Anomalous diffusion in nonlinear transformations of the noisy voter model

Rytis Kazakevičius [*] and Aleksejus Kononovicius [†]
Institute of Theoretical Physics and Astronomy, Vilnius University, Saulėtekio 3, LT-10257 Vilnius, Lithuania

(Received 9 November 2020; accepted 4 March 2021; published 30 March 2021)

Voter models are well known in the interdisciplinary community, yet they have not been studied from the perspective of anomalous diffusion. In this paper, we show that the original voter model exhibits a ballistic regime. Nonlinear transformations of the observation variable and time scale allow us to observe other regimes of anomalous diffusion as well as normal diffusion. We show that numerical simulation results coincide with derived analytical approximations describing the temporal evolution of the raw moments.

DOI: 10.1103/PhysRevE.103.032154

I. INTRODUCTION

There is a broad range of systems where the time dependence of the centered second moment is not linear as in the classical Brownian motion. Such a family of processes is called anomalous diffusion. In one dimension the anomalous diffusion is defined by the power law time dependence of the mean square displacement $\langle(\Delta x)^2\rangle \sim t^\gamma$ [1]. When $\gamma \neq 1$, this time dependence deviates from the linear function of time characteristic for the Brownian motion. If $\gamma < 1$, this phenomenon is called subdiffusion. The occurrence of subdiffusion has been experimentally observed, for example, in the behavior of individual colloidal particles in random potential energy landscapes [2]. Superdiffusion ($1 < \gamma \leqslant 2$) has been observed in vibrated granular media [3].

Recently, in [4–7] it has been suggested that the anomalous diffusion can be a result of the heterogeneous diffusion process, where the diffusion coefficient depends on the position. Such spatially dependent diffusion can be created in thermophoresis experiments using a local variation of the temperature [8,9]. Theoretical aspects of Brownian particle diffusion in an environment with a position dependent temperature have been investigated in [10].

The voter model started as a simple model of spatial competition between two species [11,12], but over the decades it has become one of the most studied models in opinion dynamics [13–15]. As the original model is extremely simple, involving only the recruitment mechanism, it has seen a lot of modifications. Introduction of zealotry [16,17], random transitions [18,19], network topologies [20–25], and nonlinear interactions [26,27] was shown to have effects on the phase behavior of the voter model. The voter model has also seen applications in the modeling of electoral and census data [28–32] and financial markets [33–39]. Anomalous diffusion, to the best of our knowledge, has not been considered in the voter model or its modifications up until recently [40]. In [40] it was shown that anomalous diffusion can be observed by considering individual agent trajectories in a modified voter model, thus providing an explanation for the observations made in the parliamentary attendance data [41]. Here we show that after a nonlinear transformation of the observed variable or the time scale the noisy voter model is able to exhibit subdiffusion, superdiffusion, and localization phenomenon.

The paper is organized as follows. Section II is dedicated to the description of the original noisy voter model. Next, in Sec. III we consider the nonlinear transformation of the observed variable. In Sec. IV we show that nonlinear transformation of the time scale yields a similar process as the nonlinear transformation of the observed variable.

II. NOISY VOTER MODEL

In [11] a simple model of competition between two species was proposed in which a randomly selected member of species replaced another member of the same or other species. In the context of social system modeling, the core mechanism of this model is quite similar to the recruitment mechanism. Actually, this model has first become well known in the opinion dynamics community, in which it is known as the voter model [12,13].

Here we will consider modification of the original voter model, which in the literature is known as the herding model [18] or the noisy voter model [19]. While the original model accounts only for the two particle interactions (one particle adopts the state of the other particle), the noisy voter model also accounts for one particle interactions (independent switching of the state). Assuming that there are two states available and the number of particles, N, is fixed, one can write the following transition rates to describe the model:

$$\pi(X \to X+1) = \pi^{+} = (N - X)(r_1 + hX),$$
$$\pi(X \to X-1) = \pi^{-} = X(r_2 + h[N - X]). \qquad (1)$$

In the above X is the number of particles in the first state, r_i are the independent (one particle interaction) transition rates, while h is the recruitment (two particle interaction) transition rate. Numerically this model can be simulated using the Gillespie method [42]. In our numerical simulations, we use the Gillespie method with a modification, which allows us to quickly obtain the results even with a large N (see Appendix A for more details).

*rytis.kazakevicius@tfai.vu.lt
†aleksejus.kononovicius@tfai.vu.lt; http://kononovicius.lt

As the rates describe one step transitions, the birth-death process framework can be used to derive a stochastic differential equation (SDE) approximating the discrete process [43]. In the thermodynamic limit, $N \to \infty$, for $x = \frac{X}{N}$ the following SDE is obtained:

$$dx = \frac{\pi^+ - \pi^-}{N} dt + \sqrt{\frac{\pi^+ + \pi^-}{N^2}} dW$$
$$\approx h[\varepsilon_1(1-x) - \varepsilon_2 x]dt + \sqrt{2hx(1-x)}dW. \quad (2)$$

In the above $\varepsilon_i = \frac{r_i}{h}$ are the relative independent transition rates and W is a standard Wiener process. Without loss of generality, we can set $h = 1$ as this parameter controls the base event rate of the process as a whole.

The steady state distribution of x is a beta distribution, the probability density function (PDF) of which is given by

$$P_{st}(x) = \frac{\Gamma(\varepsilon_1 + \varepsilon_2)}{\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)} x^{\varepsilon_1 - 1}(1-x)^{\varepsilon_2 - 1}. \quad (3)$$

The beta distribution is observed in a variety of socioeconomic scenarios. Electoral [24,30,31,44], religious adherence [45], and census data [32] seem to follow the beta distribution. While beta distribution is not commonly observed in finance, it was shown [35,46] that x can be transformed into another variable, $y = \frac{x}{1-x}$, which has a meaning of modulating (long-term) return. The new variable y has a power law steady state distribution and its time series has a long-range memory property, both of which are financial market stylized facts [47]. These empirical observations indicate that the noisy voter model might be a reasonable model of opinion dynamics in socioeconomic scenarios.

III. TRANSFORMATION OF THE OBSERVED VARIABLE

Here we will consider a more general form of nonlinear transformation of x described by SDE (2):

$$y = \left(\frac{x}{1-x}\right)^{1/\alpha}, \quad (4)$$

with $\alpha \neq 0$. This transformation, with $\alpha = 1$, was previously derived in the financial market context and was assumed to correspond to the long-term varying component of return [35,46]. The original derivation of y (see Sec. 3 in [35]) assumed that excess demands of two different types of traders grow linearly with the number of traders, but in general this dependence might be nonlinear. In a financial market scenario nonlinear dependence might arise due to the latent liquidity phenomenon [48–50], which arises because some traders hide their intentions from other traders. With $\alpha > 1$, traders are assumed to hide their intentions as the market goes out of equilibrium (excess demand grows). With $\alpha < 1$, traders are assumed to hide their intentions as the market approaches equilibrium (excess demand goes to zero).

