcal{E}^0}{x} \simeq f_0(\tan \phi) - a_1 - \frac{\sigma^2}{a_1 r_0} \tan \phi - H \tan \phi,$$

$$\tan \phi < a_1^2 r_0/\sigma^2. \qquad \text{(A7)}$$

The minimization equation is now $H = f_0'(t) - \min(\frac{\sigma^2}{a_1 r_0}, \frac{\sigma}{\sqrt{r_0 t}})$, with $t = \tan \phi$. Let us recall that from (A5) $f_0'(0) = E_k/r_0 = H_c^0$ and $f_0'(t) \simeq \gamma t$ for $t \gg \sqrt{U_0/\gamma}$. We see that now there is a threshold field

$$H_c = H_c^0 - \frac{\sigma^2}{a_1 r_0} = \frac{E_k}{r_0}\left(1 - \frac{\sigma^2}{a_1 E_k}\right), \qquad \text{(A8)}$$

provided $H_c > 0$, that is, $a_1 > \sigma^2/E_k$. For $H < H_c$ the minimum is at $\phi = 0$ and there is a true localized phase, with zero tilt, i.e., with transverse Meissner effect. For $H > H_c$ the tilt angle versus field curve can be obtained from the minimization equation. Since for small ϕ the disorder term in (A7) is linear in $\tan \phi$ we expect a similar behavior to that in the two previous subsections: either a jump from $\phi = 0$ to $\phi = \phi_c$, or, for a smooth potential, a sharp but slightly smoothed jump of amplitude ϕ_c, with $\tan \phi_c = a_1^2 r_0/\sigma^2$. In the fixed ϕ ensemble (which can handle both cases), for $\phi < \phi_c$ the line occupies a finite fraction $\ell^0/x = O(1)$ of the single active column (this fraction being unity for $\phi = 0$), and this fraction vanishes at the delocalization transition at $\phi = \phi_c$ [where ϕ_c is the value of $\phi = \phi(H)$ at $H = H_c$ in the fixed H ensemble]. These arguments are of course quite heuristic and it would be of great interest to analyze these observables in more detail.

Note that in the localized phase, the macroscopic segment of the line along the active column a_1 is stable to a large size single kink-antikink pair, since its energy cost is now $2E_k + a_1 x_p - \omega \sigma \sqrt{x_p}$, where ω a Gaussian random variable. These pairs are never favorable for large x_p and if $E_k \gg \sigma^2/a_1$ they are never favorable at any scale x_p [156].

4. Exact solution for an elastic line in a columnar potential at $T = 0$

Here we study a single elastic line, described by the model (1), in a general potential $U(u)$ at $T = 0$ and in the absence of point disorder, following closely the analysis in [157] and adding a few remarks and results useful in the present context. This method allows us to determine the function $f_0(t)$ introduced in the previous subsections. For fixed end points, the minimization equation

$$E_0[u_f, x, U, \gamma]$$
$$= \min_{u(z),u(0)=0,u(x)=u_f} \int_0^x du\left[\frac{\gamma}{2}\left(\frac{du(z)}{dz}\right)^2 + U(u(z))\right] \qquad \text{(A9)}$$

has the solution

$$E_0[u_f, x, U, \gamma] = \sqrt{\gamma} \int_0^{u_f} du \frac{\epsilon + 2U(u)}{\sqrt{2[\epsilon + U(u)]}}$$
$$= -\epsilon x + \sqrt{\gamma} \int_0^{u_f} du\sqrt{2[\epsilon + U(u)]}, \qquad \text{(A10)}$$

where ϵ is determined by the condition

$$\sqrt{\gamma} \int_0^{u_f} du \frac{1}{\sqrt{2[\epsilon + U(u)]}} = x, \qquad \text{(A11)}$$

and the optimal $u(z)$ is the solution of $\gamma \frac{d^2 u}{dz^2} = U'(u)$, i.e., of the classical mechanics problem in the inverted potential $-U(u)$, at total "energy" ϵ, thus determined by $\sqrt{\gamma} \int_0^{u(z)} du \frac{1}{\sqrt{2(\epsilon + U(u))}} = z$. In the fixed H ensemble, where u_f is free [we keep $u(0) = 0$], one minimizes $E_0 - Hu_f$ over u_f. This is equivalent to optimize over ϵ, which leads to $H = \sqrt{\gamma}\sqrt{2[\epsilon + U(u_f)]}$. There may be multiple solutions to this equation, and a more useful equation will be given below.

Periodic potential. Consider first the periodic potential $U(u)$ described above. Let us start with $u_f = r_0$ and $x \to +\infty$, which leads to the standard "kink energy" E_k for an ideal jump between 2 columns. In that case $\epsilon = -U_{\min} = U_0$ and $E_0 = -U_0 x + E_k$ with $E_k = \sqrt{\gamma} \int_0^{r_0} du\sqrt{2[U_0 + U(u)]}$. Let us consider now $u_f = (N-1)r_0 + \delta u_f$ where $0 < \delta u_f < r_0$. One can rewrite exactly (A10) and (A11) by replacing $\int_0^{u_f} du \to (N-1)\int_0^{r_0} du + \int_0^{\delta u_f} du$. We are interested in the limit where both x, N are large, with a fixed tilt angle $t = \tan \phi \simeq Nr_0/x$, for which one obtains

$$\frac{1}{t} \simeq \frac{\sqrt{\gamma}}{r_0} \int_0^{r_0} \frac{du}{\sqrt{2[\epsilon + U(u)]}},$$

$$\frac{E_0 - Hu_f}{x} \simeq -\epsilon + \frac{t\sqrt{\gamma}}{r_0} \int_0^{r_0} du\sqrt{2[\epsilon + U(u)]} - Ht. \quad \text{(A12)}$$