Using the Ito lemma for variable transformation, the following SDE for y is obtained:

$$dy = \frac{1}{\alpha^2}[(1 + \alpha - \alpha\varepsilon_2) + (1 - \alpha + \alpha\varepsilon_1)y^{-\alpha}]y(1 + y^\alpha)dt$$
$$+ \sqrt{\frac{2}{\alpha^2}} y^{1-\frac{\alpha}{2}}(1 + y^\alpha)dW. \quad (5)$$

Note that SDE (2) is qualitatively invariant to $x \to 1 - x$ transformation. Namely, the transformed SDE would have exactly the same form as SDE (2) with an exception that the parameters ε_1 and ε_2 exchange their places. This is a rather natural result as such transformation means simply relabeling the states. Yet this has an important consequence on the SDE (5): this SDE is invariant to $y \to \frac{1}{y}$ transformation with the same caveat. This property is particularly useful because SDEs with the same $|\alpha|$ can be rearranged to have the form given by Eq. (5). Thus, without loss of generality further in this section, we consider only the $\alpha > 0$ case.

The steady state PDF of y is given by

$$P_{st}(y) = \alpha \frac{\Gamma(\varepsilon_1 + \varepsilon_2)}{\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)} \times \frac{y^{\alpha\varepsilon_1 - 1}}{(1 + y^\alpha)^{\varepsilon_1 + \varepsilon_2}}. \quad (6)$$

It is easy to see that the PDF of y has power law asymptotic behavior for $y \gg 1$: $P_{st}(y) \sim y^{-\alpha\varepsilon_2 - 1}$. Raw steady state moments of y are given by

$$\langle y^k \rangle_{st} = \frac{\Gamma\left(\varepsilon_1 + \frac{k}{\alpha}\right)\Gamma\left(\varepsilon_2 - \frac{k}{\alpha}\right)}{\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)}. \quad (7)$$

Note that for the kth moment to exist $\alpha\varepsilon_2 > k$ must hold. These describe the moments we expect to get as $t \to \infty$. In what will soon follow we will derive an approximation for the time evolution of the first two moments for finite t.

For convenience sake let us introduce the following notation:

$$\eta_\pm = 1 \pm \frac{\alpha}{2}, \quad \lambda_+ = 1 + \alpha\varepsilon_2, \quad \lambda_- = 1 - \alpha\varepsilon_1,$$
$$\mu = 1 + \frac{\alpha}{2}(\varepsilon_1 - \varepsilon_2), \quad \sigma^2 = \frac{2}{\alpha^2}. \quad (8)$$

Then Eq. (5) can be rearranged into the following form:

$$dy = \sigma^2\left[\mu y + \left(\eta_+ - \frac{\lambda_+}{2}\right)y^{2\eta_+ - 1} + \left(\eta_- - \frac{\lambda_-}{2}\right)y^{2\eta_- - 1}\right]dt$$
$$+ \sigma(y^{\eta_+} + y^{\eta_-})dW. \quad (9)$$

From this form, it is evident that the SDE for y belongs to a general class of SDEs exhibiting $1/f$ noise [51], which is also known to exhibit anomalous diffusion [52]. For the results of [52] to be applicable we need to assume that $y \ll 1$ or $y \gg 1$ and approximate SDE (9) by

$$dy = \sigma^2\left[\mu y + \left(\eta_- - \frac{\lambda_-}{2}\right)y^{2\eta_- - 1}\right]$$
$$\times dt + \sigma y^{\eta_-}dW \quad \text{for } y \ll 1, \quad (10)$$

$$dy = \sigma^2\left[\mu y + \left(\eta_+ - \frac{\lambda_+}{2}\right)y^{2\eta_+ - 1}\right]$$
$$\times dt + \sigma y^{\eta_+}dW_t \quad \text{for } y \gg 1. \quad (11)$$

In [52] it was shown that the μ term influences the moments of y only for times close to the transition to a steady state value. The μ term should not have a noticeable effect on times when the anomalous diffusion is observed, thus let us ignore it and use the following simplified SDE to obtain analytical approximations for the temporal evolution of the first two

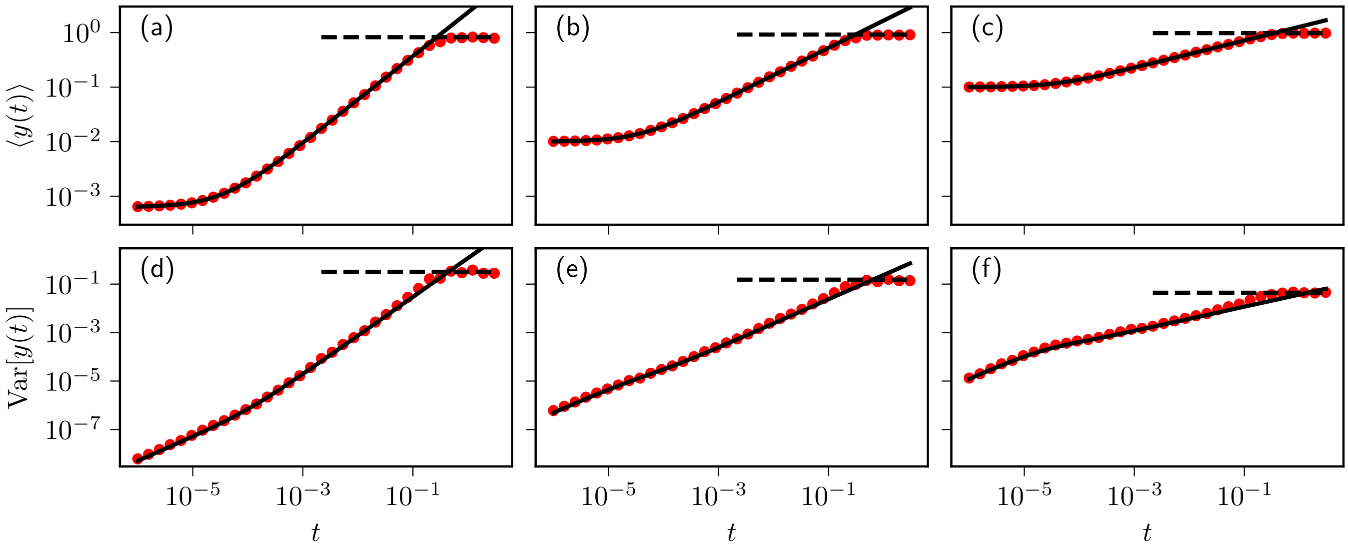

FIG. 1. First two time-dependent moments of the noisy voter model with transformed observable for the $\mu = 0$ and $y_0 \ll 1$ case: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (14) (solid black curve), and stationary value derived from Eq. (7) (dashed black curve). Discrete model parameters: $N = 10^5$, $X_0 = 10$, $\varepsilon_1 = 3$, $\varepsilon_2 = \varepsilon_1 + \frac{2}{\alpha}$ (all cases), $\alpha = 1.25$ (a, d), 2 (b, e), and 4 (c, f). Respective SDE (12) parameters: $y_0 = 9999^{-\frac{1}{\alpha}}$, $\eta = 1 - \frac{\alpha}{2}$, $\lambda = 1 - 3\alpha$, $\sigma^2 = \frac{2}{\alpha^2}$, $\mu = 0$.

moments:

$$dy = \sigma^2\left(\eta - \frac{\lambda}{2}\right)y^{2\eta-1}dt + \sigma y^{\eta}dW. \qquad (12)$$

In the above λ is a parameter describing the additional drift term. SDE (12) has been used to study the influence of external potential on the heterogeneous diffusion process [10]. The cases with $\lambda = 0$ have been studied extensively in [4–7].

The time-dependent PDF of the process given by SDE (12) was obtained in [52]:

$$P(y,t|y_0,0) = \frac{y^{\frac{1-2\eta-\lambda}{2}} y_0^{\frac{1-2\eta+\lambda}{2}}}{|\eta-1|\sigma^2 t}\exp\left(-\frac{y^{2(1-\eta)}+y_0^{2(1-\eta)}}{2(\eta-1)^2\sigma^2 t}\right) \times I_{\frac{\lambda+1-2\eta}{2(\eta-1)}}\left(\frac{y^{(1-\eta)}y_0^{(1-\eta)}}{(\eta-1)^2\sigma^2 t}\right). \qquad (13)$$

Here $I_n(\dots)$ is the modified Bessel function of the first kind of order n. The time-dependent PDF satisfies the initial condition $P(y,t|y_0,0) = \delta(y - y_0)$ (with y_0 being the initial value). To ensure absorption at the boundaries $\frac{\lambda+1-2\eta}{2(\eta-1)} > -1$ must hold. This condition holds whenever $\lambda > 1$ and $\eta > 1$ or $\lambda < 1$ and $\eta < 1$. By looking at Eq. (8) it is evident that one of these conditions is always satisfied as the relative individual transition rates, ε_i, are always positive (due to this one pair of λ and η is always larger than 1, while the other is always smaller). In this regard the steady state moment existence condition is a bit more restrictive.

From Eq. (13) we can calculate the kth time-dependent raw moment of y:

$$\langle y^k(t,y_0)\rangle = \int_0^{\infty} y^k P(y,t|y_0,0)dy = \frac{\Gamma\left(\frac{\lambda-1-k}{2(\eta-1)}\right)}{\Gamma\left(\frac{\lambda-1}{2(\eta-1)}\right)}(2(\eta-1)^2\sigma^2 t)^{\frac{k}{2(1-\eta)}} \times {}_1F_1\left(\frac{k}{2(\eta-1)};\frac{\lambda-1}{2(\eta-1)};-\frac{y_0^{2(1-\eta)}}{2(\eta-1)^2\sigma^2 t}\right). \qquad (14)$$

In the above ${}_1F_1(\dots)$ is the Kummer confluent hypergeometric function. The time-dependent moments will be finite as long as respective steady state moments exist. For the intermediate times [52],

$$\frac{y_0^{2(1-\eta)}}{2(\eta-1)^2\sigma^2} \ll t, \qquad (15)$$

the hypergeometric function of Eq. (14) is approximately equal to 1. Thus, for the intermediate times the time-dependent raw moments will grow as a power law function of time:

$$\langle y^k(t,y_0)\rangle \approx \langle y^k(t)\rangle \sim t^{\gamma}. \qquad (16)$$

Here $\gamma = \frac{k}{2(1-\eta)}$. Therefore, by manipulating η, or the power of transformation α, we can select model parameters so that we would observe subdiffusion, normal diffusion, superdiffusion, or localization. From Eq. (14) follows that the sign of power law exponent γ depends on whether the initial condition satisfies inequality $y_0 \ll 1$ ($\gamma > 0$) or $y_0 \gg 1$ ($\gamma < 0$). In Figs. 1–3 we have set $y_0 \ll 1$ and selected values of α specifically to show the three diffusive behaviors: $\alpha = 1.25$ for superdiffusion ($\gamma = 1.6$), $\alpha = 2$ for normal diffusion ($\gamma = 1$), and $\alpha = 4$ for subdiffusion ($\gamma = 0.5$). In Figs. 4–6 we have set $y_0 \gg 1$ with the same α values to show different inverse power law diffusive behaviors with exponents $\gamma = -1.6, -1,$ and -0.5, respectively, to $\alpha = 1.25, 2,$ and 4. In previous figures observed variance slow return to steady state is called localization and it is also a property of the heterogeneous diffusion process [7]. So, we can conclude that

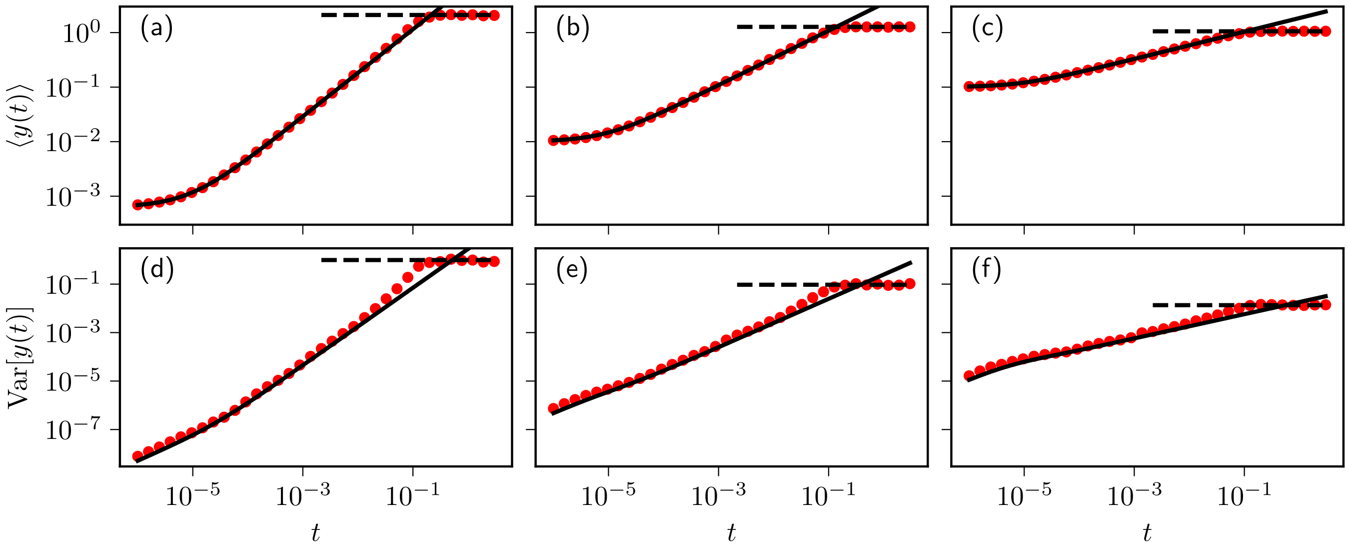

FIG. 2. First two time-dependent moments of the noisy voter model with transformed observable for the $\mu > 0$ and $y_0 \ll 1$ case: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (14) (solid black curve), and stationary value derived from Eq. (7) (dashed black curve). Discrete model parameters: $N = 10^5$, $X_0 = 10$, $\varepsilon_1 = 12$, $\varepsilon_2 = \varepsilon_1 - \frac{8}{\alpha}$ (all cases), $\alpha = 1.25$ (a, d), 2 (b, e), and 4 (c, f). Respective SDE (12) parameters: $y_0 = 9999^{-\frac{1}{\alpha}}$, $\eta = 1 - \frac{\alpha}{2}$, $\lambda = 1 - 12\alpha$, $\sigma^2 = \frac{2}{\alpha^2}$, $\mu = 5$.

nonlinear transformations of the noisy voter model exhibit the same diffusive properties as the heterogeneous diffusion process for intermediate times.

Note that analytical approximations derived from Eq. (14) are rather good up to the larger times where the steady state behavior takes over. The analytical approximations hold reasonably well even for $\mu \neq 0$, though the disagreement between numerical and analytical results around larger times is a bit more apparent in anomalous variance growth [see Figs. 1(d)–1(f) and 2(d)–2(f)].