The minimization over $t = \tan \phi$ leads to

$$H = \frac{\sqrt{\gamma}}{r_0} \int_0^{r_0} du\sqrt{2[\epsilon + U(u)]}, \qquad \text{(A13)}$$

which is different from the result $H = \sqrt{\gamma}\sqrt{2[\epsilon + U(\delta u_f)]}$ obtained if one takes the derivative with respect to δu_f. This is because there are multiple extrema and one should minimize over both N (choose the column) and δu_f (fine structure inside one column). Equation (A13) corresponds to the former, and together with (A12) determines $t = \tan \phi$ as a function of H (the δu_f then adjusts to be at a local minimum). As $\tan \phi \to 0$ one has $\epsilon \to U_0$ and one recovers the small ϕ estimate for $f_0(t)$ in (A5). For large $\tan \phi$ one has $\epsilon \gg |U(u)|, U_0$ and one can neglect the potential $U(u)$ altogether, and one finds $E_0 \simeq \epsilon x$ with $\epsilon \simeq \frac{\gamma}{2}(\tan \phi)^2$ recovering the second estimate for $f_0(t)$ in (A5).

The Eq. (A13) determines ϵ as a function of H. Since $\epsilon \geqslant U_0 = -U_{\min}$, there are no solutions to (A13) for $H < H_c$ where $H_c = \frac{\sqrt{\gamma}}{r_0} \int_0^{r_0} du\sqrt{2[U_0 + U(u)]} = E_k/r_0$, a signature of the tilting transition at H_c. The precise behavior of the tilt angle (jump or no jump) depends on the exact form of the potential. If ϵ in (A12) is a constant equal to $-U_0$, the energy is exactly linear and a jump occurs. For a smooth potential $U(u) = -U_0 + U_2 u^2$ one obtains $r_0/\sqrt{\tilde{\epsilon}} = \sinh(\frac{r_0}{t}\sqrt{\frac{2U_2}{\gamma}})$, where $\tilde{\epsilon} = (\epsilon - U_0)/U_2$. Hence at small angle, $\tilde{\epsilon} \simeq 4r_0^2 \exp(-\frac{2r_0}{t}\sqrt{\frac{2U_2}{\gamma}})$. We can use the general relation $\frac{dH}{d\epsilon} = \frac{1}{t}$ from (A13) and (A12). Integrating we find

for $H > H_c$

$$H - H_c \simeq \frac{4\tilde{U}_2}{t} \exp\left(-\frac{2}{t}\sqrt{\frac{2\tilde{U}_2}{\gamma}}\right),$$

$$t = \tan\phi \simeq \frac{2\sqrt{2\tilde{U}_2/\gamma}}{\left|\ln\left(\frac{H-H_c}{4\tilde{U}_2}\right)\right|} \qquad (A14)$$

and $\phi = 0$ for $H < H_c$, where $\tilde{U}_2 = r_0^2 U_2$ is an energy scale (per unit length) characteristic of the local curvature of the well near its minimum. The jump in the tilt angle is smoothed on this scale.

Random potential. Consider now $U(u)$ a random potential. If ergodicity applies for large x and u_f at fixed angle $\theta = x/u_f = 1/t$ with $t = \tan\phi$, it means that we can replace the translational averages over the potential $U(u)$ by disorder averages noted $\langle\ldots\rangle_U$. To mimic the notations in the OY model, we can introduce the one-point density $\rho(a)$ of minus the potential $a = -U$, as $\rho(a) = \frac{1}{u_f}\int_0^{u_f} du\delta(U(u) + a) = \langle\delta(U(u) + a)\rangle_U$. We assume that this density has an upper edge $a_e = -U_{\min}$. The above equations then lead to the system of equations

$$\theta = \sqrt{\gamma}\int da\frac{\rho(a)}{\sqrt{2(\epsilon - a)}},$$

$$\frac{E_0}{x} = -\epsilon + \sqrt{\gamma}\frac{1}{\theta}\int da\rho(a)\sqrt{2(\epsilon - a)} - \frac{H}{\theta},$$

$$H = \sqrt{\gamma}\int da\rho(a)\sqrt{2(\epsilon - a)}, \qquad (A15)$$

where $\epsilon > a_e = -U_{\min}$ is determined from the first equation as a function of θ. The first two of these equations are reminiscent of the system (81) and (82) for the OY model at fixed tilt angle, with the identification $z^* = \epsilon$ and $\mu = \frac{E_0}{N}$. They are of course different since the models are different (here an elastic line without point disorder and there a discrete one-way model with point disorder). The last equation in (A15) leads to a tilting transition in the fixed H ensemble at $H_c = \sqrt{\gamma}\int da\rho(a)\sqrt{2(a_e - a)}$. For $H < H_c$ the tilt angle is zero, and one cannot use Eqs. (A15) (one cannot use the ergodicity). For $H \geqslant H_c$ the behavior of the tilt angle depends on how $\rho(a)$ vanishes near its upper edge a_e. For $\rho(a) \sim (a_e - a)^{2k+1}$ there is a finite jump from $\phi = 0$ to ϕ_c with $1/\tan\phi_c = \theta_c = \sqrt{\gamma}\int da\frac{\rho(a)}{\sqrt{2(a_e - a)}}$ which is finite for $k > -1/2$, and one has the critical behavior [157] $\theta - \theta_c \propto \phi - \phi_c \propto (H - H_c)^{\max(1, 2k+1)}$ in the delocalized phase. As $k \to 0$ the jump in ϕ vanishes and for $k = 0$ (which corresponds to the smooth potential) one finds again a behavior similar to (A14) with $\phi_c = 0$, $\theta_c = +\infty$.

Note that for $k > 0$, as in (94) and (93), one can define an occupation length measure which acquires a delta contribution for $\phi < \phi_c$,

$$\ell(a) = \begin{cases} N\frac{\rho(a)}{\sqrt{\epsilon - a}}, & \phi > \phi_c, \\ N\frac{\rho(a)}{\sqrt{a_e - a}} + (x - N\theta_c)\delta(a - a_e), & \phi < \phi_c, \end{cases} \qquad (A16)$$

so that $\int da\,\ell(a) = x$.