IV. TRANSFORMATION OF THE TIME SCALE

In [35,46] the variable event time scale was introduced into the SDE of the noisy voter model by the means of the $\tau(x)$ function. Such inclusion of the variable event time scale stems from empirical observation in the financial data [36,53]: absolute returns correlate most with the square of the trading volume. Coincidentally such an assumption allows easy reproduction of the long-range memory phenomenon across a broad range of frequencies [35,36,46]. We include $\tau(x)$ into SDE (2) by replacing the base event rate of the process h by

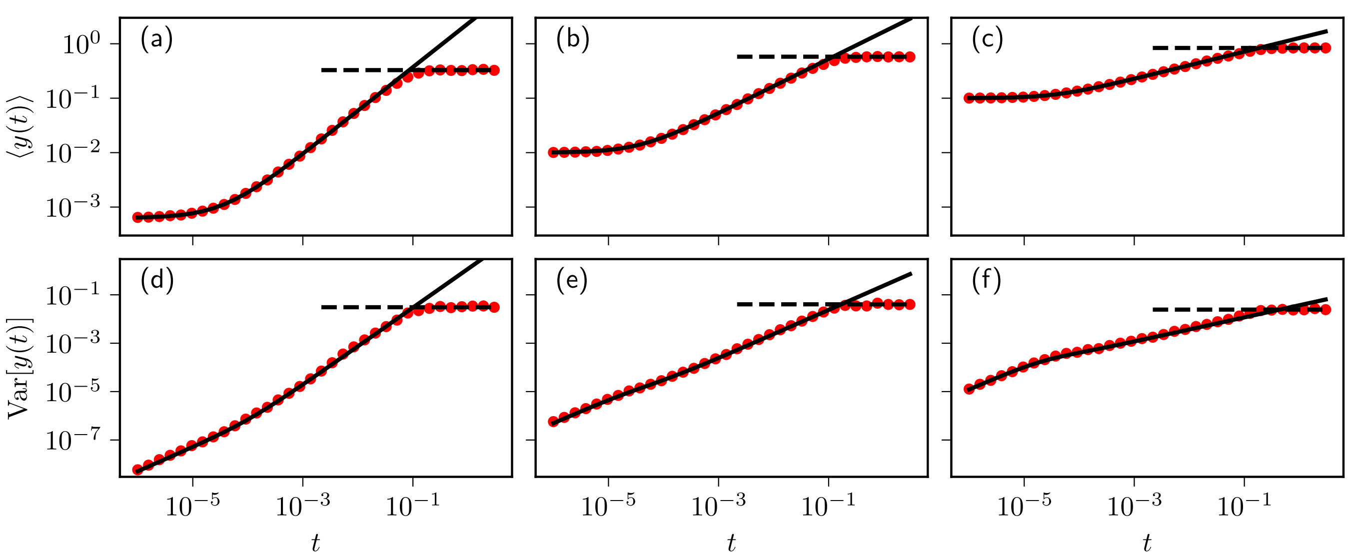

FIG. 3. First two time-dependent moments of the noisy voter model with transformed observable for the $\mu < 0$ and $y_0 \ll 1$ case: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (14) (solid black curve), and stationary value derived from Eq. (7) (dashed black curve). Discrete model parameters: $N = 10^5$, $X_0 = 10$, $\varepsilon_1 = 3$, $\varepsilon_2 = \varepsilon_1 - \frac{8}{\alpha}$ (all cases), $\alpha = 1.25$ (a, d), 2 (b, e), and 4 (c, f). Respective SDE parameters: $y_0 = 9999^{-\frac{1}{\alpha}}$, $\eta = 1 - \frac{\alpha}{2}$, $\lambda = 1 - 3\alpha$, $\sigma^2 = \frac{2}{\alpha^2}$, $\mu = -5$.

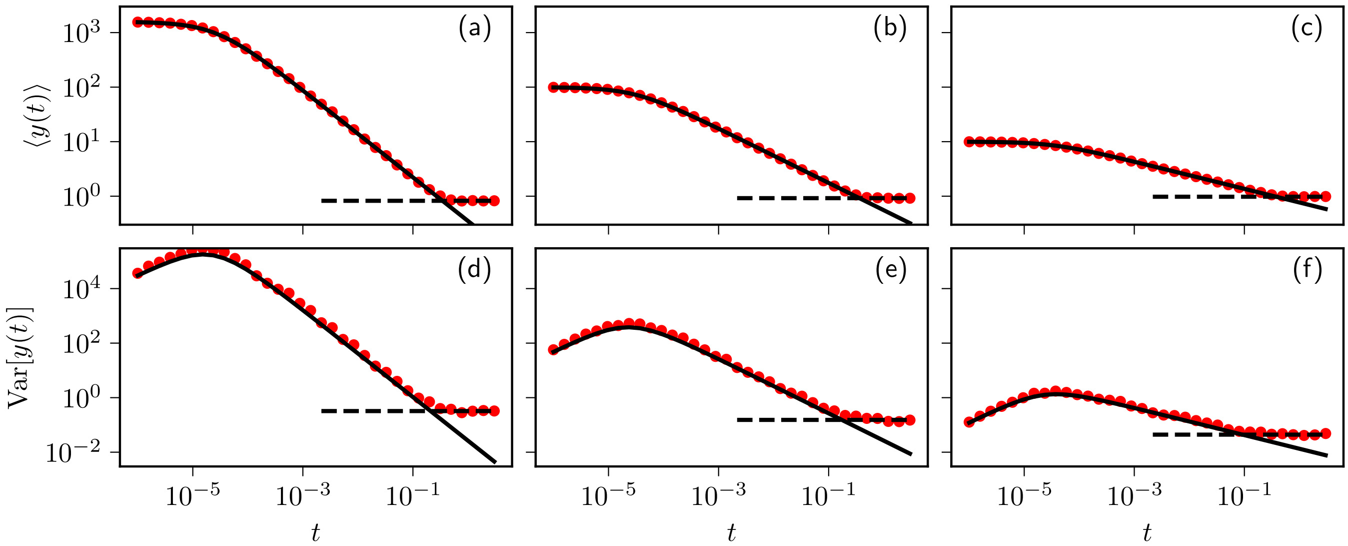

FIG. 4. First two time-dependent moments of the noisy voter model with transformed observable for the $\mu = 0$ and $y_0 \gg 1$ case: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (14) (solid black curve), and stationary value derived from Eq. (7) (dashed black curve). Discrete model parameters: $N = 10^5$, $X_0 = N - 10$, $\varepsilon_1 = 3$, $\varepsilon_2 = \varepsilon_1 + \frac{2}{\alpha}$ (all cases), $\alpha = 1.25$ (a, d), 2 (b, e), and 4 (c, f). Respective SDE (12) parameters: $y_0 = 9999^{\frac{1}{\alpha}}$, $\eta = 1 + \frac{\alpha}{2}$, $\lambda = 3 + 3\alpha$, $\sigma^2 = \frac{2}{\alpha^2}$, $\mu = 0$.

$1/\tau(x)$:

$$dx = [\varepsilon_1(1-x) - \varepsilon_2 x]\frac{dt}{\tau(x)} + \sqrt{\frac{2x(1-x)}{\tau(x)}}dW. \quad (17)$$

Based on the aforementioned empirical observation and the experience reproducing the long-range memory phenomenon, let us consider the following power law form of $\tau(x)$:

$$\tau(x) = x^{1-2\eta}. \quad (18)$$

Then the SDE (17) would take the following form:

$$dx = [\varepsilon_1(1-x) - \varepsilon_2 x]x^{2\eta-1}dt + \sqrt{2x^{2\eta}(1-x)}dW. \quad (19)$$

Note that SDE (19) can be interpreted as noisy voter model interactions occurring in the internal time τ, which is related to the physical time t by the means of transformation $dt = x(\tau)^{1-2\eta}d\tau$ (for more details see Appendix B).

The steady state distribution of x is still a beta distribution, though its parameters are slightly different:

$$P_{\mathrm{st}}(x) = \frac{\Gamma(\varepsilon_\eta + \varepsilon_2)}{\Gamma(\varepsilon_\eta)\Gamma(\varepsilon_2)}x^{\varepsilon_\eta - 1}x^{\varepsilon_2 - 1}, \quad (20)$$

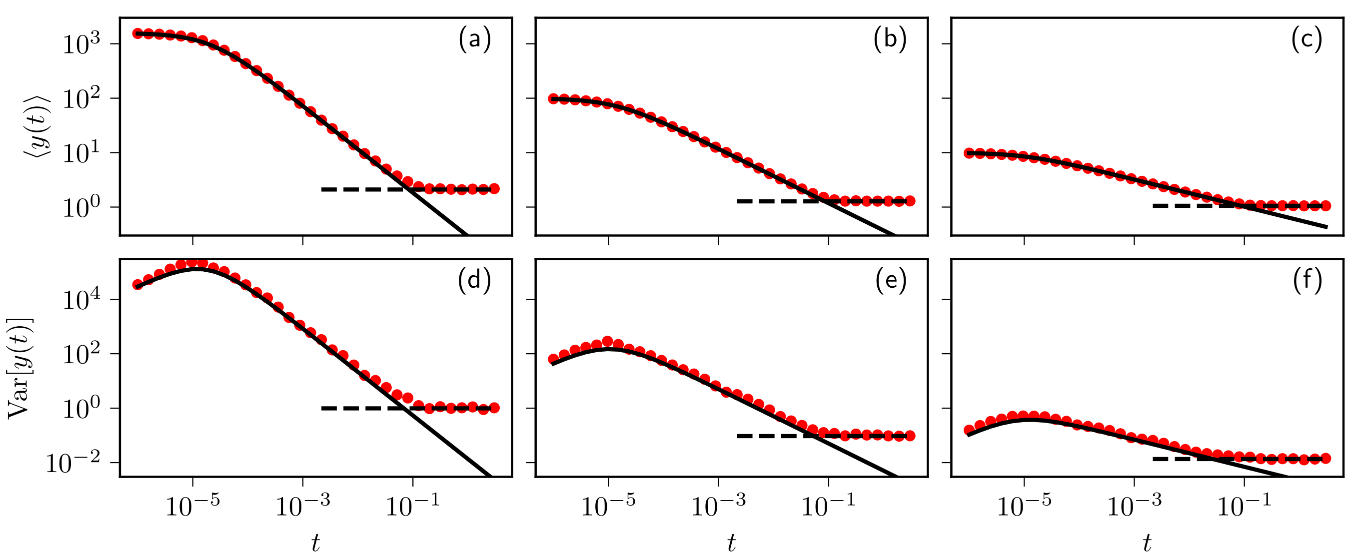

FIG. 5. First two time-dependent moments of the noisy voter model with transformed observable for the $\mu > 0$ and $y_0 \gg 1$ case: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (14) (solid black curve), and stationary value derived from Eq. (7) (dashed black curve). Discrete model parameters: $N = 10^5$, $X_0 = N - 10$, $\varepsilon_1 = 12$, $\varepsilon_2 = \varepsilon_1 - \frac{8}{\alpha}$ (all cases), $\alpha = 1.25$ (a, d), 2 (b, e), and 4 (c, f). Respective SDE (12) parameters: $y_0 = 9999^{\frac{1}{\alpha}}$, $\eta = 1 + \frac{\alpha}{2}$, $\lambda = 3\alpha - 7$, $\sigma^2 = \frac{2}{\alpha^2}$, $\mu = 5$.

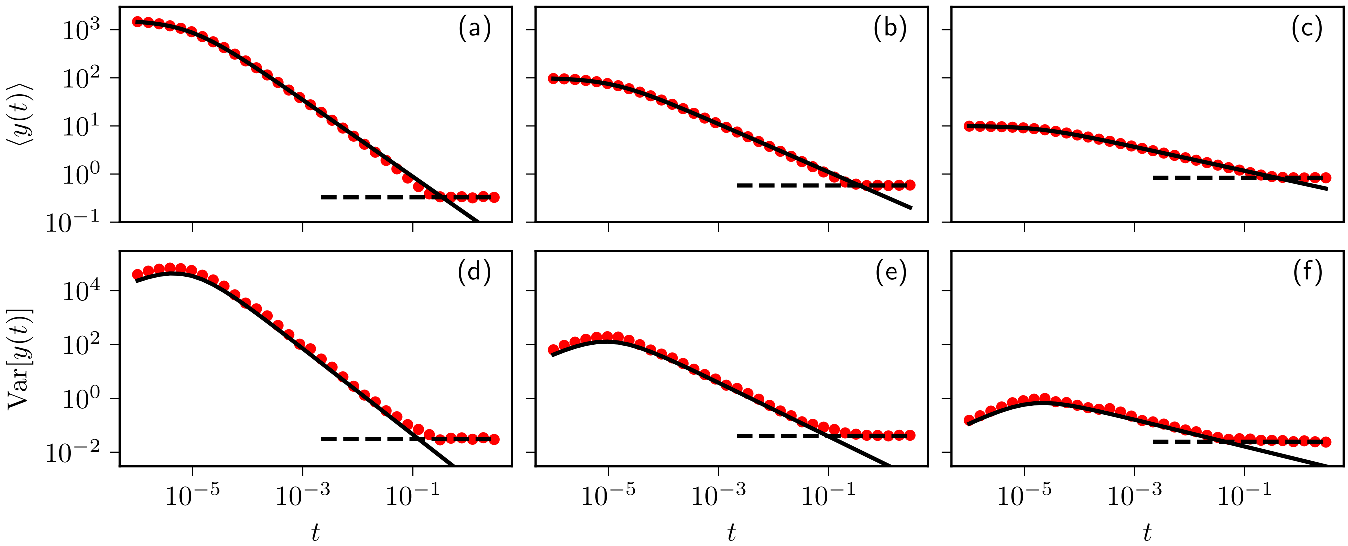

FIG. 6. First two time-dependent moments of the noisy voter model with transformed observable for the $\mu < 0$ and $y_0 \gg 1$ case: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (14) (solid black curve), and stationary value derived from Eq. (7) (dashed black curve). Discrete model parameters: $N = 10^5$, $X_0 = N - 10$, $\varepsilon_1 = 3$, $\varepsilon_2 = \varepsilon_1 - \frac{8}{\alpha}$ (all cases), $\alpha = 1.25$ (a, d), 2 (b, e), and 4 (c, f). Respective SDE (12) parameters: $y_0 = 9999^{\frac{1}{\alpha}}$, $\eta = 1 + \frac{\alpha}{2}$, $\lambda = 3\alpha - 7$, $\sigma^2 = \frac{2}{\alpha^2}$, $\mu = -5$.