APPENDIX B: DYSON BROWNIAN MOTION AND AIRY PROCESS

1. DBM without drift and the Gaussian β ensemble

Consider $W(x)$ a Hermitian Brownian motion in x, or Brownian motion in the space of $N \times N$ Hermitian matrices. The stochastic evolution equation for the process of the eigenvalues $\lambda_i(x)$, $i = 1, \ldots, N$, of $W(x)$, i.e., the Dyson Brownian motion, reads

$$d\lambda_i(x) = a\sum_{j\neq i}\frac{dx}{\lambda_i(x) - \lambda_j(x)} + b\,db_i(x), \qquad (B1)$$

where $b_i(x)$ are N independent unit Brownian motions. This is the nonstationary DBM. We have introduced two arbitrary parameters a, b so that (B1) is actually the β-DBM; the case $\beta = 2$ relevant for Hermitian random matrices corresponds to $a = b^2$ (the general case is $\beta = 2a/b^2$; see below). The choice made here in the text is $a = b = 1$, which corresponds to a choice of normalization for $W(x)$. Here we define $W(0) = 0$, i.e., $\lambda_i(0) = 0$ (see below for nonzero initial condition). In the large-N limit, the density of eigenvalues normalized to unity corresponding to (B1) is a semicircle with the edges at $\pm 2\sqrt{Nax}$.

Let us also recall the stationary version of the DBM, or Orstein-Uhlenbeck version, which can be obtained from the nonstationary one via a Lamperti transformation. Defining $\Lambda_i(X) = \frac{\lambda_i(x)}{\sqrt{x}}$ and performing the "time change" $x = e^{cX}$ one finds that

$$d\Lambda_i(X) = -\frac{1}{2}c\Lambda_i(X)dX + ac\sum_{j\neq i}\frac{dX}{\Lambda_i(X) - \Lambda_j(X)} + b\sqrt{c}\,d\mathsf{b}_i(X), \qquad (B2)$$

where $\mathsf{b}_i(X)$ are are N independent unit Brownian motions in X. The stationary JPDF of the $\Lambda_i(X)$ is, for any N, the equilibrium measure

$$P_0(\Lambda) = \frac{1}{Z_N}\prod_{1\leqslant i<j\leqslant N}|\Lambda_i - \Lambda_j|^\beta e^{-\frac{1}{2b^2}\sum_i\Lambda_i^2}, \quad \beta = \frac{2a}{b^2}, \qquad (B3)$$

where Z_N is a normalization. For large N, the density has a semicircle shape with the edges at $\pm 2\sqrt{aN}$; see Fig. 13. Note that to recover the DBM associated with the Gaussian β ensemble with support at equilibrium $[-2, 2]$ one would choose instead $a = \frac{1}{N}$ and $b^2 = \frac{2}{\beta N}$.

The equilibrium solution (B3) for the stationary DBM also provides the x-dependent JPDF $P(\lambda, x)$ for the nonstationary DBM in (B1) with initial condition $\lambda_i(0) = 0$, via the simple scaling $P(\lambda, x) = x^{-N/2}P_0(\lambda/\sqrt{x})$.

Note that the probability that the DBM remains below a barrier $\lambda_1(t) < W\sqrt{t}$ for all $t < x$ decays as $t^{-\beta_c(N,W)}$, where $\beta_c(N, W)$ was calculated in [158]. This thus has a direct translation for the polymer model with $a_i = 0$ in terms of events such that the ground state energy remains larger than $-W\sqrt{t}$ for all $t < x$.

$$\nu_{\rm sc}(\Lambda) = \frac{\sqrt{4Na - \Lambda^2}}{2\pi Na}$$
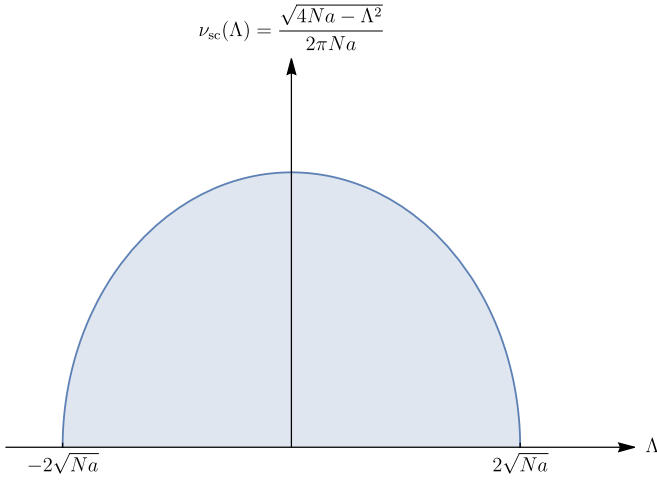
FIG. 13. Semicircular law with support $[-2\sqrt{Na}, 2\sqrt{Na}]$.

2. DBM with initial condition and the deformed Gaussian unitary ensemble

Consider now $\mathcal{M}(x)$ defined as

$$\mathcal{M}(x) = \text{diag}(c_1, \ldots, c_N) + W(x), \tag{B4}$$

and denote now $\lambda_i(x)$ the eigenvalues of $\mathcal{M}(x)$. Then the $\lambda_i(x)$ follow the DBM evolution equation (B1) with initial conditions $\lambda_i(0) = c_i$. For a fixed x, the eigenvalues of $\mathcal{M}(x)$ have the same law as the law at time x of a standard DBM with initial condition $c_1, \ldots c_N$. It is given by (137) with $a_j = c_j/x$.