where $\varepsilon_\eta = \varepsilon_1 + 1 - 2\eta$. Raw steady state moments of x are given by

$$\langle x^k \rangle_{\mathrm{st}} = \frac{\Gamma(\varepsilon_\eta + \varepsilon_2)\Gamma(\varepsilon_\eta + k)}{\Gamma(\varepsilon_\eta)\Gamma(\varepsilon_\eta + \varepsilon_2 + k)}. \tag{21}$$

Let us introduce the following notation:

$$\lambda = 2\eta - \varepsilon_1, \quad x_c = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}. \tag{22}$$

Using this notation, we can rewrite SDE (19) as

$$dx = 2\left(\eta - \frac{\lambda}{2}\right)x^{2\eta-1}\left[1 - \frac{x}{x_c}\right]dt + x^\eta\sqrt{2(1-x)}dW. \tag{23}$$

If x is sufficiently small, $x \ll x_c$, then we can approximate SDE (19) by

$$dx = 2\left(\eta - \frac{\lambda}{2}\right)x^{2\eta-1}dt + \sqrt{2}x^\eta dW. \tag{24}$$

This SDE has an identical form as SDE (12), which we have considered in a previous section. Therefore, given that the initial value x_0 is much smaller than x_c, we can use the result from the previous section, Eq. (14), to approximate the kth time-dependent raw moment of x as follows:

$$\langle x^k(t, x_0) \rangle = \frac{\Gamma\left(\frac{\lambda-1-k}{2(\eta-1)}\right)}{\Gamma\left(\frac{\lambda-1}{2(\eta-1)}\right)}(4(\eta-1)^2 t)^{\frac{k}{2(1-\eta)}}\,{}_1F_1 \times \left(\frac{k}{2(\eta-1)}; \frac{\lambda-1}{2(\eta-1)}; -\frac{x_0^{2(1-\eta)}}{4(\eta-1)^2 t}\right). \tag{25}$$

As previously, the asymptotic behavior of the moments for intermediate times will be power law function t^γ, with exponent $\gamma = \frac{k}{2(1-\eta)}$. Consequently, for variance ($k = 2$) we will observe superdiffusion when $0 < \eta \leqslant 0.5$ [Fig. 7(d)] and subdiffusion when $\eta < 0$ [Fig. 7(d)]. Note that the unmodified noisy voter model corresponds to $\eta = 0.5$ and the ballistic regime is expected to be observed. In Fig. 7 we have shown that the proposed approximation works well until steady state behavior takes over.

V. CONCLUSIONS

In this paper, we have considered nonlinear transformations of the noisy voter model. We have shown that transformations of both the observation variable and time scale lead to a process, which is able to reproduce various regimes of diffusion: subdiffusion, normal diffusion, and superdiffusion. In the case of observation variable transformation, we have also observed the localization phenomenon for the intermediate times. To the best of our knowledge anomalous diffusion was not previously examined in the context of the voter models.

Nonlinear transformations we consider here are related to the latent liquidity phenomenon observed in the financial markets [48–50]. This phenomenon arises because some traders hide their intentions from their peers. Hence nonlinear transformation parameters could serve as a measure of latent liquidity, but the proper characterization of anomalous diffusion of single trajectories still eludes the scientific community [54].

We have derived analytical approximations for the temporal evolution of the raw moments. Using these approximations one can easily approximate the temporal evolution of mean, variance, and other higher moments. Our derivations were based on the approximation of the time-dependent PDF of the processes, which could be used to derive first passage time distribution or to do Bayesian inference using the noisy voter model. As far as it is known to us, there is no closed form expression for the time-dependent PDF of the noisy voter model. There exists only an expression involving an infinite sum [55].

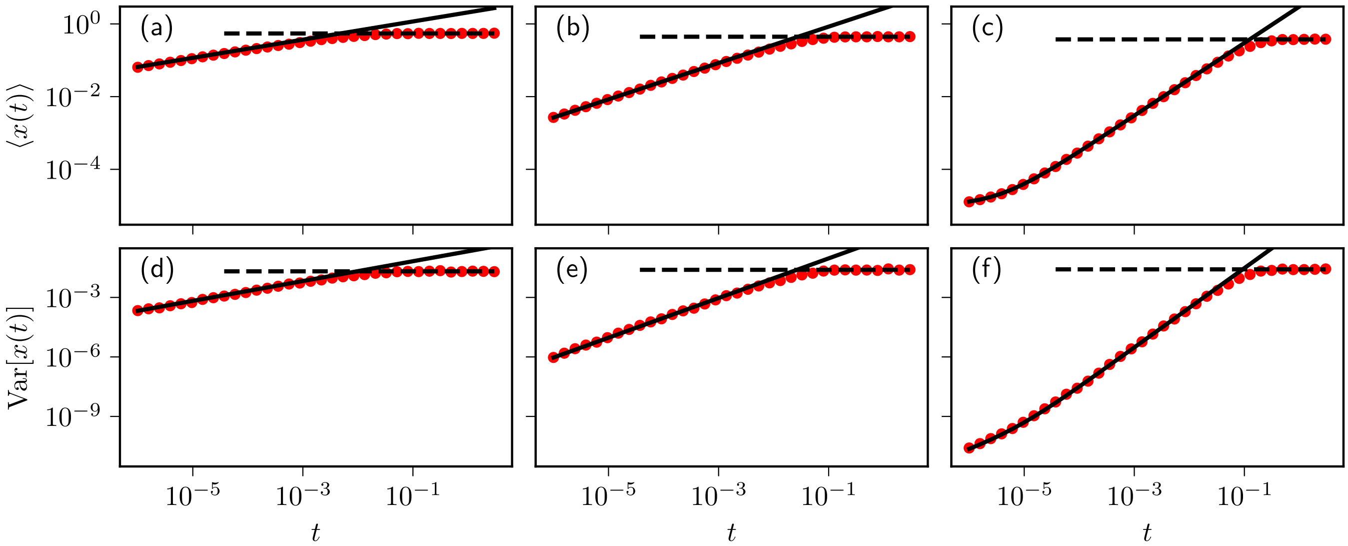

FIG. 7. First two time-dependent moments of the noisy voter model with transformed time scale: numerical results from the discrete model (red circles), analytical approximation derived from Eq. (25) (solid black curve), and stationary value derived from Eq. (21) (dashed black curve). Model parameters: $N = 10^6$, $X_0 = 10$, $\varepsilon_1 = 3$, $\varepsilon_2 = 5$ (all cases), $\eta = -1$ (a, d), 0 (b, e), and 0.5 (c, f). Respective SDE (24) parameters: $\lambda = 2\eta - 3$.

Conceptualization, R.K.; methodology, R.K. and A.K.; software, A.K.; writing—original draft, R.K. and A.K.; writing—review and editing, R.K. and A.K.; visualization, A.K.

APPENDIX A: SIMULATION METHOD

In this section, we briefly discuss the numerical simulation method used in this paper.

Our approach to numerical simulation relies on the Gillespie method [42]. Originally the Gillespie method was developed to generate statistically correct discrete trajectories of chemical reactions involving a finite number of individual molecules. This method relies on the fact that sojourn times between the chemical reactions are exponentially distributed and independent. Therefore instead of considering sojourn times of every individual reaction, we can consider only the shortest sojourn time (of the reaction that will happen first). Due to the nature of the order statistics of an exponential distribution, instead of sampling sojourn times for each individual reaction, we can sample just the shortest sojourn time.

In the noisy voter model there are just two possible types of events, either $X \to X + 1$ or $X - 1$. To model either of the events happening we sample sojourn time from an exponential distribution the rate of which is given by a sum of both event rates, Eq. (1):

$$\Delta t_i \sim \exp(\pi^+ + \pi^-). \tag{A1}$$

After Δt_i time, X is incremented with probability $p^+ = \frac{\pi^+}{\pi^+ + \pi^-}$, otherwise (with complementary probability $p^- = 1 - p^+$) X is decremented. After some event happens the rates are updated and a new sojourn time can be generated to further sample the trajectory.

To get a value of X at time t we simply run the model until the internal clock of the model passes t. As the model runs, we keep both the new and the old value of X. As soon as the internal clock passes t we return the old value of X as $X(t)$.

With small N the numerical simulation is rather quick, but in general its time complexity scales as N^2, therefore for larger N the simulations become too long, especially as we need 10^3 runs for each of the parameter sets. On our hardware simulations with $N = 10^5$ ran for a few hours (using a single parameter set). Yet in certain cases, we needed $N = 10^6$ for the anomalous diffusion to be properly observable and not distorted by the discreteness effects. Simulations with $N = 10^6$ would take a couple of weeks to complete for a single parameter set.

Yet we can improve the speed of simulations by noticing that time complexity is actually $X(N - X)$. In other words, simulation proceeds faster when X is close to zero or N than when X is intermediate. Also, we are mostly interested in the behavior when X is close to these boundaries and we care less about the intermediate values (as they correspond to the steady state behavior). Therefore, we can decrease N as X approaches the intermediate values. Whenever we decrease N we keep the ratio $\frac{X}{N}$ constant. In our implementation, we divide X and N by 10 when the threshold, $X = 100$ or $N - 100$, is reached; we do so until N drops to 10^3. The threshold and divisor were selected arbitrarily as for these values we feel that such resolution is enough for the discreteness effects not to be too obvious.

Note that we have implemented only the downscaling of N, so our modification works when $\varepsilon_i > 1$. This limitation is not important in the scope of this paper, as for the first two steady state moments to exist we need to have $\varepsilon_i > 2$. To allow for any positive ε_i one should also implement the upscaling of N.

The modified implementation runs orders of magnitude faster than the pure Gillespie implementation. For $N = 10^6$ the simulation runs not for weeks, but for a couple of minutes. The improvement in speed would be even more noticeable for even larger N.

To make sure that our implementation produces correct results we have run a single batch of simulations with $N = 10^5$ and compared the results. The differences were too small to be actually visible.

We have made our code, written using PYTHON, C, and SHELL, available from GitHub [56]. Some of our SHELL scripts rely on GNU parallel utility [57], which allows us to run multiple simulations at once.

APPENDIX B: TIME TRANSFORMATION OF THE NOISY VOTER MODEL

In this section, we derive SDE (19), which describes the noisy voter model with variable event time scale, from the time subordinated original noisy voter model described by SDE (2).

In general, a stochastic process in operational time τ is described by

$$dx_\tau = a(x_\tau)d\tau + b(x_\tau)dW_\tau. \tag{B1}$$

Here we assume that the small increments of the physical time t are deterministic and are proportional to the increments of the operational time τ. Therefore, the physical time t is related to the internal time τ via the equation

$$dt_\tau = g(x_\tau)d\tau. \tag{B2}$$

The positive function $g(x)$ is the intensity of random time that depends on the intensity of the stochastic process x. In general, the relationship between the physical time t and the operational time τ can be nondeterministic. Namely, Eq. (B2) can also include a stochastic term [58]. Such a procedure is commonly known as subordination.

To understand Eq. (B2) influence on the stochastic process x it is useful to interpret Eq. (B1) as describing the diffusion of a particle in a nonhomogeneous medium. If so, then the function $g(x)$ models the position of the structures responsible for either slowing down or accelerating the particle. Large values of $g(x)$ correspond to slowing the particle and small $g(x)$ leads to the acceleration of the diffusion.

In [59] it was shown that, corresponding to SDE (B1), the SDE describing the stochastic process x in physical time is

$$dx_t = \frac{a(x_t)}{g(x_t)}d\tau + \frac{b(x_t)}{\sqrt{g(x_t)}}dW_t. \tag{B3}$$

If we assume that the underlying stochastic process in operational time τ is governed by the noisy voter model (2) from (B3) it follows that

$$dx_t = [\varepsilon_1(1-x_t) - \varepsilon_2 x_t]\frac{d\tau}{g(x_t)} + \sqrt{\frac{2x_t(1-x_t)}{g(x_t)}}dW_t. \tag{B4}$$

If we assume that an intensity of random time is power law function $g(x) = x^{1-2\eta}$ we obtain the same SDE (19).

It is straightforward to include $g(x)$ into the noisy voter model event rates, Eq. (1). As this function changes the speed of diffusion, it scales the event rates:

$$\pi^+ = \frac{1}{g\left(\frac{X}{N}\right)}(N-X)(\sigma_1 + hX),$$

$$\pi^- = \frac{1}{g\left(\frac{X}{N}\right)}(\sigma_2 + h[N-X]). \tag{B5}$$

By treating the rates as in Eq. (2), we get SDE identical to SDE (B4). By selecting the power law form of $g(x)$ we obtain the same SDE (19).

[1] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[2] F. Evers, C. Zunke, R. D. L. Hanes, J. Bewerunge, I. Ladadwa, A. Heuer, and S. U. Egelhaaf, Phys. Rev. E **88**, 022125 (2013).
[3] C. Scalliet, A. Gnoli, A. Puglisi, and A. Vulpiani, Phys. Rev. Lett. **114**, 198001 (2015).
[4] A. G. Cherstvy, A. V. Chechkin, and R. Metzler, New J. Phys. **15**, 083039 (2013).
[5] A. G. Cherstvy and R. Metzler, Phys. Chem. Chem. Phys. **15**, 20220 (2013).
[6] A. G. Cherstvy, A. V. Chechkin, and R. Metzler, Soft Matter **10**, 1591 (2014).
[7] A. G. Cherstvy and R. Metzler, Phys. Rev. E **90**, 012134 (2014).
[8] Y. T. Maeda, T. Tlusty, and A. Libchaber, Proc. Natl. Acad. Sci. USA **109**, 17972 (2012).
[9] C. B. Mast, S. Schink, U. Gerland, and D. Braun, Proc. Natl. Acad. Sci. USA **110**, 8030 (2013).
[10] R. Kazakevičius and J. Ruseckas, J. Stat. Mech. (2015) P02021.
[11] P. Clifford and A. Sudbury, A model for spatial conflict, Biometrika **60**, 581 (1973).
[12] T. Liggett, *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes* (Springer, New York, 1999).
[13] C. Castellano, S. Fortunato, and V. Loreto, Statistical physics of social dynamics, Rev. Mod. Phys. **81**, 591 (2009).
[14] A. Jedrzejewski and K. Sznajd-Weron, Statistical physics of opinion formation: Is it a SPOOF? C. R. Phys. **20**, 244 (2019).
[15] S. Redner, Reality inspired voter models: A mini-review, C. R. Phys. **20**, 275 (2019).
[16] M. Mobilia, A. Petersen, and S. Redner, On the role of zealotry in the voter model, J. Stat. Mech. (2007) P08029.
[17] N. Khalil, M. San Miguel, and R. Toral, Zealots in the mean-field noisy voter model, Phys. Rev. E **97**, 012310 (2018).
[18] A. P. Kirman, Ants, rationality and recruitment, Quarterly Journal of Economics **108**, 137 (1993).
[19] L. B. Granovsky and N. Madras, The noisy voter model, Stochastic Processes and their Applications **55**, 23 (1995).
[20] S. Alfarano and M. Milakovic, Network structure and N-dependence in agent-based herding models, J. Econ. Dyn. Control **33**, 78 (2009).
[21] A. Kononovicius and J. Ruseckas, Continuous transition from the extensive to the non-extensive statistics in an agent-based herding model, Eur. Phys. J. B **87**, 169 (2014).
[22] A. Carro, R. Toral, and M. San Miguel, The noisy voter model on complex networks, Sci. Rep. **6**, 24775 (2016).
[23] A. F. Peralta, A. Carro, M. San Miguel, and R. Toral, Analytical and numerical study of the non-linear noisy voter model on complex networks, Chaos **28**, 075516 (2018).