Note that the DBM for $\beta = 2$ is equivalent to N Brownian walks conditioned not to cross until infinite time [68,158,160]. In the text we have chosen $c_i = x a_i$. That case can equivalently be seen as N Brownian motions all started at zero, conditioned never to intersect, and with drifts a_i in the time interval $[0, x]$. Indeed the formula (137) has a simple interpretation. The determinant containing the a_i's is the expression of the Karlin-McGregor formula [159] for nonintersecting paths on $[0, x]$, which is still valid in the presence of particle-dependent drifts (as can be seen in the simplest way from the path integral formula [161]). The second determinant, i.e., the Vandermonde $\Delta_N(\lambda)$, arises from the nonintersection conditioning for all later times.

In the text we have also introduced in (143) the eigenvalues $\tilde{\lambda}(x) = \lambda(x)/x$ of the matrix $\tilde{M}(x) = \mathcal{M}(x)/x$. The Eq. (143) can be interpreted as a process in x, and its right-hand side as a process in the variable $\tau = N/x$ as

$$\tilde{M}(x) = \text{diag}(a_1, \ldots a_j) + \tilde{W}(\tau), \tag{B5}$$

where we have replaced $\sqrt{\tau}\tilde{V} \to \tilde{W}(\tau)$, where $\tilde{W}(\tau)$ is a Hermitian Brownian motion in τ, with parameters $a = b^2 = 1/N$. Hence the process $\tau \to \tilde{\lambda}(x = N/\tau)$ is a DBM with initial conditions at $\tau = 0$ given by the $\{a_i\}$ [which is the final condition for the process $x \to \tilde{\lambda}(x)$].

3. Resolvent of the Dyson Brownian motion and the Burgers equation

The DBM $\lambda_i(x)$ is often studied using the resolvent $g_x(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \lambda_i(x)}$. Using Ito's rule, the stochastic equation (B1)

leads to [denoting $\lambda_i(x)$ simply as λ_i]

$$dg_x(z) = \frac{1}{N} \sum_i d\lambda_i \, \partial_{\lambda_i} \frac{1}{z - \lambda_i} + \frac{b^2 dx}{2N} \sum_i \partial_{\lambda_i}^2 \frac{1}{z - \lambda_i}. \tag{B6}$$

After standard manipulations, i.e., $\partial_{\lambda_i} \equiv -\partial_z$, $\frac{1}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \frac{1}{z - \lambda_i} = \frac{1}{2}[N g_x(z)^2 + \partial_z g_x(z)]$, one obtains

$$\partial_x g_x(z) = -\frac{1}{2}aN \partial_z g_x(z)^2 + \frac{1}{2}(b^2 - a)\partial_z^2 g_x(z) - \frac{b}{\sqrt{N}} \partial_z \eta_{z,x}, \tag{B7}$$

where $\eta_{z,x} dx = \frac{1}{\sqrt{N}} \sum_i \frac{db_i(x)}{z - \lambda_i}$ is Gaussian of correlator $\text{Cov}(\eta_{z,x} \eta_{z',x'}) = -\delta(x - x') \frac{g_x(z) - g_x(z')}{z - z'}$. Note that for $\beta = 2$ one has $a = b^2$ and the diffusion term is absent. Considering for instance the Gaussian β ensemble with support $[-2\sqrt{x}, 2\sqrt{x}]$, i.e., the choice $a = \frac{1}{N}$ and $b^2 = \frac{2}{\beta N}$, it is clear that in the large-N limit (for fixed β) all terms on the right-hand side of (B7) except the first one are subdominant in N. In the text, to study the $\beta = 2$ DBM of the $\lambda(x)$, we choose instead $a = b = 1$, but the conclusion remains (it is a simple change in scale); i.e., at large N one has $\partial_x g_x(z) = -N g_x(z) \partial_z g_x(z)^2$.

In the text of the paper we study a regime where $x = N\theta$ with $\theta = O(1)$. We can thus define $\lambda_i = N\hat{\lambda}_i$ where $\hat{\lambda}_i = O(1)$. Then if one defines $\hat{g}_\theta(\hat{z}) = \frac{1}{N} \sum_i \frac{1}{\hat{z} - \hat{\lambda}_i}$ we have $g_x(z) = \frac{1}{N}\hat{g}_\theta(\hat{z} = z/N)$ which leads to

$$\partial_\theta \hat{g}_\theta(\hat{z}) = -\hat{g}_\theta(\hat{z}) \partial_{\hat{z}} \hat{g}_\theta(\hat{z}) - \frac{1}{N} \partial_{\hat{z}} \hat{\eta}_{\hat{z},\theta}, \tag{B8}$$

with $\text{Cov}(\hat{\eta}_{\hat{z},\theta} \hat{\eta}_{\hat{z},\theta}) = -\delta(\theta - \theta') \frac{\hat{g}_\theta(\hat{z}) - \hat{g}_\theta(\hat{z}')}{\hat{z} - \hat{z}'}$. The rescaled process $\theta \to \hat{\lambda}_i$ is thus a $\beta = 2$ DBM with $a = b^2 = \frac{1}{N}$.

On the other hand, from the discussion around (B5) we see that the process $\tau \to \tilde{\lambda}_i(x = N/\tau)$ is also a $\beta = 2$ DBM with the same parameter $a = b^2 = \frac{1}{N}$. Its resolvent $G_\tau(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \tilde{\lambda}_i(x)}$ thus also satisfies Eq. (B7) with these parameters; i.e., it satisfies the same equation as (B8) with $g_\theta(\hat{z})$ replaced by $G_\tau(z)$. This is true for any N, and for large N it yields Eq. (147) of the text. This is quite remarkable since the process $\tau \to \tilde{\lambda}_i(x = N/\tau)$ is the "time-inverted" process of the DBM process $\theta \to \hat{\lambda}_i$. The fact that it is also a DBM with the same parameters originates from the following property, which we quote with slightly different notations:

Time inversion of the Dyson Brownian motion. Let $t \to \mu_i(t)$, $t \geq 0$, a DBM (B1) (with parameters a, b), and initial condition $\mu_i(0) = 0$. Then $\tau \to \rho_i(\tau) = \tau \mu_i(1/\tau)$, $\tau \geq 0$, is also a DBM with the same parameters and the same initial condition $\rho_i(0) = 0$.