[24] S. Mori, M. Hisakado, and K. Nakayama, Voter model on networks and the multivariate beta distribution, Phys. Rev. E **99**, 052307 (2019).
[25] M. T. Gastner and K. Ishida, Voter model on networks partitioned into two cliques of arbitrary sizes, J. Phys. A **52**, 505701 (2019).
[26] O. Artime, A. Carro, A. F. Peralta, J. J. Ramasco, M. San Miguel, and R. Toral, Herding and idiosyncratic choices: Nonlinearity and aging-induced transitions in the noisy voter model, C. R. Phys. **20**, 262 (2019).
[27] C. Castellano, M. A. Munoz, and R. Pastor-Satorras, The non-linear q-voter model, Phys. Rev. E **80**, 041129 (2009).
[28] J. Fernandez-Gracia, K. Suchecki, J. J. Ramasco, M. San Miguel, and V. M. Eguiluz, Is the voter model a model for voters? Phys. Rev. Lett. **112**, 158701 (2014).
[29] F. Sano, M. Hisakado, and S. Mori, Mean field voter model of election to the house of representatives in Japan, JPS Conf. Proc. **16**, 011016 (2017).
[30] A. Kononovicius, Empirical analysis and agent-based modeling of Lithuanian parliamentary elections, Complexity **2017**, 7354642 (2017).
[31] D. Braha and M. A. M. de Aguiar, Voting contagion: Modeling and analysis of a century of U.S. presidential elections, PLoS ONE **12**, 1 (2017).
[32] A. Kononovicius, Compartmental voter model, J. Stat. Mech. (2019) 103402.
[33] S. Alfarano, T. Lux, and F. Wagner, Estimation of agent-based models: The case of an asymmetric herding model, Comput. Econ. **26**, 19 (2005).
[34] S. Alfarano, T. Lux, and F. Wagner, Time variation of higher moments in a financial market with heterogeneous agents: An analytical approach, J. Econ. Dyn. Control **32**, 101 (2008).
[35] A. Kononovicius and V. Gontis, Agent based reasoning for the non-linear stochastic models of long-range memory, Phys. A **391**, 1309 (2012).
[36] V. Gontis and A. Kononovicius, Consentaneous agent-based and stochastic model of the financial markets, PLoS ONE **9**, e102201 (2014).
[37] R. Franke and F. Westerhoff, Different compositions of animal spirits and their impact on macroeconomic stability, working paper at University of Bamberg, doi:10.13140/RG.2.2.26068.09609 (2018).
[38] A. Kononovicius and J. Ruseckas, Order book model with herding behavior exhibiting long-range memory, Phys. A **525**, 171 (2019).
[39] A. L. M. Vilela, C. Wang, K. P. Nelson, and H. E. Stanley, Majority-vote model for financial markets, Phys. A **515**, 762 (2019).
[40] A. Kononovicius, Noisy voter model for the anomalous diffusion of parliamentary presence, J. Stat. Mech. (2020) 063405.
[41] D. S. Vieira, J. M. E. Riveros, M. Jauregui, and R. S. Mendes, Anomalous diffusion behavior in parliamentary presence, Phys. Rev. E **99**, 042141 (2019).
[42] D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, New York, 1992).
[43] N. G. van Kampen, *Stochastic Process in Physics and Chemistry* (North-Holland, Amsterdam, 2007).
[44] S. E. Rigdon, S. H. Jacobson, W. K. T. Cho, E. C. Sewell, and C. J. Rigdon, A Bayesian prediction model for the U.S. presidential election, American Politics Research **37**, 700 (2009).
[45] M. Ausloos and F. Petroni, Statistical dynamics of religions and adherents, EPL **77**, 38002 (2007).
[46] J. Ruseckas, B. Kaulakys, and V. Gontis, Herding model and $1/f$ noise, EPL **96**, 60007 (2011).
[47] R. Cont, Empirical properties of asset returns: Stylized facts and statistical issues, Quant. Financ. **1**, 1 (2001).
[48] I. Mastromatteo, B. Toth, and J. P. Bouchaud, Agent-based models for latent liquidity and concave price impact, Phys. Rev. E **89**, 042805 (2014).
[49] L. Dall'Amico, A. Fosset, J. Bouchaud, and M. Benzaquen, How does latent liquidity get revealed in the limit order book? J. Stat. Mech. (2019) 013404.
[50] I. Lemhadri, Price impact in a latent order book, Market Microstructure and Liquidity (2020).
[51] J. Ruseckas and B. Kaulakys, Tsallis distributions and 1/f noise from nonlinear stochastic differential equations, Phys. Rev. E **84**, 051125 (2011).
[52] R. Kazakevičius and J. Ruseckas, Influence of external potentials on heterogeneous diffusion processes, Phys. Rev. E **94**, 032109 (2016).
[53] R. Rak, S. Drozdz, J. Kwapien, and P. Oswiecimka, Stock returns versus trading volume: Is the correspondence more general? Acta Phys. Pol. B **44**, 2035 (2013).
[54] G. Muñoz-Gil, G. Volpe, M. A. García-March, R. Metzler, M. Lewenstein, and C. Manzo, The anomalous diffusion challenge: Single trajectory characterisation as a competition, in *Emerging Topics in Artificial Intelligence 2020*, edited by G. Volpe, J. B. Pereira, D. Brunner, and A. Ozcan (SPIE, 2020).
[55] V. Lánská, P. Lánský, and C. E. Smith, Synaptic transmission in a diffusion model for neural activity, J. Theor. Biol. **166**, 393 (1994).
[56] A. Kononovicius, Anomalous diffusion in nonlinear transformations of the noisy voter model, GitHub repository, https://github.com/akononovicius/anomalous-diffusion-in-nonlinear-transformations-of-the-noisy-voter-model.
[57] O. Tange, *GNU Parallel 2018* (Ole Tange, 2018).
[58] R. Kazakevičius and J. Ruseckas, Anomalous diffusion in nonhomogeneous media: Power spectral density of signals generated by time-subordinated nonlinear Langevin equations, Phys. A **438**, 210 (2015).
[59] J. Ruseckas, R. Kazakevičius, and B. Kaulakys, $1/f$ noise from point process and time-subordinated Langevin equations, J. Stat. Mech. (2015) 054022.