For $N = 1$ and $\mu_1(t) = b(t)$ a Brownian motion, this property is well known as the time inversion of the Brownian [[95], Sec. IV. 2] (the same property extends to a Bessel process of any index [162]). If $\mu_1(t) = b(t) + at$ is a Brownian with a drift, the property is that $\rho_1(\tau) = \tau \mu_1(1/\tau)$ is a Brownian (without drift) started at a, i.e., $\rho_1(\tau = 0) = a$. One can then use the fact that the $\beta = 2$ DBM is equivalent to a collection of N Brownians conditioned to never cross, and one sees that the property immediately carries through. Note that this property extends to any β. For $N = 2$, since the sum is a Brownian motion (BM), and the difference is an independent Bessel process, it follows from [162].

Note that under the Lamperti transformation discussed above this property of the nonstationary DBM becomes simply the time reversal of the stationary DBM, i.e., that the process $T \to \Lambda_i(-T)$ is an identical copy of the stationary DBM $\Lambda_i(T)$.

4. Convergence to the Airy process

In this section we recall how the process $x \mapsto \lambda_1(x)$ of the largest eigenvalue of the matrix $\mathcal{M}(x)$ studied in the text can be described, as a random function, by the Airy$_2$ process. We will make connection and use the properties of the Airy$_2$ process with respect to the stationary DBM given in Ref. [163]. In Ref. [163] the stationary Hermitian BM for a Hermitian matrix $\mathsf{M}(X)$ was considered,

$$d\mathsf{M}(X) = -\frac{1}{N}\mathsf{M}(X)dX + \frac{1}{\sqrt{N}}d\mathcal{B}(X), \quad (\text{B9})$$

where X is the Lamperti variable (the same as in Appendix B 1), with $d\mathcal{B}_{ij}(X)d\mathcal{B}_{kl}(X) = \frac{dX}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Note that $\mathsf{M}(X) = \frac{\mathcal{M}(x)}{\sqrt{N}}$ where $\mathcal{M}(x)$ obeys Eq. (1.6) in [163] (with $X \equiv t$). One easily sees that the eigenvalues $\Lambda_i(X)$ of $\mathsf{M}(X)$ satisfy (B2) with $c = \frac{1}{N}$, $a = b = 1$. As claimed there, at large N one has the following convergence of the largest eigenvalue [noted $\Lambda_1(X)$] of the DBM to the Airy$_2$ process:

$$\Lambda_1(X + \delta X) \simeq 2\sqrt{N} + N^{-1/6}\mathcal{A}_2\left(\frac{\delta X}{N^{2/3}}\right). \quad (\text{B10})$$

We can now perform the Lamperti mapping in reverse, $\lambda(x) = \sqrt{x}\Lambda(\frac{\ln x}{c})$, and we obtain, upon expanding to leading order in the fluctuations,

$$\lambda_1(x + \delta x) \simeq 2\sqrt{N(x + \delta x)} + \sqrt{x}N^{-1/6}\mathcal{A}_2\left(\frac{\delta x}{cxN^{2/3}}\right)$$

$$= 2\sqrt{N(x + \delta x)} + \sqrt{x}N^{-1/6}\mathcal{A}_2\left(N^{1/3}\frac{\delta x}{2x}\right), \quad (\text{B11})$$

which holds for large N in the scaling region in x, N, δx where the argument of $\mathcal{A}_2$ is of order $O(1)$. Note that if we set $x = N^{1/3}$ and $\delta x = 2s$, this is consistent with [[166], Theorem 5.3]. The result is more general and the convergence to the extended Airy$_2$ point process holds for all eigenvalues.

APPENDIX C: O'CONNELL-YOR POLYMER MODELS

1. Stochastic equation

Let us for completeness give the stochastic evolution equation for the free energy of the OY model. In the OY model (minus) the free energy $F_N(t)$ satisfies the Ito evolution

$$dF_N(t) = e^{F_{N-1}(t)-F_N(t)}dt + a_N dt + dB_N(t). \quad (\text{C1})$$

One can define the partition sum $z_N(t) = e^{-\alpha t}e^{F_N(t)}$ which, using Ito rule, satisfies

$$dz_N(t) = [z_{N-1}(t) + (\tfrac{1}{2} - \alpha)z_N(t)]dt$$
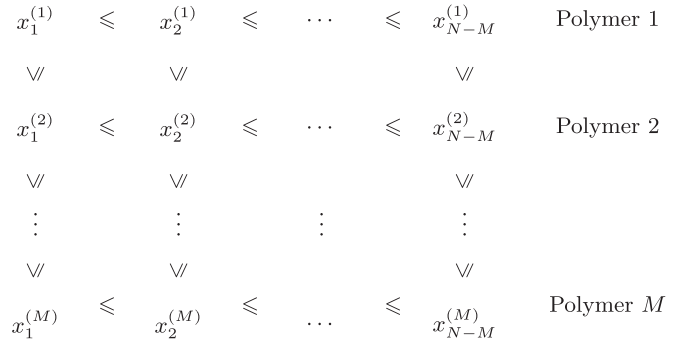$$+ z_N(t)[a_N dt + dB_N(t)], \quad (\text{C2})$$

FIG. 14. Interlacing property of the jumping positions $x_i^{(k)}$.

with $z_0(t) = 0$ [equivalently $dF_1(t) = dB_1(t)$]. The initial condition studied here is $z_N(0) = \delta_{N1}$. Convenient choices for α are $\alpha = 1/2$ or $\alpha = 3/2$.

2. Definition of the many line model

The O'Connell-Yor model with M lines, extended to arbitrary drifts, is defined as follows. One consider M paths $\pi_1, \ldots, \pi_M$ which live only on the columns $j = 1, \ldots, N$. The paths are noncrossing. Their starting points are on columns $1, \ldots, M$ and their end points are on columns $N - M + 1, \ldots, N$. The path π_k jumps from column j to $j + 1$ at $x_j^{(k)}$. There are no backward jumps, i.e., from $j \to j - 1$. One has

$$x_{k-1}^{(k)} = 0 < x_k^{(k)} < \cdots < x_{N-M+k}^{(k)} = x, \quad k = 1, \ldots, M. \quad (\text{C3})$$

The noncrossing conditions on the paths furthermore imply that the paths starting upward must jump before, hence

$$x_j^{(k)} < x_{j-1}^{(k+1)} < x_j^{(k+1)}, \quad j = 1, \ldots, N - M + 1. \quad (\text{C4})$$

The set of paths is now parametrized by $\mathbf{x} = \{x^{(k)}\}_{0 \leqslant j \leqslant N-M+1, 1 \leqslant k \leqslant M}$ and one defines its total energy as

$$E_{N,M}(\mathbf{x}) = \sum_{k=1}^{M} \sum_{j=k}^{N-M+k} \left[B_j\left(x_j^{(k)}, x_{j-1}^{(k)}\right) - a_j\left(x_j^{(k)} - x_{j-1}^{(k)}\right)\right]. \quad (\text{C5})$$

Finally the optimal energy, i.e., the ground state energy for the system of M lines, is

$$\mathcal{E}_{N,M}^0(x) = \min_{\mathbf{x}} E_{N,M}(\mathbf{x}), \quad (\text{C6})$$

where the minimum is over all allowed configurations. Figure 14 shows the interlacing condition obeyed by the $x_j^{(k)}$.

APPENDIX D: BAIK–BEN AROUS–PÉCHÉ KERNEL

It is useful to recall the formula for the BBP kernel mentioned in the text, which describes a rank-m perturbation, in

the critical regime. Let $\mathbf{b} = (b_1 \leqslant \cdots \leqslant b_m) \in \mathbb{R}^m$,

$$K_{\mathrm{BBP},\mathbf{b}}(v, v') = \frac{1}{(2\mathbf{i}\pi)^2} \int\limits_{e^{-2\mathbf{i}\pi/3\infty}}^{e^{2\mathbf{i}\pi/3\infty}} dw \int\limits_{e^{-\pi i/3\infty}}^{e^{\pi i/3\infty}} dz \frac{1}{z-w} \frac{e^{z^3/3-zv}}{e^{w^3/3-wv'}}$$

$$\times \prod_{k=1}^{m} \frac{z - b_k}{w - b_k}. \tag{D1}$$

Note that for $m = 1$, the kernel reads

$$K_{\mathrm{BBP},b_1}(\eta, \eta') = K_{Ai}(\eta, \eta') + Ai(\eta) \int_0^{+\infty} du Ai(\eta' + u) e^{b_1 u}. \tag{D2}$$

APPENDIX E: GROUND STATE ENERGY: CRITICAL BEHAVIOR AT THE LOCALIZATION TRANSITION

Here we give details on the derivation of some of the results displayed in Sec. III A. We recall that $\theta = x/N$ and that the transition from the delocalized phase to localized phase occurs at $\theta = \theta_c$ where θ_c is given by (85). We assume that the density $\rho(a)$ vanishes fast enough, i.e., $k > 1/4$ in (76), so that this transition exists. We are interested in the ground state energy per column $\mu = -\lim_{N\to+\infty} \frac{1}{N} \mathcal{E}_N^0$ and its dependence as a function of θ, $\mu(\theta)$. Let us denote

$$\mu_e(\theta) = (\theta - \theta_c)a_e + \mu(\theta_c), \tag{E1}$$

which, according to (86), is the energy per column in the localized phase for $\theta > \theta_c$, but can be defined from (E1) for any value of θ. We want to calculate the difference $\mu(\theta) - \mu_e(\theta)$ in the delocalized phase, which by definition vanishes in the localized phase (hence provides an order parameter). To this aim we must first calculate z^*, and we focus now on the region near the transition $\theta \to \theta_c^-$. Let us define

$$b = a_e - a, \quad z^* - a_e = \epsilon, \tag{E2}$$

and subtract (82) and (85) and rearrange. We obtain

$$\theta_c - \theta = \epsilon \int_0^{+\infty} db \frac{2b + \epsilon}{b^2(b + \epsilon)^2} \rho(a_e - b), \tag{E3}$$

and we recall that $\rho(a_e - b) \simeq Ab^{2k+1/2}$. We see that if $k > 3/4$ the integral $J_3 = \int_0^{+\infty} db \frac{1}{b^3} \rho(a_e - b)$ converges and one finds $\theta_c - \theta \sim 2J_3\epsilon$ leading to the linear behavior (89). However, for $1/4 < k < 3/4$, J_3 diverges and the critical behavior changes. At small ϵ the leading behavior of the right-hand side of (E3) is obtained by rescaling $b \to \epsilon b$ leading to

$$\theta_c - \theta \simeq AC_k \epsilon^{2k-1/2},$$
$$C_k = \int_0^{+\infty} db \frac{2b + 1}{b^2(b + 1)^2} b^{2k+1/2}$$
$$= \frac{-\pi}{\cos(2\pi k)} \frac{1 + 4k}{2}, \tag{E4}$$

where the integral is convergent for $\frac{1}{4} < k < \frac{3}{4}$. Hence we obtain $z^* - a_e \sim [(\theta_c - \theta)/AC_k]^{\frac{2}{4k-1}}$ as given in (91), and by integration using (87)

$$\mu(\theta) - \mu_e(\theta) \simeq -\frac{4k - 1}{4k + 1} AC_k \left(\frac{\theta_c - \theta}{AC_k}\right)^{\frac{4k+1}{4k-1}}. \tag{E5}$$

APPENDIX F: MORE ON THE VARIATIONAL CALCULATION

1. Single active column in the bulk to leading order: Test of the permutation invariance

Let us consider the case of a single active column in position $j = n$, $\epsilon_n = -a_n < 0$, and all other $a_j = 0$. Let us determine the ground state energy $\mathcal{E}_N^0(x)$ and the occupation length ℓ_j^0 to leading order in N at large N. In that case one must minimize

$$\mathcal{E}_N^0(x) = \min_{x_{n-1} < x_n \in [0,x]} [B_n(x_n) - B_n(x_{n-1}) - a_n(x_n - x_{n-1})$$
$$+ G_n(0, x_{n-1}) + G_{N-n}(x_n, x)], \tag{F1}$$

where $G_m(y, z) = \min_{y = x_1 < x_2 < \cdots < x_{m-1} < x_m = z} \sum_{i=2}^{m} B_i(x_i, x_{i-1})$ represents the ground state energy of a segment of polymer with no active column, and is defined in (19). For the simplest application of the variational calculation we now assume that $n \sim N$ as $N \to \infty$. We use that $G_m(y, z) \simeq -2\sqrt{m(z-y)} + o(m)$ at large m. One can neglect the fluctuations and other subleading terms and obtain to leading order in N

$$\mathcal{E}_N^0(x) \simeq \min_{x_{n-1} < x_n \in [0,x]} [-a_n(x_n - x_{n-1})$$
$$- 2\sqrt{nx_{n-1}} - 2\sqrt{(N-n)x_n}]. \tag{F2}$$

One finds that the minimum is attained for

$$x_{n-1}^0 = \frac{n}{a_n^2}, \quad x_n^0 = x - \frac{N-n}{a_n^2}. \tag{F3}$$

Hence the occupation length is $\ell_n^0 = x_n^0 - x_{n-1}^0 = (x - \frac{N}{a_n^2})_+$, i.e., it is the same (with $a_n \to a_1$) as for the case $n = 1$ studied in the text, see Eq. (22), and the energy

$$\mathcal{E}_N^0(x) \simeq -\left(a_n x + \frac{N}{a_n}\right) \tag{F4}$$

is also the same as for $n = 1$. The above is valid in the localized phase, $a_n > \sqrt{\frac{N}{x}}$. The simple argument is thus in agreement with the general property of invariance by permutation of the columns.

2. Two macroscopic groups of columns

Let us turn to another interesting example where $N_1 = Np$ columns have the same $\epsilon_j = -a_1$ and $N_2 = N(1 - p)$ have $\epsilon_j = -a_2$. Then denoting X the total length in the region a_1 one obtains to leading order in N

$$\mathcal{E}_N^0(x) = -\max_{0 < X < x} [a_1 X - G_{N_1}(0, X) + a_2(x - X)$$
$$- G_{N_2}(X, x)] \tag{F5}$$

$$\simeq -\max_{0 < X < x} [a_1 X + 2\sqrt{pNX} + a_2(x - X)$$
$$+ 2\sqrt{(1 - p)N(x - X)}]. \tag{F6}$$

With no loss of generality we can choose $a_2 = 0$ and $a_1 > 0$, and study

$$\mathcal{E}_N^0(x) \simeq -N \max_{0 < \tilde{X} < \theta} [a_1\tilde{X} + 2\sqrt{p\tilde{X}} + 2\sqrt{(1 - p)(\theta - \tilde{X})}], \tag{F7}$$

where $\tilde{X} = X/N$ and $\theta = x/N$. The minimization equation is

$$\frac{\sqrt{1-p}}{\sqrt{\theta - \tilde{X}}} - \frac{\sqrt{p}}{\sqrt{\tilde{X}}} = a_1. \tag{F8}$$

Since the derivative of the left-hand side is increasing from $]-\infty, +\infty[$ there is always a unique root $\tilde{X} \in [0, \theta]$ which, reported in (F7), gives the ground state energy as a function of a_1 and θ. There is thus no localization phase transition in that case. This is because one is looking here at the largest eigenvalue of the matrix $\mathcal{M}(x)$. However it is known that there can be a phase transition in the middle of the spectrum (see [129] for references), which is thus relevant for the $M = O(N)$ line problem.

3. Delocalized phase: Distribution of the occupation length

Preliminary remark. Let us define the positive random variables v, ω and their scaled versions $\tilde{v}, \tilde{\omega}$ as

$$v = \frac{\sigma^2}{\mu} \tilde{v} = \max_{y \geqslant 0}[\sigma B(y) - \mu y],$$

$$\omega = \frac{\sigma^2}{\mu^2} \tilde{\omega} = \arg\max_{y \geqslant 0}[\sigma B(y) - \mu y]. \tag{F9}$$

The JPDF of v and ω is known. One has [[95], Chapter IV, item 32] the Laplace transform

$$\mathbb{E}[e^{-r\tilde{v} - s\tilde{\omega}}] = \frac{2}{\sqrt{2s+1} + r + 1}, \tag{F10}$$

which leads to the JPDF and the marginals, with $\tilde{v} > 0$, $\tilde{\omega} > 0$,

$$P(\tilde{v}, \tilde{\omega}) = \sqrt{\frac{2}{\pi}} \frac{\tilde{v}}{\tilde{\omega}^{3/2}} e^{-\frac{(\tilde{v}+\tilde{\omega})^2}{2\tilde{\omega}}},$$

$$P(\tilde{\omega}) = \sqrt{\frac{2}{\pi \tilde{\omega}}} e^{-\frac{\tilde{\omega}}{2}} - \mathrm{Erfc}\left(\sqrt{\frac{\tilde{\omega}}{2}}\right), \quad P(\tilde{v}) = 2e^{-2\tilde{v}}, \tag{F11}$$

which gives the formula of the text (35). Note that the PDF of the value of the maximum, v, is simply an exponential distribution of parameter 2μ.

Let us now ask what is the distribution of the occupation length in the delocalized phase. Let us start with the case where all $a_j = 0$ and focus on the first column. The calculation has similarities with the one in Sec. II A 2, but is different. Using the estimate (16) with $\delta x = -\ell_1$, one has [anticipating that $\ell_1 = O(1)$]

$$\ell_1^0 = \arg\min_{\ell_1 \in [0,x]}[B_1(\ell_1) + G_{N-1}(\ell_1, x)] \tag{F12}$$

$$\simeq \arg\min_{\ell_1 \in [0,x]}\left[B_1(\ell_1) - 2\sqrt{N(x - \ell_1)}\right.$$

$$\left. - \frac{\sqrt{x}}{N^{1/6}} \mathcal{A}_2\left(-\frac{N^{1/3}\ell_1}{2x}\right)\right] \tag{F13}$$

$$\simeq \arg\min_{\ell_1 \in [0,x]}\left[B_1(\ell_1) + \frac{1}{\sqrt{\theta}}\ell_1 + \tilde{B}(\ell_1)\right] \tag{F14}$$

$$= \arg\min_{\ell_1 \in [0,x]}\left[\sqrt{2}B(\ell_1) + \frac{1}{\sqrt{\theta}}\ell_1\right], \tag{F15}$$

where in the first equation of the last line the second term comes from the expansion of the second one in the line above, and the Brownian motion $\tilde{B}(\ell_1)$ comes from the usual estimate of the Airy process near zero. Hence using the preliminary remark above we find that the occupation length has the same distribution as was found in the text in (36) in some limit of the critical regime [but here with a different scale $O(1)$],

$$\ell_1^0 \simeq 2\tilde{\omega}\theta = 2\tilde{\omega}\frac{x}{N}, \quad P(\tilde{\omega}) = \sqrt{\frac{2}{\pi \tilde{\omega}}} e^{-\frac{\tilde{\omega}}{2}} - \mathrm{Erfc}\left(\sqrt{\frac{\tilde{\omega}}{2}}\right). \tag{F16}$$

We note that $\mathbb{E}[\tilde{\omega}] = \frac{1}{2}$ and hence

$$\overline{\ell_1^0}^B = \frac{x}{N}, \tag{F17}$$

which is consistent with all columns having the same mean occupation length in that case. Although we will not do it in detail, it is clear that the same variational formula with two Brownians on each side will arise if one looks at any other column, with the same result.

It is more difficult to study the same question in the presence of many active columns, e.g., when all a_j are nonzero, in the delocalized phase. However in the case of a single active column it is easy to obtain the result. One can indeed extend the above calculation to the case where $a_1 > 0$ and all other $a_{j \geqslant 2} = 0$. It amounts to adding the term $-a_1\ell_1$ into (F16) and we see that it simply changes $\frac{1}{\sqrt{\theta}} \to \frac{1}{\sqrt{\theta}} - a_1$ in the last line; this leads to

$$\ell_1^0 \simeq \frac{2\theta}{(1 - a_1\sqrt{\theta})^2} \tilde{\omega}, \tag{F18}$$

which is valid for $\theta < \theta_c = \frac{1}{a_1^2}$, i.e., in the delocalized phase. It shows how the occupation length diverges upon approaching the transition from the delocalized phase side.

If we now get closer to the transition and set $\frac{\theta}{\theta_c} = 1 + \frac{\delta}{N^{1/3}}$, we find

$$\ell_1^0 \simeq \frac{8N^{2/3}}{a_1^2\delta^2} \tilde{\omega}, \tag{F19}$$

which perfectly agrees with the result obtained in (36), using that ω there equals $4\tilde{\omega}$. The two regimes, (i) inside the delocalized phase where $\ell_1^0 = O(1)$, and (ii) inside the critical regime where $\ell_1^0 = O(N^{2/3})$, thus match very smoothly, with the same random variable $\tilde{\omega}$.

Remark. Consider now the overlap $|\psi_1(j)|^2$, whose mean value is related to the average occupation length via Eq. (14). Consider the case where all $a_j = 0$, where $\mathcal{M}(x)$ is a GUE matrix. The PDF of the overlap can be obtained remembering that for the GUE, the eigenvectors are independent from the spectrum and are uniformly distributed on the unit sphere of $\mathbb{C}^N$. As $N \to +\infty$, the real and imaginary parts of their components become independent Gaussians, hence

$$|\psi_1(j)|^2 \overset{\text{in law}}{=} \frac{u_j^2 + v_j^2}{2N} \overset{\text{in law}}{=} \frac{1}{2N}\chi, \quad \chi = \chi_{\beta=2}^2,$$

$$P(\chi) = \frac{1}{2}e^{-\chi/2}\Theta(\chi), \tag{F20}$$

where u_j, v_j are independent standard Gaussians and the normalizing factor is determined from the mean of the constraint

$\sum_{j=1}^{N} |\psi_1(j)|^2 = 1$. Hence at large N the overlap $|\psi_1(j)|^2$ is $1/(2N)$ times a chi-square χ_β^2 distributed random variable with a parameter $\beta = 2$ (see Refs. [164,165] for more details). It is useful to recall the Laplace transform $\mathbb{E}[e^{-z\chi_\beta^2}] =$ $\frac{1}{(1+2z)^{\frac{\beta}{2}}}$ for $z > 0$. Since $\mathbb{E}[\chi] = 2$ one finds that $\overline{|\psi_1(j)|^2}^V = \frac{1}{N}$ which is consistent with (14) and $\overline{\ell_j^0}^B = \frac{x}{N}$. However we see explicitly that the PDFs of $N|\psi_1(j)|^2$ and of ℓ_j^0 in (F16) are different.

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[153] Note that the constant c is missing in Theorem 1.3 (b) in [152]. It can be restored from the derivation in Sec. 2.1 there.

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[155] Note that this does not guarantee the stability to larger scale excitations. The effect of the distribution of interkink distance has also been neglected.

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